Topics on mean-field and McKean–Vlasov backward stochastic differential equations, and the backward propagation of chaos

by

Stefanos Theodorakopoulos

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Department of Mathematics
School of Applied Mathematics and Physical Sciences
National Technical University of Athens

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Advisory committee:

Antonis Papapantoleon	Associate Professor, NTUA	(Supervisor)
	Professor, TU Delft	
Michail Loulakis	Professor, NTUA	(Member)
Michael Kupper	Professor, University of Konstanz	(Member)

Examination committee:

1	Antonis Papapantoleon	Associate Professor, NTUA
		Professor, TU Delft
2	Michail Loulakis	Professor, NTUA
3	Michael Kupper	Professor, University of Konstanz
4	Apostolos Giannopoulos	Professor, NTUA
5	Dimitris Cheliotis	Professor, University of Athens
6	Ludovic Tangpi	Assistant Professor, Princeton University
7	Panayotis G. Michaelides	Professor, NTUA

I would like to dedicate this thesis to my loving parents. I hope I did not disappoint them, much.

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Abstract

Backward propagation of chaos is referring to the phenomenon where the behavior of interactive agents (or particles), described from a system of backward stochastic differential equations (BSDEs), progressively resembles the one as if they where independent, while the number of agents increases to infinity. This thesis aims to study backward propagation of chaos in a setting as general as possible, and also to introduce the notion of stability of backward propagation of chaos. Here stability is understood as the continuity property of backward propagation of chaos with respect to the data sets.

The interaction between the different agents is expressed through their empirical measure. In order to identify the asymptotic behaviour of the mean-field systems of BSDEs we are going to use the McKean–Vlasov BSDE. We consider two instances of backward propagation of chaos, when we have path dependence in the generator and when we have the usual instantaneous dependence. So, we begin by establishing the existence and uniqueness for the solutions of the mean-field system and McKean–Vlasov BSDE, under an appropriate framework. Next, we introduce a new way of proving backward propagation of chaos which allows for asymmetric terminal conditions for the mean-field systems, and general square-integrable martingales with independent increments as drivers. Furthermore, we also show that the known convergence rates for the backward propagation of chaos extend to our general setting. Finally, we introduce the notion of stability of backward propagation of chaos with respect to data sets, and prove its validity under a natural setting, for the usual dependence case. First we establish the uniform convergence of the mean-field systems to the McKean–Vlasov BSDEs with respect to the data sets, and then we naturally expand the known stability of BSDEs to include the McKean–Vlasov BSDEs. Their conjunction gives us the stability result. Because our setting incorporates both continuous and discontinuous cases, it allows the development of numerical schemes for the backward propagation of chaos under $\mathbb{L}^2-$type approximations.

Additionally to the treatment of backward propagation of chaos, in chapter 1 we present a new way for proving the main section theorems of stochastic calculus. Some merits of our approach are that it requires minimum prerequisites, avoids any direct mention of capacities and works directly with the predictable section while measurable section is an immediate corollary, thus also avoids the double work that is hidden in the background of the usual proofs. Last but not least, optional (resp. accessible) section follows from an intuitive approximation argument based on the dichotomy of predictable and totally inaccessible times that further clarifies the relationship between these concepts. The chapter closes with an interesting short proof of a disintegration theorem about measures, provided for completeness reasons.

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Chapter 1

The section theorems and disintegration of measures

This chapter has mainly a pedagogical purpose. First we are going to give new, short, elementary and intuitive proofs of the basic results of the general theory of stochastic processes, *i.e.* the measurable, optional and predictable section theorems. As an immediate application of them we will prove the optional and predictable projection theorems. Finally, we will give a short and relatively elementary proof of an important theorem about the disintegration of measures, that is going to be used many times later on.

1.1 Section theorems

In the following we prove the predictable section theorem directly. We show that all one really needs from analytic set theory is the almost trivial Lemma 1.1.2, which connects Souslin classes with $\sigma-$algebras. Additionally, our approach allows without much trouble to avoid any mention of capacities and rely only to the immediate properties of $\mathbb{P}^*$, see Theorem 1.1.3 and Theorem 1.1.5. Then, measurable projection and section are an immediate corollary of predictable section, see Theorem 1.1.6. Our final insight is that optional section also follows directly from predictable section, as long as a suitable approximation result of optional sets from predictable is used, see Lemma 1.1.11.

1.1.1 Souslin operation

First we will use the Souslin operation to get very efficiently, Theorem 1.1.2, a description for the structure of the Borel sets "from the inside".

Definition 1.1.1. *Let E be an arbitrary nonempty set and $\emptyset \subset \mathcal{E} \subseteq 2^E$.*
A function $A_{\{n_1,\dots,n_k\}} : \bigcup_{k=1}^{\infty} \mathbb{N}^k \longrightarrow \mathcal{E} \cup \{E\}$ is said to be a Souslin scheme *with values in $\mathcal{E}$.*
The Souslin operation *given a Souslin scheme $A_{\{n_1,\dots,n_k\}}$ produces the set $A := \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k}$.*
The collection of all these sets is denoted by $\mathcal{S}(\mathcal{E})$ and called the Souslin class *of $\mathcal{E}$.*
Finally, a Souslin scheme $A_{\{n_1,\dots,n_k\}}$ is called monotone*, when for every $n \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, $A_{n_1,\dots,n_k,n_{k+1}} \subseteq A_{n_1,\dots,n_k}$ for its initial segments.*

Lemma 1.1.2. *Let E be an arbitrary nonempty set and $\emptyset \subset \mathcal{E} \subseteq 2^E$. The following are true.*

i. *$\mathcal{S}(\mathcal{E})$ is closed with respect to countable unions and intersections.*

ii. *If for every $D \in \mathcal{E} \cup \{E\}$ we have $D^c \in \mathcal{S}(\mathcal{E})$, then $\sigma(\mathcal{E}) \subseteq \mathcal{S}(\mathcal{E})$.*

iii. *Let $A_{\{n_1,...,n_k\}}$ be a monotone Souslin scheme with values in $\mathcal{E}$ and $m^* \in \mathbb{N}^{\mathbb{N}}$. If for every $k \in \mathbb{N}$ we define as $S_k := \bigcup_{n_1=1}^{m_1^*} ... \bigcup_{n_k=1}^{m_k^*} A_{n_1,...,n_k}$, then $\bigcap_{k=1}^{\infty} S_k \subseteq \bigcup_{n \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,...,n_k}$.*

Proof. For *i.* let $\{A^m\}_{m\in\mathbb{N}} \subseteq \mathcal{S}(\mathcal{E})$, then $\bigcup_{m=1}^{\infty} A^m = \bigcup_{m=1}^{\infty} \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A^m_{n_1,...,n_k} = \bigcup_{h\in\mathbb{N}^{\mathbb{N}}} \bigcap_{l=1}^{\infty} D_{h_1,...,h_l}$, where for every $l \in \mathbb{N} \setminus \{1\}$ and $(h_1,...,h_l) \in \mathbb{N}^l$ we defined $D_{h_1} := E$ and $D_{h_1,...,h_l} := A^{h_1}_{h_2,...,h_{l-1}}$. Next, for the intersection we will need an increasing per coordinate bijection $\vartheta : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$.[1] Hence, we can decompose $\mathbb{N}$ into a sequence of disjoint sets as, $\mathbb{N} = \bigcup_{m=1}^{\infty} \{\vartheta(k,m) : k \in \mathbb{N}\}$. So, $\bigcap_{m=1}^{\infty} A^m = \bigcap_{m=1}^{\infty} \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A^m_{n_1,...,n_k} = \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A^m_{n_{\vartheta(1,m)},...,n_{\vartheta(k,m)}} = \bigcup_{h\in\mathbb{N}^{\mathbb{N}}} \bigcap_{l=1}^{\infty} D_{h_1,...,h_l}$, where for every $l \in \mathbb{N}$ and $(h_1,...,h_l) \in \mathbb{N}^l$ we defined $D_{h_1,...,h_l} := A^{\pi_2\circ\vartheta^{-1}(l)}_{h_{\vartheta(1,\pi_2\circ\vartheta^{-1}(l))},...,h_{\vartheta(\pi_1\circ\vartheta^{-1}(l),\pi_2\circ\vartheta^{-1}(l))}}$.[2]

For *ii.*, if we define the class $\mathcal{F} := \{D \in \mathcal{S}(\mathcal{E}) : D^c \in \mathcal{S}(\mathcal{E})\}$ we can easily check with the help of *i.* that is a $\sigma-$algebra, which contains $\mathcal{E}$.

Lastly for *iii.*, let $x \in \bigcap_{k=1}^{\infty} S_k$. For every $k \in \mathbb{N}$ and $(n_1,...,n_k) \in \mathbb{N}^k$ such that $n_1 \leq m_1^*,...,n_k \leq m_k^*$ we say that $(n_1,...,n_k)$ is *x-admissible* if and only if it is the initial segment of infinite (countable) other finite sequences $(n_1,...,n_k,\phi_{k+1}...,\phi_\gamma)$ with $n_1 \leq m_1^*,...,\phi_\gamma \leq m_\gamma^*$ such that $x \in A_{n_1,...,n_k,\phi_{k+1}...,\phi_\gamma}$. By the definition of $\{S_k\}_{k\in\mathbb{N}}$ and the finiteness of the set $\{1,...,m_1^*\}$, we easily see that for $k = 1$ we can choose an $n_1^0 \in \{1,...,m_1^*\}$ such that (n_1^0) is *x-admissible*. For $k = 2$, again by the finiteness of $\{1,...,m_2^*\}$, we can choose an $n_2^0 \in \{1,...,m_2^*\}$ such that (n_1^0,n_2^0) is *x-admissible*. Continuing this way, by induction, we find a sequence $\{n_k^0\}_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ the string $(n_1^0,...,n_k^0)$ is *x-admissible*. From the above and the monotonicity of the Souslin scheme we have $x \in \bigcap_{k=1}^{\infty} A_{n_1^0,...,n_k^0}$. □

Because the images of projections generally are not measurable we need to make a trivial extension of $\mathbb{P}$ to the whole power set.

Theorem 1.1.3. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Then the set function $\mathbb{P}^* : 2^\Omega \to \mathbb{R}_+$ where $\mathbb{P}^*(A) := \inf\{\mathbb{P}(E) : A \subseteq E$ and $E \in \mathcal{F}\}$ is an extension of $\mathbb{P}$, monotone and continuous on increasing sequences. Furthermore, $\mathbb{P}^*$ is countable subadditive, and for all $A \in 2^\Omega$ there exists $E_A \in \mathcal{F}$ with $A \subseteq E_A$ and $\mathbb{P}^*(A) = \mathbb{P}(E_A)$.*

Proof. The fact that is monotone and extension is obvious, as is also obvious that for all $A \in 2^\Omega$ exists $E_A \in \mathcal{F}$ with $A \subseteq E_A$ and $\mathbb{P}^*(A) = \mathbb{P}(E_A)$. From these follows immediately the countable subadditivity property. Now, let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(\Omega)$ be an increasing sequence, then there exists a corresponding sequence $\{E_{A_n}\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that for all $n \geq 1$ it is true that $A_n \subseteq E_{A_n}$ and $\mathbb{P}(E_{A_n}) = \mathbb{P}^*(A_n)$. We define the sequence $\{B_n\}_{n\in\mathbb{N}}$ where $B_n := \bigcap_{m=n}^{\infty} E_{A_m}$, this is increasing with $\mathbb{P}(B_n) = \mathbb{P}^*(A_n)$ and $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$. Hence, we have $\mathbb{P}\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \mathbb{P}^*(A_n) \leq \mathbb{P}^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \mathbb{P}\Big(\bigcup_{n=1}^{\infty} B_n\Big)$. □

[1]For example we can pick as $\vartheta(k,m) := \begin{cases} (m-1)^2 + m - 1 + k & \text{if } k \leq m \\ (k-1)^2 + m & \text{if } m < k. \end{cases}$

[2]$\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $\pi_1(k,m) := k$, $\pi_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $\pi_2(k,m) := m$.

1.1.2 Measurable, predictable and optional section

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{P}$ a probability measure on $\mathcal{F}$. We remind that given a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ (without the usual conditions) on $\Omega \times \mathbb{R}_+$ such that $\bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t \subseteq \mathcal{F}$, and assuming that one is familiar with the *optional* (stopping) times denoted by $\mathcal{O}$ and their basic properties, we say that an optional time ρ is *predictable* if and only if there exists a non decreasing sequence of optional times $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_n < \infty, \rho_n \leq \rho$ and $\rho_n < \rho$ on $\{0 < \rho\}$, for all $n \in \mathbb{N}$, with the property $\rho_n \nearrow \rho$. By abusing notation, as we did with the optional times and the optional $\sigma-$algebra, we denote the set of predictable times as $\mathcal{P}$. It will be clear from the context when $\mathcal{P}$ symbolizes the predictable times and when the predictable $\sigma-$algebra of $\Omega \times \mathbb{R}_+$.

Definition 1.1.4. *For every $S \in \Omega \times \mathbb{R}_+$ the* debut *of S is a function $\Omega \to \mathbb{R}_+ \cup \{\infty\}$ which is denoted by $\mathcal{D}[S]$ and is defined as*

$$\mathcal{D}[S](\omega) := \inf\{s \in \mathbb{R} : (\omega, s) \in S\},$$

with the convention $\inf \emptyset = \infty$. *In addition, for every function $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ we denote by $[\![\tau]\!]$ its* graph $\{(\omega, \tau(\omega)) : \omega \in \Omega$ *and* $\tau(\omega) < \infty\}$, *and for every $A \in \mathcal{F}$ we denote by τ_A the function $\Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that $\tau_A := \tau \mathbb{1}_A + \infty \mathbb{1}_{A^c}$.*

Observe that above the predictable times were defined without any reference to a specific probability measure by demanding the convergence to hold for all ω. The following properties of predictable times are basic and their proofs are trivial so they are omitted.

- For all $\tau \in \mathcal{O}$ and $t \in \mathbb{R}_+ \setminus \{0\}$ we have that $\tau + t \in \mathcal{P}$.
- For all $A \in \mathcal{F}_0$ we have $0_A \in \mathcal{P}$.
- For all $\rho_1, \rho_2 \in \mathcal{P}$ we have that $\rho_1 \wedge \rho_2 \in \mathcal{P}$.
- For every sequence $\{\rho_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}$ we have that $\sup_{n\in\mathbb{N}}\{\rho_n\} \in \mathcal{P}$.
- For all $\tau \in \mathcal{O}$ and $\rho \in \mathcal{P}$ we have that $\rho_{\{\rho\leq\tau\}} \in \mathcal{P}$.

At this point we note that when we write $[\![\rho, \tau]\!]$ for $\rho, \tau \in \mathcal{O}$ we mean $[\![\rho, \infty[\![\cap [\![0, \tau]\!]$ but we do not demand $\rho \leq \tau$. Also $[\![\infty, \tau]\!] = \emptyset$, for every $\tau \in \mathcal{O}$.

Theorem 1.1.5 (Predictable Section)**.** *For every predictable set P in $\mathcal{P}$ and $\epsilon > 0$ there exists a predictable time $\rho^{P,\epsilon}$ such that $[\![\rho^{P,\epsilon}]\!] \subseteq P$ and $\mathbb{P}^* (\pi_\Omega(P)) - \mathbb{P}\left(\{\rho^{P,\epsilon} < \infty\}\right) \leq \epsilon$.*

Proof. Let $\mathcal{E}^* := \{\bigcup_{k=1}^n [\![\rho_k, \tau_k]\!] : \text{for } n \in \mathbb{N}, \rho_k \in \mathcal{P} \text{ and } \tau_k \in \mathcal{O} \text{ with } \tau_k < \infty\} \cup \{\emptyset\}$, obviously $\mathcal{E}^*$ is closed with respect to finite unions and intersections. For $A \in \mathcal{E}^*$ let $A = \bigcup_{i=1}^{m_A} [\![\rho_i^A, \tau_i^A]\!]$ according to definition, then $\mathcal{D}[A] = \min_{1\leq i\leq m_A}\{\rho^A_{i\,\{\rho_i^A\leq\tau_i^A\}}\} \in \mathcal{P}$ and $\pi_\Omega(A) = \{\mathcal{D}[A] < \infty\} \in \mathcal{F}$. Next, for every $\rho \in \mathcal{P}$ and $\tau \in \mathcal{O}$ we have $([\![\rho, \tau]\!])^c = [\![0, \rho[\![\cup]\!]\tau, \infty[\![$. Hence, by Lemma 1.1.2, because $[\![0, \rho[\![= \bigcup_{n=1}^\infty [\![\frac{1}{n}, \rho_n]\!] \cup [\![0_{\{\rho>0\}}, 0_{\{\rho>0\}} \wedge 1]\!]$, where $\{\rho_n\}_{n\in\mathbb{N}} \subseteq \mathcal{O}$ such that $\rho_n \nearrow \rho$ as in the definition of a predictable time, and $]\!]\tau, \infty[\![= \bigcup_{n=1}^\infty [\![\tau + \frac{1}{n}, n]\!]$, we get that $([\![\rho, \tau]\!])^c \in \mathcal{S}(\mathcal{E}^*)$. Thus we have $\mathcal{P} := \sigma(\{[\![\tau, \infty[\![: \tau \text{ predictable time}\}) = \sigma(\mathcal{E}^*) \subseteq \mathcal{S}(\mathcal{E}^*)$. So, because $\mathcal{E}^*$ is closed with respect to finite intersections, there exists a monotone Souslin scheme $P_{\{n_1,\dots,n_k\}}$ with values in $\mathcal{E}^*$ such that

$$P = \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} P_{n_1,\dots,n_k}.$$

Fix $\epsilon > 0$, for every $k \in \mathbb{N}$ and $(m_1, \dots, m_k) \in \mathbb{N}^k$ we define

$$M_{m_1,\dots,m_k} := \bigcup_{\{n\in\mathbb{N}^{\mathbb{N}}:\ n_1\le m_1,\dots,n_k\le m_k\}} \bigcap_{i=1}^{\infty} P_{n_1,\dots,n_i}.$$

It is straightforward that if $m_1 \to \infty$ then $M_{m_1} \uparrow P$, and more generally, if $m_{k+1} \to \infty$ then $M_{m_1,\dots,m_k,m_{k+1}} \uparrow M_{m_1,\dots,m_k}$.Thus $\pi_\Omega(M_{m_1,\dots,m_k,m_{k+1}}) \uparrow \pi_\Omega(M_{m_1,\dots,m_k})$ and from Theorem 1.1.3 we can choose a sequence $m^* \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbb{P}^*(\pi_\Omega(M_{m_1^*,\dots,m_k^*})) > \mathbb{P}^*(\pi_\Omega(P)) - \epsilon$, for every $k \in \mathbb{N}$. Next, we define the sequence of sets

$$S_k := \bigcup_{n_1=1}^{m_1^*} \dots \bigcup_{n_k=1}^{m_k^*} P_{n_1,\dots,n_k}.$$

Because $\mathcal{E}^*$ is closed with respect to finite unions we have $\{S_k\}_{k\in\mathbb{N}} \subseteq \mathcal{E}^*$. So, $\mathcal{D}[S_k] \in \mathcal{P}$ and $\pi_\Omega(S_k) = \{\mathcal{D}[S_k] < \infty\} \in \mathcal{F}$, for every $k \in \mathbb{N}$. From the monotonicity of the Souslin scheme $\{S_k\}_{k\in\mathbb{N}}$ is decreasing and $M_{m_1^*,\dots,m_k^*} \subseteq S_k$, it follows that $\mathbb{P}(\pi_\Omega(S_k)) > \mathbb{P}^*(\pi_\Omega(P)) - \epsilon$, for every $k \in \mathbb{N}$. Now, by the definition of $\mathcal{E}^*$, for every $\omega \in \Omega$ the sets $\{\pi_{\mathbb{R}_+}((\{\omega\} \times \mathbb{R}_+) \cap S_k)\}_{k\in\mathbb{N}}$ are compact, hence $\pi_\Omega(\bigcap_{k=1}^\infty S_k) = \bigcap_{k=1}^\infty \pi_\Omega(S_k)$ and $(\omega, \sup_{k\in\mathbb{N}}\{\mathcal{D}[S_k](\omega)\}) \in \bigcap_{k=1}^\infty S_k$ for every $\omega \in \pi_\Omega(\bigcap_{k=1}^\infty S_k)$. So, $\mathcal{D}[\bigcap_{k=1}^\infty S_k] = \sup_{k\in\mathbb{N}}\{\mathcal{D}[S_k]\} \in \mathcal{P}$, $[\![\mathcal{D}[\bigcap_{k=1}^\infty S_k]]\!] \subseteq \bigcap_{k=1}^\infty S_k$ and $\pi_\Omega(\bigcap_{k=1}^\infty S_k) = \{\mathcal{D}[\bigcap_{k=1}^\infty S_k] < \infty\}$. Finally we have $\mathbb{P}(\pi_\Omega(\bigcap_{k=1}^\infty S_k)) = \lim_{k\to\infty} \mathbb{P}(\pi_\Omega(S_k)) \ge \mathbb{P}^*(\pi_\Omega(P)) - \epsilon$. This leads us to define as $\rho^{P,\epsilon}$ the debut of $\bigcap_{k=1}^\infty S_k$, in other words, $\rho^{P,\epsilon} := \mathcal{D}[\bigcap_{k=1}^\infty S_k]$. To finish the proof note that from Lemma 1.1.2 we have $\bigcap_{k=1}^\infty S_k \subseteq P$. □

By choosing to work with the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ where $\mathcal{F}_t = \mathcal{F}$ for all $t \in \mathbb{R}_+$ we get that $\mathcal{P} = \mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$. To see this note that for $A \in \mathcal{F}$ and $s_1, s_2 \in \mathbb{R}_+$ the functions $\tau_1 := s_1\mathbb{1}_A + \infty\mathbb{1}_{A^c}$ and $\tau_2 := s_2\mathbb{1}_A + \infty\mathbb{1}_{A^c}$ are predictable times. So, $A \times [s_1, s_2) = [\![\tau_1, \infty[\![\setminus [\![\tau_2, \infty[\![\in \mathcal{P}$.

Theorem 1.1.6 (**Measurable Section and Projection**). *For every $S \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ there exists an $\mathcal{F}-$measurable function $\tau_S : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ and an $A^S \in \mathcal{F}$ such that $[\![\tau_S]\!] \subseteq S$, $\{\tau_S < \infty\} \subseteq \pi_\Omega(S) \subseteq A^S$ and $\mathbb{P}\left(A^S\right) = \mathbb{P}(\{\tau_S < \infty\}) = \mathbb{P}^*(\pi_\Omega(S))$.*

Proof. The existence of A^S is immediate from the way the extension (see Theorem 1.1.3) was constructed. Now, assuming we work under the aforementioned filtration, *i.e.* $\mathcal{F}_t = \mathcal{F}$ for every $t \in \mathbb{R}_+$, from Theorem 1.1.5 we get for every $k \in \mathbb{N}$ a predictable time $\rho^{S,\frac{1}{k}}$ such that $[\![\rho^{S,\frac{1}{k}}]\!] \subseteq S$ and $\mathbb{P}^*(\pi_\Omega(S)) - \mathbb{P}\left(\{\rho^{S,\frac{1}{k}} < \infty\}\right) \le \frac{1}{k}$. We define $A_n := \bigcup_{k=1}^n\{\rho^{S,\frac{1}{k}} < \infty\}$. Then, we define as

$$\tau_S(\omega) := \begin{cases} \rho^{S,\frac{1}{n}}(\omega) & \text{if } \omega \in A_n \setminus A_{n-1} \\ \infty & \text{if } \omega \in \left(\bigcup_{n=1}^\infty A_n\right)^c \end{cases},$$

with the convention $A_0 := \emptyset$. From the monotonicity of $\mathbb{P}^*$ the proof is complete. □

Remark 1.1.7. *The results of Theorem 1.1.6 generalize immediately to any Borel space X (e.g. Polish) in place of $\mathbb{R}_+$ and $\mathcal{F} \otimes \mathcal{B}(X)$ instead of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$.*

In order to prove the optional section theorem let us return to a random filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$. We will need the next definitions.

Definition 1.1.8. *An optional time τ is called*

i. total inaccessible *if and only if for every predictable time ρ we have $\mathbb{P}(\tau=\rho<\infty)=0$,*

ii. accessible *if and only if there exists a sequence of predictable times $\{\rho_m\}_{m\in\mathbb{N}}$ such that $[\![\tau]\!]\subseteq\bigcup_{m=1}^{\infty}[\![\rho_m]\!]$.*

Before we continue we remind the following, their proofs are trivial and so are omitted.

- For every $\tau\in\mathcal{O}$ and progressively measurable set S the function with graph $[\![\tau]\!]\cap S$ is an optional time.
- Every set $O\in\mathcal{O}$ is progressively measurable.
- For every $\tau\in\mathcal{O}$ there exist τ^1 total inaccessible and τ^2 accessible such that $[\![\tau]\!]=[\![\tau^1]\!]\cup[\![\tau^2]\!]$.

Remark 1.1.9. *From the last bullet it is straight forward that, for every $\tau\in\mathcal{O}$ there exists a total inaccessible time τ^1 and a sequence of predictable times $\{\rho_m\}_{m\in\mathbb{N}}$ such that $[\![\tau]\!]\subseteq[\![\tau^1]\!]\cup\left(\bigcup_{m=1}^{\infty}[\![\rho_m]\!]\right)$.*

Definition 1.1.10. *A set $S\subseteq\Omega\times\mathbb{R}_+$ is called* thin set *if and only if there exists a sequence of optional times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $S=\bigcup_{n=1}^{\infty}[\![\tau_n]\!]$. Specifically, S is called* total inaccessible thin set *if and only if every τ_n can be chosen to be total inaccessible.*

Lemma 1.1.11. *For every optional set $O\in\mathcal{O}$ there exists a predictable set $P\in\mathcal{P}$ such that $O\setminus P$ is a thin set and $P\setminus O$ is a total inaccessible thin set.*

Proof. For the optional $\sigma-$algebra we have that $\mathcal{O}:=\sigma\left(\{[\![\tau,\infty[\![\,:\tau\text{ optional time}\}\right)$. Obviously, for every τ optional time we have $[\![\tau,\infty[\![\setminus]\!]\tau,\infty[\![=[\![\tau]\!]$. So, because the family $\{[\![\tau,\infty[\![\,:\tau\text{ optional time}\}$ is closed with respect to finite intersections, by an easy Dynkin class argument we have that exist a predictable set P' and a thin set S' such that $O\triangle P'$[3] $\subseteq S'$. Because $O\triangle P'$ is progressively measurable it follows that is also a thin set. Next, let $\{\tau_n^1\}_{n\in\mathbb{N}}$ and $\{\rho_m\}_{m\in\mathbb{N}}$ be sequences of total inaccessible and predictable times respectively such that $O\triangle P'\subseteq(\bigcup_{n=1}^{\infty}[\![\tau_n^1]\!])\cup(\bigcup_{m=1}^{\infty}[\![\rho_m]\!])$. It is immediate that the set $P:=P'\setminus(\bigcup_{m=1}^{\infty}[\![\rho_m]\!])$ satisfies what we want. □

Theorem 1.1.12 (Optional Section)**.** *For every optional set $O\in\mathcal{O}$ and $\epsilon>0$ there exists an optional time $\tau^{O,\epsilon}$ such that $[\![\tau^{O,\epsilon}]\!]\subseteq O$ and $\mathbb{P}^*(\pi_\Omega(O))-\mathbb{P}\left(\{\tau^{O,\epsilon}<\infty\}\right)\le\epsilon$.*

Proof. From Lemma 1.1.11 there exists a predictable set P such that $O\setminus P$ is a thin set and $P\setminus O$ is a total inaccessible thin set. From Theorem 1.1.5 there is a predictable stopping time $\rho^{P,\frac{\epsilon}{2}}$ such that $[\![\rho^{P,\frac{\epsilon}{2}}]\!]\subseteq P$ and $\mathbb{P}^*(\pi_\Omega(P))-\mathbb{P}\left(\{\rho^{P,\frac{\epsilon}{2}}<\infty\}\right)\le\frac{\epsilon}{2}$. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be optional times such that $O\setminus P=\bigcup_{n=1}^{\infty}[\![\tau_n]\!]$. We have $\pi_\Omega(O\setminus P)=\bigcup_{n=1}^{\infty}\{\tau_n<\infty\}\in\mathcal{F}$. So, we pick $N\in\mathbb{N}$ large enough such that

[3] $O\triangle P:=(O\setminus P)\cup(P\setminus O)$.

$\mathbb{P}\left(\pi_\Omega(O \setminus P)\right) - \mathbb{P}\left(\bigcup_{n=1}^N \{\tau_n < \infty\}\right) \leq \frac{\epsilon}{2}$. Next, we define as ρ the optional time with graph $[\![\rho^{P,\frac{\epsilon}{2}}]\!] \cap (P \setminus O)^c$ and $\tau := \min_{n \in \{1,\dots N\}}\{\tau_n\}$. We have that $\mathbb{P}^*\left(\pi_\Omega(O \cap P)\right) - \mathbb{P}\left(\{\rho < \infty\}\right) \leq \frac{\epsilon}{2}$, and

$$\begin{aligned}\mathbb{P}^*\left(\pi_\Omega(O)\right) &\leq \mathbb{P}^*\left(\pi_\Omega(O \cap P)\right) + \mathbb{P}\left(\pi_\Omega(O \setminus P) \setminus (\{\rho < \infty\} \cap \{\tau < \infty\})\right) \\ &\leq \mathbb{P}\left(\{\rho < \infty\}\right) + \mathbb{P}\left(\{\tau < \infty\} \setminus \{\rho < \infty\}\right) + \epsilon = \mathbb{P}\left(\{\rho < \infty\} \cup \{\tau < \infty\}\right) + \epsilon.\end{aligned}$$

Hence, we can define as $\tau^{O,\epsilon} := \rho \wedge \tau$. □

Remark 1.1.13. *One can prove the accessible section theorem, i.e. the analog of Theorem 1.1.12 for the accessible $\sigma-$algebra, $\mathcal{A} := \sigma\left(\{[\![\tau, \infty[\![: \tau \text{ accessible time}\}\right)$, and the accessible times exactly the same way as above.*

1.2 Projection theorems

To define the projections we will need a regularization theorem, with respect to the continuity of the paths. So, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ a filtration on $\Omega \times \mathbb{R}_+$ such that $\bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t \subseteq \mathcal{F}$ and $\mathcal{F}_0$ contains all sets of $\mathcal{F}$ with zero probability.

Theorem 1.2.1. *Let X be an adapted, with respect to $\mathbb{F}$, real-valued process. If*

i. X_t *is integrable for every* $t \in \mathbb{R}_+$,

ii. *for every* $t \in \mathbb{R}_+$

$$\sup\left\{\left|\mathbb{E}\left[\sum_{i=1}^n \mathbb{1}_{A_i}\left(X_{s_i} - X_{s_{i-1}}\right)\right]\right| : n \in \mathbb{N}, 0 = s_0 \leq \dots \leq s_n = t, A_i \in \mathcal{F}_{s_{i-1}}\right\} < \infty,$$

then X has a version Y with left and right limits for every $(\omega, t) \in \Omega \times \mathbb{R}_+$. Moreover, there exists a countable set $\mathcal{C} \subseteq \mathbb{R}_+$ such that for every $t \in \mathcal{C}^c$ we have

$$\lim_{s \to t^+} Y_s(\omega) = Y_t(\omega),$$

for every $\omega \in \Omega$.

Remark 1.2.2. *Note that a real-valued function X defined on $\mathbb{R}_+$ has finite variation if and only if for every $t \in \mathbb{R}_+$ we have*

$$\sup\left\{\left|\sum_{i=1}^n a_i\left(X_{s_i} - X_{s_{i-1}}\right)\right| : n \in \mathbb{N}, 0 = s_0 \leq \dots \leq s_n = t, (a_1, \dots, a_n) \in \{0,1\}^n\right\} < \infty.$$

Hence, because the finite variation functions are the archetype of a function with left and right limits everywhere, the conditions of theorem 1.2.1 can be considered natural.

Proof of Theorem 1.2.1. Let $t \in \mathbb{R}_+$ and $a, b \in \mathbb{Q}$ with $a < b$. Furthermore, let $S := \{t_0, \dots, t_m, \dots\}$ be a countable and dense subset of $\mathbb{R}_+$ such that $t_0 = 0$. First we are going to show that the number of upcrossings of X between a and b in the interval $[0, t]$ with respect to S, denoted by $U_{[a,b]}^{S,t}(X)$, is finite.

And second, that the sequence of numbers $\{X_s(\omega)\}_{s\in S\cap[0,t]}$ is bounded $\mathbb{P}-$a.e. So, start with the first n members of S (in respect to the ordering of its elements) and t, $\{t_0,...,t_{n-1},t\}\subseteq S\cup\{t\}$ and reorder it if need be such that $\{t_0,...,t_{n-1},t\}=\{s_0,...,s_n\}$ and $0=s_0\leq s_1\leq...\leq s_n=t$. Define

$$A_1:=\{X_{s_0}<a\} \text{ and } A_i:=\left(A_{i-1}\cap\{X_{s_{i-1}}\leq b\}\right)\cup\left(A_{i-1}^c\cap\{X_{s_{i-1}}<a\}\right), i\in\{2,...,n\}.$$

Then, define

$$U_{[a,b]}^{\{t_0,...,t_n\},t}(X):=\sum_{i=2}^{n}\mathbb{1}_{A_{i-1}\cap A_i^c}+\mathbb{1}_{A_n\cap\{X_t>b\}}.$$

We have for every $n_1,n_2\in\mathbb{N}$ with $n_1\leq n_2$ that

$$U_{[a,b]}^{\{t_0,...,t_{n_1}\},t}(X)\leq U_{[a,b]}^{\{t_0,...,t_{n_2}\},t}(X)$$

and

$$U_{[a,b]}^{S,t}(X)=\lim_{n\to\infty}U_{[a,b]}^{\{t_0,...,t_n\},t}(X).$$

Next, observe that

$$(b-a)\,U_{[a,b]}^{\{t_0,...,t_n\},t}(X)\leq\sum_{i=1}^{n}\mathbb{1}_{A_i}\left(X_{s_i}-X_{s_{i-1}}\right)+(a-X_t)\vee 0,$$

hence

$$\mathbb{E}\left[U_{[a,b]}^{\{t_0,...,t_n\},t}(X)\right]\leq\frac{1}{b-a}\left(\mathbb{E}\left[\sum_{i=1}^{n}\mathbb{1}_{A_i}\left(X_{s_i}-X_{s_{i-1}}\right)\right]+|a|+\mathbb{E}\left[|X_t|\right]\right).$$

By the theorem assumptions and the monotone convergence theorem we get that $\mathbb{E}\left[U_{[a,b]}^{S,t}(X)\right]<\infty$, in other words $\mathbb{P}\left(\left\{U_{[a,b]}^{S,t}(X)=\infty\right\}\right)=0$.

Now, let $N\in\mathbb{N}$ and define

$$B_1:=\{X_{s_0}<N\} \text{ and } B_i:=B_{i-1}\cup\left(B_{i-1}^c\cap\{X_{s_{i-1}}>N\}\right), i\in\{2,...,n\}.$$

We get that

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{1}_{B_i}\left(X_{s_i}-X_{s_{i-1}}\right)&\leq\mathbb{1}_{\left\{\sup_{i\in\{0,...,n\}}\left\{X_{t_i}\right\}>N\right\}}(X_t-N)\\&\leq|X_t|-N\mathbb{1}_{\left\{\sup_{i\in\{0,...,n\}}\left\{X_{t_i}\right\}>N\right\}}.\end{aligned}$$

So, by taking the expectations we have

$$\mathbb{P}\left(\left\{\sup_{i\in\{0,...,n\}}\{X_{s_i}\}>N\right\}\right)\leq\frac{1}{N}\left(\mathbb{E}\left[\sum_{i=1}^{n}\mathbb{1}_{B_i}\left(X_{s_i}-X_{s_{i-1}}\right)\right]+\mathbb{E}\left[|X_t|\right]\right).$$

By applying the same procedure to the process $-X$ we also get that

$$\mathbb{P}\left(\left\{\inf_{i\in\{0,\dots,n\}}\{X_{s_i}\}<-N\right\}\right)\le\frac{1}{N}\left(-\mathbb{E}\left[\sum_{i=1}^{n}\mathbb{1}_{B_i'}\left(X_{s_i}-X_{s_{i-1}}\right)\right]+\mathbb{E}\left[|X_t|\right]\right)$$

and finally from the theorem assumptions and the above that

$$\mathbb{P}\left(\left\{\sup_{s\in S\cap[0,t]}\{|X_s|\}=\infty\right\}\right)=0.$$

So, we can deduce that there exists a set $D\in\mathcal{F}$ with $\mathbb{P}(D)=0$ such that, for every $\omega\in D^c$ and for every $t\in\mathbb{R}_+$ the left and right limits

$$\lim_{\substack{s\to t^+\\ s\in S}}X_s(\omega)\quad\text{and}\quad\lim_{\substack{s\to t^-\\ s\in S}}X_s(\omega)$$

exist. Hence, if we define the process

$$\widetilde{X}_t(\omega):=\lim_{\substack{s\to t^+\\ s\in S}}X_s(\omega)=\inf_{n\in\mathbb{N}}\left\{\sup_{s\in S\cap\left(t,t+\frac{1}{n}\right)}\{X_s(\omega)\mathbb{1}_{D^c}(\omega)\}\right\}$$

we see immediately that is right continuous with left limits.

At this point it is important to note the following obvious fact. Let $\widehat{S}\supseteq S$ be an countable, dense subset of $\mathbb{R}_+$ and we define similarly to before

$$\widehat{X}_t(\omega):=\lim_{\substack{s\to t^+\\ s\in\widehat{S}}}X_s(\omega)=\inf_{n\in\mathbb{N}}\left\{\sup_{s\in\widehat{S}\cap\left(t,t+\frac{1}{n}\right)}\left\{X_s(\omega)\mathbb{1}_{\widehat{D}^c}(\omega)\right\}\right\},$$

then we will have for every $t\in\mathbb{R}_+$ that

$$\mathbb{P}\left(\left\{\widetilde{X}_t\neq\widehat{X}_t\right\}\right)=0.$$

Now we will investigate at which points we have $\mathbb{P}\left(\left\{\widetilde{X}_t\neq X_t\right\}\right)>0$. Of course if for a point $t\in\mathbb{R}_+$ the above relation is true, then there will be $n_t\in\mathbb{N}$ such that $\mathbb{P}\left(\left\{\left|\widetilde{X}_t-X_t\right|>\frac{1}{n_t}\right\}\right)>\frac{1}{n_t}$. We will show that for every $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$ we have $\mathbb{P}\left(\left\{\left|\widetilde{X}_s-X_s\right|>\frac{1}{n}\right\}\right)>\frac{1}{n}$ only for finitely many points $s\in[0,t]$. Assume instead that there exist $n_0\in\mathbb{N}$, $t_0\in\mathbb{R}_+$ and an increasing or decreasing sequence $\{s_m\}_{m\in\mathbb{N}}\subseteq[0,t]$ with $\mathbb{P}\left(\left\{\left|\widetilde{X}_{s_m}-X_{s_m}\right|>\frac{1}{n_0}\right\}\right)>\frac{1}{n_0}$ for every $m\in\mathbb{N}$ and $s_m\to s_\infty$. If we define $\widehat{S}:=S\cup(\{s_m\}_{m\in\mathbb{N}})$ and $\widehat{X}$ as above we will get that

$$\lim_{m\to\infty}X_{s_m}=\lim_{m\to\infty}\widehat{X}_{s_m}=\lim_{m\to\infty}\widetilde{X}_{s_m},\ \ \mathbb{P}-a.e.$$

This is a contradiction, so indeed, for every $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$ we have that $\mathbb{P}\left(\left\{\left|\widetilde{X}_s-X_s\right|>\frac{1}{n}\right\}\right)>\frac{1}{n}$ only for finitely many points $s\in[0,t]$. In other words, we have $\mathbb{P}\left(\left\{\widetilde{X}_t\neq X_t\right\}\right)>0$ only for a countable subset of $\mathbb{R}_+$, let's denote this set with $\{s_m\}_{m\in\mathbb{N}}\subseteq\mathbb{R}_+$. Hence, a final note is that the set where $\mathbb{P}\left(\left\{\widetilde{X}_t\neq X_t\right\}\right)=0$

is a dense subset of $\mathbb{R}_+$. To finish the proof define $\mathcal{C} := S \cup (\{s_m\}_{m\in\mathbb{N}})$ and

$$Y_t(\omega) := \begin{cases} \widehat{X}_t(\omega), & t \in \mathbb{R}_+ \setminus \{s_m\}_{m\in\mathbb{N}} \\ X_t(\omega), & t \in \{s_m\}_{m\in\mathbb{N}}, \end{cases}$$

where $\widehat{X}$ the process defined with respect to $\widehat{S} := \mathcal{C}$ as before. □

Corollary 1.2.3. *Let $\xi \in \mathbb{L}^1(\mathcal{F})$. If the filtration $\mathbb{F}$ is right continuous, then the martingale $X_t := \mathbb{E}[\xi|\mathcal{F}_t]$ has a càdlàg version.*

Proof. It is obvious that the process X satisfies the conditions of Theorem 1.2.1. Hence, there exists a version Y of X that has left and right limits everywhere. By hypothesis we have $\mathcal{F}_{t^+} = \mathcal{F}_t$ for every $t \in \mathbb{R}_+$, so from the dominated convergence theorem for the conditional expectation we get that

$$Y_{t^+} := \lim_{s\to t^+} Y_s = \mathbb{E}[\xi|\mathcal{F}_{t^+}] = \mathbb{E}[\xi|\mathcal{F}_t] = Y_t, \;\; \mathbb{P} - a.e.$$

Because from theorem 1.2.1 the set of times where Y may not be right continuous is countable the proof is complete. □

Now we are ready to state and prove the projection theorems. For presentation reasons in the next results we are going to assume that $\mathbb{F}$ is a right continuous filtration, although that is not a necessary condition for the results to hold.

Theorem 1.2.4 (**Optional projection**). *Let X be an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)-$measurable process such that*

$$\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_\tau] < \infty, \;\; \mathbb{P} - a.e.,$$

for every optional time τ. Then, there exists a unique, up to indistinguishability, optional process, denoted by ${}^{\circ}X$ and referred to as the optional projection of X, such that for every optional time τ we have

$$\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_\tau] = {}^{\circ}X_\tau \mathbb{1}_{\{\tau<\infty\}}, \;\; \mathbb{P} - a.e.$$

Proof. The uniqueness of the optional projection is secured from Theorem 1.1.12. So, from now on we will focus on its construction. From Doob's optional sampling theorem we get that for every $X = \xi$, with $\xi \in \mathbb{L}^1(\mathcal{F})$, we can define as ${}^{\circ}X_t := \mathbb{E}[\xi|\mathcal{F}_t]$, where we chose the càdlàg version of the above martingale, provided from Corollary 1.2.3. Next, because for every $s \in \mathbb{R}_+$ and optional time τ we have that $\{\tau \leq s\} \in \mathcal{F}_\tau$ we immediately get that for $X := \mathbb{1}_{[0,s]}\xi$ we can define as ${}^{\circ}X_t := \mathbb{1}_{[0,s]}\mathbb{E}[\xi|\mathcal{F}_t]$. The set $\{[0,s] \times A : s \in \mathbb{R}_+, A \in \mathcal{F}\}$ is a $\pi-$system which generates $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$, so from an easy Dynkin-class argument we get that we can define the optional projection for every process $X := \mathbb{1}_S$, where $S \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. Then, from an easy monotone-class argument we can extend the set of processes X for which we can define the projection to that of all non-negative $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)-$measurable processes. From this is obvious that if a process X satisfies the condition of the theorem, then we can define as ${}^{\circ}X := {}^{\circ}X_+ - {}^{\circ}X_-$. □

Theorem 1.2.5 (**Predictable projection**). *Let X be a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)-$measurable process such that*

$$\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}] < \infty, \;\; \mathbb{P} - a.e.,$$

for every predictable time τ. Then, there exists a unique, up to indistinguishability, predictable process, denoted with ${}^{p}X$ and referred to as the predictable projection of X, such that for every predictable time τ we have

$$\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}] = {}^{p}X_\tau \mathbb{1}_{\{\tau<\infty\}},\ \ \mathbb{P}-a.e.$$

Proof. Exactly the same with that of Theorem 1.2.4, under the appropriate modifications, with the only exception being that we define for $\xi \in \mathbb{L}^1(\mathcal{F})$ and $X := \xi$ its predictable projection as ${}^{p}X_t := {}^{\circ}X_{t-}$. ☐

1.3 Disintegration of measures

For the proof of corollary 1.3.6 we are going to follow Cohn [13, Section 8.6].

Definition 1.3.1. *Let $(E, \mathcal{E})$ be a measurable space. We say that $\mathcal{E}$ is* countably generated *if and only if there exists a countable family $\mathcal{C} \subseteq \mathcal{P}(E)$ such that $\mathcal{E} = \sigma(\mathcal{C})$. Furthermore, we say that $\mathcal{C}$* separates *E if and only if for every pair x, y of distinct points of E there exists $A \in \mathcal{C}$ such that $x \in A$ and $y \in A^c$. Then, if there exists a family $\mathcal{C}$ which generates $\mathcal{E}$ and seperates E we say that $(E, \mathcal{E})$ is separated and countably generated.*

Remark 1.3.2. *For every second countable Hausdorff space X the Borel $\sigma-$algebra $\mathcal{B}(X)$ is separated and countably generated.*

Definition 1.3.3. *Let $(E^1, \mathcal{E}^1), (E^2, \mathcal{E}^2)$ be two measurable spaces. A function $F : E^1 \longrightarrow E^2$ is said to be a* measurable bijection *if and only if it is injective, surjective and for every $(A^1, A^2) \in \mathcal{E}^1 \times \mathcal{E}^2$ we have $(F(A^1), F(A^2)) \in \mathcal{E}^2 \times \mathcal{E}^1$. Moreover, if E^1, E^2 are Polish spaces and $\mathcal{E}^1 = \mathcal{B}(E^1), \mathcal{E}^2 = \mathcal{B}(E^2)$, then we say that F is a* Borel-isomorphism *and $(E^1, \mathcal{E}^1), (E^2, \mathcal{E}^2)$ are* Borel-isomorphic.

We give the following elementary construction that reveals the usefulness of the above definitions.

Theorem 1.3.4. *Let $(E, \mathcal{E})$ be separated and countably generated. Then, there exists a subset A of $\{0,1\}^{\mathbb{N}}$ such that there exists a measurable bijection between $(E, \mathcal{E})$ and $(A, \mathcal{B}(A))$.*

Proof. Let $\mathcal{C} := \{C_n\}_{n\in\mathbb{N}}$ be a sequence that satisfies the properties of Definition 1.3.1 for $\mathcal{E}$. We define the function $F : E \longrightarrow \{0,1\}^{\mathbb{N}}$ where for every $n \in \mathbb{N}$ we have that,

$$(F(x))_n = 0 \ \text{ if } \ x \in C_n^c \quad \text{or} \quad (F(x))_n = 1 \ \text{ if } \ x \in C_n.$$

By the separation property of $\mathcal{C}$ we have that F is injective. Next, because for every $n \in \mathbb{N}$

$$C_n = F^{-1}\left(\left\{a \in \{0,1\}^{\mathbb{N}} : a_n = 1\right\}\right) \quad \text{and} \quad F(C_n) = F(E) \cap \left\{a \in \{0,1\}^{\mathbb{N}} : a_n = 1\right\},$$

we define $A := F(E)$ and get what we want. ☐

Theorem 1.3.5. *The measurable space $\left(\{0,1\}^{\mathbb{N}}, \mathcal{B}\left(\{0,1\}^{\mathbb{N}}\right)\right)$ is Borel-isomorphic to $([0,1], \mathcal{B}([0,1]))$.*

Proof. We define the function $F : \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]$ with

$$F(a) := \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Let $D := \bigcup_{n=1}^{\infty} \left\{ \frac{m}{2^n} : m \in \{0, ..., 2^n\} \right\}$. Trivially we see that $D, F^{-1}(D)$ are countable, and the function $F' : \{0,1\}^{\mathbb{N}} \setminus F^{-1}(D) \longrightarrow [0,1] \setminus D$, with $F'(x) = F(x)$, is a homeomorphism. So, the wanted result follows by patching together F' and F'', where $F'' : F^{-1}(D) \longrightarrow D$ a bijection of our choice between the countable sets. □

Corollary 1.3.6. *Let $(E, \mathcal{E})$ be separated and countably generated. Then, there exists a measurable bijection between $(E, \mathcal{E})$ and $(A, \mathcal{B}(A))$, where A is a subset of $[0,1]$.*

Remark 1.3.7. *Of course $\mathcal{B}(A) = \mathcal{B}([0,1]) \cap A = \sigma\left([0,t] \cap A : t \in \mathbb{Q}_+ \cap [0,1]\right)$.*

Definition 1.3.8. *Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces. A function $\mu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ will be called* finite random measure *if and only if the following are true:*

i. *For every $\omega \in \Omega$ $\mu(\omega, \cdot)$ is a finite measure on $(E, \mathcal{E})$.*

ii. *For every $S \in \mathcal{E}$ the function $\mu(\cdot, S)$ is $\mathcal{F}-$measurable.*

Furthermore, μ will be called random measure[4] *if and only if $\mu = \sum_{n=1}^{\infty} \mu_n$ and μ_n is a finite random measure for every $n \in \mathbb{N}$.*

Now we are ready to prove our disintegration theorem.

Theorem 1.3.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ a separated and countably generated measurable space. Additionally, let $m : \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathbb{R}_+$ be a finite measure on $(\Omega \times E, \mathcal{F} \otimes \mathcal{E})$. Define the measure $m_1 : \mathcal{F} \longrightarrow \mathbb{R}_+$ with $m_1(S) := m(S \times E)$. If $m_1 \ll \mathbb{P}$, then there exists a finite random measure $\mu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+$ such that for every $S \in \mathcal{F} \otimes \mathcal{E}$ we have*

$$m(S) = \int_{\Omega} \int_{E} \mathbb{1}_S(\omega, x) \mu(\omega, \mathrm{d}x) \mathbb{P}(\mathrm{d}\omega). \tag{1.1}$$

Moreover, the finite random measure μ is unique $\mathbb{P}-$a.e.

Proof. First, from Corolarry 1.3.6 we have that there exists a measurable bijection $F : (E, \mathcal{E}) \longrightarrow (A, \mathcal{B}(A))$ with $A \subseteq [0,1]$. So, we have that

$$\mathcal{F} \otimes \mathcal{E} = \mathcal{F} \otimes F^{-1}(\mathcal{B}(A)) = \sigma\left(\left\{S \times F^{-1}([0,t] \cap A) : S \in \mathcal{F} \text{ and } t \in \mathbb{Q} \cap [0,1]\right\}\right).$$

Now, for every $t \in [0,1]$ define the measures $m_{1,t} : \mathcal{F} \longrightarrow \mathbb{R}_+$ with $m_{1,t}(S) := m(S \times F^{-1}([0,t] \cap A))$. Then, for every $t \in \mathbb{Q} \cap [0,1]$ define as

$$\mu^0(\cdot, F^{-1}([0,t] \cap A)) := \frac{\mathrm{d}m_{1,t}}{\mathrm{d}\mathbb{P}},$$

[4]This is what Kallenberg in [31, p. 30] calls s-finite kernel.

a version of the Radon-Nikodym derivative of $m_{1,t}$ with respect to $\mathbb{P}$. By definition we get that, for every $q_1, q_2 \in \mathbb{Q} \cap [0,1]$ with $q_1 \leq q_2$ the set

$$S_{q_1,q_2} := \Big\{\omega \in \Omega : \mu^0(\omega, F^{-1}([0,q_1] \cap A)) > \mu^0(\omega, F^{-1}([0,q_2] \cap A))\Big\} \in \mathcal{F}$$

and also

$$S_{1,\infty} := \Big\{\omega \in \Omega : \mu^0(\omega, F^{-1}([0,1] \cap A)) = \infty\Big\} \in \mathcal{F},$$

with probabilities $\mathbb{P}(S_{q_1,q_2}) = \mathbb{P}(S_{1,\infty}) = 0$, the second equality comes from the fact that $\mu^0(\cdot, F^{-1}([0,1] \cap A))$ is integrable with respect to $\mathbb{P}$. Let

$$S_0 := \Bigg(\bigcup_{\substack{(q_1,q_2) \in \mathbb{Q}^2 \cap [0,1]^2 \\ q_1 \leq q_2}} S_{q_1,q_2} \Bigg) \cup S_{1,\infty},$$

obviously $S_0 \in \mathcal{F}$ and $\mathbb{P}(S_0) = 0$. Hence now define

$$\mu(\cdot, F^{-1}([0,t] \cap A)) := \mathbb{1}_{S_0^c}(\cdot)\, \mu^0(\cdot, F^{-1}([0,t] \cap A)),$$

for every $t \in \mathbb{Q} \cap [0,1]$. To expand our definition for every $t \in [0,1]$ we will use that $\mu(\cdot, F^{-1}([0,1] \cap A))$ is integrable with respect to $\mathbb{P}$, so by dominated convergence define as

$$\mu(\cdot, F^{-1}([0,t] \cap A)) := \lim_{\substack{s \to t^+ \\ s \in \mathbb{Q} \cap [0,1]}} \mu(\cdot, F^{-1}([0,s] \cap A))$$

and get that

$$\mu(\cdot, F^{-1}([0,t] \cap A)) = \frac{\mathrm{d}m^*_{1,t}}{\mathrm{d}\mathbb{P}}, \ \mathbb{P} - a.e.,$$

for every $t \in [0,1]$. Finally, with the usual procedure of defining measures from càdlàg, non-negative, non-decreasing functions on $[0,1]$ we can define the random measure $\mu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+$. Note that property ii. of definition 1.3.8 is proved with an easy Dynkin-class argument, by using the $\pi-$system $\{F^{-1}([0,t] \cap A) : t \in \mathbb{Q} \cap [0,1]\}$ which generates $\mathcal{E}$. Similarly (1.1) is proved by using the $\pi-$system $\{S \times F^{-1}([0,t] \cap A) : S \in \mathcal{F} \text{ and } t \in \mathbb{Q} \cap [0,1]\}$ which generates $\mathcal{F} \otimes \mathcal{E}$.

The last claim about uniqueness follows again from an easy Dynkin-class argument, by using again the $\pi-$system $\{F^{-1}([0,t] \cap A) : t \in \mathbb{Q} \cap [0,1]\}$ which generates $\mathcal{E}$, and noting that for every random measure $\nu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+$ which satisfies (1.1) we must have

$$\nu\Big(\cdot, F^{-1}([0,t] \cap A)\Big) = \frac{\mathrm{d}m_{1,t}}{\mathrm{d}\mathbb{P}}, \quad \mathbb{P} - a.e.,$$

for every $t \in \mathbb{Q} \cap [0,1]$. □

Chapter 2

Stochastic prerequisites and more

In this chapter, we are going to introduce the notation, as well as to give a short overview of known results, which will be useful in the current work. Furthermore, we present the Γ function, basic ingredient of our framework. The chapter then closes with the a priori estimates and an examination of their coefficients.

Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ denote a complete stochastic basis in the sense of Jacod and Shiryaev [29, I.1.3]. Once there is no ambiguity about the reference filtration, we are going to conceal the dependence on $\mathbb{G}$. The letters p, n and d will denote arbitrary natural numbers. For every $(m,q) \in \mathbb{N}^* \times \mathbb{N}^*$ we will denote with $\mathbb{R}^{m\times q}$ the $m \times q-$matrices with real entries. Also, we will denote with $\|\cdot\|_2$ the Euclidean norm on $\mathbb{R}^{m\times q}$, *i.e.* for $z \in \mathbb{R}^{m\times q}$ we have $\|z\|_2^2 := \mathrm{Tr}[z^T z]$. Note that we identify $\mathbb{R}^q$ as $\mathbb{R}^{q\times 1}$, in this case we will use the notation $|\cdot|$ instead of $\|\cdot\|_2$.

2.1 Martingales

Let us denote by $\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ the set of square integrable $\mathbb{G}-$martingales, *i.e.*,

$$\mathcal{H}^2(\mathbb{G};\mathbb{R}^p) := \Big\{X : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^p, X \text{ is a } \mathbb{G}\text{−martingale with } \sup_{t\in\mathbb{R}_+} \mathbb{E}[|X_t|^2] < \infty\Big\},$$

equipped with its usual norm

$$\|X\|^2_{\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)} := \mathbb{E}[|X_\infty|^2] = \mathbb{E}\left[\mathrm{Tr}[\langle X\rangle^{\mathbb{G}}_\infty]\right],$$

where $\langle X\rangle^{\mathbb{G}}$ denotes the $\mathbb{G}-$predictable quadratic variation of X. In other words, $\langle X\rangle^{\mathbb{G}}$ is the $\mathbb{G}-$predictable compensator of the $\mathbb{G}-$optional quadratic variation $[X]$.

Let us also define a notion of orthogonality between two square integrable martingales. More precisely, we say that $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ and $Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^q)$ are (mutually) orthogonal if and only if $\langle X, Y\rangle^{\mathbb{G}} = 0$, and denote this relation by $X \perp_{\mathbb{G}} Y$. Moreover, for a subset $\mathcal{X}$ of $\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$, we denote the space of martingales orthogonal to each component of every element of $\mathcal{X}$ by $\mathcal{X}^{\perp_{\mathbb{G}}}$, *i.e.*,

$$\mathcal{X}^{\perp_{\mathbb{G}}} := \{Y \in \mathcal{H}^2(\mathbb{G};\mathbb{R}),\ \langle Y, X\rangle^{\mathbb{G}} = 0 \text{ for every } X \in \mathcal{X}\}.$$

A martingale $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ will be called a *purely discontinuous* martingale if $X_0 = 0$ and if each of its components is orthogonal to all continuous martingales of $\mathcal{H}^2(\mathbb{G};\mathbb{R})$. Using [29, Corollary I.4.16] we can decompose $\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ as follows

$$\mathcal{H}^2(\mathbb{G};\mathbb{R}^p) = \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^p) \oplus \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^p), \tag{2.1}$$

where $\mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^p)$ denotes the subspace of $\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ consisting of all continuous square–integrable martingales and $\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^p)$ denotes the subspace of $\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ consisting of all purely discontinuous square–integrable martingales.

Let us also provide a classical example of the decomposition of the space of square–integrable martingales; we will later expand this result to a more general setting. Using [29, Theorem I.4.18], any square–integrable $\mathbb{G}-$martingale $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$ admits a unique (up to $\mathbb{P}-$indistinguishability) decomposition

$$X_\cdot = X_0 + X^c_\cdot + X^d_\cdot, \tag{2.2}$$

where $X^c_0 = X^d_0 = 0$. The process $X^c \in \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^p)$ will be called the *continuous martingale part of* X, while the process $X^d \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^p)$ will be called the *purely discontinuous martingale part of* X. The pair (X^c, X^d) is called the *natural pair of* X *(under* $\mathbb{G}$*).*

2.1.1 Itō stochastic integrals

Using [29, Section III.6.a], in order to define the stochastic integral with respect to a square–integrable martingale $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)$, we need to select a $\mathbb{G}-$predictable, non-decreasing and right continuous process $C^{\mathbb{G}}$ with the property that

$$\langle X\rangle^{\mathbb{G}} = \int_{(0,\cdot]} \frac{\mathrm{d}\langle X\rangle^{\mathbb{G}}_s}{\mathrm{d}C^{\mathbb{G}}_s}\mathrm{d}C^{\mathbb{G}}_s, \tag{2.3}$$

where the equality is understood component-wise. That is to say, $\frac{\mathrm{d}\langle X\rangle^{\mathbb{G}}}{\mathrm{d}C^{\mathbb{G}}}$ is a predictable process with values in the set of all symmetric, non–negative definite $p \times p$ matrices. Then, we define the set of integrable processes to be

$$\mathbb{H}^2(\mathbb{G}, X;\mathbb{R}^{d\times p}) := \left\{Z : (\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}) \longrightarrow (\mathbb{R}^{d\times p}, \mathcal{B}(\mathbb{R}^{d\times p})), \mathbb{E}\left[\int_0^\infty \mathrm{Tr}\left[Z_t\frac{\mathrm{d}\langle X\rangle^{\mathbb{G}}_s}{\mathrm{d}C^{\mathbb{G}}_s}Z_t^\top\right]\mathrm{d}C^{\mathbb{G}}_t\right] < \infty\right\},$$

where $\mathcal{P}^{\mathbb{G}}$ denotes the $\mathbb{G}-$predictable $\sigma-$field on $\Omega\times\mathbb{R}_+$; see [29, Definition I.2.1]. The associated stochastic integrals will be denoted either by $Z \cdot X$ or by $\int Z_s\mathrm{d}X_s$. In case we need to underline the filtration under which the Itō stochastic integral is defined, we will write either $(Z \cdot X)^{\mathbb{G}}$ or $(\int Z_s\mathrm{d}X_s)^{\mathbb{G}}$. The most important relation of the stochastic integral is the following formula for its predictable quadratic variation (see [29, Theorem III.6.4.c)])

$$\left(Z\frac{\mathrm{d}\langle X\rangle^{\mathbb{G}}}{\mathrm{d}C^{\mathbb{G}}}Z^\top\right) \cdot C^{\mathbb{G}} = \langle Z \cdot X\rangle^{\mathbb{G}}.$$

Hence, we have the analogon of the usual Itō isometry

$$\|Z\|^2_{\mathbb{H}^2(\mathbb{G},X;\mathbb{R}^{d\times p})} := \mathbb{E}\left[\int_0^\infty \operatorname{Tr}\left[Z_t \frac{\mathrm{d}\langle X\rangle^{\mathbb{G}}_s}{\mathrm{d}C^{\mathbb{G}}_s} Z_t^\top\right] \mathrm{d}C^{\mathbb{G}}_t\right] = \mathbb{E}\left[\operatorname{Tr}[\langle Z\cdot X\rangle^{\mathbb{G}}_\infty]\right].$$

We will denote the space of Itō stochastic integrals of processes in $\mathbb{H}^2(\mathbb{G},X)$ with respect to X by $\mathcal{L}^2(\mathbb{G},X)$. In particular, for $X^c \in \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^d)$ we remind the reader that, by [29, Theorem III.4.5], $Z\cdot X^c \in \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^d)$ for every $Z\in\mathbb{H}^2(X^c,\mathbb{G})$, *i.e.*, $\mathcal{L}^2(X^c,\mathbb{G}) \subseteq \mathcal{H}^{2,c}(\mathbb{G};\mathbb{R}^d)$.

2.1.2 Integrals with respect to an integer–valued random measure

Let us now expand the space, and accordingly the predictable σ–algebra, in order to construct measures that depend also on the height of the jumps of a stochastic process, that is

$$\left(\widetilde{\Omega},\widetilde{\mathcal{P}}^{\mathbb{G}}\right) := \left(\Omega\times\mathbb{R}_+\times\mathbb{R}^n, \mathcal{P}^{\mathbb{G}}\otimes\mathcal{B}(\mathbb{R}^n)\right).$$

A measurable function $U:\left(\widetilde{\Omega},\widetilde{\mathcal{P}}^{\mathbb{G}}\right)\longmapsto\left(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\right)$ is called $\widetilde{\mathcal{P}}^{\mathbb{G}}-$*measurable function* and, abusing notation, the space of these functions will also be denoted by $\widetilde{\mathcal{P}}^{\mathbb{G}}$. In particular, we will denote by $\widetilde{\mathcal{P}}^{\mathbb{G}}_+$ the space of non-negative $\widetilde{\mathcal{P}}^{\mathbb{G}}-$measurable functions.[1]

We say that μ is a random measure if $\mu := \{\mu(\omega;\mathrm{d}t,\mathrm{d}x)\}_{\omega\in\Omega}$ is a family of non–negative measures defined on $(\mathbb{R}_+\times\mathbb{R}^n,\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^n))$, satisfying identically $\mu(\omega;\{0\}\times\mathbb{R}^n)=0$. Consider a function $U\in\widetilde{\mathcal{P}}^{\mathbb{G}}$, then we define the process

$$U*\mu_\cdot(\omega) := \begin{cases}\displaystyle\int_{(0,\cdot]\times\mathbb{R}^n} U(\omega,s,x)\,\mu(\omega;\mathrm{d}s,\mathrm{d}x), \text{ if } \int_{(0,\cdot]\times\mathbb{R}^n} |U(\omega,s,x)|\mu(\omega,\mathrm{d}s,\mathrm{d}x)<\infty,\\ \infty, \text{ otherwise.}\end{cases}$$

Let $X\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$, we associate to it the $\mathbb{G}-$optional integer–valued random measure μ^X on $\mathbb{R}_+\times\mathbb{R}^n$ defined by its jumps via the formula

$$\mu^X(\omega;\mathrm{d}t,\mathrm{d}x) := \sum_{s>0}\mathbb{1}_{\{\Delta X_s(\omega)\neq 0\}}\delta_{(s,\Delta X_s(\omega))}(\mathrm{d}t,\mathrm{d}x), \tag{2.4}$$

where, for any $z\in\mathbb{R}_+\times\mathbb{R}^n$, δ_z denotes the Dirac measure at the point z; see also [29, Proposition II.1.16] which verifies that μ^X is $\mathbb{G}-$optional and $\mathcal{P}^{\mathbb{G}}-\sigma-$finite. Notice that $\mu^X(\omega;\mathbb{R}_+\times\{0\})=0$. Moreover, for a $\mathbb{G}-$predictable stopping time σ we define the random variable

$$\int_{\mathbb{R}^n} U(\omega,\sigma,x)\mu^X(\omega;\{\sigma\}\times\mathrm{d}x) := U(\omega,\sigma(\omega),\Delta X_{\sigma(\omega)}(\omega))\mathbb{1}_{\{\Delta X_\sigma\neq 0, |U(\omega,\sigma(\omega),\Delta X_{\sigma(\omega)}(\omega))|<\infty\}}.$$

Since $X\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$, the $\mathbb{G}-$compensator of μ^X under $\mathbb{P}$ exists, see [29, Theorem II.1.8]. This is the unique, up to a $\mathbb{P}-$null set, $\mathbb{G}-$predictable random measure $\nu^{(\mathbb{G},X)}$ on $\mathbb{R}_+\times\mathbb{R}^n$, for which

$$\mathbb{E}\left[U*\mu^X_\infty\right] = \mathbb{E}\left[U*\nu^{(\mathbb{G},X)}_\infty\right]$$

[1] Analogous notation we will adopt for any non-negative measurable function, *e.g.*, for a $\sigma-$algebra $\mathcal{A}$, the set $\mathcal{A}_+$ denotes the set of non-negative $\mathcal{A}-$measurable functions.

holds for every non–negative function $U \in \widetilde{\mathcal{P}}^{\mathbb{G}}$, where we have defined

$$U * \nu.(\omega) := \begin{cases} \displaystyle\int_{(0,\cdot]\times\mathbb{R}^n} U(\omega,s,x)\,\nu(\omega;\mathrm{d}s,\mathrm{d}x), \text{ if } \int_{(0,\cdot]\times\mathbb{R}^n} |U(\omega,s,x)|\nu(\omega,\mathrm{d}s,\mathrm{d}x) < \infty, \\ \infty, \text{ otherwise.} \end{cases}$$

Let $U \in \widetilde{\mathcal{P}}^{\mathbb{G}}_+$ and consider a $\mathbb{G}-$predictable time σ, whose graph is denoted by $[\![\sigma]\!]$ (see [29, Notation I.1.22] and the comments afterwards); we define the random variable

$$\int_{\mathbb{R}^n} U(\omega,\sigma,x)\nu^{(\mathbb{G},X)}(\omega;\{\sigma\}\times\mathrm{d}x) := \int_{\mathbb{R}_+\times\mathbb{R}^n} U(\omega,\sigma(\omega),x)\mathbb{1}_{[\![\sigma]\!]}\,\nu^{(\mathbb{G},X)}(\omega;\mathrm{d}s,\mathrm{d}x)$$

if $\int_{\mathbb{R}_+\times\mathbb{R}n}|U(\omega,\sigma(\omega),x)|\mathbb{1}_{[\![\sigma]\!]}\,\nu^{(\mathbb{G},X)}(\omega;\mathrm{d}s,\mathrm{d}x) < \infty$, otherwise it equals ∞. Using [29, Property II.1.11], we have

$$\int_{\mathbb{R}^n} U(\omega,\sigma,x)\nu^{(\mathbb{G},X)}(\omega;\{\sigma\}\times\mathrm{d}x) = \mathbb{E}\left[\int_{\mathbb{R}^n} U(\omega,\sigma,x)\mu^X(\omega;\{\sigma\}\times\mathrm{d}x)\bigg|\mathcal{G}_{\sigma-}\right]. \tag{2.5}$$

In order to simplify the notation, let us denote for any $\mathbb{G}-$predictable time σ

$$\widehat{U}^{(\mathbb{G},X)}_\sigma(\omega) := \int_{\mathbb{R}^n} U(\omega,\sigma,x)\nu^{(\mathbb{G},X)}(\omega;\{\sigma\}\times\mathrm{d}x). \tag{2.6}$$

In particular, for $U = 1$ we define

$$\zeta^{(\mathbb{G},X)}_\sigma(\omega) := \int_{\mathbb{R}^n} \nu^{(\mathbb{G},X)}(\omega;\{\sigma\}\times\mathrm{d}x). \tag{2.7}$$

In order to define the stochastic integral of a function $U \in \widetilde{\mathcal{P}}^{\mathbb{G}}$ with respect to the $\mathbb{G}-$*compensated integer–valued random measure* $\widetilde{\mu}^{(\mathbb{G},X)} := \mu^X - \nu^{(\mathbb{G},X)}$, we will consider the following class

$$G_2(\mathbb{G},\mu^X) := \left\{U : \left(\widetilde{\Omega},\widetilde{\mathcal{P}}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\right), \mathbb{E}\left[\sum_{t>0}\left|U(t,\Delta X_t)\mathbb{1}_{\{\Delta X_t\neq 0\}} - \widehat{U}^{(\mathbb{G},X)}_t\right|^2\right] < \infty\right\}.$$

The elements of $G_2(\mathbb{G},\mu^X)$ and $\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^d)$ can be associated, uniquely up to $\mathbb{P}-$indistinguishability, via

$$G_2(\mathbb{G},\mu^X) \ni U \longmapsto U \star \widetilde{\mu}^{(\mathbb{G},X)} \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^d),$$

see [29, Definition II.1.27, Proposition II.1.33.a] and He et al. [21, Theorem XI.11.21]. We call $U \star \widetilde{\mu}^{(\mathbb{G},X)}$ the *stochastic integral of* U *with respect to* $\widetilde{\mu}^{(\mathbb{G},X)}$. Let us point out that for an arbitrary function of $G_2(\mathbb{G},\mu^X)$ the two processes $U * (\mu^X - \nu^{(\mathbb{G},X)})$ and $U \star \widetilde{\mu}^{(\mathbb{G},X)}$ are not equal. We will make use of the following notation for the space of stochastic integrals with respect to $\widetilde{\mu}^X$ which are square integrable martingales

$$\mathcal{K}^2(\mathbb{G},\mu^X) := \left\{U \star \widetilde{\mu}^{(\mathbb{G},X)},\ U \in G_2(\mathbb{G},\mu^X)\right\}.$$

Moreover, by [29, Theorem II.1.33] or [21, Theorem 11.21], we have

$$\mathbb{E}\left[\langle U \star \widetilde{\mu}^{(\mathbb{G},X)}\rangle^{\mathbb{G}}_\infty\right] < \infty \text{ if and only if } U \in G_2(\mathbb{G},\mu^X),$$

which enables us to define the following more convenient space

$$\mathbb{H}^2(\mathbb{G},X) := \Big\{U : \big(\widetilde{\Omega},\widetilde{\mathcal{P}}^{\mathbb{G}}\big) \longrightarrow \big(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\big),\ \mathbb{E}\Big[\mathrm{Tr}\Big[\langle U\star\widetilde{\mu}^{(\mathbb{G},X)}\rangle_t^{\mathbb{G}}\Big]\Big] < \infty\Big\},$$

and we emphasize that we have the direct identification

$$\mathbb{H}^2(\mathbb{G},X) = G_2(\mathbb{G},\mu^X).$$

Let us finish this subsection with the following useful formulas

$$\begin{aligned}\mathbb{E}\Big[\mathrm{Tr}\langle U\star\widetilde{\mu}^{(\mathbb{G},X)}\rangle_\infty^{\mathbb{G}}\Big] &= \mathbb{E}\Bigg[\sum_{t>0}\Big|U(t,\Delta X_t)\mathbb{1}_{\{\Delta X_t\neq 0\}} - \widehat{U}_t^{(\mathbb{G},X)}\Big|^2\Bigg] \\ &= \mathbb{E}\Bigg[\sum_{t>0}\bigg|\int_{\mathbb{R}^n} U(t,x)\,\mu^X(t,\mathrm{d}x) - \int_{\mathbb{R}^n} U(t,x)\,\nu^{(\mathbb{G},X)}(t,\mathrm{d}x)\bigg|^2\Bigg].\end{aligned}$$

2.1.3 Orthogonal decompositions

Let us now state the decomposition results that will be used to solve the BSDEs of interest; for more details, we refer to Papapantoleon et al. [42, Section 2.2.1].

Let $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ with $M_{\mu^{X^\natural}}[\Delta X^\circ|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$. Using this assumption, we get that for $Y^1 \in \mathcal{L}^2(X^\circ,\mathbb{G})$ and $Y^2 \in \mathcal{K}^2(\mu^{X^\natural},\mathbb{G})$ it holds that $\langle Y^1, Y^2\rangle = 0$; see *e.g.* Cohen and Elliott [12, Theorem 13.3.16]. Then we define

$$\mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}}) := \Big(\mathcal{L}^2(X^\circ,\mathbb{G})\oplus\mathcal{K}^2(\mu^{X^\natural},\mathbb{G})\Big)^{\perp_{\mathbb{G}}}.$$

Subsequently, we have the following description for $\mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}})$, which is [42, Proposition 2.6].

Proposition 2.1.1. *Let $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ be a pair of square integrable martingales with $M_{\mu^{X^\natural}}[\Delta X^\circ|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$. Then,*

$$\mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}}) = \Big\{L\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^d),\ \langle X^\circ, L\rangle^{\mathbb{G}} = 0 \text{ and } M_{\mu^{X^\natural}}[\Delta L|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0\Big\}.$$

Moreover, the space $\Big(\mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}}), \|\cdot\|_{\mathcal{H}^2(\mathbb{R}^d)}\Big)$ is closed.

Summing up the previous results, we arrive at the decomposition result that is going to dictate the structure of the BSDEs in our setting

$$\mathcal{H}^2(\mathbb{G};\mathbb{R}^p) = \mathcal{L}^2(X^\circ,\mathbb{G})\oplus\mathcal{K}^2(\mu^{X^\natural},\mathbb{G})\oplus\mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}}),$$

where each of the spaces appearing in the identity above is closed.

2.2 Doléans-Dade measure and disintegration

Assume that we are given a square integrable $\mathbb{G}$–martingale $X \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$ along with its associated random measures μ^X and $\nu^{(\mathbb{G},X)}$. In $\Big(\widetilde{\Omega}, \mathcal{G}_\infty\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^n)\Big)$ we can define the Doléans-Dade measures of μ^X, resp. of $\nu^{(\mathbb{G},X)}$, as follows

$$M_{\mu^X}(A) := \mathbb{E}\left[\mathbb{1}_A * \mu^X_\infty\right],$$
$$\text{resp. } M_{\nu^{(\mathbb{G},X)}}(A) := \mathbb{E}\left[\mathbb{1}_A * \nu^{(\mathbb{G},X)}_\infty\right],$$

for every $A \in \mathcal{G}_\infty \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^n)$. Because M_{μ^X} is σ-integrable with respect to $\widetilde{\mathcal{P}}^{\mathbb{G}}$, we can define, for every non negative $\mathcal{G}_\infty \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^n)$–measurable function W, its conditional expectation with respect to $\widetilde{\mathcal{P}}^{\mathbb{G}}$ using M_{μ^X}, which we denote by $M_{\mu^X}[W|\widetilde{\mathcal{P}}]$. Furthermore, since $\nu^{(\mathbb{G},X)}$ is the $\mathbb{G}-$compensator of μ^X under $\mathbb{P}$, by definition we have

$$M_{\mu^X}(W) = M_{\nu^{(\mathbb{G},X)}}(W), \tag{2.8}$$

for every non–negative, $\widetilde{\mathcal{P}}^{\mathbb{G}}-$measurable function W. Let us denote with $|I|$ the map in $\mathbb{R}^n$ where $|I|(x) := |x| + \mathbb{1}_{\{0\}}(x)$. Then, using the facts that $M_{\mu^X}(|I|^2) < \infty$, $M_{\mu^X}(\Omega \times \mathbb{R}_+ \times \{0\}) = 0 = M_{\nu^X}(\Omega \times \mathbb{R}_+ \times \{0\})$ and that $|I|^2$ is $\widetilde{\mathcal{P}}^{\mathbb{G}}-$measurable, we can define the new measures $|I|^2_\cdot\mu^X$, resp. $|I|^2_\cdot\nu^{(\mathbb{G},X)}$, as

$$M_{|I|^2_\cdot\mu^X}(A) := M_{\mu^X}\left(|I|^2\mathbb{1}_A\right),$$
$$\text{resp. } M_{|I|^2_\cdot\nu^{(\mathbb{G},X)}}(A) := M_{\nu^{(\mathbb{G},X)}}\left(|I|^2\mathbb{1}_A\right),$$

for every $A \in \mathcal{G}_\infty \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^n)$. Then, for every predictable, increasing process C such that $|I|^2 * \nu^{(\mathbb{G},X)} \ll C$ $\mathbb{P}$–a.s., we get from [21, Theorem 5.14] that $\mathbb{P} \otimes (|I|^2 * \nu^{(\mathbb{G},X)}) \ll \mathbb{P} \otimes C$.

Consider a pair of martingales $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$, and define

$$C^{(\mathbb{G},\overline{X})} := \mathrm{Tr}\left[\langle X^\circ\rangle^{\mathbb{G}}\right] + |I|^2 * \nu^{(\mathbb{G},X^\natural)}. \tag{2.9}$$

Using the Kunita–Watanabe inequality, we can easily verify that $C^{(\mathbb{G},\overline{X})}$ possesses the property described in (2.3). Furthermore, note that from [21, 6.23 Theorem 2)] if $X^\natural \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ then we have

$$\mathrm{Tr}\left[\langle X^\natural\rangle^{\mathbb{G}}\right] = |I|^2 * \nu^{(\mathbb{G},X^\natural)}.$$

At this point let us make an immediate, yet crucial for the current work, remark.

Remark 2.2.1. *Assuming that $\mathbb{G}$ is immersed in $\mathbb{H}$,* i.e., *$\mathbb{H}$ is a filtration such that $\mathcal{G}_t \subseteq \mathcal{H}_t$ for every $t \in \mathbb{R}_+$ and, additionally, it possesses the property that every $\mathbb{G}-$martingale is an $\mathbb{H}-$martingale, then $C^{(\mathbb{G},\overline{X})}$ and $C^{(\mathbb{H},\overline{X})}$ are indistinguishable. Indeed, one immediately has that*

$$\langle X^\circ\rangle^{\mathbb{G}} - \langle X^\circ\rangle^{\mathbb{H}} = (\langle X^\circ\rangle^{\mathbb{G}} - X^\circ \cdot (X^\circ)^\top) + (X^\circ \cdot (X^\circ)^\top - \langle X^\circ\rangle^{\mathbb{H}})$$

is an $\mathbb{H}-$martingale, $\mathbb{H}-$predictable and of finite variation. Hence, since its initial value is 0, it is the zero martingale, which proves that $\langle X^\circ\rangle^{\mathbb{G}}$ and $\langle X^\circ\rangle^{\mathbb{H}}$ are indistinguishable. One can argue analogously for the processes $|I|^2 * \nu^{(\mathbb{G},X^\natural)}$ and $|I|^2 * \nu^{(\mathbb{H},X^\natural)}$. In other words, we are allowed to interchange the filtration symbol in the notation of (2.9)*, or even omit it.*

Returning to (2.9), one notices that we can disintegrate $\nu^{(\mathbb{G},X^\natural)}$, *i.e.*, we can determine kernels $K^{(\mathbb{G},\overline{X})}$: $\left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \mathcal{R}\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$, where $\mathcal{R}\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$ are the Radon measures on $\mathbb{R}^n$, such that

$$\nu^{(\mathbb{G},X^\natural)}(\omega, \mathrm{d}t, \mathrm{d}x) = K^{(\mathbb{G},\overline{X})}(\omega, t, \mathrm{d}x)\mathrm{d}C_t^{(\mathbb{G},\overline{X})}(\omega). \tag{2.10}$$

The kernels $K^{(\mathbb{G},\overline{X})}$ are $\mathbb{P} \otimes C^{(\mathbb{G},\overline{X})}-$unique, as one can deduce by a straightforward Dynkin class argument.

Moreover, let us define

$$c^{(\mathbb{G},\overline{X})} := \left(\frac{\mathrm{d}\langle X^\circ\rangle^{\mathbb{G}}}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}\right)^{\frac{1}{2}}. \tag{2.11}$$

The reader may observe that $\frac{\mathrm{d}\langle X^\circ\rangle^{\mathbb{G}}}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}$ is a $\mathbb{G}-$predictable process with values in the set of all symmetric, non-negative definite $p \times p$ matrices. Using the diagonalization property of these matrices and results from Azoff [2], one can easily show that $c^{(\mathbb{G},\overline{X})}$ will also be a $\mathbb{G}-$predictable process with values in the set of all symmetric, non-negative definite $p \times p$ matrices.

2.3 Stochastic exponential

Let A be a finite variation process, and define the process

$$\mathcal{E}(A)_\cdot := \mathrm{e}^{A_\cdot} \prod_{s \leq \cdot} \frac{1 + \Delta A_s}{\mathrm{e}^{\Delta A_s}}, \tag{2.12}$$

which is called the stochastic exponential of A. Using the trivial inequalities $0 \leq 1 + x \leq \mathrm{e}^x$, for all $x \geq -1$, and the usual properties of the jumps of finite variation processes, we can easily see that the above process is well defined, adapted, càdlàg and locally bounded. The main functionality of the above process is that it satisfies the SDE

$$\mathcal{E}(A)_t = 1 + \int_0^t \mathcal{E}(A)_{s-}\mathrm{d}A_s. \tag{2.13}$$

This fact is proved using Itō's formula, and yields that $\mathcal{E}(A)$ also has finite variation; see *e.g.* [12, Section 15.1].

In the sequel, we will need additional properties for a process expressed as a stochastic exponential, therefore we collect them in the next lemma. In order to ease notation, we adopt the following convention: whenever we write $\Delta A \neq -1$, we mean that the set $\{\Delta A_t = -1 \text{ for some } t \in \mathbb{R}_+\}$ is evanescent. Analogous will be the understanding for $\Delta A \geq a$, for any $a \in \mathbb{R}_+$ as well as for $\mathcal{E}(A) \neq 0$, *etc.*

Lemma 2.3.1. *Let A be a càdlàg process of finite variation.*

(i) *If $\Delta A \neq -1$, then $\mathcal{E}(A) \neq 0$.*

(ii) *If $\Delta A \neq -1$, then $\mathcal{E}(A)^{-1} = \mathcal{E}\left(-\overline{A}\right)$, where $\overline{A}_\cdot := A_\cdot - \sum_{s \leq \cdot} \frac{(\Delta A_s)^2}{(1+\Delta A_s)}$.*

(iii) *If $\Delta A \geq -1$, then $0 \leq \mathcal{E}(A)_\cdot \leq \mathrm{e}^{A_\cdot}$.*

(iv) *If A is non-decreasing, then $\mathcal{E}(A)$ is non-decreasing and if A is non-increasing, then $\mathcal{E}(A)$ is non-increasing.*

(v) *If B is another finite variation process, then we have the identity $\mathcal{E}(A)\mathcal{E}(B) = \mathcal{E}(A + B + [A, B])$, where $[A, B]_\cdot := \sum_{s\leq\cdot} \Delta A_s \Delta B_s$.*

(vi) *Let $\overline{A}$ be as defined in (ii), then we have the identity*

$$\Delta\overline{A}_\cdot = \frac{\Delta A_\cdot}{1 + \Delta A_\cdot}$$

and

$$\overline{A}_\cdot = A_0 + \int_0^\cdot \frac{1}{1 + \Delta A_s} \mathrm{d}A_s.$$

(vii) *Let $\gamma, \delta \geq 0$ and $\overline{A}$ as defined in (ii). Define*

$$\widetilde{A}^{\delta,\gamma}_\cdot := \delta A_\cdot - \overline{\gamma A}_\cdot - [\delta A, \overline{\gamma A}]_\cdot;$$

then, from (ii) and (v) it trivially holds $\mathcal{E}(\delta A)\mathcal{E}(\gamma A)^{-1} = \mathcal{E}(\widetilde{A}^{\delta,\gamma})$. Therefore,

$$\Delta\widetilde{A}^{\delta,\gamma}_\cdot = \frac{\Delta((\delta - \gamma)A)_\cdot}{1 + \Delta(\gamma A)_\cdot}$$

and

$$\widetilde{A}^{\delta,\gamma}_\cdot = A_0 + \int_0^\cdot \frac{1}{1 + \Delta(\gamma A)_s} \mathrm{d}((\delta - \gamma)A)_s.$$

If A is non-decreasing, then $\Delta(\widetilde{A}^{\delta,\gamma}) > -1$.

Proof. The above properties are fairly standard and follow from relatively simple calculations; one may consult [12, Section 15.1] for (i)–(v). Nevertheless, we also briefly argue about them for the convenience of the reader, since some of the arguments will be used for the proofs of (vi) and (vii).

We present some preparatory computations, which will allow us to immediately conclude the required properties. Let us fix an arbitrary $t \geq 0$. The first step is to write (2.12) in the form

$$\mathcal{E}(A)_t := \mathrm{e}^{A_t} \prod_{s\leq t} \frac{1 + \Delta A_s}{\mathrm{e}^{-\Delta A_s}} = \mathrm{e}^{A^c_t} \prod_{s\leq t} (1 + \Delta A_s). \tag{2.14}$$

Consider a finite variation process, then the multitude of its jumps that have magnitude greater than a given $a \in \mathbb{R}_+ \setminus \{0\}$ is finite, in any given interval $[0, t]$. Hence, if we write

$$\prod_{s\leq t}(1 + \Delta A_s) = \prod_{\{s\leq t: |\Delta A_s|\geq 1\}} (1 + \Delta A_s) \prod_{\{s\leq t: |\Delta A_s|<1\}} (1 + \Delta A_s), \tag{2.15}$$

then the first term on the product on the right side of (2.15) is what determines the sign, because this is a finite product, while the second term is always a non-negative number. As for the second term, we additionally have

$$\prod_{\{s\leq t: |\Delta A_s|<1\}} (1 + \Delta A_s) = \prod_{\{s\leq t:\ -1<\Delta A_s<0\}} (1 - |\Delta A_s|) \prod_{\{s\leq t:\ 0\leq\Delta A_s<1\}} (1 + \Delta A_s) \tag{2.16}$$

Now, the first term on the right hand side of (2.16) is the limit of a decreasing sequence of positive numbers and the second term is one of an increasing sequence of positive numbers. Using the classical inequality $1 + x \leq \mathrm{e}^x$ for $x \in \mathbb{R}$, we can extract an upper bound for the latter term, as follows:

$$\prod_{\{s \leq t:\ 0 \leq \Delta A_s < 1\}} (1 + \Delta A_s) \leq \prod_{\{s \leq t:\ 0 \leq \Delta A_s < 1\}} \mathrm{e}^{\Delta A_s} = \exp\Big\{ \sum_{\{s \leq t:\ 0 \leq \Delta A_s < 1\}} \Delta A_s \Big\} \leq \exp\Big\{ \mathrm{Var(A)}_t \Big\},$$

where Var(A) denotes the total variation process associated to A. We also need to find a lower bound for the former term. We have identically $(1 - |\Delta A_s|)(1 + |\Delta A_s|) = 1 - |\Delta A_s|^2$, which implies that

$$(1 - |\Delta A_s|)(1 + |\Delta A_s|) \geq \frac{3}{4}, \quad \text{for} \quad |\Delta A_s| < \frac{1}{2}.$$

Therefore, one gets

$$\begin{aligned}
\prod_{\{s \leq t:\ -1 < \Delta A_s < 0\}} (1 - |\Delta A_s|) &= \prod_{\left\{s \leq t:\ -1 < \Delta A_s < -\frac{1}{2}\right\}} (1 - |\Delta A_s|) \prod_{\left\{s \leq t:\ -\frac{1}{2} < \Delta A_s < 0\right\}} (1 - |\Delta A_s|) \\
&\geq \frac{3}{4} \prod_{\left\{s \leq t:\ -1 < \Delta A_s < -\frac{1}{2}\right\}} (1 - |\Delta A_s|) \prod_{\left\{s \leq t:\ -1 < \Delta A_s < -\frac{1}{2}\right\}} \frac{1}{(1 + |\Delta A_s|)} \\
&\geq \frac{3}{4}\, \mathrm{e}^{-\mathrm{Var(A)}_t} \prod_{\left\{s \leq t:\ -1 < \Delta A_s < -\frac{1}{2}\right\}} (1 - |\Delta A_s|) > 0.
\end{aligned}$$

Indeed, the term $\prod_{\left\{s \leq t:\ -1 < \Delta A_s < -\frac{1}{2}\right\}} (1 - |\Delta A_s|)$ is a finite product. In total, (i) is proved. As for (ii), we shall use the fact that the function $x \mapsto \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$ as well as that $\prod_{s \leq t}(1 + \Delta A_s)$ is a well defined non-zero limit. Hence, we have that

$$\left(\prod_{s \leq t} (1 + \Delta A_s) \right)^{-1} = \prod_{s \leq t} \frac{1}{(1 + \Delta A_s)}.$$

The reader should observe that the continuous parts of $\mathcal{E}(A)^{-1}$ and $\mathcal{E}(-\overline{A})$ are identical. Therefore, we only need to compare the associated jump processes. To this end, we immediately have

$$1 + \Delta(-\overline{A}) = 1 - \Delta A + \frac{(\Delta A)^2}{(1 + \Delta A)} = \frac{(1 + \Delta A)}{(1 + \Delta A)} - \frac{\Delta A}{(1 + \Delta A)} = \frac{1}{(1 + \Delta A_s)},$$

which is the desired identity. The claims in (iii) and (iv) are obvious in view of equation (2.14) and the classical inequality $0 \leq 1 + x \leq \mathrm{e}^x$, for $x \geq -1$. Analogously to (ii), one immediately derives (v), (vi) and (vii) once the respective continuous and discontinuous parts are compared. Indeed, regarding (vi) we have, on the one hand, that the continuous part of $\overline{A}$ is A^c, while $\Delta \overline{A} = \frac{(\Delta A)^2}{1 + \Delta A}$. On the other hand,

$$\left(\int_0^t \frac{1}{1 + \Delta A_s} \mathrm{d}A_s \right)^c = \int_0^t \frac{1}{1 + \Delta A_s} \mathrm{d}A_s^c = A_t^c,$$

because the process ΔA is non-zero only countably many times. Moreover, when one compares the respective jump processes for every $t \geq 0$

$$\Delta\left(\int_0^\cdot \frac{1}{1+\Delta A_s}\mathrm{d}A_s\right)_t = \frac{\Delta A_t}{1+\Delta A_t} = \frac{\Delta A_t + (\Delta A_t)^2 - (\Delta A_t)^2}{1+\Delta A_t} = \Delta A_t - \frac{(\Delta A_t)^2}{1+\Delta A_t} = \Delta \overline{A}_t.$$

Let us focus now on (vii). On the one hand, the continuous parts of $\widetilde{A}^{\delta,\gamma}$ and $(\delta-\gamma)A$ are identical. On the other hand, we have for their discontinuous parts

$$\begin{aligned}
\Delta(\widetilde{A}^{\delta,\gamma}) &= \Delta(\delta A) - \Delta(\overline{\gamma A}) - \Delta(\delta A)\Delta(\overline{\gamma A}) \\
&= \Delta(\delta A) - \frac{\Delta(\gamma A)}{1+\Delta(\gamma A)} - \Delta(\delta A)\frac{\Delta(\gamma A)}{1+\Delta(\gamma A)} \\
&= \frac{\Delta(\delta A)}{1+\Delta(\gamma A)} - \frac{\Delta(\gamma A)}{1+\Delta(\gamma A)} \\
&= \frac{\Delta((\delta-\gamma)A)}{1+\Delta(\gamma A)}.
\end{aligned} \tag{2.17}$$

Using the fact that ΔA is non-zero only countably many times, we can conclude as before that

$$\widetilde{A}^{\delta,\gamma}_\cdot = A_0 + \int_0^\cdot \frac{1}{1+\Delta(\gamma A)_s}\mathrm{d}\Big((\delta-\gamma)A\Big)_s.$$

Finally, we only need to verify that $\Delta(\widetilde{A}^{\delta,\gamma}) > -1$, which is trivial in view of the following equivalences

$$-1 < \Delta(\widetilde{A}^{\delta,\gamma}) \overset{(2.17)}{=} \frac{\Delta\Big((\delta-\gamma)A\Big)}{1+\Delta(\gamma A)} \Leftrightarrow -1 - \Delta(\gamma A) < \Delta\Big((\delta-\gamma)A\Big) \Leftrightarrow -1 < \Delta(\delta A). \qquad \square$$

2.4 Norms and spaces

We will largely follow the notation of [42, Section 2.3] with regards to norms and spaces of stochastic processes. However, we will need to additionally keep track of the filtration under which we are working, given that later many filtrations will appear in our framework.

Let $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ with $M_{\mu^{X^\natural}}[\Delta X^\circ|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$, and $A, C : (\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}) \longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ càdlàg and increasing. The following spaces will appear in the analysis throughout this work, for $\beta \geq 0$ and T a $\mathbb{G}-$stopping time:

$$\mathbb{L}^2_\beta(\mathcal{G}_T, A; \mathbb{R}^d) := \Big\{\xi,\ \mathbb{R}^d\text{–valued, } \mathcal{G}_T\text{–measurable, } \|\xi\|^2_{\mathbb{L}^2_\beta(\mathcal{G}_T,A;\mathbb{R}^d)} := \mathbb{E}\left[\mathcal{E}(\beta A)_{T-}|\xi|^2\right] < \infty\Big\},$$

$$\mathcal{H}^2_\beta(\mathbb{G}, A; \mathbb{R}^d) := \Big\{M \in \mathcal{H}^2(\mathbb{G}, A; \mathbb{R}^d), \|M\|^2_{\mathcal{H}^2_\beta(\mathbb{G},A;\mathbb{R}^d)} := \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{t-}\,\mathrm{dTr}\big[\langle M\rangle^{\mathbb{G}}\big]_t\right] < \infty\Big\},$$

$$\mathbb{H}^2_\beta(\mathbb{G}, A, C; \mathbb{R}^d) := \Big\{\phi,\ \mathbb{R}^d\text{–valued, } \mathbb{G}\text{–optional,}$$
$$\|\phi\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,C;\mathbb{R}^d)} := \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{t-}|\phi|^2_t\,\mathrm{d}C_t\right] < \infty\Big\},$$

$$\mathcal{S}^2_\beta(\mathbb{G}, A; \mathbb{R}^d) := \Big\{\phi,\ \mathbb{R}^d\text{–valued, } \mathbb{G}\text{–optional, } \|\phi\|^2_{\mathcal{S}^2_\beta(\mathbb{G},A;\mathbb{R}^d)} := \mathbb{E}\Big[\sup_{t\in[0,T]}\Big\{\mathcal{E}(\beta A)_{t-}|\phi|^2_t\Big\}\Big] < \infty\Big\},$$

$$\mathbb{H}^2_\beta(\mathbb{G}, A, X^\circ; \mathbb{R}^{d\times p}) := \Big\{Z \in \mathbb{H}^2(\mathbb{G}, X^\circ; \mathbb{R}^{d\times p}),$$
$$\|Z\|^2_{\mathbb{H}^2_\beta((\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p}))} := \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{t-}\,\mathrm{dTr}\big[\langle Z\cdot X^\circ\rangle^{\mathbb{G}}\big]_t\right] < \infty\Big\},$$

$$\mathbb{H}^2_\beta(\mathbb{G}, A, X^\natural; \mathbb{R}^d) := \Big\{U \in \mathbb{H}^2(\mathbb{G}, X^\natural; \mathbb{R}^d),$$
$$\|U\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)} := \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{t-}\,\mathrm{dTr}\big[\langle U\star\tilde{\mu}^{X^\natural}\rangle^{\mathbb{G}}\big]_t\right] < \infty\Big\},$$

and

$$\mathcal{H}^2_\beta(\mathbb{G}, A, \overline{X}^{\perp_{\mathbb{G}}}; \mathbb{R}^d) := \Big\{M \in \mathcal{H}^2(\overline{X}^{\perp_{\mathbb{G}}}), \|M\|^2_{\mathcal{H}^2_\beta(\mathbb{G},A,\overline{X}^{\perp_{\mathbb{G}}};\mathbb{R}^d)} := \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{t-}\,\mathrm{dTr}\big[\langle M\rangle^{\mathbb{G}}\big]_t\right] < \infty\Big\}.$$

Moreover, for

$$(Y, Z, U, M) \in \mathcal{S}^2_\beta(\mathbb{G}, A; \mathbb{R}^d) \times \mathbb{H}^2_\beta(\mathbb{G}, A, X^\circ; \mathbb{R}^{d\times p}) \times \mathbb{H}^2_\beta(\mathbb{G}, A, X^\natural; \mathbb{R}^d) \times \mathcal{H}^2_\beta(\mathbb{G}, A, \overline{X}^{\perp_{\mathbb{G}}}; \mathbb{R}^d)$$

we define

$$\|\,(Y, Z, U, M)\,\|^2_{\star,\beta,\mathbb{G},A,\overline{X}}$$
$$:= \|Y\|^2_{\mathcal{S}^2_\beta(\mathbb{G},A;\mathbb{R}^d)} + \|Z\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} + \|U\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)} + \|M\|^2_{\mathcal{H}^2_\beta(\mathbb{G},A,\overline{X}^{\perp_{\mathbb{G}}};\mathbb{R}^d)}.$$

Later on, we will need to rewrite the norms associated to the spaces $\mathbb{H}^2_\beta(\mathbb{G}, A, X^\circ; \mathbb{R}^{d\times p})$ and $\mathbb{H}^2_\beta(\mathbb{G}, A, X^\natural; \mathbb{R}^d)$ in terms of Lebesgue–Stieltjes integrals with respect to $C^{(\mathbb{G},\overline{X})}$, for $C^{(\mathbb{G},\overline{X})}$ as defined in (2.9); one may consult [42, Lemma 2.13] for the details. Hence, for $(Z,U) \in \mathbb{H}^2_\beta(\mathbb{G}, A, X^\circ; \mathbb{R}^d) \times \mathbb{H}^2_\beta(\mathbb{G}, A, X^\natural; \mathbb{R}^d)$ and $C^{(\mathbb{G},\overline{X})}$ as defined in (2.9), we have

$$\|Z\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\circ;\mathbb{R}^d)} = \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{s-}\|Z_s c_s^{(\mathbb{G},\overline{X})}\|^2\mathrm{d}C_s^{(\mathbb{G},\overline{X})}\right] \tag{2.18}$$

and

$$\|U\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)} = \mathbb{E}\left[\int_0^T \mathcal{E}(\beta A)_{s-}\left(|||U_s(\cdot)|||_s^{(\mathbb{G},\overline{X})}\right)^2 \mathrm{d}C_s^{(\mathbb{G},\overline{X})}\right], \tag{2.19}$$

where

$$\begin{aligned}\left(|||U_t(\omega;\cdot)|||_t^{(\mathbb{G},\overline{X})}(\omega)\right)^2 := \int_{\mathbb{R}^n}\left|U(\omega,s,x) - \widehat{U}_s^{(\mathbb{G},\overline{X})}(\omega)\right|^2 K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)\\ + \left(1-\zeta_s^{(\mathbb{G},X^\natural)}(\omega)\right)\Delta C_s^{(\mathbb{G},\overline{X})}(\omega)\left|\int_{\mathbb{R}^n} U(\omega,s,x)\,K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)\right|^2,\end{aligned} \tag{2.20}$$

with $K^{(\mathbb{G},\overline{X})}$ satisfying (2.10), *i.e.*,

$$\nu^{(\mathbb{G},X^\natural)}(\omega,\mathrm{d}t,\mathrm{d}x) = K^{(\mathbb{G},\overline{X})}(\omega,t,\mathrm{d}x)\mathrm{d}C_t^{(\mathbb{G},\overline{X})}(\omega).$$

Finally, because of the assumption $M_{\mu^{X^\natural}}[\Delta X^\circ|\widetilde{\mathcal{P}}^{\mathbb{G}}]=0$, and in conjunction with [12, Theorem 13.3.16], we have

$$\|Z\cdot X^\circ + U\star\widetilde{\mu}^{X^\natural}\|^2_{\mathcal{H}^2_\beta(\mathbb{G},A;\mathbb{R}^d)} = \|Z\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} + \|U\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)}. \tag{2.21}$$

Remark 2.4.1. *In order to simplify the notation whenever possible, if we consider one of the aforementioned spaces for $\beta=0$, then we will omit 0. As a result, the dependence on the process A is redundant, hence we will also omit the process A from the notation of the respective space. As an example, $\mathbb{L}^2(\mathcal{G}_T;\mathbb{R}^d)$ denotes the space $\mathbb{L}^2_0(\mathcal{G}_T,A;\mathbb{R}^d)$, which is the classical Lebesgue space, and so forth.*

In case we have to deal with a system of $N\in\mathbb{N}$ couples of martingales, we will need to introduce norms associated to the respective product space. To this end, let $\{\overline{X}^i\}_{i\in\{1,\dots,N\}}$ be a family of couples of martingales, *i.e.*, $\overline{X}^i := (X^{i,\circ},X^{i,\natural})\in\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$, such that $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{G}}]=0$, for all $i\in\{1,\dots,N\}$. Moreover, let $A^i:(\Omega\times\mathbb{R}_+,\mathcal{P}^{\mathbb{G}})\longrightarrow(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ be càdlàg and increasing, for every $i\in\{1,\dots,N\}$. Let us denote

$$(\mathbf{Y}^N,\mathbf{Z}^N,\mathbf{U}^N,\mathbf{M}^N) := (Y^{i,N},Z^{i,N},U^{i,N},M^{i,N})_{i\in\{1,\dots,N\}}.$$

Then, for

$$(\mathbf{Y}^N,\mathbf{Z}^N,\mathbf{U}^N,\mathbf{M}^N)\in \prod_{i=1}^N \mathcal{S}^2_{\hat\beta}(\mathbb{G},A^i;\mathbb{R}^d)\times\mathbb{H}^2_{\hat\beta}(\mathbb{G},A^i,X^{i,\circ};\mathbb{R}^{d\times p})\times\mathbb{H}^2_{\hat\beta}(\mathbb{G},A^i,X^{i,\natural};\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}(\mathbb{G},A^i,\overline{X}^{i\perp_{\mathbb{G}}};\mathbb{R}^d)$$

we define

$$\left\|(\mathbf{Y}^N,\mathbf{Z}^N,\mathbf{U}^N,\mathbf{M}^N)\right\|^2_{\star,\beta,\mathbb{G},\{A^i\}_{i\in\{1,\dots,N\}},\{\overline{X}^i\}_{i\in\{1,\dots,N\}}} := \sum_{i=1}^N \|(Y^i,Z^i,U^i,M^i)\|^2_{\star,\beta,\mathbb{G},A^i,\overline{X}^i}.$$

We conclude this section with the following important definitions. Let $E:=\mathbb{R}_+\times\mathbb{R}^n$ and $(F,\|\cdot\|_2),(G,\|\cdot\|_2)$ two Euclidean finite-dimensional spaces.

- Set $E_0 := (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}^n)$ and $\widetilde{E} := E \setminus E_0$.

- Let $f : (F, \|\cdot\|_2) \longrightarrow (G, \|\cdot\|_2)$. We will call the support of f the set $\mathrm{supp}(f) := \overline{\{f \neq 0\}}^{\|\cdot\|_2}$, where for a set A we denote with $\overline{A}^{\|\cdot\|_2}$ the closure of it, under the metric corresponding to the norm $\|\cdot\|_2$.

- $C_c(F;G) := \{f : (F, \|\cdot\|_2) \longrightarrow (G, \|\cdot\|_2) : f \text{ continuous with compact support}\}$.

- $C_{c|\widetilde{E}}(E;\mathbb{R}^d) := \{f \in C_c(E;\mathbb{R}^d) : \mathrm{supp}(f) \subseteq \widetilde{E}\}$.

- $D^{\circ,d\times m} \subseteq C_c(\mathbb{R}_+;\mathbb{R}^{d\times p})$, is a fixed with respect to $\omega \in \Omega$, countable set which is dense in the space $\mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \langle X^\circ\rangle(\omega))$ for $\mathbb{P} - a.e.\ \omega \in \Omega$. For the existence of such set one can see [44, Lemma A.14.].

- $D^\natural \subseteq C_{c|\widetilde{E}}(E;\mathbb{R}^d)$ is a fixed with respect to $\omega \in \Omega$, countable set which is dense in the space $\mathbb{L}^2(E, \mathcal{B}(E), \nu^{(\mathbb{G},X^\natural)}(\omega))$ for $\mathbb{P}-a.e.\ \omega \in \Omega$. For the existence of such set one can see [44, Lemma A.15.]. Also, keep in mind that $\nu^{(\mathbb{G},X^\natural)}(\omega)(E_0) = 0$, $\mathbb{P} - a.e.$ and $\int_E \|x\|_2^2\, \nu^{(\mathbb{G},X^\natural)}(\omega)(\mathrm{d}s, \mathrm{d}x) < \infty$, $\mathbb{P} - a.e.$

2.5 The Γ function

One of the key characteristics about the backward propagation of chaos is the fact that the $Y-$components of the solutions of McKean–Vlasov BSDEs need to be identically distributed. Ideally we would like to work in a setting as general as the one of [42], however this creates some difficulties. It is evident from (2.20) that the triple bar norm depends from the driver pair $\overline{X}$. So, to get the Lipschitz conditions of [42] while pass on the uniqueness in law from the pairs $\{\overline{X}^i\}_{i\in\mathbb{N}}$ to the $\{Y^i\}_{i\in\mathbb{N}}$'s, we have to define an appropriate map from the pair of drivers $\overline{X}$ to the generators $f^{\overline{X}}$. To do this we will start from a base function f, with usual Lipschitz conditions, and compose it with a couple of functions depending from $\overline{X}$. The first member of the couple will be the map which corresponds to the matrix $c^{(\mathbb{G},\overline{X})}$ (see (2.11)), and the second is the following Γ function, which in view of (2.20) and remark 2.5.2 (iii) its definition can be considered natural.

Definition 2.5.1. *Let $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$, let $C^{(\mathbb{G},\overline{X})}$ be as defined in* (2.9) *and $K^{(\mathbb{G},\overline{X})}$ that satisfies* (2.10)*. Additionally, let Θ be an $\mathbb{R}$–valued, $\widetilde{\mathcal{P}}^{\mathbb{G}}-$measurable function such that $|\Theta| \leq |I|$, for $|I|(x) := |x| + \mathbb{1}_{\{0\}}(x)$. Define the process $\Gamma^{(\mathbb{G},\overline{X},\Theta)} : \mathbb{H}^2(\mathbb{G}, X^\natural;\mathbb{R}^d) \longrightarrow \mathcal{P}^{\mathbb{G}}(\mathbb{R}^d)$ such that, for every $s \in \mathbb{R}_+$, holds*

$$\begin{aligned}\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s(\omega) := &\int_{\mathbb{R}^n} \Big(U(\omega,s,x) - \widehat{U}_s^{(\mathbb{G},X^\natural)}(\omega)\Big)\Big(\Theta(\omega,s,x) - \widehat{\Theta}_s^{(\mathbb{G},X^\natural)}(\omega)\Big)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)\\ &+(1-\zeta_s^{(\mathbb{G},X^\natural)}(\omega))\Delta C_s^{(\mathbb{G},\overline{X})}(\omega)\int_{\mathbb{R}^n} U(\omega,s,x)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)\int_{\mathbb{R}^n}\Theta(\omega,s,x)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x).\end{aligned}$$

Remark 2.5.2. *(i) Given the square–integrability of the martingale $X^\natural$, it is immediate that the process Θ as defined above lies in $\mathbb{H}^2(\mathbb{G}, X^\natural;\mathbb{R})$. Therefore, for any $U \in \mathbb{H}^2(\mathbb{G}, X^\natural;\mathbb{R}^d)$, the process Γ is well-defined $\mathbb{P}\otimes C^{(\mathbb{G},\overline{X})}-a.e.$*

(ii) *One would expect in* Γ *a notational dependence on the kernel. However, for* $K_1^{(\mathbb{G},\overline{X})}, K_2^{(\mathbb{G},\overline{X})}$ *satisfying* (2.10) *and* $U \in \mathbb{H}^2(\mathbb{G}, X^\natural; \mathbb{R}^d)$ *we have* $\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s = \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s$, $\mathbb{P}\otimes C^{(\mathbb{G},\overline{X})}-$*a.e; here, implicitly, the left hand side is defined with respect to* K_1 *and the right hand side with respect to* K_2*. In that way, we have uniqueness of the kernels that satisfy* (2.10)*. Since in the respective computations all the appearing equalities will be taken under* $\mathbb{P}\otimes C^{(\mathbb{G},\overline{X})}$*, we have suppressed the notational dependence on the kernels.*

(iii) *The choice of* Γ *was based on, and inspired by, applications. The reader may recall, for example, the connection between BSDEs and partial integro-differential equations, and the special structure that is required for the generator, see* e.g. *Barles et al. [3] or Delong [17, Section 4.2]. Moreover, one can easily verify that* Γ *is equal to*

$$\frac{\mathrm{d}\langle U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}, \Theta \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}.$$

The next lemma will be useful and, essentially, justifies the definition of the process Γ. In other words, the function Γ is Lipschitz in the sense described below. However, note that the inner characteristics of the function Γ play no part in what comes in the remaining sections. If one can prove that results similar to Lemma 2.5.3 and Lemma A.2.6 hold, then the rest will remain valid, under the appropriate modifications.

Lemma 2.5.3. *Let* $\overline{X} \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ *and* $\Theta \in \widetilde{\mathcal{P}}^{\mathbb{G}}$ *be an* $\mathbb{R}$*–valued function such that* $|\Theta| \leq |I|$*. Then, for every* $U^1, U^2 \in \mathbb{H}^2(\mathbb{G}, X^\natural; \mathbb{R}^d)$*, we have*

$$|\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^1)_t(\omega) - \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^2)_t(\omega)|^2 \leq 2\left(\left|\!\left|\!\left| U^1_t(\omega;\cdot) - U^2_t(\omega;\cdot)\right|\!\right|\!\right|_t^{(\mathbb{G},\overline{X})}(\omega)\right)^2, \;\; \mathbb{P}\otimes C^{(\mathbb{G},\overline{X})} - a.e.$$

Proof. Let $\delta U := U^1 - U^2$ then, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\left|\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^1)_t - \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^2)_t\right|^2 \\
&\leq 2\left|\int_{\mathbb{R}^n}\left[\delta U_t(x) - \widehat{\delta U}_t^{(\mathbb{G},\overline{X})}\right]\left[\Theta_t(x) - \widehat{\Theta}_t^{(\mathbb{G},\overline{X})}\right] K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\right|^2 \\
&\qquad + 2\,(1-\zeta_t^{(\mathbb{G},X^\natural)})^2\left(\Delta C_t^{(\mathbb{G},\overline{X})}\right)^2\left|\int_{\mathbb{R}^n}\delta U_t(x)\,K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\right|^2\left|\int_{\mathbb{R}^n}\Theta_t(x)\,K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\right|^2 \\
&\leq 2\int_{\mathbb{R}^n}\left|\delta U_t(x) - \widehat{\delta U}_t^{(\mathbb{G},\overline{X})}\right|^2 K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\int_{\mathbb{R}^n}\left|\Theta_t(x) - \widehat{\Theta}_t^{(\mathbb{G},\overline{X})}\right|^2 K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x) \\
&\qquad + 2(1-\zeta_t^{(\mathbb{G},X^\natural)})^2\left(\Delta C_t^{(\mathbb{G},\overline{X})}\right)^2\left|\int_{\mathbb{R}^n}\delta U_t(x)\,K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\right|^2\left|\int_{\mathbb{R}^n}\Theta_t(x)\,K_t^{(\mathbb{G},\overline{X})}(\mathrm{d}x)\right|^2 \\
&\leq 2\Big(\left|\!\left|\!\left| U^2_t(\omega\,;\cdot) - U^1_t(\omega\,;\cdot)\right|\!\right|\!\right|_t^{(\mathbb{G},\overline{X})}(\omega)\Big)^2\Big(\left|\!\left|\!\left|\Theta_t(\cdot)\right|\!\right|\!\right|_t^{(\mathbb{G},\overline{X})}(\omega)\Big)^2, \;\; \mathbb{P}\otimes C^{(\mathbb{G},\overline{X})} - \text{a.e.}
\end{aligned}$$

Using [21, 11.21 Theorem, Part 3)], because $|\Theta| \leq |I|$ and $X^\natural \in \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$, we have

$$\langle \Theta \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}_{\cdot} = |\Theta|^2 * \nu^{(\mathbb{G},X^\natural)}_{\cdot} - \sum_{s\leq\cdot}\left|\widehat{\Theta}_s^{(\mathbb{G},\overline{X})}\right|^2.$$

Hence, for every $s,u\in\mathbb{Q}_+$ with $s<u$, we get

$$\begin{aligned}\langle\Theta\star\widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}_u-\langle\Theta\star\widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}_s&=|\Theta|^2*\nu^{(\mathbb{G},X^\natural)}_u-|\Theta|^2*\nu^{(\mathbb{G},X^\natural)}_s-\sum_{s<r\leq u}\Big|\widehat{\Theta}^{(\mathbb{G},\overline{X})}_r\Big|^2\\&\leq|\Theta|^2*\nu^{(\mathbb{G},X^\natural)}_u-|\Theta|^2*\nu^{(\mathbb{G},X^\natural)}_s=\Big(|\Theta|^2\mathbb{1}_{\rrbracket s,u\rrbracket}\Big)*\nu^{(\mathbb{G},X^\natural)}_\infty\\&\leq\Big(|I|^2\mathbb{1}_{\rrbracket s,u\rrbracket}\Big)*\nu^{(\mathbb{G},X^\natural)}_\infty=|I|^2*\nu^{(\mathbb{G},X^\natural)}_u-|I|^2*\nu^{(\mathbb{G},X^\natural)}_s,\ \ \mathbb{P}-\text{a.e.}\end{aligned}$$

By a straightforward monotone class argument, we have

$$\mathrm{d}\langle\Theta\star\widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}\leq\mathrm{d}\left(|I|^2*\nu^{(\mathbb{G},X^\natural)}\right),\ \ \mathbb{P}-\text{a.e.}$$

Therefore, using the above, we get

$$\left(|||\Theta_t(\cdot)|||^{(\mathbb{G},\overline{X})}_t(\omega)\right)^2=\frac{\mathrm{d}\langle\Theta\star\widetilde{\mu}^{(\mathbb{G},X^\natural)}\rangle^{\mathbb{G}}}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}\leq\frac{\mathrm{d}\left(|I|^2*\nu^{(\mathbb{G},X^\natural)}\right)}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}\leq 1,\ \ \mathbb{P}\otimes C^{(\mathbb{G},\overline{X})}-\text{a.e.},$$

and the proof is thus complete. □

2.6 Wasserstein distance

Let $\mathbb{X}$ be a Polish space endowed with a metric ρ, then we denote by $\mathcal{P}(\mathbb{X})$ the space of probability measures on $(\mathbb{X},\rho)$. Moreover, for every real $q\in[1,\infty)$, we define the probability measures on $\mathbb{X}$ with finite q moment to be

$$\mathcal{P}_q(\mathbb{X}):=\left\{\mu:\int_{\mathbb{X}}\rho(x_0,x)^q\,\mu(\mathrm{d}x)<\infty\right\},\ \text{ for some } x_0\in\mathbb{X}.$$

By the triangle inequality and the fact that we consider probability (*i.e.* finite) measures, it is immediate that the space $\mathcal{P}_q(\mathbb{X})$ is independent of the choice of x_0. The space $\mathcal{P}(\mathbb{X})$ is equipped with the usual weak topology,[2] which we denote by $\mathcal{T}$. Let us recall the form of the elements for the usual basis of $\mathcal{T}$.

Definition 2.6.1. *Let $f:\mathbb{X}\longrightarrow\mathbb{R}$ be any continuous and bounded function (i.e. $f\in C_b(\mathbb{X})$), then we denote by $I^f:\mathcal{P}(\mathbb{X})\to\mathbb{R}$ the function where*

$$\mathcal{P}(\mathbb{X})\ni\mu\longmapsto\int_{\mathbb{X}}f(x)\mu(\mathrm{d}x)\in\mathbb{R}.$$

We can now give a description of the usual basis of $\mathcal{T}$

$$\mathbf{B}(\mathcal{T}):=\left\{\bigcap_{i=1}^m\left(I^{f_i}\right)^{-1}(A_i):m\in\mathbb{N},f_i\in C_b(X),A_i\text{ open sets of }\mathbb{R}\right\}.\tag{2.22}$$

[2] We use the term as the probabilists do, *i.e.*, the topological dual space of the set of probability measures is the set of continuous and bounded functions defined on $\mathbb{X}$. In other words, from the point of view of functional analysis, this is the weak$^*-$ topology.

The weak topology is metrizable and, in fact, $(\mathcal{P}(\mathbb{X}), \mathcal{T})$ is a Polish space; see Aliprantis and Border [1, 15.15 Theorem]. Moreover, it is well-known, see *e.g.* [1, 15.3 Theorem], that

$$\begin{gathered}\text{a sequence of probability measures } \{\mu_m\}_{m\in\mathbb{N}}\\ \text{converges weakly to a probability measure } \mu\\ \text{if and only if for every bounded and continuous function } f \text{ we have}\\ \int_{\mathbb{X}} f(x)\,\mu_m(\mathrm{d}x) \xrightarrow[m\to\infty]{} \int_{\mathbb{X}} f(x)\,\mu(\mathrm{d}x).\end{gathered}$$

In view of the above remarks, we have the following result using (2.22). A generalization of this result appears in Varadarajan [48].

Lemma 2.6.2. *Let $\mu \in \mathcal{P}(\mathbb{X})$, then there exists a sequence $\{f_k^\mu\}_{k\in\mathbb{N}} \subseteq C_b(\mathbb{X})$ such that a sequence of probability measures $\{\mu_m\}_{m\in\mathbb{N}}$ converges weakly to μ if and only if*

$$\int_{\mathbb{X}} f_k^\mu(x)\,\mu_m(\mathrm{d}x) \xrightarrow[m\to\infty]{} \int_{\mathbb{X}} f_k^\mu(x)\,\mu(\mathrm{d}x), \text{ for all } k \in \mathbb{N}.$$

Proof. Let $\rho_{\mathcal{T}}$ be a metric that makes $(\mathcal{P}(\mathbb{X}), \mathcal{T})$ Polish. Then for every open ball with center μ and radius $\frac{1}{m}$ for $m \in \mathbb{N}$, denoted by $B_{\rho_{\mathcal{T}}}(\mu, \frac{1}{m})$, there exists $D_m \in \mathbf{B}(\mathcal{T})$ where $\mu \in D_m \subseteq B_{\rho_{\mathcal{T}}}(\mu, \frac{1}{m})$. Using (2.22), we can conclude the proof. ☐

On $\mathcal{P}_q(\mathbb{X})$ we can define an even stronger mode of convergence, that allows for more functions to be tested. We will simply call it weak convergence in $\mathcal{P}_q(\mathbb{X})$ and this mode says that

$$\begin{gathered}\text{a sequence of probability measures } \{\mu_m\}_{m\in\mathbb{N}} \text{ converges weakly in } \mathcal{P}_q(\mathbb{X})\\ \text{to a probability measure } \mu\\ \text{if and only if for every continuous function } f \text{ such that}\\ |f(x)| \leq C\,(1+\rho(x_0,x)^q), \text{ where } C := C(f) \in \mathbb{R}_+,\\ \text{we have } \int_{\mathbb{X}} f(x)\,\mu_m(\mathrm{d}x) \xrightarrow[m\to\infty]{} \int_{\mathbb{X}} f(x)\,\mu(\mathrm{d}x).\end{gathered} \tag{2.23}$$

Of course, as before, it is immediate that in the above definition it does not matter which x_0 we choose. Now, the topology that is induced from this stronger mode of convergence is metrizable from a metric with nice properties; this metric is called the Wasserstein distance of order q. More precisely, given two probability measures $\mu, \nu \in \mathcal{P}_q(\mathbb{X})$ we define the Wasserstein distance of order q between them to be

$$\mathcal{W}_{q,\rho}^q(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \left\{ \int_{\mathbb{X}\times\mathbb{X}} \rho(x,y)^q\,\pi(\mathrm{d}x,\mathrm{d}y) \right\}, \tag{2.24}$$

where $\Pi(\mu,\nu)$ are the probability measures on $\mathbb{X}\times\mathbb{X}$ with marginals $\pi_1 = \mu$ and $\pi_2 = \nu$. The interested reader may consult Villani [49, Theorem 6.9] for the fact that $\mathcal{W}_{q,\rho}$ metrizes $\mathcal{P}_q(\mathbb{X})$. Moreover, $\mathcal{P}_q(\mathbb{X})$ with this mode of convergence is a Polish space, as one can see from [49, Theorem 6.18].

Remark 2.6.3. *With the possibility of taking infinite values, one can see the Wasserstein distance of order q as a non-negative function on the polish space $(\mathcal{P}(\mathbb{X}), \mathcal{T}) \times (\mathcal{P}(\mathbb{X}), \mathcal{T})$. Then, from [49, Remark 6.12] we have that $\mathcal{W}_q$ is lower semi-continuous, hence measurable.*

A useful inequality about the Wasserstein distance of order q, that is going to be used multiple times hereinafter, concerns the distance between two empirical measures on $\mathbb{X}$. Given an $N \in \mathbb{N}$ and $\mathbf{x}^N := (x_1, ..., x_N), \mathbf{y}^N := (y_1, ..., y_N) \in \mathbb{X}^N$, we have the empirical measures

$$L^N(\mathbf{x}^N) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \quad \text{and} \quad L^N(\mathbf{y}^N) := \frac{1}{N}\sum_{i=1}^N \delta_{y_i},$$

where $\delta_\cdot$ is the Dirac measure on $\mathbb{X}$. Then we have

$$\mathcal{W}_{q,\rho}^q\Big(L^N(\mathbf{x}^N), L^N(\mathbf{y}^N)\Big) \leq \frac{1}{N}\sum_{i=1}^N \rho(x_i, y_i)^q. \tag{2.25}$$

The above inequality is immediate if in the definition of the Wasserstein distance (2.24) we choose the probability measure

$$\pi := \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,y_i)},$$

where $\delta_{(\cdot,\cdot)}$ is the Dirac measure over $\mathbb{X} \times \mathbb{X}$. One can immediately see that $\pi_1 = L^N(\mathbf{x}^N)$ and $\pi_2 = L^N(\mathbf{y}^N)$.

Remark 2.6.4. *Last but not least, note that if ρ is a bounded metric, then for every $q \in [1, \infty)$ we have $\mathcal{P}(\mathbb{X}) = \mathcal{P}_q(\mathbb{X})$, and every function f as in* (2.23) *belongs to $C_b(\mathbb{X})$. Hence, we get that $\mathcal{W}_{q,\rho}$ metrizes the weak convergence on $\mathcal{P}(\mathbb{X})$, see [49, Corollary 6.13].*

2.7 Skorokhod space

Let $\mathbb{X}$ be a Polish space endowed with a metric ρ. We will denote by $\mathbb{D}(\mathbb{X}) := \{f : [0, \infty) \longrightarrow \mathbb{X} : f \text{ càdlàg }\}$ the space of càdlàg paths with values in $\mathbb{X}$. We supply $\mathbb{D}(\mathbb{X})$ with its usual J_1-metric, which we denote by $\rho_{\mathrm{J}_1^{\mathbb{X}}}$. Endowed with this metric, $\mathbb{D}(\mathbb{X})$ becomes a Polish space. We are not going to get into the specifics of $\rho_{\mathrm{J}_1^{\mathbb{X}}}$, as we will only need a couple of its basic properties. Firstly, for every $x, y \in \mathbb{D}(\mathbb{X})$, we have

$$\rho_{\mathrm{J}_1^{\mathbb{X}}}(x, y) \leq \sup_{s\in[0,\infty)} \{\rho(x_s, y_s)\} \wedge 1. \tag{2.26}$$

Secondly, the Borel $\sigma-$algebra that $\rho_{\mathrm{J}_1^{\mathbb{X}}}$ generates coincides with the usual product $\sigma-$algebra on $\mathbb{X}^{[0,\infty)}$ that the projections generate. To be more precise, we have

$$\mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}(\mathbb{X}) = \sigma\Big(\mathrm{Proj}_s^{-1}(A) : A \in \mathcal{B}(\mathbb{X}), s \in [0, \infty)\Big) \cap \mathbb{D}(\mathbb{X}), \tag{2.27}$$

where $\mathbb{X}^{[0,\infty)} \ni x \overset{\mathrm{Proj}_s}{\longmapsto} x(s) \in \mathbb{X}$, for every $s \in [0, \infty)$. Additional results on the Skorokhod space are available in [21, Chapter 15] or [29, Chapter VI].

Using that the $\sigma-$algebra on Ω is $\mathcal{G}$, it is obvious from (2.27) that every $\mathcal{G}\otimes\mathcal{B}([0, \infty))-$jointly measurable càdlàg process X can be seen as a function with domain Ω and taking values in $\mathbb{D}(\mathbb{X})$ such that X is $\Big(\mathcal{G}/\mathcal{B}_{\rho_{\mathrm{J}_1^{\mathbb{X}}}}(\mathbb{D}(\mathbb{X}))\Big)-$measurable and *vice versa*: every $\Big(\mathcal{G}/\mathcal{B}_{\rho_{\mathrm{J}_1^{\mathbb{X}}}}(\mathbb{D}(\mathbb{X}))\Big)-$measurable random variable X can be seen as a $\mathcal{G} \otimes \mathcal{B}([0, \infty))-$jointly measurable càdlàg process.

In the special cases where $\mathbb{X} = \mathbb{R}^{m\times q}$ or $\mathbb{X} = \mathbb{R}^m$ for $m, q \in \mathbb{N}$, we simplify notation and denote as $\mathbb{D}^{m\times q} := \mathbb{D}(\mathbb{X})$, $\mathbb{D}^m := \mathbb{D}(\mathbb{X})$ and $\rho_{\mathrm{J}_1^{m\times q}} := \rho_{\mathrm{J}_1^{\mathbb{X}}}$, $\rho_{\mathrm{J}_1^m} := \rho_{\mathrm{J}_1^{\mathbb{X}}}$ respectively.

Remark 2.7.1. *Later on, when we say that a collection of càdlàg processes is independent or is identically distributed or is exchangeable, they will be understood as* $\Big(\mathcal{G}/\mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}^d)\Big)-$*measurable random variables.*

Finally, for every $a \in (0,\infty)$ and $x \in \mathbb{D}^d$, the initial segment $x|_{[0,a]}$, resp. $x|_{[0,a-]}$, will be understood as an element of $\mathbb{D}^d$, using the convention

$$\begin{aligned} x|_{[0,a]}(s) &:= x(s)\mathbb{1}_{[0,a)}(s) + x(a)\mathbb{1}_{[a,\infty)}(s), \\ \text{resp. } x|_{[0,a-]}(s) &:= x(s)\mathbb{1}_{[0,a)}(s) + x(a-)\mathbb{1}_{[a,\infty)}(s). \end{aligned} \tag{2.28}$$

For $a \in [0,\infty)$, we define $\mathbb{D}^d_a := \{x_{[0,a]} : x \in \mathbb{D}^d\}$, resp. $\mathbb{D}^d_{a-} := \{x_{[0,a-]} : x \in \mathbb{D}^d\}$, and we naturally have

$$\mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}^d_a) = \mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}^d) \cap \mathbb{D}^d_a, \text{ resp. } \mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}^d_{a-}) = \mathcal{B}_{\rho_{\mathrm{J}_1^d}}(\mathbb{D}^d) \cap \mathbb{D}^d_{a-}.$$

2.8 Weak convergence of filtrations

Definition 2.8.1. *Let* $(m,q) \in \mathbb{N}\times\mathbb{N}$ *and assume that* $\{a^k\}_{k\in\overline{\mathbb{N}}}$ *is a sequence of* $\Big(\mathcal{G}/\mathcal{B}_{\rho_{\mathrm{J}_1^{m\times q}}}(\mathbb{D}^{m\times q})\Big)-$ *measurable random variables.*

(i) *We say that the sequence* $\{a^k\}_{k\in\mathbb{N}}$ *converges in probability under the* $\mathrm{J}_1^{m\times q}-$*metric to* a^∞*, denoted by* $a^k \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{m\times q}),\mathbb{P})} a^\infty$*, if and only if, for every* $\varepsilon > 0$ *we have*

$$\mathbb{P}\Big(\rho_{\mathrm{J}_1^{m\times q}}(a^k, a^\infty) > \varepsilon\Big) \xrightarrow[k\to\infty]{|\cdot|} 0.$$

(ii) *For every* $\vartheta \in [1,\infty)$*, we say that the sequence* $\{a^k\}_{k\in\mathbb{N}}$ *converges in* $\mathbb{L}^\vartheta-$*mean under the* $\mathrm{J}_1^{m\times q}-$*metric to* a^∞*, denoted by* $a^k \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{m\times q}),\mathbb{L}^\vartheta)} a^\infty$*, if and only if, we have*

$$\mathbb{E}\Big[\Big(\rho_{\mathrm{J}_1^{m\times q}}(a^k, a^\infty)\Big)^\vartheta\Big] \xrightarrow[k\to\infty]{|\cdot|} 0.$$

Now, let $\{\mathbb{G}^k := (\mathcal{G}^k_t)_{t\in\mathbb{R}_+}\}_{k\in\mathbb{N}}$ and $\mathbb{G}^\infty := (\mathcal{G}^\infty_t)_{t\in\mathbb{R}_+}$ be subfiltrations of $\mathbb{G}$ that satisfy the usual conditions. Also, define $\mathcal{G}^\infty_\infty := \bigvee_{t\in\mathbb{R}_+}\mathcal{G}^\infty_t$.

Definition 2.8.2. *The sequence* $\{\mathbb{G}^k\}_{k\in\mathbb{N}}$ *converges weakly to* $\mathbb{G}^\infty$*, denoted by* $\mathbb{G}^k \xrightarrow[k\to\infty]{\mathrm{w}} \mathbb{G}^\infty$*, if and only if, for every set* $S \in \mathcal{G}^\infty_\infty$ *we have*

$$\mathbb{E}\Big[\mathbb{1}_S\Big|\mathcal{G}^k_\cdot\Big] \xrightarrow[k\to\infty]{(\mathrm{J}_1(R),\mathbb{P})} \mathbb{E}\Big[\mathbb{1}_S\Big|\mathcal{G}^\infty_\cdot\Big].$$

2.9 *A priori* estimates

The following Theorem will be our primary tool for making estimations. Intuitively the last term in (2.29) on average is zero, as martingales are the stochastic analog of deterministic constant functions. So because the norms are defined using the expectation, it should be possible to bound the norm of y with the norms of f and ξ. Then would also follow that one could bound the norm of η with the norms of f and ξ just by switching places in (2.29). The results expand those in [42, Section 3.4] by replacing the deterministic exponential with the stochastic one. Also, we would like to underline that recently in Possamaï and Rodrigues [45] similar a-priori estimates have been extracted, *i.e.*, the norms are defined by means of the stochastic exponential. The two works are totally independent. As a last remark of the paragraph, one should note that the discrepancy of the coefficients between proposition 2.9.2 and [45, Proposition 5.4.] is due to the difference in the definition of the norms and the focus in [45] on the special case when $\xi = 0$.

For the current section we fix a stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ satisfying the usual conditions. Additionally, we fix a predictable, càdlàg, non-decreasing process C, a predictable, real-valued process α and define $A_\cdot := \int_0^\cdot \alpha_s \, \mathrm{d}C_s$.

Finally, in this section we ease the introduced notation for the spaces by dropping the dependence on $\mathbb{G}$, C and A. More precisely, $\mathbb{L}^2_\beta(\mathcal{G}_T, A; \mathbb{R}^d)$ for a stopping time T, resp. $\mathbb{H}^2_\beta(\mathbb{G}, A, C; \mathbb{R}^d)$, $\mathcal{S}^2_\beta(\mathbb{G}, A; \mathbb{R}^d)$, $\mathcal{H}^2_\beta(\mathbb{G}, A; \mathbb{R}^d)$, will be simply denoted by $\mathbb{L}^2_\beta(\mathcal{G}_T; \mathbb{R}^d)$, resp. $\mathbb{H}^2_\beta(\mathbb{R}^d)$, $\mathcal{S}^2_\beta(\mathbb{R}^d)$, $\mathcal{H}^2_\beta(\mathbb{R}^d)$.

Lemma 2.9.1. *Assume that we are given a d-dimensional semimartingale y of the form*

$$y_t = \xi + \int_t^T f_s \, \mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s, \tag{2.29}$$

where T is a stopping time, $\xi \in \mathbb{L}^2(\mathcal{G}_T; \mathbb{R}^d)$, f is a d-dimensional optional process, and $\eta \in \mathcal{H}^2(\mathbb{R}^d)$. In addition, assume there exists some $\Phi \geq 0$ such that $\Delta A \leq \Phi, \mathbb{P} \otimes C-$almost everywhere. Finally, suppose there exists $\beta \in (0, \infty)$ such that

$$\|\xi\|_{\mathbb{L}^2_\beta(\mathcal{G}_T; \mathbb{R}^d)} + \left\| \frac{f}{\alpha} \right\|_{\mathbb{H}^2_\beta(\mathbb{R}^d)} < \infty. \tag{2.30}$$

Then, for any $(\gamma, \delta) \in (0, \beta]^2$ with $\gamma \neq \delta$ we have

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} \leq \frac{2(1+\delta\Phi)}{\delta} \|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T; \mathbb{R}^d)} + 2\Lambda^{\gamma,\delta,\Phi} \left\| \frac{f}{\alpha} \right\|^2_{\mathbb{H}^2_{\gamma \vee \delta}(\mathbb{R}^d)},$$

$$\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} \leq 8\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T; \mathbb{R}^d)} + 8\frac{1+\gamma\Phi}{\gamma} \left\| \frac{f}{\alpha} \right\|^2_{\mathbb{H}^2_{\gamma \vee \delta}(\mathbb{R}^d)}$$

and

$$\|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq 9(2+\delta\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T; \mathbb{R}^d)} + 9\Big(\frac{1}{\gamma \vee \delta} + \delta\Lambda^{\gamma,\delta,\Phi}\Big) \left\| \frac{f}{\alpha} \right\|^2_{\mathbb{H}^2_{\gamma \vee \delta}(\mathbb{R}^d)},$$

where

$$\Lambda^{\gamma,\delta,\Phi} := \frac{(1+\gamma\Phi)^2}{\gamma|\delta-\gamma|}.$$

Putting the pieces together we have

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq \Big(18 + \frac{2}{\delta} + (9\delta+2)\Phi\Big)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + \Big(\frac{9}{\gamma\vee\delta} + (9\delta+2)\Lambda^{\gamma,\delta,\Phi}\Big)\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)},$$

$$\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq (26+9\delta\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathbb{R}^d)} + \Big(\frac{8}{\gamma} + 8\Phi + \frac{9}{\gamma\vee\delta} + 9\delta\Lambda^{\gamma,\delta,\Phi}\Big)\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}$$

and

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} + \|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq \Big(26 + \frac{2}{\delta} + (9\delta+2)\Phi\Big)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + \Big(\frac{8}{\gamma} + 8\Phi + \frac{9}{\gamma\vee\delta} + (9\delta+2)\Lambda^{\gamma,\delta,\Phi}\Big)\Big\|\frac{f}{\alpha}\Big\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.$$

Proof. By definition we have $\int_t^T \eta_s\,\mathrm{d}s = \eta_T - \eta_{T\wedge t}$. Because y is adapted and $\eta \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^d)$, we have for every $t \geq 0$

$$y_t = \mathbb{E}\left[y_t|\mathcal{G}_t\right] = \mathbb{E}\left[\xi + \int_t^T f_s\,\mathrm{d}C_s\middle|\mathcal{G}_t\right]. \tag{2.31}$$

From the above identity is evident that we need to study the following process

$$F(t) := \int_t^T f_s \mathrm{d}C_s. \tag{2.32}$$

For $\gamma \in \mathbb{R}_+$, we have from Cauchy–Schwarz inequality that

$$|F(t)|^2 \leq \int_t^T \mathcal{E}(\gamma A)^{-1}_{s-}\,\mathrm{d}A_s \int_t^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s, \tag{2.33}$$

which dictates further focusing on the first factor of the right-hand side of the inequality. From Lemma 2.3.1.(ii), for $\overline{A}_\cdot := A_\cdot - \sum_{s\leq\cdot}\frac{(\Delta A_s)^2}{(1+\Delta A_s)}$, we have

$$\int_t^T \mathcal{E}(\gamma A)^{-1}_{s-}\,\mathrm{d}A_s = \int_t^T \mathcal{E}(-\overline{\gamma A})_{s-}\,\mathrm{d}A_s.$$

For the jumps of $\overline{\gamma A}$ from Lemma 2.3.1.(vii), we have that $-1 < \Delta(-\overline{\gamma A}) \leq 0$, which implies that $\mathcal{E}(-\overline{\gamma A}) > 0$; see (i) and (iii) of Lemma 2.3.1. Then, from Lemma 2.3.1.(iv),

$$\begin{aligned}\int_t^T \mathcal{E}(-\overline{\gamma A})_{s-}\,\mathrm{d}A_s &= \frac{1}{\gamma}\int_t^T (1+\Delta(\gamma A_s))\mathcal{E}(-\overline{\gamma A})_{s-}\,\mathrm{d}(\overline{\gamma A})_s \\ &\leq \frac{1+\gamma\Phi}{\gamma}\int_t^T \mathcal{E}(-\overline{\gamma A})_{s-}\,\mathrm{d}(\overline{\gamma A})_s = -\frac{1+\gamma\Phi}{\gamma}\int_t^T \mathcal{E}(-\overline{\gamma A})_{s-}\,\mathrm{d}(-\overline{\gamma A})_s \\ &= -\frac{1+\gamma\Phi}{\gamma}\mathcal{E}(-\overline{\gamma A})\Big|_t^T = \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\gamma A)^{-1}\Big|_T^t \leq \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\gamma A)_t^{-1} \\ &\leq \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\gamma A)_{t-}^{-1},\end{aligned}$$

where the last inequality is validated by the fact that $\mathcal{E}(\gamma A)_\cdot^{-1}$ is non-increasing. Finally, combining the above results and returning to (2.33), we get

$$|F(t)|^2 \leq \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\gamma A)_{t-}^{-1}\int_t^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s. \tag{2.34}$$

From the assumption (2.30), for $\gamma \in (0,\beta]$ we have

$$\mathbb{E}\left[|F(0)|^2\right] < \infty.$$

Next, for $\delta \in (0,\beta]$, we will integrate $|F(t)|^2$ with respect to $\mathcal{E}(\delta A)_-\mathrm{d}A$. Before we proceed, we underline that we are going to use the fact that $\mathcal{E}(\widetilde{A}^{\delta,\gamma})$ is (strictly) positive. Indeed, this is straightforward from (i),(iii) and (vii) of Lemma 2.3.1. Now, we return to our aim and with the aid of Lemma 2.3.1.(vii), inequality (2.34) and Tonelli's theorem, we get

$$\begin{aligned}\int_0^T \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t &\leq \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\delta A)_{t-}\mathcal{E}(\gamma A)_{t-}^{-1}\int_t^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\,\mathrm{d}A_t \\ &= \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\int_0^T \mathbb{1}_{]\!]t,T]\!]}(s)\mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\,\mathrm{d}A_t \\ &= \frac{1+\gamma\Phi}{\gamma}\int_0^T\int_0^T \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\mathbb{1}_{]\!]t,T]\!]}(s)\mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\,\mathrm{d}A_t \\ &= \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\int_0^T \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\mathbb{1}_{]\!]t,T]\!]}(s)\,\mathrm{d}A_t\,\mathrm{d}C_s \\ &= \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\int_0^{s-} \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}A_t\,\mathrm{d}C_s.\end{aligned} \tag{2.35}$$

For a moment, we are going to concentrate on the term $\int_0^{s-}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\mathrm{d}A_t$, by considering the two cases $\delta > \gamma$ and $\delta < \gamma$:

- $\delta > \gamma$: From Lemma 2.3.1.(vii) we derive the inequality

$$\begin{aligned}\int_0^{s-} \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}A_t &= \frac{1}{\delta-\gamma}\int_0^{s-}(1+\Delta(\gamma A)_t)\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}\widetilde{A}_t^{\delta,\gamma} \\ &\leq \frac{(1+\gamma\Phi)}{\delta-\gamma}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{s-}.\end{aligned} \tag{2.36}$$

Thus, returning to (2.35), we have

$$\begin{aligned}\int_0^T \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t &\overset{(2.35)}{\leq} \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\int_0^{s-}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}A_t\,\mathrm{d}C_s\\ &\overset{(2.36)}{\leq} \frac{(1+\gamma\Phi)^2}{\gamma(\delta-\gamma)}\int_0^T \mathcal{E}(\gamma A)_{s-}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &= \frac{(1+\gamma\Phi)^2}{\gamma(\delta-\gamma)}\int_0^T \mathcal{E}(\gamma A)_{s-}\mathcal{E}(\gamma A)_{s-}^{-1}\mathcal{E}(\delta A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\\ &= \frac{(1+\gamma\Phi)^2}{\gamma(\delta-\gamma)}\int_0^T \mathcal{E}(\delta A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s,\end{aligned}$$

which is integrable for $\delta \leq \beta$.

• $\delta < \gamma$: From Lemma 2.3.1.(vii) we deduce

$$\begin{aligned}\int_0^{s-}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}A_t &= \frac{1}{\delta-\gamma}\int_0^{s-}(1+\Delta(\gamma A)_t)\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}\widetilde{A}_t^{\delta,\gamma}\\ &= \frac{1}{|\delta-\gamma|}\int_0^{s-}(1+\Delta(\gamma A)_t)\,\mathrm{d}\left(-\mathcal{E}(\widetilde{A}^{\delta,\gamma})\right)_t\\ &\leq \frac{1+\gamma\Phi}{|\delta-\gamma|},\end{aligned} \tag{2.37}$$

where in the inequality we used that $\mathcal{E}(\widetilde{A}^{\delta,\gamma})$ is non-increasing; see (iv) and (vii) of Lemma 2.3.1. Thus, returning to (2.35), we have

$$\begin{aligned}\int_0^T \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t &\overset{(2.35)}{\leq} \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\int_0^{s-}\mathcal{E}(\widetilde{A}^{\delta,\gamma})_{t-}\,\mathrm{d}A_t\,\mathrm{d}C_s\\ &\overset{(2.37)}{\leq} \frac{(1+\gamma\Phi)^2}{\gamma|\delta-\gamma|}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s.\end{aligned}$$

In total, summing up the conclusions of the two cases, we have that for every $(\gamma,\delta)\in(0,\beta]^2$ with $\gamma\neq\delta$ we can rewrite (2.35) as

$$\int_0^T \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t \leq \frac{(1+\gamma\Phi)^2}{\gamma|\delta-\gamma|}\int_0^T \mathcal{E}((\gamma\vee\delta)A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s$$

or, equivalently -in terms of the introduced notation- as

$$\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t\right] \leq \Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}. \tag{2.38}$$

We are ready to estimate $\|\alpha y\|_{\mathbb{H}^2_\delta(\mathbb{R}^d)}$. Using (2.31), (2.32), Jensen's inequality and the inequality $(a+b)^2 \leq 2(a^2+b^2)$ in conjunction with the fact that A is predictable, we have

$$\begin{aligned}
\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} &= \mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}|y_t|^2\,\mathrm{d}A_t\right] \leq \mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}\left[|\xi+F(t)|^2\Big|\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \mathbb{E}\left[\mathcal{E}(\delta A)_{t-}|\xi|^2 + \mathcal{E}(\delta A)_{t-}\Big|F(t)\Big|^2\Big|\mathcal{G}_t\right]\,\mathrm{d}A_t\right] \\
&= 2\mathbb{E}\left[\int_{\mathbb{R}_+} \mathbb{E}\left[\mathcal{E}(\delta A)_{t-}|\xi|^2 + \mathcal{E}(\delta A)_{t-}\Big|F(t)\Big|^2\Big|\mathcal{G}_t\right]\,\mathrm{d}A_{T\wedge t}\right] \\
&= 2\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}|\xi|^2 + \mathcal{E}(\delta A)_{t-}|F(t)|^2\,\mathrm{d}A_t\right] \\
&\leq 2\mathbb{E}\left[|\xi|^2\int_0^T \mathcal{E}(\delta A)_{t-}\,\mathrm{d}A_t\right] + 2\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}} \\
&\leq \frac{2(1+\delta\Phi)}{\delta}\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 2\,\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.
\end{aligned}$$

We move on with the estimate of $\|y\|_{\mathcal{S}^2_\delta(\mathbb{R}^d)}$, Once again, we will use (2.31), (2.32), (2.34), Jensen's inequality and $(a+b)^2 \leq 2(a^2+b^2)$. Furthermore, we will need Doob's inequality and the vector analogue of the triangle inequality for conditional expectations. By definition

$$\begin{aligned}
\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} &= \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathcal{E}(\delta A)_{t-}|y_t|^2\Big)\Big] = \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathcal{E}(\delta A)^{\frac{1}{2}}_{t-}|y_t|\Big)^2\Big] \\
&= \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathcal{E}(\delta A)^{\frac{1}{2}}_{t-}\Big|\mathbb{E}\Big[\xi+F(t)\Big|\mathcal{G}_t\Big]\Big|\Big)^2\Big] \\
&\leq \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathcal{E}(\delta A)^{\frac{1}{2}}_{t-}\mathbb{E}\Big[\Big|\xi+F(t)\,\mathrm{d}C_s\Big|\Big|\mathcal{G}_t\Big]\Big)^2\Big] \\
&\leq 2\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathbb{E}\Big[\sqrt{\mathcal{E}(\delta A)_{t-}|\xi|^2+\mathcal{E}(\delta A)_{t-}|F(t)|^2}\Big|\mathcal{G}_t\Big]\Big)^2\Big] \\
&\leq 2\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\mathbb{E}\Big[\Big(\mathcal{E}(\delta A)_{t-}|\xi|^2 + \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\delta A)_{t-}\mathcal{E}(\gamma A)^{-1}_{t-}\int_t^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\Big)^{\frac{1}{2}}\Big|\mathcal{G}_t\Big]\Big)^2\Big].
\end{aligned}$$

At this point, we will split again our analysis into two cases:

• $\delta < \gamma$: By definition of the stochastic exponential, see (2.14), for A increasing we have $0 < \mathcal{E}(\delta A)_{t-} \leq \mathcal{E}(\gamma A)_{t-}$ or equivalently $0 < \mathcal{E}(\delta A)_{t-}\mathcal{E}(\gamma A)_{t-}^{-1} \leq 1$. So we get

$$
\begin{aligned}
&2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\mathbb{E}\left[\sqrt{\mathcal{E}(\delta A)_{t-}|\xi|^2 + \frac{1+\gamma\Phi}{\gamma}\mathcal{E}(\delta A)_{t-}\mathcal{E}(\gamma A)_{t-}^{-1}\int_t^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s}\middle|\mathcal{G}_t\right]\right)^2\right] \qquad (2.39)\\
&\leq 2\mathbb{E}\left[\sup_{0\leq t}\left(\mathbb{E}\left[\sqrt{\mathcal{E}(\delta A)_{T-}|\xi|^2 + \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s}\middle|\mathcal{G}_t\right]\right)^2\right]\\
&\leq 8\mathbb{E}\left[\mathcal{E}(\delta A)_{T-}|\xi|^2 + \frac{1+\gamma\Phi}{\gamma}\int_0^T \mathcal{E}(\gamma A)_{s-}\frac{|f_s|^2}{\alpha_s^2}\,\mathrm{d}C_s\right]\\
&\leq 8\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 8\,\frac{1+\gamma\Phi}{\gamma}\,\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_\gamma(\mathbb{R}^d)}.
\end{aligned}
$$

• $\delta > \gamma$: We will use the fact that $\mathcal{E}(\delta A)_{\cdot}\mathcal{E}(\gamma A)_{\cdot}^{-1} = \mathcal{E}(\widetilde{A}^{\delta,\gamma})_{\cdot}$ is non-decreasing; see (iv) and (vii) Lemma 2.3.1. Starting at the left-hand side of (2.39), we can proceed exactly like in the previous case except that now we transfer $\mathcal{E}(\delta A)_{t-}\mathcal{E}(\gamma A)_{t-}^{-1}$ inside the integral and we bound it from above by $\mathcal{E}(\delta A)_{s-}\mathcal{E}(\gamma A)_{s-}^{-1}$, for $s \geq t$. After the simplification we have the same formulas with the difference that we have $\mathcal{E}(\delta A)_{s-}$ in the place of $\mathcal{E}(\gamma A)_{s-}$ inside the Lebesgue–Stieltjes integral.

Combining the two cases we get

$$
\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} \leq 8\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 8\,\frac{1+\gamma\Phi}{\gamma}\,\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.
$$

What remains is a bound for $\|\eta\|_{\mathcal{H}^2_\delta(\mathbb{R}^d)}$. For this we are going to use the identity $\int_t^T \mathrm{d}\eta_s = \xi - y_t + F(t)$. So, let $\eta = (\eta^1, ..., \eta^d)$. We have per coordinate $i \in \{1, ..., d\}$ that

$$
\begin{aligned}
(\eta^i_T - \eta^i_{T\wedge t})^2 &= (\eta^i_T)^2 - (\eta^i_{T\wedge t})^2 - 2\eta^i_{T\wedge t}(\eta^i_T - \eta^i_{T\wedge t})\\
&= ((\eta^i_T)^2 - \langle\eta^i\rangle_T) - ((\eta^i_{T\wedge t})^2 - \langle\eta^i\rangle_{T\wedge t}) - 2\eta^i_{T\wedge t}(\eta^i_T - \eta^i_{T\wedge t}) + \langle\eta^i\rangle_T - \langle\eta^i\rangle_{T\wedge t}.
\end{aligned}
$$

Hence, because for every $\mathbb{G}-$martingale M and $t \in \mathbb{R}_+$ holds that $\mathbb{E}\left[M_T\middle|\mathcal{G}_t\right] = M_{T\wedge t}$ and $\eta^i, (\eta^i)^2 - \langle\eta^i\rangle$ are $\mathbb{G}-$martingales, we have from the linearity property of the conditional expectation that

$$
\mathbb{E}[|\xi - y_t + F(t)|^2|\mathcal{G}_t] = \mathbb{E}[|\eta_T - \eta_{T\wedge t}|^2|\mathcal{G}_t] = \mathbb{E}\left[\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\middle|\mathcal{G}_t\right]. \qquad (2.40)
$$

Calculating with the help of (2.13) we get

$$\begin{aligned}\int_0^T \mathcal{E}(\delta A)_{s-}\mathrm{d}\mathrm{Tr}[\langle\eta\rangle_s] &= \delta\int_0^T\int_0^{s-}\mathcal{E}(\delta A)_{t-}\,\mathrm{d}A_t\,\mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s + \mathrm{Tr}[\langle\eta\rangle]_T - \mathrm{Tr}[\langle\eta\rangle]_0\\ &= \delta\int_0^T\int_0^T \mathbb{1}(s)_{\rrbracket t,T\rrbracket}\mathcal{E}(\delta A)_{t-}\,\mathrm{d}A_t\,\mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s + \mathrm{Tr}[\langle\eta\rangle]_T - \mathrm{Tr}[\langle\eta\rangle]_0\\ &= \delta\int_0^T \mathcal{E}(\delta A)_{t-}\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\,\mathrm{d}A_t + \mathrm{Tr}[\langle\eta\rangle]_T - \mathrm{Tr}[\langle\eta\rangle]_0\\ &\leq \delta\int_0^T \mathcal{E}(\delta A)_{t-}\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\,\mathrm{d}A_t + \mathrm{Tr}[\langle\eta\rangle]_T.\end{aligned}$$

Hence, we have

$$\|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} = \mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{s-}\mathrm{d}\mathrm{Tr}[\langle\eta\rangle_s]\right] \leq \delta\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\,\mathrm{d}A_t\right] + \mathbb{E}\left[\mathrm{Tr}[\langle\eta\rangle]_T\right].$$

For the first term in the right side of the above inequality, using the fact that A is predictable and from (2.13), (2.31), (2.40) we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\,\mathrm{d}A_t\right] &= \mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}\left[\int_t^T \mathrm{d}\mathrm{Tr}[\langle\eta\rangle]_s\middle|\mathcal{G}_t\right]\,\mathrm{d}A_t\right]\\ &= \mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}[|\xi - y_t + F(t)|^2|\mathcal{G}_t]\,\mathrm{d}A_t\right]\\ &\leq 3\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}[|\xi|^2 + |y_t|^2 + |F(t)|^2|\mathcal{G}_t]\,\mathrm{d}A_t\right]\\ &\leq 3\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}|\xi|^2\,\mathrm{d}A_t\right] + 3\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}[|F(t)|^2|\mathcal{G}_t]\,\mathrm{d}A_t\right]\\ &\quad +6\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}[|\xi|^2 + |F(t)|^2|\mathcal{G}_t]\,\mathrm{d}A_t\right]\\ &= 9\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}|\xi|^2\,\mathrm{d}A_t\right] + 9\,\mathbb{E}\left[\int_0^T \mathcal{E}(\delta A)_{t-}\mathbb{E}[|F(t)|^2|\mathcal{G}_t]\,\mathrm{d}A_t\right]\\ &\leq \frac{9(1+\delta\Phi)}{\delta}\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 9\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.\end{aligned}$$

For the second term from (2.31) and (2.34) we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}[\langle\eta\rangle]_T\right] &= \mathbb{E}\left[|\xi - y_0 + F(0)|^2\right] \leq 3\,\mathbb{E}\left[|\xi|^2\right] + 3\,\mathbb{E}\left[|y_0|^2\right] + 3\,\mathbb{E}\left[F(0)|^2\right]\\ &\leq 9\,\mathbb{E}\left[|\xi|^2\right] + 9\,\mathbb{E}\left[F(0)|^2\right]\\ &\leq 9\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + \frac{9}{\gamma\vee\delta}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.\end{aligned}$$

Combining the above we have

$$\|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq 9(2+\delta\Phi)\,\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 9\left(\frac{1}{\gamma\vee\delta} + \delta\Lambda^{\gamma,\delta,\Phi}\right)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.$$

□

Let $\mathcal{C}_\beta := \{(\gamma,\delta)\in(0,\beta]^2 : \gamma<\delta\}$. We define

$$M^\Phi_\star(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta}\left\{\frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\delta}{\delta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\}$$

and

$$\widetilde{M}^\Phi(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta}\left\{\frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\delta}{\delta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\}.$$

To complete our analysis we give asymptotic bounds for $M^\Phi_\star(\beta)$ and $\widetilde{M}^\Phi(\beta)$ with respect to Φ.

Proposition 2.9.2. *For $\Phi\geq 0$ and $\beta\in(0,\infty)$ we have*

$$\begin{aligned} M^\Phi_\star(\beta) &= \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\} \\ &= \frac{6\sqrt{17}+35}{\beta} + \left(6\sqrt{17}+26\right)\Phi \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} \widetilde{M}^\Phi(\beta) &= \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\} \\ &= \frac{2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17} + \frac{4}{\beta} + 35}{\beta} + \left(2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17} + \frac{4}{\beta} + 26\right)\Phi. \end{aligned} \tag{2.42}$$

Hence, we get that

$$\lim_{\beta\to\infty} M^\Phi_\star(\beta) = \lim_{\beta\to\infty}\widetilde{M}^\Phi(\beta) = \left(6\sqrt{17}+26\right)\Phi. \tag{2.43}$$

Proof. Here we will present the part of the results that is needed in the main text, specifically (2.41),(2.42). The remainder of the proof will be given in Appendix A.1.

From the defining formulas of the $M^\Phi_\star(\beta), \widetilde{M}^\Phi(\beta)$ we make a couple of observations. The first one is that we should only examine the case $\delta=\beta$, because for every pair $(\gamma,\delta)\in\mathcal{C}_\beta$ we have

$$\begin{aligned} \frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\delta}{\delta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma} &\geq \frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}, \\ \frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\delta}{\delta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma} &\geq \frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}. \end{aligned}$$

The second one is that

$$\lim_{\gamma\to 0^+}\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma} = \lim_{\gamma\to\beta^-}\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma} = \infty$$

and

$$\lim_{\gamma\to 0^+} \frac{9}{\beta} \;+\; 8\frac{(1+\gamma\Phi)}{\gamma} \;+\; \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma} \quad=\quad \lim_{\gamma\to\beta^-} \frac{9}{\beta} \;+\; 8\frac{(1+\gamma\Phi)}{\gamma} \;+\; \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma} \quad=\quad \infty.$$

So, we have

$$M_\star^\Phi(\beta) = \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma}\right\}$$

and

$$\widetilde{M}^\Phi(\beta) = \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\}.$$

□

Chapter 3

Existence and uniqueness of solutions for mean-field systems of BSDEs and McKean–Vlasov BSDEs

We model the motions of $n-$particles in a closed system as the solution $\mathbf{Y}^N := (Y^{i,N})_{1\leq i\leq N}$ of the following BSDE system

$$\begin{aligned} Y_t^{i,N} = \xi^{i,N} + \int_t^T f\left(s, Y_s^{i,N}, Z_s^{i,N} c_s^i, \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta)}(U^{i,N})_s, L^N(\mathbf{Y}_s^N)\right) \mathrm{d}C_s^{\overline{X}^i} \\ - \int_t^T Z_s^{i,N}\,\mathrm{d}X_s^{i,\circ} - \int_t^T \int_{\mathbb{R}^n} U_s^{i,N}(x)\,\widetilde{\mu}^{(\mathbb{F}^{1,\dots,N},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s^{i,N}, \quad i = 1,\dots,N. \end{aligned} \tag{3.1}$$

The interaction of the solutions $\{Y^i\}_{1\leq i\leq n}$ appears in the last argument of the generator f as the respective (random) empirical measure. Given the comments in the introduction of chapter 4, one expects that, by an appropriate application of the law of large numbers, the empirical measures converge to a (deterministic) law, rendering thus the interactions weaker and weaker. Obviously, the absence of the empirical measure translates the above system to n non-interacting equations of the same type. In order to identify the asymptotic, as N tends to ∞, behaviour of the solution $\mathbf{Y}^N$ of the aforementioned mean-field system, we will need the McKean–Vlasov BSDE of the form

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y_s)\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s\,\mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^n} U_s\,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s. \end{aligned} \tag{3.2}$$

For the well-posedness of the aforementioned BSDEs we will make use of conditions corresponding to square-integrability of the data. One of the first works that employed coupling techniques of this kind was Sznitman [46], some recent references can be found in Cardaliaguet et al. [10], Delarue et al. [16].

In our work, central role play the a priori estimates of Lemma 2.9.1. They are an improvement of the a priori estimates of Papapantoleon et al. [42] to the case when the stochastic exponential is used instead of the classical one. Not only do we use them to prove the well-posedness of the mean-field BSDE as well as the McKean–Vlasov BSDE, but also to prove the backward propagation of chaos in the next chapter.

Now we are going to provide general existence and uniqueness results for McKean–Vlasov and mean-field BSDEs, in a setting where the filtrations can be stochastically discontinuous and the stochastic integrals are defined with respect to general $\mathbb{L}^2$–martingales. More specifically, we are going to consider first a "path-dependent" version of McKean–Vlasov BSDEs and mean-field systems of BSDEs, where the generator depends on the initial segment of the solution Y, see (3.3) and (3.9). Then, we will also provide existence and uniqueness results for "classical" McKean–Vlasov BSDEs and systems of mean-field BSDEs that depend only on the instantaneous value of Y, see (3.2) and (3.1), under weaker assumptions.

3.1 McKean–Vlasov BSDE

At this point we are ready to introduce our setting for the first existence and uniqueness theorem. We are given a stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ that satisfies the usual conditions and supports the following:

(F1) A couple of martingales $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ that satisfy $M_{\mu^{X^\natural}}[\Delta X^\circ | \widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$, where $\mu^{X^\natural}$ is the random measure generated by the jumps of $X^\natural$.[1]

(F2) A $\mathbb{G}-$stopping time T and terminal condition $\xi \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{G}_T, A^{(\mathbb{G},\overline{X},f)};\mathbb{R}^d)$, for a $\hat{\beta} > 0$ and $A^{(\mathbb{G},\overline{X},f)}$ the one defined in **(F5)** below.

(F3) Functions Θ, Γ as in Definition 2.5.1, where the data for the definition are the pair $(\mathbb{G}, \overline{X})$, the process $C^{(\mathbb{G},\overline{X})}$ and the kernels $K^{(\mathbb{G},\overline{X})}$.

(F4) A generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y, z, u, \mu) \in \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d)$, the map

$$(\omega, t) \longmapsto f(\omega, t, y, z, u, \mu) \text{ is } \mathbb{G} - \text{progressively measurable}$$

and satisfies the following Lipschitz condition

$$\begin{aligned}&|f(\omega,t,y,z,u,\mu) - f(\omega,t,y',z',u',\mu')|^2 \\ &\quad\le r(\omega,t)\,\rho^2_{J_1^d}(y,y') + \vartheta^o(\omega,t)\,|z-z'|^2 + \vartheta^\natural(\omega,t)\,|u-u'|^2 + \vartheta^*(\omega,t)\,W^2_{2,\rho_{J_1^d}}(\mu,\mu'),\end{aligned}$$

where $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : \left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(F5) Define $\alpha^2 := \max\{\sqrt{r}, \vartheta^o, \vartheta^\natural, \sqrt{\vartheta^*}\}$. For the $\mathbb{G}$-predictable and càdlàg process

$$A^{(\mathbb{G},\overline{X},f)}_\cdot := \int_0^\cdot \alpha_s^2 \,\mathrm{d}C_s^{(\mathbb{G},\overline{X})}$$

there exists $\Phi > 0$ such that $\Delta A^{(\mathbb{G},\overline{X},f)} \le \Phi$, $\mathbb{P} \otimes C^{(\mathbb{G},\overline{X})} -$ a.e.

(F6) For the same $\hat{\beta}$ as in **(F2)** there exists $\Lambda_{\hat{\beta}} > 0$ such that $\mathcal{E}\left(\hat{\beta} A^{(\mathbb{G},\overline{X},f)}\right)_T \le \Lambda_{\hat{\beta}}$ $\mathbb{P}-$a.s.

[1] Since the filtration $\mathbb{G}$ is given as well as the pair $\overline{X}$, we will make use of $C^{(\mathbb{G},\overline{X})}$, resp. $c^{(\mathbb{G},\overline{X})}$, as defined in (2.9), resp. (2.11). Moreover, we will use the kernels $K^{(\mathbb{G},\overline{X})}$ as determined by (2.10).

(F7) For the same $\hat{\beta}$ as in **(F2)** we have

$$\mathbb{E}\left[\int_0^T \mathcal{E}\left(\hat{\beta} A^{(\mathbb{G},\overline{X},f)}\right)_{s-} \frac{|f(s,0,0,0,\delta_0)|^2}{\alpha_s^2} \mathrm{d}C_s^{(\mathbb{G},\overline{X})}\right] < \infty,$$

where δ_0 is the Dirac measure on the domain of the last variable concentrated at 0, the neutral element of the addition.

Remark 3.1.1. *Let us provide a couple of remarks regarding the notation and description we used in the conditions we imposed:*

(i) *In* **(F5)***, at the notation of* α *we have suppressed the dependence on* $(\mathbb{G}, f)$*, but we have carried it on the notation of* $A^{(\mathbb{G},\overline{X},f)}$*.*

(ii) *In* **(F7)***, and in view of* ***(F6)****, the integrability condition in* **(F7)** *could be equivalently described by* $\|\frac{f}{\alpha}\|_{\mathbb{H}^2(\mathbb{G},C^{(\mathbb{G},\overline{X})};\mathbb{R}^d)} < \infty$*. Indeed, under* **(F6)** *every* $\hat{\beta}-$*norm is equivalent to its* $0-$*counterpart. However, later we will weaken* **(F6)***, hence we prefer to write the integrability condition by means of the stochastic exponential.*

(iii) *In* **(F7)***, and in view of* **(F4)** *where the probability measures are defined on the Skorokhod space, the neutral element of the addition is the constant function which equals to 0. Later, we will deal with generators whose last variable will be probability measures defined on the Euclidean space* $\mathbb{R}^d$*, see* ***(F4′)*** *defined below. Hence, in this case* 0 *will denote the origin of* $\mathbb{R}^d$*.*

Now, we consider the McKean–Vlasov BSDE of the form (see (2.28))

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y|_{[0,s]}, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s \,\mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^n} U_s \,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s. \end{aligned} \tag{3.3}$$

Definition 3.1.2. *A set of data* $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ *that satisfies the assumptions* **(F1)-(F7)** *will be called* standard under $\hat{\beta}$ *for the path dependent McKean–Vlasov BSDE* (3.3)*.*

It follows the existence and uniqueness result for the solution of the McKean–Vlasov BSDE (3.3) under the path dependence.

Theorem 3.1.3. *Let* $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ *be standard data under* $\hat{\beta}$ *for the path dependent McKean–Vlasov BSDE* (3.3)*. If*

$$\max\left\{2, \frac{2\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M_\star^\Phi(\hat{\beta}) < 1,$$

then the McKean–Vlasov BSDE

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y|_{[0,s]}, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s \,\mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^n} U_s \,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s \end{aligned} \tag{3.3}$$

admits a unique solution

$$(Y,Z,U,M)\in\mathcal{S}^2(\mathbb{G};\mathbb{R}^d)\times\mathbb{H}^2(\mathbb{G},X^\circ;\mathbb{R}^{d\times p})\times\mathbb{H}^2(\mathbb{G},X^\natural;\mathbb{R}^d)\times\mathcal{H}^2(\mathbb{G},\overline{X}^{\perp_{\mathbb{G}}};\mathbb{R}^d).^2$$

Proof. Let us initially (re-)mention that assumption **(F6)** leads to equivalent norms. However, we will need the $\hat{\beta}-$norms in order to construct the contraction, which will ultimately provide the fixed-point we are seeking.

Regarding the notation we will use in the rest of the proof, since we have fixed the set of standard data under $\hat{\beta}$, for the convenience of the reader we will ease the notation by dropping the dependence on $\mathbb{G},\overline{X}$ and f. More precisely, the objects $C^{(\mathbb{G},\overline{X})}$, $c^{(\mathbb{G},\overline{X})}$, $A^{(\mathbb{G},\overline{X},f)}$, $K^{(\mathbb{G},\overline{X})}$, $\Gamma^{(\mathbb{G},\overline{X},\Theta)}$, and $|||\cdot|||^{(\mathbb{G},\overline{X})}$ will be simply denoted by, respectively, C, c, A, K, Γ^Θ, and $|||\cdot|||$. Additionally, we introduce the symbol $\mathcal{H}^2_{\hat{\beta}}$ for the product space

$$\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})\times\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,X^\natural;\mathbb{R}^d)\times\mathcal{H}^2_{\hat{\beta}}(\mathbb{G},A,\overline{X}^{\perp_{\mathbb{G}}};\mathbb{R}^d),$$

whose norm corresponds to the sum of the respective norms.

Let us begin with a quadruple $(y,z,u,m)\in\mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d)\times\mathcal{H}^2_{\hat{\beta}}$. Following the classical approach, see *e.g.* [42, Theorem 3.5], for the given (y,z,u,m) we get from the representation of the martingale

$$\mathbb{E}\left[\xi+\int_0^T f\left(s,y|_{[0,s]},z_sc_s,\Gamma^\Theta(u)_s,\mathcal{L}(y|_{[0,s]})\right)\,\mathrm{d}C_s\middle|\mathcal{G}_\cdot\right]$$

a unique[3] triple of processes $(Z,U,M)\in\mathcal{H}^2_{\hat{\beta}}$, as long as

$$\left\|\frac{g}{\alpha}\right\|_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)}<\infty \text{ for } g_\cdot:=f(\cdot,y|_{[0,\cdot]},z_\cdot c_\cdot,\Gamma^\Theta(u)_\cdot,\mathcal{L}(y|_{[0,\cdot]})). \tag{3.4}$$

Then, we define the $\mathbb{G}-$semimartingale

$$Y_\cdot:=\mathbb{E}\left[\xi+\int_\cdot^T f\left(s,y|_{[0,s]},z_sc_s,\Gamma^\Theta(u)_s,\mathcal{L}(y|_{[0,s]})\right)\,\mathrm{d}C_s\middle|\mathcal{G}_\cdot\right],$$

where we use its càdlàg version, and we obtain the identity

$$Y_t=\xi+\int_t^T f\left(s,y|_{[0,s]},z_sc_s,\Gamma^\Theta(u)_s,\mathcal{L}(y|_{[0,s]})\right)\,\mathrm{d}C_s \\ -\int_t^T Z_s\,\mathrm{d}X^\circ_s-\int_t^T\int_{\mathbb{R}^n}U_s\,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x)-\int_t^T\mathrm{d}M_s.$$

[2]The reader may recall remark 2.4.1 and the fact that under **(F5)** the $\hat{\beta}-$norms are equivalent to their counterparts.
[3]Of course, we use the convention that a class is represented by its elements.

We have postponed the verification of (3.4), whose validity we present now: by using the trivial inequality $(a+b)^2 \leq 2a^2 + 2b^2$ one derives

$$\int_0^T \mathcal{E}(\hat{\beta}A)_{s-} \frac{\left|f\left(s, y|_{[0,s]}, z_s c_s, \Gamma^\Theta(u)_s, \mathcal{L}(y|_{[0,s]})\right)\right|^2}{\alpha_s^2} \mathrm{d}C_s$$
$$\leq 2\int_0^T \mathcal{E}(\hat{\beta}A)_{s-} \frac{\left|f(s,0,0,0,\delta_0) - f\left(s, y|_{[0,s]}, z_s c_s, \Gamma^\Theta(u)_s, \mathcal{L}(y|_{[0,s]})\right)\right|^2}{\alpha_s^2} \mathrm{d}C_s$$
$$+ 2\int_0^T \mathcal{E}(\hat{\beta}A)_{s-} \frac{|f(s,0,0,0,\delta_0)|^2}{\alpha_s^2} \mathrm{d}C_s$$

and afterwards, from **(F7)** and an application of the Lipschitz property as described in **(F4)** one gets[4]

$$\left\| \frac{f\left(\cdot, y|_{[0,\cdot]}, z_\cdot c_\cdot, \Gamma^\Theta(u)_\cdot, \mathcal{L}(y|_{[0,\cdot]})\right)}{\alpha_\cdot} \right\|_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)} < \infty.$$

So, the above computations, Lemma 2.9.1 and Proposition 2.9.2 provide that

$$\|Y\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d)} \leq 8\|\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{G}_T,A;\mathbb{R}^d)} + M^\Phi_\star(\hat{\beta}) \left\| \frac{f\left(\cdot, y|_{[0,\cdot]}, z_\cdot c_\cdot, \Gamma^\Theta(u)_\cdot, \mathcal{L}(y|_{[0,\cdot]})\right)}{\alpha_\cdot} \right\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)}.$$

Summing up the above arguments, to each quadruple $(y,z,u,m) \in \mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}$ we have uniquely associated a new one (Y,Z,U,M) lying in the same space. Hence, we can define the function

$$S : \mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}} \longrightarrow \mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}$$

with

$$S(y,z,u,m) := (Y,Z,U,M).$$

We proceed to prove that under the assumption $\max\left\{2, \frac{2\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M^\Phi_\star(\hat{\beta}) < 1$ the function S is a contraction, so that by Banach's fixed point theorem we get the unique solution that we want. Let $(y^j, z^j, u^j, m^j) \in \mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A;\mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}$ for $j = 1,2$. For $t \in [0,T]$ we define

$$\psi_t := f\left(t, y^2|_{[0,t]}, z^2_t c_t, \Gamma^\Theta(u^2)_t, \mathcal{L}\left(y^2|_{[0,t]}\right)\right) - f\left(t, y^1|_{[0,t]}, z^1_t c_t, \Gamma^\Theta(u^1)_t, \mathcal{L}\left(y^1|_{[0,t]}\right)\right).$$

[4]We provide similar computations below in (3.6). Hence, for the sake of compactness, we omit at this point the detailed computations.

From **(F4)** and Lemma 2.5.3 we have[5]

$$\begin{aligned}\left|\frac{\psi_t}{\alpha_t}\right|^2 &\leq \alpha_t^2\, \rho^2_{J_1^d}(y^2|_{[0,t]}, y^1|_{[0,t]}) + \|(z_t^2 - z_t^1)c_t\|^2 + 2\, \left|\!\left|\!\left| u_t^2 - u_t^1 \right|\!\right|\!\right|_t^2 \\ &\quad + \alpha_t^2\, W^2_{2,\rho_{J_1^d}}(\mathcal{L}(y^2|_{[0,t]}), \mathcal{L}(y^1|_{[0,t]})) \\ &\leq \alpha_t^2 \sup_{s\in[0,t]} \{|y_s^2 - y_s^1|^2\} + \|(z_t^2 - z_t^1)c_t\|^2 + 2\, \left|\!\left|\!\left| u_t^2 - u_t^1 \right|\!\right|\!\right|_t^2 + \alpha_t^2\, \mathbb{E}\Big[\sup_{s\in[0,t]} \{|y_s^2 - y_s^1|^2\}\Big].\end{aligned}$$

In the last inequality we used the fact that

$$\begin{aligned}W^2_{2,\rho_{J_1^d}}\left(\mathcal{L}(y^2|_{[0,t]}), \mathcal{L}(y^1|_{[0,t]})\right) &\leq \int_{\mathbb{D}^d\times\mathbb{D}^d} \rho_{J_1^d}(x,z)^2 \pi(\mathrm{d}x, \mathrm{d}z) = \mathbb{E}\left[\rho_{J_1^d}\left(y^2|_{[0,t]}, y^1|_{[0,t]}\right)^2\right] \\ &\overset{(2.26)}{\leq} \mathbb{E}\Big[\sup_{s\in[0,t]} \{|y_s^2 - y_s^1|^2\}\Big],\end{aligned}$$

where we chose π to be the image measure on $\mathbb{D}^d \times \mathbb{D}^d$ produced by the measurable function $(y^1|_{[0,t]}, y^2|_{[0,t]}) : \Omega \longrightarrow \mathbb{D}^d \times \mathbb{D}^d$. Hence,

$$\begin{aligned}\mathcal{E}(\hat\beta A)_{t-} \left|\frac{\psi_t}{\alpha_t}\right|^2 &\leq \alpha_t^2\, \mathcal{E}(\hat\beta A)_{t-} \sup_{s\in[0,t]} \{|y_s^2 - y_s^1|^2\} + \mathcal{E}(\hat\beta A)_{t-} \|(z_t^2 - z_t^1)c_t\|^2 \\ &\quad + 2\,\mathcal{E}(\hat\beta A)_{t-} \left|\!\left|\!\left| u_t^2 - u_t^1 \right|\!\right|\!\right|_t^2 + \alpha_t^2\, \mathcal{E}(\hat\beta A)_{t-} \mathbb{E}\Big[\sup_{s\in[0,t]} \{|y_s^2 - y_s^1|^2\}\Big].\end{aligned} \tag{3.5}$$

Then, we integrate with respect to the measure $\mathbb{P}\otimes C$ in order to get from **(F6)**, (2.18) and (2.19) that

$$\begin{aligned}\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)} &\leq \frac{\Lambda_{\hat\beta}}{\hat\beta}\, \|y^2 - y^1\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)} + \|z^2 - z^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} \\ &\quad + 2\,\|u^2 - u^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\natural;\mathbb{R}^d)} + \frac{1}{\hat\beta}\, \mathbb{E}\Big[\mathcal{E}(\hat\beta A)_T\Big] \|y^2 - y^1\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)} \\ &\leq \frac{2\Lambda_{\hat\beta}}{\hat\beta} \|y^2 - y^1\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)} + \|z^2 - z^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} + 2\,\|u^2 - u^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\natural;\mathbb{R}^d)} \\ &\leq \max\left\{2, \frac{2\Lambda_{\hat\beta}}{\hat\beta}\right\} \|(y^2 - y^1, z^2 - z^1, u^2 - u^1, m^2 - m^1)\|^2_{\star,\hat\beta,\mathbb{G},A,\overline{X}}.\end{aligned} \tag{3.6}$$

[5]We use the fact that $\frac{r}{\alpha^2} \leq \alpha^2$, $\vartheta^\circ \leq \alpha^2$, $\vartheta^\natural \leq \alpha^2$ and $\frac{\vartheta^*}{\alpha^2} \leq \alpha^2$.

Let now $S(y^i,z^i,u^i,m^i)=(Y^i,Z^i,U^i,M^i)$, for $i=1,2$. We will apply lemma 2.9.1 for Y^2-Y^1; the reader should observe that $Y^2_T-Y^1_T=0$. Consequently,

$$\begin{aligned}
&\|S(y^2,z^2,u^2,m^2)-S(y^1,z^1,u^1,m^1)\|^2_{\star,\hat\beta,\mathbb{G},A,\overline{X}}\\
&\quad= \|Y^2-Y^1\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)}+\|(Z^2-Z^1,U^2-U^1,M^2-M^1)\|_{\mathcal{H}^2_{\hat\beta}}\\
&\quad\le M^\Phi_\star(\hat\beta)\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)}\\
&\quad\overset{(3.6)}{\le} \max\left\{2,\frac{2\Lambda_{\hat\beta}}{\hat\beta}\right\}M^\Phi_\star(\hat\beta)\|(y^2-y^1,z^2-z^1,u^2-u^1,m^2-m^1\|^2_{\star,\hat\beta,\mathbb{G},A,\overline{X}}.
\end{aligned}$$

Hence, we obtain the desired contraction if $\max\left\{2,\frac{2\Lambda_{\hat\beta}}{\hat\beta}\right\}M^\Phi_\star(\hat\beta)<1$. □

Remark 3.1.4. *Let us provide some remarks related to specific points of the proof of Theorem 3.1.3:*

(i) *For $y\in\mathcal{S}^2_{\hat\beta}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)};\mathbb{R}^d)$ it is easy to show from the inequality*

$$W^2_{2,\rho_{J^d_1}}\Big(\mathcal{L}(y|_{[0,t_2]}),\mathcal{L}(y|_{[0,t_1]})\Big)\le\mathbb{E}\Big[\sup_{s\in(t_1,t_2]}\{|y_s-y_{t_1}|^2\}\Big],$$

for (real) numbers $0\le t_1<t_2$, that $\mathcal{L}(y|_{[0,\cdot]})$ is a càdlàg (deterministic) process.

(ii) *It seems that it is inevitable the imposition of the bound $\Lambda_{\hat\beta}$, if we want to consider BSDEs whose generator f depends on the initial segment of the paths of the solution Y. Indeed, in* (3.5) *we need to multiply the stochastic exponential with the (square of the) running maximum of y^2-y^1. One could possible be able to proceed without an assumption on boundedness of the stochastic exponential, if it was possible to extract* a priori *estimates for the running maximum analogous to lemma 2.9.1. In this case, one expects in lemma 2.9.1 an integrability condition of the form $\mathbb{E}[\mathcal{E}(\hat\beta A_{T-})\sup_{s\in[0,T]}|y_s|^2]<\infty$ to appear, which is clearly stronger than the $\mathcal{S}^2-$norm we are using. Unfortunately, we were not able to extract such an* a priori *estimate.*

(iii) *If we consider a BSDE whose generator f depends at time s on the instantaneous value of the soultion Y,* e.g., *Y_s or Y_{s-}, and not on the initial -up to time s- segment of its paths,* e.g., *$Y|_{[0,s]}$ or $Y|_{[0,s-]}$, then we can proceed under the assumption that $\mathcal{E}(\hat\beta A^{(\mathbb{G},\overline{X},f)})_T$ is integrable, for $\hat\beta$ the one determining the standard data. However, in this case we will need to seek a solution such that*

$$Y\in\mathcal{S}^2_{\hat\beta}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)};\mathbb{R}^d)\ \textit{and}\ \alpha Y\in\mathbb{H}^2_{\hat\beta}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)},C^{(\mathbb{G},\overline{X})};\mathbb{R}^d),$$

instead of simply $Y\in\mathcal{S}^2_{\hat\beta}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)};\mathbb{R}^d)$.

In view of the previous remarks, we will close this subsection by considering the McKean–Vlasov BSDE whose generator depends at each time $s\in[0,T]$ only on the instantaneous value of Y, *e.g.*, Y_s or Y_{s-}.

To this end, we need to reformulate assumptions **(F4)** and **(F6)** as follows:

(F4′) A generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y, z, u, \mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the map

$$(\omega, t) \longmapsto f(\omega, t, y, z, u, \mu) \text{ is } \mathbb{G} - \text{progressively measurable}$$

and satisfies the following Lipschitz condition

$$|f(\omega, t, y, z, u, \mu) - f(\omega, t, y', z', u', \mu')|^2 \\ \leq r(\omega, t)\, |y - y'|^2 + \vartheta^o(\omega, t)\, |z - z'|^2 + \vartheta^\natural(\omega, t)\, |u - u'|^2 + \vartheta^*(\omega, t)\, W^2_{2,|\cdot|}(\mu, \mu'),$$

where $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : \left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(F6′) For the same $\hat{\beta}$ as in **(F2)** the process $\mathcal{E}\left(\hat{\beta} A^{(\mathbb{G}, \overline{X}, f)}\right)$ is integrable.

We introduce the following convenient notation, where α is the process determined in **(F5)**:

$$\mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, \alpha, C^{(\mathbb{G},\overline{X})}; \mathbb{R}^d) := \{y \in \mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X},f)}; \mathbb{R}^d) : \|\alpha y\|_{\mathbb{H}_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X},f)}, C^{(\mathbb{G},\overline{X})}; \mathbb{R}^d)} < \infty\}$$

with associated norm defined by

$$\|\cdot\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{G},\alpha,C^{(\mathbb{G},\overline{X})};\mathbb{R}^d)} := \|\cdot\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)};\mathbb{R}^d)} + \|\alpha\cdot\|^2_{\mathbb{H}_{\hat{\beta}}(\mathbb{G},A^{(\mathbb{G},\overline{X},f)},C^{(\mathbb{G},\overline{X})};\mathbb{R}^d)}. \tag{3.7}$$

Definition 3.1.5. *A set of data* $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ *that satisfies the assumptions* **(F1)-(F3)**, **(F4′)**, **(F5)**, **(F6′)** *and* **(F7)** *will be called* standard under $\hat{\beta}$ *for the McKean–Vlasov BSDE* (3.2).

Theorem 3.1.6. *Let* $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ *be standard data under* $\hat{\beta}$ *for the McKean–Vlasov BSDE* (3.2). *If*

$$\max\left\{2, \frac{\mathbb{E}[\mathcal{E}(\hat{\beta} A_T)]}{\hat{\beta}}\right\} \widetilde{M}^{\Phi}(\hat{\beta}) < 1,$$

then the McKean–Vlasov BSDE

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y_s)\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s\, \mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^n} U_s\, \widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}M_s \tag{3.2}$$

admits a unique solution

$$(Y, Z, U, M) \in \mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, \alpha, C^{(\mathbb{G},\overline{X})}; \mathbb{R}^d) \times \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, X^\circ; \mathbb{R}^{d\times p}) \times \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, X^\natural; \mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, \overline{X}^{\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Proof. Adopting the notation of the proof of Theorem 3.1.3, we closely follow the arguments presented and we omit the steps that are identical or can be immediately adapted to the present case. We endow the product space $\mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, \alpha, C; \mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}$ with the norm which comes from the sum of the respective norms.

So, we begin with a quadruple $(y,z,u,m)\in\mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}$, for which we can prove that

$$\left\|\frac{f\left(\cdot,y_\cdot,z_\cdot c_\cdot,\Gamma^\Theta(u)_\cdot,\mathcal{L}(y_\cdot)\right)}{\alpha_\cdot}\right\|_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)}<\infty.$$

Hence, for the triple (Z,U,M) obtained from the martingale representation and the associated càdlàg vesrion of the $\mathbb{G}-$semimartingale Y, we have from Lemma 2.9.1 and Proposition 2.9.2 that

$$\begin{aligned}&\|Y\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)}+\|\alpha Y\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)}+\|(Z,U,M)\|_{\mathcal{H}^2_{\hat\beta}}\\&\qquad\le\left(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi\right)\|\xi\|^2_{\mathbb{L}^2_{\hat\beta}(\mathcal{G}_T,A;\mathbb{R}^d)}+\widetilde{M}^\Phi(\hat\beta)\left\|\frac{f\left(\cdot,y_\cdot,z_\cdot c_\cdot,\Gamma^\Theta(u)_\cdot,\mathcal{L}(y_\cdot)\right)}{\alpha_\cdot}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)},\end{aligned}$$

where we used (2.21) to equivalently write the left-hand side of the last inequality. In other words, the function

$$S:\mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}\longrightarrow\mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}$$

with

$$S(y,z,u,m):=(Y,Z,U,M)$$

is well-defined.

We proceed to prove that the function S is a contraction. Let $(y^j,z^j,u^j,m^j)\in\mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}$ for $j=1,2$. For $t\in[\![0,T]\!]$ we define

$$\psi_t:=f\left(t,y^2_t,z^2_tc_t,\Gamma^\Theta(u^2)_t,\mathcal{L}\left(y^2_t\right)\right)-f\left(t,y^1_t,z^1_tc_t,\Gamma^\Theta(u^1)_t,\mathcal{L}\left(y^1_t\right)\right).$$

From **(F4′)** and Lemma 2.5.3 we have

$$\begin{aligned}\mathcal{E}(\hat\beta A)_{t-}\left|\frac{\psi_t}{\alpha_t}\right|^2&\le\alpha_t^2\,\mathcal{E}(\hat\beta A)_{t-}|y^2_t-y^1_t|^2+\mathcal{E}(\hat\beta A)_{t-}\|(z^2_t-z^1_t)c_t\|^2\\&\qquad+2\,\mathcal{E}(\hat\beta A)_{t-}\left\|u^2_t-u^1_t\right\|^2_t+\alpha_t^2\,\mathcal{E}(\hat\beta A)_{t-}\mathbb{E}[|y^2_t-y^1_t|^2],\end{aligned}\tag{3.8}$$

which is the analogous to (3.5).

Then, we integrate with respect to the measure $\mathbb{P}\otimes C$ in order to get from **(F6′)**, (2.18) and (2.19) that

$$\begin{aligned}\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)}&\le\|\alpha(y^2-y^1)\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)}+\|z^2-z^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})}\\&\qquad+2\,\|u^2-u^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\natural;\mathbb{R}^d)}+\frac{1}{\hat\beta}\,\mathbb{E}\big[\mathcal{E}(\hat\beta A)_T\big]\|y^2-y^1\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)}\\&\le\max\left\{2,\frac{\mathbb{E}[\mathcal{E}(\hat\beta A)_T]}{\hat\beta}\right\}\left(\|y^2-y^1\|_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},A;\mathbb{R}^d)}+\|(z^2-z^1,u^2-u^1,m^2-m^1)\|^2_{\mathcal{H}^2_{\hat\beta}}\right).\end{aligned}$$

Let now $S(y^i, z^i, u^i, m^i) = (Y^i, Z^i, U^i, M^i)$, for $i = 1, 2$. We will apply Lemma 2.9.1 for $Y^2 - Y^1$; the reader should observe that $Y^2_T - Y^1_T = 0$. Consequently,

$$\begin{aligned}\|Y^2 - Y^1\|^2_{\mathbb{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)} &+ \|(Z^2 - Z^1, U^2 - U^1, M^2 - M^1)\|_{\mathcal{H}^2_{\hat\beta}} \\ &\leq \widetilde{M}^{\Phi}(\hat\beta)\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)} \\ \leq \max\left\{2, \frac{\mathbb{E}[\mathcal{E}(\hat\beta A_T)]}{\hat\beta}\right\}&\widetilde{M}^{\Phi}(\hat\beta)\|(y^2 - y^1, z^2 - z^1, u^2 - u^1, m^2 - m^1)\|^2_{\mathbb{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)\times\mathcal{H}^2_{\hat\beta}}.\end{aligned}$$

Hence, we obtain the desired contraction if $\max\left\{2, \frac{\mathbb{E}[\mathcal{E}(\hat\beta A_T)]}{\hat\beta}\right\}\widetilde{M}^{\Phi}(\hat\beta) < 1$. □

We may improve the condition under which one can prove the existence and the uniqueness of the solution of the McKean–Vlasov BSDE (3.2) by imposing stronger properties on $A^{(\mathbb{G},\overline{X},f)}$ and on the time horizon T. For the former we introduce the following condition

(F6″) The process $A^{(\mathbb{G},\overline{X},f)}$ is deterministic.

Remark 3.1.7. *If T in **(F2)** is deterministic and finite, then Condition* **(F6)** *is satisfied. However, when $T = \infty$ it does not necessarily hold. Either way,* **(F6)** *is not required in the method we are going to use .*

Theorem 3.1.8. *Let $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ satisfy* **(F1)-(F3)**,**(F4′)**,**(F5)**,**(F6″)** *and* **(F7)** *under $\hat\beta$. Let, additionally, T in* **(F2)** *being deterministic. If*

$$2\widetilde{M}^{\Phi}(\hat\beta) < 1,$$

then the McKean–Vlasov BSDE

$$\begin{aligned}Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y_s)\right)\,\mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s\,\mathrm{d}X^\circ - \int_t^T\int_{\mathbb{R}^n} U_s\,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s.\end{aligned} \tag{3.2}$$

admits a unique solution

$$(Y, Z, U, M) \in \mathbb{S}^2_{\hat\beta}(\mathbb{G}, \alpha, C^{(\mathbb{G},\overline{X})}; \mathbb{R}^d) \times \mathbb{H}^2_{\hat\beta}(\mathbb{G}, X^\circ; \mathbb{R}^{d\times p}) \times \mathbb{H}^2_{\hat\beta}(\mathbb{G}, X^\natural; \mathbb{R}^d) \times \mathcal{H}^2_{\hat\beta}(\mathbb{G}, \overline{X}^{\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Proof. Adopting the notation and the arguments used in the proof of theorem 3.1.6 we proceed from (3.8), *i.e.*,

$$\begin{aligned}\mathcal{E}(\hat\beta A)_{t-}\left|\frac{\psi_t}{\alpha_t}\right|^2 \leq \alpha_t^2\,\mathcal{E}(\hat\beta A)_{t-}|y_t^2 - y_t^1|^2 + \mathcal{E}(\hat\beta A)_{t-}\|(z_t^2 - z_t^1)c_t\|^2 \\ + 2\,\mathcal{E}(\hat\beta A)_{t-}\left\|\!\left| u_t^2 - u_t^1 \right|\!\right\|_t^2 + \alpha_t^2\,\mathcal{E}(\hat\beta A)_{t-}\mathbb{E}[|y_t^2 - y_t^1|^2].\end{aligned}$$

From **(F6″)**, by integrating with respect to $\mathbb{P}\otimes C$ and by applying Tonelli's theorem[6] in the last summand of the right-hand side of the inequality, we obtain

$$\begin{aligned}\left\|\frac{\psi}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)} &\leq 2\left\|\alpha(y^2-y^1)\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,C;\mathbb{R}^d)} + \left\|z^2-z^1\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} \\ &\quad +2\left\|u^2-u^1\right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A,X^\natural;\mathbb{R}^d)} \\ &\leq 2\|y^2-y^1\|_{\mathbb{S}^2_{\hat\beta}(\mathbb{G},\alpha,C;\mathbb{R}^d)} + 2\|(z^2-z^1,u^2-u^1,m^2-m^1)\|^2_{\mathcal{H}^2_{\hat\beta}}.\end{aligned}$$

Hence, by applying Lemma 2.9.1 the contraction is obtained for $2\widetilde{M}^\Phi(\hat\beta)<1$. □

Remark 3.1.9. *In Theorem 3.1.3 and Theorem 3.1.6 the maximum is taken over the number* 2 *and another quantity. The number* 2 *essentially appears because of the Lipschitz constant of Lemma 2.5.3. So the conditions for these theorems can be improved, if one uses a different form of a* Γ *function, whose Lipschitz constant is smaller. However, this is not the case for Theorem 3.1.8, in which we have twice the same term after the application of Tonelli. So, the number 2 in the condition of Theorem 3.1.8 can only worsen for a different form of a* Γ *function.*

3.2 Mean-field system of BSDEs

In the current subsection we are given a stochastic basis $(\Omega,\mathcal{G},\mathbb{G},\mathbb{P})$ that satisfies the usual conditions and supports the following:

(G1) N couples of martingales $\{\overline{X}^i := (X^{i,\circ},X^{i,\natural})\}_{i\in\{1,\dots,N\}} \in \Big(\mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)\Big)^N$ that satisfy $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{G}}]=0$, for $i\in\{1,\dots,N\}$, where $\mu^{X^{i,\natural}}$ is the random measure generated by the jumps of $X^{i,\natural}$.[7]

(G2) A $\mathbb{G}$ stopping time T and terminal conditions $\{\xi^{i,N}\}_{i\in\{1,\dots,N\}}\in\prod_{i=1}^N\mathbb{L}^2_{\hat\beta}(\mathcal{G}_T,A^{(\mathbb{G},\overline{X}^i,f)};\mathbb{R}^d)$, for a $\hat\beta>0$ and $\{A^{(\mathbb{G},\overline{X}^i,f)}\}_{i\in\{1,\dots,N\}}$ the ones defined in **(G5)**.

(G3) Functions $\{\Theta^i\}_{i\in\{1,\dots,N\}},\Gamma$ as in Definition 2.5.1, where for each $i\in\{1,\dots,N\}$ the data for the definition are the pair $(\mathbb{G},\overline{X}^i)$, the process $C^{(\mathbb{G},\overline{X}^i)}$ and the kernels $K^{(\mathbb{G},\overline{X}^i)}$.

(G4) A generator $f:\Omega\times\mathbb{R}_+\times\mathbb{D}^d\times\mathbb{R}^{d\times p}\times\mathbb{R}^d\times\mathcal{P}(\mathbb{D}^d)\longrightarrow\mathbb{R}^d$ such that for any $(y,z,u,\mu)\in\mathbb{D}^d\times\mathbb{R}^{d\times p}\times\mathbb{R}^d\times\mathcal{P}(\mathbb{D}^d)$, the map

$$(\omega,t)\longmapsto f(\omega,t,y,z,u,\mu)\ \text{ is }\ \mathbb{G}-\text{progressively measurable}$$

[6] It is at this point that we use the fact that T is deterministic.

[7] Since the filtration $\mathbb{G}$ is given as well as the pairs $\overline{X}^i$, for $i\in\{1,\dots,N\}$, we will make use of $C^{(\mathbb{G},\overline{X}^i)}$, resp. $c^{(\mathbb{G},\overline{X}^i)}$, as defined in (2.9), resp. (2.11). Moreover, we will use the kernels $K^{(\mathbb{G},\overline{X}^i)}$ as determined by (2.10).

and satisfies the following Lipschitz condition

$$
\begin{aligned}
&|f(\omega,t,y,z,u,\mu)-f(\omega,t,y',z',u',\mu')|^2\\
&\qquad\le\ r(\omega,t)\,\rho^2_{J_1^d}(y,y')+\ \vartheta^o(\omega,t)\,|z-z'|^2+\ \vartheta^\natural(\omega,t)\,|u-u'|^2+\vartheta^*(\omega,t)\,W^2_{2,\rho_{J_1^d}}(\mu,\mu')\,,
\end{aligned}
$$

where $(r,\vartheta^o,\vartheta^\natural,\vartheta^*):\left(\Omega\times\mathbb{R}_+,\mathcal{P}^{\mathbb{G}}\right)\longrightarrow\left(\mathbb{R}^4_+,\mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(G5) Define $\alpha^2:=\max\{\sqrt{r},\vartheta^o,\vartheta^\natural,\sqrt{\vartheta^*}\}$. For the $\mathbb{G}$-predictable and càdlàg processes

$$
A_t^{(\mathbb{G},\overline{X}^i,f)}:=\int_0^t\alpha_s^2\mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)}
$$

there exists $\Phi>0$ such that $\Delta A_t^{(\mathbb{G},\overline{X}^i,f)}(\omega)\le\Phi$, $\mathbb{P}\otimes C^{(\mathbb{G},\overline{X}^i)}$ – a.e, $\quad i\in\{1,...,N\}$.

(G6) For the same $\hat\beta$ as in **(G2)** there exists $\Lambda_{\hat\beta}>0$ such that $\max_{i\in\{1,\dots,N\}}\left\{\mathcal{E}\left(\hat\beta A^{(\mathbb{G},\overline{X}^i,f)}\right)_T\right\}\le\Lambda_{\hat\beta}$.

(G7) For the same $\hat\beta$ as in **(G2)** we have

$$
\mathbb{E}\left[\int_0^T\mathcal{E}\left(\hat\beta A^{(\mathbb{G},\overline{X}^i,f)}\right)_{s-}\frac{|f(s,0,0,0,\delta_0)|^2}{\alpha_s^2}\mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)}\right]<\infty,\qquad i\in\{1,...,N\},
$$

where δ_0 is the Dirac measure on the domain of the last variable concentrated at 0, the neutral element of the addition.

Now, we consider a mean-field system of BSDEs of the form (see (2.28))

$$
\begin{aligned}
Y_t^{i,N}=\xi^{i,N}+\int_t^T f\left(s,Y^{i,N}|_{[0,s]},Z_s^{i,N}c_s^{(\mathbb{G},\overline{X}^i)},\Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(U^{i,N})_s,L^N(\mathbf{Y}^N|_{[0,s]})\right)\mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)}&\\
-\int_t^T Z_s^{i,N}\,\mathrm{d}X_s^{i,\circ}-\int_t^T\int_{\mathbb{R}^n}U_s^{i,N}(x)\,\widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x)-\int_t^T\mathrm{d}M_s^{i,N},\quad i=1,...,N,&
\end{aligned}
\tag{3.9}
$$

Definition 3.2.1. *A set of data* $\left(\mathbb{G},\{\overline{X}^i\}_{i\in\{1,\dots,N\}},T,\left\{\xi^{i,N}\right\}_{i\in\{1,\dots,N\}},\{\Theta^i\}_{i\in\{1,\dots,N\}},\Gamma,f\right)$ *that satisfies the assumptions* **(G1)-(G7)** *will be called* standard under $\hat\beta$ *for the mean-field path dependent BSDE* (3.9).

It follows the existence and uniqueness result for the solution of the mean-field system of BSDEs (3.9) under the path dependence. Before we proceed to the statement of the announced theorem, we mention that we adopt the following convention hereinafter: whenever we consider a (finite) Cartesian product of normed spaces, the norm on this product will be the sum of the norms of the normed spaces used to construct the Cartesian product and it will be simply denoted by $\|\cdot\|$.

Theorem 3.2.2. *Let* $\left(\mathbb{G},\{\overline{X}^i\}_{i\in\{1,\dots,N\}},T,\left\{\xi^{i,N}\right\}_{i\in\{1,\dots,N\}},\{\Theta^i\}_{i\in\{1,\dots,N\}},\Gamma,f\right)$ *be standard data under* $\hat\beta$ *for the path dependent mean-field BSDE* (3.9)*. If*

$$
\max\left\{2,\frac{2\Lambda_{\hat\beta}}{\hat\beta}\right\}M_\star^\Phi(\hat\beta)<1,
$$

then the system of $N-$BSDEs

$$\begin{aligned} Y_t^{i,N} = \xi^{i,N} + \int_t^T f\left(s, Y^{i,N}|_{[0,s]}, Z_s^{i,N} c_s^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(U^{i,N})_s, L^N(\boldsymbol{Y}^N|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \\ - \int_t^T Z_s^{i,N}\,\mathrm{d}X_s^{i,\circ} - \int_t^T \int_{\mathbb{R}^n} U_s^{i,N}(x)\,\widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s^{i,N}, \quad i = 1,...,N, \end{aligned} \tag{3.9}$$

admits a unique $N-$quadruple $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$ as solution, such that

$$\boldsymbol{Y}^N := (Y^{1,N}, \dots, Y^{N,N}) \in \prod_{i=1}^N \mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}; \mathbb{R}^d),$$

$$\boldsymbol{Z}^N := (Z^{1,N}, \dots, Z^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ}; \mathbb{R}^{d\times p}),$$

$$\boldsymbol{U}^N := (U^{1,N}, \dots, U^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural}; \mathbb{R}^d)$$

and

$$\boldsymbol{M}^N := (M^{1,N}, \dots, M^{N,N}) \in \prod_{i=1}^N \mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Proof. Not surprisingly, we will follow analogous arguments as in Theorem 3.1.3 which dealt with the McKean–Vlasov BSDE (3.3). However, there are points which differentiate from the previous proofs. So, for the reader's convenience, we will present here the proof in a compact, yet sufficiently clear, way. To this end, let us introduce the following more compact notation:

$$\mathcal{S}^2_{\hat{\beta},N} := \prod_{i=1}^N \mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}; \mathbb{R}^d) \quad \mathbb{H}^{2,\circ}_{\hat{\beta},N} := \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ}; \mathbb{R}^{d\times p})$$

$$\mathbb{H}^{2,\natural}_{\hat{\beta},N} := \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural}; \mathbb{R}^d) \quad \text{and} \quad \mathcal{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N} := \prod_{i=1}^N \mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Let $(\mathbf{y}^N, \mathbf{z}^N, \mathbf{u}^N, \mathbf{m}^N) \in \mathcal{S}^2_{\hat{\beta},N} \times \mathbb{H}^{2,\circ}_{\hat{\beta},N} \times \mathbb{H}^{2,\natural}_{\hat{\beta},N} \times \mathcal{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N}$ with

$$\mathbf{y}^N := (y^{1,N}, \dots, y^{N,N}), \mathbf{z}^N := (z^{1,N}, \dots, z^{N,N}), \mathbf{u}^N := (u^{1,N}, \dots, u^{N,N}) \text{ and } \mathbf{m}^N := (m^{1,N}, \dots, m^{N,N}).$$

Working per coordinate as in the proof of Theorem 3.1.3 and following analogous arguments to those provided in the extraction of (3.12) below, it permits us to conclude that we get unique processes $(\mathbf{Y}^N, \mathbf{Z}^N, \mathbf{U}^N, \mathbf{M}^N) \in \mathcal{S}^2_{\hat{\beta},N} \times \mathbb{H}^{2,\circ}_{\hat{\beta},N} \times \mathbb{H}^{2,\natural}_{\hat{\beta},N} \times \mathcal{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N}$ such that for $i = 1,...,N$

$$\begin{aligned} Y_t^{i,N} = \xi^{i,N} + \int_t^T f\left(s, y^{i,N}|_{[0,s]}, z_s^{i,N} c_s^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(u^{i,N})_s, L^N(\mathbf{y}^N|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \\ - \int_t^T Z_s^{i,N}\,\mathrm{d}X_s^{i,\circ} - \int_t^T \int_{\mathbb{R}^n} U_s^{i,N}(x)\,\widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s^{i,N}, \end{aligned}$$

with

$$Y_t^{i,N} := \mathbb{E}\left[\xi^{i,N} + \int_t^T f\left(s, y^{i,N}|_{[0,s]}, z_s^{i,N} c_s^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(u^{i,N})_s, L^N(\mathbf{y}^N|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \middle| \mathcal{G}_t\right].$$

The reader may observe that for each $i \in \{1, \dots, N\}$ we have represented the martingale part of the semimartingale Y^i as a sum of stochastic integrals with respect to the elements of the pair $\overline{X}^i$ and an element of the orthogonal space $\mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i\perp_{\mathbb{G}}}; \mathbb{R}^d)$.

Hence, we define the function

$$\mathbf{S}^N : \mathcal{S}^2_{\hat{\beta},N} \times \mathbb{H}^{2,\circ}_{\hat{\beta},N} \times \mathbb{H}^{2,\natural}_{\hat{\beta},N} \times \mathbb{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N} \longrightarrow \mathcal{S}^2_{\hat{\beta},N} \times \mathbb{H}^{2,\circ}_{\hat{\beta},N} \times \mathbb{H}^{2,\natural}_{\hat{\beta},N} \times \mathbb{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N}$$

with

$$\mathbf{S}^N(\mathbf{y}^N, \mathbf{z}^N, \mathbf{u}^N, \mathbf{m}^N) := (\mathbf{Y}^N, \mathbf{Z}^N, \mathbf{U}^N, \mathbf{M}^N).$$

As before we want to prove that $\mathbf{S}^N$ is a contraction, so that by Banach's fixed point theorem we get the unique solution that we want.

For $j = 1, 2$, let $(\mathbf{y}^{N,j}, \mathbf{z}^{N,j}, \mathbf{u}^{N,j}, \mathbf{m}^{N,j}) \in \mathcal{S}^2_{\hat{\beta},N} \times \mathbb{H}^{2,\circ}_{\hat{\beta},N} \times \mathbb{H}^{2,\natural}_{\hat{\beta},N} \times \mathcal{H}^{2,\perp_{\mathbb{G}}}_{\hat{\beta},N}$ with

$$\mathbf{y}^{N,j} := (y^{1,N,j}, \dots, y^{N,N,j}), \quad \mathbf{z}^{N,j} := (z^{1,N,j}, \dots, z^{N,N,j}),$$
$$\mathbf{u}^{N,j} := (u^{1,N,j}, \dots, u^{N,N,j}) \quad \text{and} \quad \mathbf{m}^{N,j} := (m^{1,N,j}, \dots, m^{N,N,j})$$

and let us, also, denote $\mathbf{S}^N(\mathbf{y}^{N,j}, \mathbf{z}^{N,j}, \mathbf{u}^{N,j}, \mathbf{m}^{N,j})$ by $\mathbf{Y}^{N,j}, \mathbf{Z}^{N,j}, \mathbf{U}^{N,j}, \mathbf{M}^{N,j}$. We proceed to define for every $i \in \{1, ..., N\}$

$$\psi_t^i := f\left(t, y^{i,N,2}|_{[0,t]}, z_t^{i,N,2} c_t^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(u^{i,N,2})_t, L^N\left(\mathbf{y}^{N,2}|_{[0,t]}\right)\right)$$
$$- f\left(t, y^{i,N,1}|_{[0,t]}, z_t^{i,N,1} c_t^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(u^{i,N,1})_t, L^N\left(\mathbf{y}^{N,1}|_{[0,t]}\right)\right).$$

In order to control the Wasserstein distance between the empirical measures we have from (2.25) and (2.26)

$$W^2_{2,\rho_{J_1^d}}\left(L^N\left(\mathbf{y}^{N,2}|_{[0,t]}\right), L^N\left(\mathbf{y}^{N,1}|_{[0,t]}\right)\right) \leq \frac{1}{N} \sum_{m=1}^N \sup_{s\in[0,t]} \{|y_s^{m,N,2} - y_s^{m,N,1}|^2\}.$$

The reader may observe that no expectation appears on the right-hand side of the inequality, which is something to be expected because of the nature of the left-hand side. Indeed, there we have a Wasserstein distance of empirical measures, which depend from ω. Hence, in general, there can be no deterministic upper bound for this random variable.

By an application of (2.18), (2.19) and Lemma 2.5.3 per coordinate we get the following estimation[8] for $i \in \{1, \dots, N\}$

$$
\begin{aligned}
&\left\| \frac{\psi^i}{\alpha} \right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, C^{(\mathbb{G},\overline{X}^i)};\mathbb{R}^d)} \\
&\leq \frac{\Lambda_{\hat\beta}}{\hat\beta} \|y^{i,N,2} - y^{i,N,1}\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)};\mathbb{R}^d)} + \|z^{i,N,2} - z^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ};\mathbb{R}^{d\times p})} \\
&\qquad + 2\|u^{i,N,2} - u^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural};\mathbb{R}^d)} \\
&\qquad + \frac{1}{N}\sum_{m=1}^N \mathbb{E}\left[\frac{1}{\hat\beta} \mathcal{E}\left(\hat\beta A^{(\mathbb{G},\overline{X}^i,f)}\right)_{T-} \sup_{s\in[0,T]} \{|y^{m,N,2}_s - y^{m,N,1}_s|^2\} \right]
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\leq \frac{\Lambda_{\hat\beta}}{\hat\beta} \|y^{i,N,2} - y^{i,N,1}\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)};\mathbb{R}^d)} + \|z^{i,N,2} - z^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ};\mathbb{R}^{d\times p})} \\
&\qquad + 2\|u^{i,N,2} - u^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural};\mathbb{R}^d)} \\
&\qquad + \frac{\Lambda_{\hat\beta}}{\hat\beta}\frac{1}{N}\sum_{m=1}^N \|y^{m,N,2} - y^{m,N,1}\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^m,f)};\mathbb{R}^d)}
\end{aligned} \tag{3.12}
$$

Then, by employing Lemma 2.9.1 per coordinate, in conjunction with the fact that $\mathbf{Y}^{N,1}_T = \mathbf{Y}^{N,2}_T$, and by summing over $i \in \{1, \dots, N\}$, we have (recall the notation for the norm on the Cartesian product)

$$
\begin{aligned}
&\left\| \mathbf{S}^N(\mathbf{y}^{N,2}, \mathbf{z}^{N,2}, \mathbf{u}^{N,2}, \mathbf{m}^{N,2}) - \mathbf{S}^N(\mathbf{y}^{N,1}, \mathbf{z}^{N,1}, \mathbf{u}^{N,1}, \mathbf{m}^{N,1}) \right\|^2 \\
&= \sum_{i=1}^N \|Y^{i,N,2} - Y^{i,N,1}\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)};\mathbb{R}^d)} \\
&\qquad + \sum_{i=1}^N \|Z^{i,N,2} - Z^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ};\mathbb{R}^{d\times p})} \\
&\qquad + \sum_{i=1}^N \|U^{i,N,2} - U^{i,N,1}\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural};\mathbb{R}^d)} \\
&\qquad + \sum_{i=1}^N \|M^{i,N,2} - M^{i,N,1}\|^2_{\mathcal{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i\perp_{\mathbb{G}}};\mathbb{R}^d)} \\
&\overset{(\mathbf{Y}^{N,1}_T = \mathbf{Y}^{N,2}_T)}{\leq} M^{\Phi}_{\star}(\hat\beta) \sum_{i=1}^N \left\| \frac{\psi^i}{\alpha} \right\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, C^{(\mathbb{G},\overline{X}^i)};\mathbb{R}^d)} \\
&\overset{(3.12)}{\leq} \max\left\{2, \frac{2\Lambda_{\hat\beta}}{\hat\beta}\right\} M^{\Phi}_{\star}(\hat\beta) \left\| \mathbf{y}^{N,2} - \mathbf{y}^{N,1}, \mathbf{z}^{N,2} - \mathbf{z}^{N,1}, \mathbf{u}^{N,2} - \mathbf{u}^{N,1}, \mathbf{m}^{N,2} - \mathbf{m}^{N,1} \right\|^2,
\end{aligned}
$$

which provides the desired contraction.[9] □

Remark 3.2.3. *Let us provide at this point some comments related to the proof of Theorem 3.2.2.*

[8] This is the analogous to (3.6) in theorem 3.1.3.

[9] The reader may observe that in (3.12) for each fixed $i \in \{1, \dots, N\}$ there are terms which correspond to $m \neq i$, whose coefficient is $1/N$. These terms sum up their coefficients up to 1 when we sum over $i \in \{1, \dots, N\}$.

(i) *For $\mathbf{y}^N \in \mathcal{S}^2_{\hat{\beta},N}$, the notation was introduced within the aforementioned proof, it is easy to show from* (2.26) *that $L^N(\mathbf{y}^N|_{[0,\cdot]})$ is an adapted, càdlàg process; see also the proof of lemma A.2.10 for similar arguments.*

(ii) *In the derivation of* (3.12) *one faces the problem of multiplying the running maximum of processes lying within $\mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)};\mathbb{R}^d)$ with the stochastic exponential associated to $A^{(\mathbb{G},\overline{X}^m,f)}$, for $m \neq i$. In general, one cannot derive such estimates, except for special cases,* e.g., *like those described by Condition* **(G6)** *or* **(G6′)** *provided below.*

(iii) *In view of remark 3.1.4.(iii), we may also consider mean-field system of BSDEs whose generator depends on the instantaneous value of the $\mathbf{Y}^N-$part of the solution. However, one is not able to prove the existence and the uniqueness of the solution under the analogous framework of Theorem 3.1.6,* i.e., *under a condition that involves the mean of the stochastic exponentials. This can be easily explained. To this end, we have to recall and combine two facts. The first one is the previous remark (ii). The second one is that in* (3.5) *the Wasserstein distance provides an expectation, which allows to integrate it with respect to a stochastic exponential, thus factorizing the respective mean values. However, this is not the case in the inequality before* (3.12).

In view of the previous remark, we will consider mean-field system of BSDEs whose generator depends on the instantaneous value of the $\mathbf{Y}^N-$part of the solution. To this end, again, we need to reformulate assumptions **(G4)** and **(G6)** as follows:

(G4′) A generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y,z,u,\mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the map

$$(\omega,t) \longmapsto f(\omega,t,y,z,u,\mu) \text{ is } \mathbb{G}-\text{progressively measurable}$$

and satisfies the following Lipschitz condition

$$|f(\omega,t,y,z,u,\mu) - f(\omega,t,y',z',u',\mu')|^2 \\ \leq r(\omega,t)\,|y-y'|^2 + \vartheta^o(\omega,t)\,|z-z'|^2 + \vartheta^\natural(\omega,t)\,|u-u'|^2 + \vartheta^*(\omega,t)\,W^2_{2,|\cdot|}(\mu,\mu'),$$

where $(r,\vartheta^o,\vartheta^\natural,\vartheta^*) : \left(\Omega\times\mathbb{R}_+,\mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^4_+,\mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(G6′) For $i,j \in \{1,...,N\}$ we have $A^{(\mathbb{G},\overline{X}^i,f)} = A^{(\mathbb{G},\overline{X}^j,f)}$.[10]

Theorem 3.2.4. *Let* $\left(\mathbb{G},\{\overline{X}^i\}_{i\in\{1,...,N\}},T,\{\xi^i\}_{i\in\{1,...,N\}},\{\Theta^i\}_{i\in\{1,...,N\}},\Gamma,f\right)$ *satisfy* **(G1)**-**(G3)**, **(G4′)**, **(G5)**, **(G6′)** *and* **(G7)**. *If*

$$2\widetilde{M}^{\Phi}(\hat{\beta}) < 1,$$

[10]The equality is understood up to evanescence. Moreover, in view of the definition of $A^{(\mathbb{G},\overline{X}^i,f)}$, for $i \in \{1,\dots,N\}$, this condition is equivalent to $C^{(\mathbb{G},\overline{X}^i)} = C^{(\mathbb{G},\overline{X}^j)}$ for $i,j \in \{1,\dots,N\}$. We prefer to present it in the way we did, because it will be more convenient for the justification of the computations.

then the system of $N-$BSDEs

$$\begin{aligned} Y_t^{i,N} = \xi^{i,N} + \int_t^T f\left(s, Y_s^{i,N}, Z_s^{i,N} c_s^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(U^{i,N})_s, L^N(\boldsymbol{Y}_s^N)\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \\ - \int_t^T Z_s^{i,N}\,\mathrm{d}X_s^{i,\circ} - \int_t^T\int_{\mathbb{R}^n} U_s^{i,N}(x)\,\widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s^{i,N}, \quad i=1,...,N, \end{aligned} \tag{3.1}$$

admits a unique solution $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$ *such that*

$$\boldsymbol{Y}^N := (Y^{1,N},\dots,Y^{N,N}) \in \prod_{i=1}^N \mathcal{S}^2_{\hat{\beta}}(\mathbb{G},\alpha,C^{(\mathbb{G},\overline{X}^i)};\mathbb{R}^d),^{11}$$

$$\boldsymbol{Z}^N := (Z^{1,N}\dots,Z^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A^{(\mathbb{G},\overline{X}^i,f)},X^{i,\circ};\mathbb{R}^{d\times p}),$$

$$\boldsymbol{U}^N := (U^{1,N},\dots,U^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A^{(\mathbb{G},\overline{X}^i,f)},X^{i,\natural};\mathbb{R}^d)$$

and

$$\boldsymbol{M}^N := (M^{1,N},\dots,M^{N,N}) \in \prod_{i=1}^N \mathcal{H}^2_{\hat{\beta}}(\mathbb{G},A^{(\mathbb{G},\overline{X}^i,f)},\overline{X}^{i\perp_{\mathbb{G}}};\mathbb{R}^d).$$

Proof. In view of **(G6′)**, we will denote every $A^{(\mathbb{G},\overline{X}^i,f)}$, resp. $C^{(\mathbb{G},\overline{X}^i)}$, for $i\in\{1,\dots,N\}$, simply by A, resp. C. Adopting the notation of the proof of Theorem 3.2.2 and following exactly the same arguments as in the aforementioned proof, we arrive to the following inequality (which is analogous to (3.11)) for $i\in\{1,\dots,N\}$

$$\begin{aligned} &\left\|\frac{\psi^i}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)} \\ &\quad\le \|\alpha(y^{i,N,2}-y^{i,N,1})\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)} + \|z^{i,N,2}-z^{i,N,1}\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,X^{i,\circ};\mathbb{R}^{d\times p})} \\ &\qquad + 2\,\|u^{i,N,2}-u^{i,N,1}\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,X^{i,\natural};\mathbb{R}^d)} + \frac{1}{N}\sum_{m=1}^N \|\alpha\delta(y^{m,N,2}-y^{m,N,1})\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A,C;\mathbb{R}^d)}. \end{aligned} \tag{3.14}$$

We underline that from **(G6′)** for $i,m\in\{1,...,N\}$ we have $\mathbb{P}-$a.s. (we return to the initial notation to demonstrate the property we used)

$$\begin{aligned} &\int_0^T \alpha_s^2\mathcal{E}\left(\hat{\beta}A^{(\mathbb{G},\overline{X}^i,f)}\right)_{s-}|y_s^{m,N,2}-y_s^{m,N,1}|^2\mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \\ &\quad= \frac{1}{\hat{\beta}}\int_0^T |y_s^{m,N,2}-y_s^{m,N,1}|^2\mathrm{d}\mathcal{E}\left(\hat{\beta}A^{(\mathbb{G},\overline{X}^i,f)}\right)_s \\ &\quad= \frac{1}{\hat{\beta}}\int_0^T |y_s^{m,N,2}-y_s^{m,N,1}|^2\mathrm{d}\mathcal{E}\left(\hat{\beta}A^{(\mathbb{G},\overline{X}^m,f)}\right)_s \\ &\quad= \int_0^T \alpha_s^2\mathcal{E}\left(\hat{\beta}A^{(\mathbb{G},\overline{X}^m,f)}\right)_{s-}|y_s^{m,N,2}-y_s^{m,N,1}|^2\mathrm{d}C_s^{(\mathbb{G},\overline{X}^m)}. \end{aligned}$$

The rest is obvious. □

[11]The $\mathcal{S}^2-$spaces have been introduced before definition 3.1.5.

Remark 3.2.5. *Assumption* **(G6′)** *can be considered natural. Indeed, this is the case when* $\{\overline{X}^i\}_{i\in\{1,\dots,N\}}$ *is a family whose elements are identically distributed and define deterministic* $C^{(\mathbb{G},\overline{X}^i)}$ *processes, e.g. from proposition 4.1.2 have independent increments. Then, from* (2.9) *and [21, 6.23 Theorem] we would have for* $i,j\in\{1,\dots,N\}$ *and* $t\in\mathbb{R}_+$ *that*

$$\begin{aligned}C_t^{(\mathbb{G},\overline{X}^i)} &= \mathbb{E}\Big[C_t^{(\mathbb{G},\overline{X}^i)}\Big] = \mathbb{E}\Big[\mathrm{Tr}\Big[\langle X^{i,\circ}\rangle_t^{\mathbb{G}}\Big]\Big] + \mathbb{E}\left[|I|^2 * \nu_t^{(\mathbb{G},X^{i,\natural})}\right]\\ &= \mathbb{E}\Big[\Big|X_t^{i,\circ}\Big|^2\Big] + \mathbb{E}\left[|I|^2 * \mu_t^{X^{i,\natural}}\right] = \mathbb{E}\Big[\Big|X_t^{i,\circ}\Big|^2\Big] + \mathbb{E}\Big[\Big|X_t^{i,\natural}\Big|^2\Big]\\ &= \mathbb{E}\Big[\Big|X_t^{j,\circ}\Big|^2\Big] + \mathbb{E}\Big[\Big|X_t^{j,\natural}\Big|^2\Big] = \mathbb{E}\Big[\Big|X_t^{j,\circ}\Big|^2\Big] + \mathbb{E}\left[|I|^2 * \mu_t^{X^{j,\natural}}\right]\\ &= \mathbb{E}\Big[\mathrm{Tr}\Big[\langle X^{j,\circ}\rangle_t^{\mathbb{G}}\Big]\Big] + \mathbb{E}\left[|I|^2 * \nu_t^{(\mathbb{G},X^{j,\natural})}\right] = \mathbb{E}\Big[C_t^{(\mathbb{G},\overline{X}^j)}\Big] = C_t^{(\mathbb{G},\overline{X}^j)}.\end{aligned}$$

Chapter 4

Backward propagation of chaos

The notion of propagation of chaos started to get a lot of attention when in a series of lectures Lasry and Lyons [35–37] used it in order to simplify the study of mean-field games. They introduced ideas from statistical physics into the study of Nash equilibria for stochastic differential games with symmetric interactions along with Malhamé and Caines in [25, 26]. Generally, problems with large number of players are notoriously difficult to control. However, as statistical physics has shown us, under the appropriate assumptions (the most important being symmetry) one can study the asymptotic behaviour of a system as the number of players grows to infinity much more easily. Of course there is not a single way to mathematically express the notion that the players, or particles in statistical physics, interact with one another, a choice must be made. In probability theory, one of the first deep theorems was the strong law of large numbers. The law gives a set of conditions under which randomness asymptotically collapses to determinism. This feature is ideal for studying systems with a large number of agents as it allows for calculating simplifications. Perhaps inspired by this, the interaction that has extensively studied in the statistical physics literature is that which emerges from the empirical measure of the states of the particles. Hence an interaction that involves the empirical measure of the states of the participants is called *mean-field interaction*.

The modern notion about propagation of chaos started in the fifties. M. Kac in the process of investigating particle system approximations for some nonlocal partial differential equations (PDE) arising in thermodynamics, see [30], made an important observation about a characteristic of large systems. Assume that the behaviour of the particles is symmetric and interact in a weak way that its magnitude decreases inversely proportional to the size of the system. Maybe due to cancellations of the contributions of different particles. Then, if the initial positions of them are chaotic, here understood as independent and identically distributed, this initial state of the system could be seen asymptotically to propagate (or spread) to the other points in time, when its size grows to infinite. This idea of propagation has been used ever since to various topics with many applications, some recent are found in Jabin and Wang [27, 28], Malrieu [39].

In this work, combining the above ideas we want to study the backward propagation of chaos for solutions of mean-field systems of backward stochastic differential equations (BSDEs) under a general setting. The backward propagation is understood as having chaotic behaviour on the terminal conditions, instead of having it on the initial conditions.

Although propagation of chaos is extensively studied for (Forward) Stochastic Differential Equations (SDEs), e.g., see the review Chaintron and Diez [11], for backward propagation of chaos only a handful of papers have been published so far Hu et al. [23], Briand and Hibon [8], Djehiche et al. [19], Buckdahn et al. [9], Laurière and Tangpi [38]. None of these papers works in a setting as general as of the work presented here. More precisely, for the propagation of chaos property the current framework allows for square–integrable martingales with independent increments as integrators of stochastic integrals, càdlàg predictable increasing processes as integrators of Lebesgue–Stieltjes integrals and also dependence in the generator from the initial segments of the paths of the solution $\mathbf{Y}$. In fact, the results of Chapter 3 and Section 4.1 are also valid without change in the case where the dependence in the generator comes from $x|_{[0,s-]}$ instead of $x|_{[0,s]}$; see the notation introduced in (2.28). Similarly in Section 4.2 we can replace Y_s with Y_{s-}.

For ease of the presentation the generator f assumed to be deterministic. However, if one wishes to work in a setting where the generators are stochastic, then as it is customary, she can assume instead that there exists a sequence of copies $\{f^i\}_{i\in\mathbb{N}}$ of a stochastic driver f^1 of a prototype probability space (see Section A.2.1), and proceed with the obvious modifications in the proofs.

In Section 4.2.2 we provide rates of convergence for the usual dependence. The theorems generalize those found in [38] for the Brownian framework. Although, in their work Lauriere and Tangpi prove also that the requirements of the theorems, *i.e.* advanced integrability of the solutions, can be satisfied under an additional specific linear growth condition for the generator. In the present work we do not study under what conditions for the generator these integrability requirements can be achieved, but we can say that they are trivially achieved when the generator is bounded, see Remark 4.2.11. Obviously, in the previous results, one has to assume also advanced integrability for the terminal conditions. Alternatively, if one wants to keep sharp square integrability conditions for the data, then needs to specialize the dependence of f from the argument of the probability measures to a type that allows it, like in [9] or [38, Proposition 2.12.].

Finally, we want to underline that our approach allows to prove backward propagation of chaos in the general case where the terminal conditions of the mean-field systems differ from the terminal conditions of the McKean–Vlasov BSDEs. Moreover, in every system, the former are not required to be identically distributed or independent.

4.1 Propagation of chaos under the path dependence

Despite the fact that we have already provided existence and uniqueness results for the respective mean-field BSDE system (3.9) and McKean–Vlasov BSDE (3.3), there are still quite a few preparatory and auxiliary results that will be required in order to fulfill our promise for the proof of the respective propagation of chaos. These auxiliary results are presented, accompanied by their proofs, in Sections A.2.2, A.2.3.

Naturally, the framework we are going to use for the propagation of chaos will be based on the common ground of the frameworks we used in the previous chapter, suitably enriched and reinforced wherever required.

4.1.1 Setting

Let a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which supports the following:

(H1) A sequence of independent and identically distributed processes $\{\overline{X}^i\}_{i\in\mathbb{N}}$ such that, for every $i \in \mathbb{N}$, $\overline{X}^i = (X^{i,\circ}, X^{i,\natural}) \in \mathcal{H}^2(\mathbb{F}^i;\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^i;\mathbb{R}^n)$ with $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^i}] = 0$, where $\mathbb{F}^i := (\mathcal{F}^i_t)_{t\geq 0}$ is the usual augmentation of the natural filtration of $\overline{X}^i$ and $\mu^{X^{i,\natural}}$ is the random measure generated by the jumps of $X^{i,\natural}$.[1]

(H2) A deterministic time T and a sequence of identically distributed terminal conditions $\{\xi^i\}_{i\in\mathbb{N}}$ and a sequence of sets of terminal conditions $\Big\{\{\xi^{i,N}\}_{i\in\{1,\dots,N\}}\Big\}_{N\in\mathbb{N}}$ such that, under a $\hat{\beta} > 0$ it holds $\xi^i, \xi^{i,N} \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^i_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)$, $\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)$[2] respectively for every $i \in \mathbb{N}$, where $\{A^{(\mathbb{F}^i,\overline{X}^i,f)}\}_{i\in\mathbb{N}}$ the ones defined in **(H5)**.

Moreover, we assume that $\|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0$, for every $i \in \mathbb{N}$, and $\frac{1}{N}\sum_{i=1}^N \|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0$.

(H3) Functions Θ, Γ as in Definition 2.5.1, where Θ is deterministic and for each $i \in \mathbb{N}$ the data for the definition are the pair $(\mathbb{F}^i, \overline{X}^i)$, the process $C^{(\mathbb{F}^i,\overline{X}^i)}$ and the kernels $K^{(\mathbb{F}^i,\overline{X}^i)}$. It is underlined that $\Theta \in \widetilde{\mathcal{P}}^{\mathbb{F}^i}$, for each $i \in \mathbb{N}$.

(H4) A generator $f : \mathbb{R}_+ \times \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y, z, u, \mu) \in \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d)$, the map

$$t \longmapsto f(t, y, z, u, \mu) \text{ is } \mathcal{B}(\mathbb{R}_+) - \text{measurable}$$

and satisfies the following Lipschitz condition

$$\begin{aligned}&|f(t, y, z, u, \mu) - f(t, y', z', u', \mu')|^2 \\ &\qquad\leq r(t)\, \rho^2_{\mathrm{J}^d_1}(y, y') + \vartheta^o(t)\, |z - z'|^2 + \vartheta^\natural(t)\, |u - u'|^2 + \vartheta^*(t)\, W^2_{2,\rho_{\mathrm{J}^d_1}}(\mu, \mu'),\end{aligned}$$

where $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(H5) Define $\alpha^2 := \max\{\sqrt{r}, \vartheta^o, \vartheta^\natural, \sqrt{\vartheta^*}\}$. For the $\mathbb{F}^i$-predictable and càdlàg processes

$$A^{(\mathbb{F}^i,\overline{X}^i,f)}_t := \int_0^t \alpha^2_s \mathrm{d}C^{(\mathbb{F}^i,\overline{X}^i)}_s \tag{4.1}$$

there exists $\Phi > 0$ such that $\Delta A^{(\mathbb{F}^i,\overline{X}^i,f)}_t(\omega) \leq \Phi$, $\mathbb{P}\otimes C^{(\mathbb{F}^i,\overline{X}^i)} -$ a.e, for every $i \in \mathbb{N}$.

(H6) For the same $\hat{\beta}$ as in **(H2)** we have

$$\mathbb{E}\left[\int_0^T \mathcal{E}\left(\hat{\beta} A^{(\mathbb{F}^i,\overline{X}^i,f)}\right)_{s-} \frac{|f(s,0,0,0,\delta_0)|^2}{\alpha^2_s} \mathrm{d}C^{(\mathbb{F}^i,\overline{X}^i)}_s\right] < \infty, \quad i \in \mathbb{N}, \tag{4.2}$$

[1]Since for every $i \in \mathbb{N}$ the filtration $\mathbb{F}^i$ is associated to $\overline{X}^i$, we will make use of $C^{(\mathbb{F}^i,\overline{X}^i)}$, resp. $c^{(\mathbb{F}^i,\overline{X}^i)}$, as defined in (2.9), resp. (2.11). Moreover, we will use the kernels $K^{(\mathbb{F}^i,\overline{X}^i)}$ as determined by (2.10).

[2]see Remark 4.1.1 *(i)*

where δ_0 is the Dirac measure on the domain of the last argument concentrated at 0, the neutral element of the addition.

(H7) There exist a non-decreasing, right continuous function Q, a Borel-measurable function γ and a family $\{b^i\}_{i\in\mathbb{N}}$, with $b^i \in \mathcal{P}_+^{\mathbb{F}^i}$ for every $i \in \mathbb{N}$, such that

$$\mathcal{E}\left(\hat{\beta} A^{(\mathbb{F}^i,\overline{X}^i,f)}\right)_{\cdot} = 1 + \int_0^{\cdot} b_s^i \,\mathrm{d}Q_s, \qquad i \in \mathbb{N}$$

and

$$\sup_{i\in\mathbb{N}}\{b^i\} \leq \gamma, \qquad Q - a.e.$$

(H8) For the same $\hat{\beta}$ as in **(H2)** and γ as in **(H7)** there exists a $\Lambda_{\hat{\beta}} > 0$ such that $1 + \int_0^T \gamma_s \,\mathrm{d}Q_s = \Lambda_{\hat{\beta}}$.

(H9) For the same $\hat{\beta}$ as in **(H2)** we have $\max\left\{2, \frac{3\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M_\star^\Phi(\hat{\beta}) < 1$.

In the following, we collect a few observations regarding the framework **(H1)**-**(H9)**. Some of these observations provide immediate properties derived from the conditions, while other observations justify the imposed conditions. Additionally, let us introduce the required notation used hereinafter. To this end, let us fix $N \in \mathbb{N}$ and assume **(H1)**-**(H9)**. For each $i \in \{1, \ldots, N\}$, the McKean–Vlasov BSDE (3.3) associated to the standard data $\left(\overline{X}^i, \mathbb{F}^i, \Theta^i, \Gamma, T, \xi^i, f\right)$ under $\hat{\beta}$ admits, by Theorem 3.1.3, a unique solution, which will be denoted by (Y^i, Z^i, U^i, M^i). For later reference, we will say that $(\widetilde{\mathbf{Y}}^N, \widetilde{\mathbf{Z}}^N, \widetilde{\mathbf{U}}^N, \widetilde{\mathbf{M}}^N)$ is the solution of the first N McKean–Vlasov BSDEs, where we define

$$\widetilde{\mathbf{Y}}^N := (Y^1, \ldots, Y^n),\ \widetilde{\mathbf{Z}}^N := (Z^1, \ldots, Z^n),\ \widetilde{\mathbf{U}}^N := (U^1, \ldots, U^n)$$
$$\text{and } \widetilde{\mathbf{M}}^N := (M^1, \ldots, M^n).$$

We underline that the symbol $(\mathbf{Y}^N, \mathbf{Z}^N, \mathbf{U}^N, \mathbf{U}^N)$ is reserved for the solution of the mean-field BSDE.

Remark 4.1.1. *(i)* *By construction, $\{\mathbb{F}^i\}_{i\in\mathbb{N}}$ is a sequence of independent filtrations on $(\Omega, \mathcal{G}, \mathbb{P})$. Moreover, for every $N \in \mathbb{N}$, we define the filtration $\mathbb{F}^{1,..,N} := \bigvee_{m=1}^N \mathbb{F}^m$. From Wu and Gang [52, Theorem 1] we have that $\mathbb{F}^{1,..,N}$ satisfies the usual conditions. For $i \in \{1, \ldots, N\}$ and $N \in \mathbb{N}$, a direct consequence of the independence of filtrations is that every $\mathbb{F}^i-$martingale, remains martingale under $\mathbb{F}^{1,\ldots,N}$,* i.e., *the filtration $\mathbb{F}^i$ is immersed in the filtration $\mathbb{F}^{1,\ldots,N}$. In particular, $X^{i,\natural} \in \mathcal{H}^{2,d}(\mathbb{F}^{1,\ldots,N};\mathbb{R}^n)$; see Corollary A.2.7. Additionally, from the assumption $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^i}] = 0$ one can deduce that it is also true $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^{1,\ldots,N}}] = 0$; one can follow the exact same arguments as in Lemma A.2.9.*

(ii) *Let $i \in \mathbb{N}$ and $N \in \mathbb{N}$. Under $\mathbb{F}^i$ we have defined via* (2.9) *and* (2.11) *the càdlàg, $\mathbb{F}^i-$predictable and increasing processes $C^{(\mathbb{F}^i,\overline{X}^i)}$ and $c^{(\mathbb{F}^i,\overline{X}^i)}$. Naturally, one can consider the respective processes under the filtration $\mathbb{F}^{1,..,N}$,* i.e., *$C^{(\mathbb{F}^{1,..,N},\overline{X}^i)}$ and $c^{(\mathbb{F}^{1,..,N},\overline{X}^i)}$. However, in view of the immersion of the filtrations and of Remark 2.2.1, we have*

$$C^{(\mathbb{F}^i,\overline{X}^i)} = C^{(\mathbb{F}^{1,..,N},\overline{X}^i)} \text{ and } c^{(\mathbb{F}^i,\overline{X}^i)} = c^{(\mathbb{F}^{1,..,N},\overline{X}^i)}. \tag{4.3}$$

This property allows us to drop the notational dependence from the filtration. Hence, under **(H1)** *where we fix the sequence* $\{\overline{X}^i\}_{n\in\mathbb{N}}$*, we will simply denote these objects as* C^i*,* c^i*, for* $i \in \mathbb{N}$*. Additionally, recalling the definition of* $A^{(\mathbb{F}^i,\overline{X}^i,f)}$ *in* (4.1)*, we can also simplify the respective notation. Hence, under* **(H1)** *and* **(H5)***, we denote by* A^i *the process* $A^{(\mathbb{F}^i,\overline{X}^i,f)} \equiv A^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,f)}$.

(iii) *Under* **(H1)** *and* **(H5)***, let us assume that there exist a non-decreasing, right continuous function* Q*, a Borel-measurable function* γ *and a family* $\{b^i\}_{i\in\mathbb{N}}$*, with* $b^i \in \mathcal{P}_+^{\mathbb{F}^i}$ *for every* $i \in \mathbb{N}$*, such that*

$$\sup_{i\in\mathbb{N}}\{\alpha^2 b^i\} \leq \gamma \ \ \textit{and} \ \ C^i_\cdot = \int_0^\cdot b^i_s \,\mathrm{d}Q_s, \ \ \textit{for every } i \in \mathbb{N}.$$

Then, this property obviously transfers through (4.1)*,* (2.12) *and* (2.13) *to the sequence* $\left\{\mathcal{E}\left(\hat{\beta}A^i\right)\right\}_{i\in\mathbb{N}}$*,* i.e.*,*

$$\mathcal{E}\left(\hat{\beta}A^i\right)_\cdot = 1 + \frac{1}{\hat{\beta}}\int_0^\cdot \mathcal{E}\left(\hat{\beta}A^i\right)_{s-} \alpha_s^2\, b_s^i \,\mathrm{d}Q_s, \ \ \textit{for every } i \in \mathbb{N}$$

and from **(H8)**

$$\mathcal{E}\left(\hat{\beta}A^i\right)_T \leq \mathrm{e}^{\hat{\beta}\Lambda_{\hat{\beta}}}.$$

In other words, **(H7)** *is fulfilled under the initial assumptions of this remark.*

(iv) *Let* $N \in \mathbb{N}$ *and* $i \in \{1,\dots,N\}$*. Under the Conditions* **(H1)**-**(H9)** *it is guaranteed that the septuple* $(\mathbb{F}^i, \overline{X}^i, T, \xi^i, \Theta, \Gamma, f)$ *consists of standard data under* $\hat{\beta}$ *for the McKean–Vlasov BSDE* (3.3)*; see Theorem 3.1.3. Hereinafter, we will denote its solution by* (Y^i, Z^i, U^i, M^i)*. Completely analogously, under the same framework, the septuple* $(\mathbb{F}^{1,\dots,N}, \{\overline{X}^i\}_{i\in\{1,\dots,N\}}, T, \{\xi^i\}_{i\in\{1,\dots,N\}}, \Theta, \Gamma, f)$ *consists of standard data under* $\hat{\beta}$ *for the mean-field BSDE system* (3.1)*; see Theorem 3.2.2. Hereinafter, we will denote its solution by* $\{(Y^{i,N}, Z^{i,N}, U^{i,N}, M^{i,N})\}_{i\in\{1,\dots,N\}}$*. In particular,* $\boldsymbol{Y}^N := (Y^{1,N},\dots,Y^{N,N})$.

(v) *The sequence of driving martingales associated to the McKean–Vlasov equations are independent and identically distributed, as well as the terminal random variables. Hence, for every* $N \in \mathbb{N}$*, from Lemma A.2.9 see the solutions of the first* N *McKean–Vlasov* $Y^1, ..., Y^N$ *as strong solutions under the larger filtration* $\mathbb{F}^{1,\dots,N}$*, and conclude the uniqueness in law. In other words, the solutions of the McKean–Vlasov are identical distributed.*

The following proposition in conjunction with **(H1)**, as well as the comments after the proposition, justify the assumptions imposed in Remark 4.1.1.(iii). These indicate that **(H7)** is by no means restrictive for applications.

Proposition 4.1.2. *Let* $\mathbb{G}$ *be a filtration on* $(\Omega, \mathcal{G}, \mathbb{P})$ *that satisfies the usual conditions. For every pair* $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$ *with independent increments the processes* $C^{(\mathbb{G},\overline{X})}$ *and* $c^{(\mathbb{G},\overline{X})}$*, as defined via* (2.9) *and* (2.11)*, are deterministic.*

Proof. Let $j \in \{1,\dots,p\}$, then $X^{\circ,j}$ denotes the $j-$element of the $p-$dimensional process X°. Analogously, x^j denotes the $j-$element of the $p-$dimensional vector x. From [21, Definition 6.27] we have that the dual predictable projection of the process $\sum_{s\leq t}|\Delta X^{\circ,j}_s|^2$ is equal with $(x^j)^2 * \nu^{(\mathbb{G},X^\circ)}$, for every $j \in \{1,...,p\}$.

From Medvegyev [40, Corollary 7.87] we have that $\langle X^{\circ}\rangle^{\mathbb{G}}$ and $\nu^{(\mathbb{G},X^{\natural})}$ are deterministic, which completes the proof. $\square$

A couple of examples where **(H7)** is satisfied are the extended Grigelionis martingales, see Kallsen [32, Definition 2.15], and affine martingales, see Kallsen [33], Kallsen et al. [34, Section 2], where we respectively have

$$C_t^{(\mathbb{G},\overline{X})} = \lambda^2\Big(t + \sum_{s\leq t}\mathbb{1}_B(s)\Big) \quad \text{and} \quad C_t^{(\mathbb{G},\overline{X})} = \int_0^t b_s\,\mathrm{d}s,$$

for some $\lambda \in \mathbb{R}, B \subseteq (0,\infty), b \in \mathcal{P}_+^{\mathbb{G}}$, with B at most countable and b appropriately bounded.

4.1.2 Main results

We are ready to prove the propagation of chaos between the mean-field BSDE (3.9) and the McKean–Vlasov BSDE (3.3). The setting consists of the conditions **(H1)-(H9)**. For fixed $N \in \mathbb{N}$, we introduce the following notation for the solution $(\mathbf{Y}^N, \mathbf{Z}^N, \mathbf{U}^N, \mathbf{M}^N)$ associated to the mean-field BSDE (3.9):[3]

$$\mathbf{Y}^N = (Y^{1,N},\dots,Y^{N,N}),\ \mathbf{Z}^N = (Z^{1,N},\dots,Z^{N,N}),$$
$$\mathbf{U}^N = (U^{1,N},\dots,U^{N,N}) \ \text{ and } \ \mathbf{M}^N = (M^{1,N},\dots,M^{N,N}).$$

Also, for $i \in \mathbb{N}$, we will call the $i-$th McKean–Vlasov BSDE (3.3) the one that corresponds to the standard data $\big(\overline{X}^i, \mathbb{F}^i, \Theta^i, \Gamma, T, \xi^i, f\big)$ under $\hat{\beta}$. Additionally, we will call the first N McKean–Vlasov BSDEs (3.3) those that correspond to the set of standard data $\Big\{\big(\overline{X}^i, \mathbb{F}^i, \Theta, \Gamma, T, \xi^i, f\big)\Big\}_{i\in\{1,\dots,N\}}$ under $\hat{\beta}$ with associated solution $(\widetilde{\mathbf{Y}}^N, \widetilde{\mathbf{Z}}^N, \widetilde{\mathbf{U}}^N, \widetilde{\mathbf{M}}^N)$, see also the comments at the beginning of Section A.2.3.

Theorem 4.1.3 (**Propagation of chaos for the system**)**.** *Assume* **(H1)-(H9)**. *For the solution of the mean-field BSDE* (3.9)*, den. by* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$*, and the solutions of the first N McKean–Vlasov BSDE* (3.3)*, den. by* $(\widetilde{\boldsymbol{Y}}^N, \widetilde{\boldsymbol{Z}}^N, \widetilde{\boldsymbol{U}}^N, \widetilde{\boldsymbol{M}}^N)$*, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \big\|\Big(Y^{i,N}-Y^i, Z^{i,N}-Z^i, U^{i,N}-U^i, M^{i,N}-M^i\Big)\big\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i} = 0.^4 \tag{4.4}$$

Proof. For fixed $N \in \mathbb{N}$, we will work under the filtration $\mathbb{F}^{1,\dots,N}$. For $i \in \{1,\dots,N\}$, Lemma A.2.9 permits us to consider the $i-$th McKean–Vlasov BSDE under the filtration $\mathbb{F}^{1,\dots,N}$ instead of under $\mathbb{F}^i$ without affecting the solution (Y^i, Z^i, U^i, M^i). This property allows us to finally consider the N first McKean-Vlasov equations under the filtration $\mathbb{F}^{1,\dots,N}$, which is also the filtration considered for the mean-field BSDE $N-$system.

[3] See also Remark 4.1.1.(iv).

[4] Recall Remark 4.1.1.(ii) for the notation A^i, for every $i \in \mathbb{N}$.

For every $i \in \{1, ..., N\}$ we subtract the solution of the $i-$th McKean–Vlasov BSDE from the $i-$th element of the solution of the mean-field BSDE in order to derive

$$\begin{aligned} Y_t^{i,N} - Y_t^i = \xi^{i,N} - \xi^i + \int_t^T f\Big(s, Y^{i,N}|_{[0,s]}, Z_s^{i,N} c^i, \Gamma^{(\mathbb{F}^{1,\dots,N}, \overline{X}^i, \Theta^i)}(U^{i,N})_s, L^N(\mathbf{Y}^N|_{[0,s]})\Big) \\ - f\Big(s, Y^i|_{[0,s]}, Z_s^i c^i, \Gamma^{(\mathbb{F}^{1,\dots,N}, \overline{X}^i, \Theta^i)}(U^i)_s, \mathcal{L}(Y^i|_{[0,s]})\Big) \, \mathrm{d}C_s^i \\ - \int_t^T \mathrm{d}\Big[\Big(Z^{i,N} - Z^i\Big) \cdot X^{i,\circ} + \Big(U^{i,N} - U^i\Big) \star \widetilde{\mu}^{(\mathbb{F}^{1,\dots,N}, X^{i,\natural})} + M^{i,N} - M^i\Big]_s, \end{aligned} \tag{4.5}$$

which finally provides a BSDE system. Hence, we can utilize Lemma 2.9.1.

Let us define $\psi := (\psi^1, \dots, \psi^N)$, where for $i \in \{1, \dots, N\}$ we have defined

$$\begin{aligned} \psi^i_\cdot := f\Big(\cdot, Y^{i,N}|_{[0,\cdot]}, Z^{i,N}_\cdot c^i_\cdot, \Gamma^{(\mathbb{F}^{1,\dots,N}, \overline{X}^i, \Theta^i)}(U^{i,N})_\cdot, L^N(\mathbf{Y}^N|_{[0,\cdot]})\Big) \\ - f\Big(\cdot, Y^i|_{[0,\cdot]}, Z^i_\cdot c^i_\cdot, \Gamma^{(\mathbb{F}^{1,\dots,N}, \overline{X}^i, \Theta^i)}(U^i)_\cdot, \mathcal{L}(Y^i|_{[0,\cdot]})\Big). \end{aligned}$$

So, from the Lipschitz condition **(H4)**, the identities (2.18) and (2.19), Lemma 2.5.3 which provides the Lipschitz property of Γ with respect to $|||\cdot|||$ and the combination of **(H7)** and **(H8)**, which in particular provides the bound of the respective stochastic exponential, we get for every $i \in \{1, ..., N\}$ that

$$\begin{aligned} \left\| \frac{\psi^i}{\alpha} \right\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N}, A^i, C^i; \mathbb{R}^d)} \leq \frac{\Lambda_{\hat{\beta}}}{\hat{\beta}} \|Y^{i,N} - Y^i\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N}, A^i; \mathbb{R}^d)} \\ + \|Z^{i,N} - Z^i\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N}, A^i, X^{i,\circ}; \mathbb{R}^{d\times p})} + 2\, \|U^{i,N} - U^i\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N}, A^i, X^{i,\natural}; \mathbb{R}^d)} \\ + \frac{1}{\hat{\beta}} \mathbb{E}\left[\int_0^T W^2_{2,\rho_{\mathbb{J}^d_1}}\left(L^N\left(\mathbf{Y}^N|_{[0,s]}\right), \mathcal{L}\left(Y^i|_{[0,s]}\right)\right) \, \mathrm{d}\mathcal{E}(\hat{\beta} A^i)_s\right]. \end{aligned} \tag{4.6}$$

At this point let us observe that, in order to proceed, we need to derive the convergence (in the sense dictated from the last term of the right-hand side of (4.6)) of the empirical mean of the mean-field solution to the common (in view of Remark 4.1.1.(v)) law of the solution of the McKean–Vlasov BSDEs. To this end, we will use the triangular inequality for the Wasserstein distance as follows

$$\begin{aligned} W^2_{2,\rho_{\mathbb{J}^d_1}}\Big(L^N(\mathbf{Y}^N|_{[0,s]}), \mathcal{L}(Y^i|_{[0,s]})\Big) \\ \leq 2\, W^2_{2,\rho_{\mathbb{J}^d_1}}\Big(L^N(\mathbf{Y}^N|_{[0,s]}), L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]})\Big) + 2\, W^2_{2,\rho_{\mathbb{J}^d_1}}\Big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}), \mathcal{L}\left(Y^i|_{[0,s]}\right)\Big), \end{aligned} \tag{4.7}$$

in order to reduce our initial problem into two easier ones: the convergence (in the sense dictated in (4.6)) to 0 of the right-hand summands of (4.7). The following computations serve this purpose.

Let us, for the time being, deal with the first summand of the right-hand side of (4.7). Then, for $i \in \{1, \ldots, N\}$ and by integrating with respect to $\mathbb{P} \otimes \mathcal{E}(\hat{\beta} A^i)$ we have by means of (2.25) and (2.26)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T W^2_{2,\rho_{\mathrm{J}^d_1}}\Big(L^N(\mathbf{Y}^N|_{[0,s]}), L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]})\Big)\mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\right]\\
&\quad\leq \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{m=1}^N \sup_{z\in[0,s]}\{|Y^{m,N}_z - Y^m_z|\}^2 \mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\right]\\
&\quad\leq \Lambda_{\hat{\beta}}\frac{1}{N}\sum_{m=1}^N \|Y^{m,N} - Y^m\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{F}^{1,\ldots,N}, A^{\overline{X}^m};\mathbb{R}^d)}.
\end{aligned} \tag{4.8}$$

Returning to the system (4.5), we utilize Lemma 2.9.1 (in conjunction with Proposition 2.9.2), which essentially amounts to adding (4.6) over $i \in \{1, \ldots, N\}$, and in conjunction with (4.7) and (4.8) we have

$$\begin{aligned}
&\sum_{i=1}^N \|\Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big)\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\ldots,N},A^i,\overline{X}^i}\\
&\quad\leq (26 + 9\hat{\beta}\Phi)\sum_{i=1}^N \|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\ldots,N}_T, A^i;\mathbb{R}^d)}\\
&+\max\left\{2, \frac{3\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M^{\Phi}_{\star}(\hat{\beta}) \sum_{i=1}^N \|\Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big)\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\ldots,N},\alpha,A^i,\overline{X}^i}\\
&\quad+\frac{2M^{\Phi}_{\star}(\hat{\beta})}{\hat{\beta}} \sum_{i=1}^N \mathbb{E}\Big[\int_0^T W^2_{2,\rho_{\mathrm{J}^d_1}}\Big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}), \mathcal{L}(Y^i|_{[0,s]})\Big)\,\mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\Big].
\end{aligned}$$

So, from **(H9)** we get

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \|\Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big)\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\ldots,N},A^i,\overline{X}^i}\\
&\leq \frac{(26 + 9\hat{\beta}\Phi)}{1 - \max\left\{2, \frac{3\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M^{\Phi}_{\star}(\hat{\beta})}\frac{1}{N}\sum_{i=1}^N \|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\ldots,N}_T, A^i;\mathbb{R}^d)}\\
&\quad+\frac{2M^{\Phi}_{\star}(\hat{\beta})}{1 - \max\left\{2, \frac{3\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M^{\Phi}_{\star}(\hat{\beta})}\frac{1}{\hat{\beta}}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N\int_0^T W^2_{2,\rho_{\mathrm{J}^d_1}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right), \mathcal{L}\left(Y^i|_{[0,s]}\right)\right)\,\mathrm{d}\mathcal{E}\left(\hat{\beta}A^i\right)_s\right].
\end{aligned}$$

In other words, from **(H2)**, we have reduced our initial problem to the one which consists of proving that

$$\lim_{N\to\infty}\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N\int_0^T W^2_{2,\rho_{\mathrm{J}^d_1}}\Big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}), \mathcal{L}(Y^i|_{[0,s]})\Big)\,\mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\Big] = 0. \tag{4.9}$$

In view of the above, we focus hereinafter on proving that (4.9) is indeed true. From Remark 4.1.1.(v) we have for every $i, j \in \mathbb{N}$ that $\mathcal{L}\left(Y^i|_{[0,s]}\right) = \mathcal{L}\left(Y^j|_{[0,s]}\right)$, for every $s \in \mathbb{R}_+$. Now, from **(H7)** and Tonelli's theorem we have

$$\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T W^2_{2,\rho_{\mathbb{J}_1^d}}\big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}),\mathcal{L}(Y^i|_{[0,s]})\big)\,\mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\Big]$$
$$=\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\int_0^T W^2_{2,\rho_{\mathbb{J}_1^d}}\big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}),\mathcal{L}(Y^i|_{[0,s]})\big)b^i_s\,\mathrm{d}Q_s\Big]$$
$$=\int_0^T \mathbb{E}\Big[W^2_{2,\rho_{\mathbb{J}_1^d}}\big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]}),\mathcal{L}(Y^1|_{[0,s]})\big)\frac{1}{N}\sum_{i=1}^{N}b^i_s\Big]\,\mathrm{d}Q_s,$$

where in the last equality we used Remark 4.1.1.(v). Then, because $W_{2,\rho_{\mathbb{J}_1^d}}(\cdot,\cdot)\leq 1$, from **(H8)** we have that the Borel-measurable function γ is $Q-$integrable and dominates the sequence

$$\Big\{\mathbb{E}\Big[W^2_{2,\rho_{\mathbb{J}_1^d}}\big(L^N(\widetilde{\mathbf{Y}}^N|_{[0,\cdot]}),\mathcal{L}(Y^1|_{[0,\cdot]})\big)\frac{1}{N}\sum_{i=1}^{N}b^i_s\Big]\Big\}_{N\in\mathbb{N}}.$$

So, it will be enough to show that for every $s\in(0,T]$ we have

$$\lim_{N\to\infty}\mathbb{E}\left[W^2_{2,\rho_{\mathbb{J}_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right),\mathcal{L}\left(Y^1|_{[0,s]}\right)\right)\frac{1}{N}\sum_{i=1}^{N}b^i_s\right]=0.$$

Obviously we have

$$W^2_{2,\rho_{\mathbb{J}_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right),\mathcal{L}\left(Y^1|_{[0,s]}\right)\right)\frac{1}{N}\sum_{i=1}^{N}b^i_s\leq\gamma_s.$$

From dominated convergence theorem our goal is reduced to show that, for every $s\in[0,T]$

$$\lim_{N\to\infty}W^2_{2,\rho_{\mathbb{J}_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right),\mathcal{L}\left(Y^1|_{[0,s]}\right)\right)=0,\ \ \mathbb{P}-\text{a.e.}$$

Recalling now the comments provided in Section 2.6, more precisely the fact that the Wasserstein distance metrizes the weak convergence of measures on $\mathbb{D}^d$, we are eligible to translate the desired convergence into weak convergence of the respective measures. Fix $s\in(0,T]$, from Lemma 2.6.2 -we follow the notation of the aforementioned lemma- we only need to hold that

$$\lim_{N\to\infty}\int_{\mathbb{D}^d}f_k^{\mathcal{L}\left(Y^1|_{[0,s]}\right)}(x)\,L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right)(\mathrm{d}x)=\int_{\mathbb{D}^d}f_k^{\mathcal{L}\left(Y^1|_{[0,s]}\right)}(x)\,\mathcal{L}\left(Y^1|_{[0,s]}\right)(\mathrm{d}x),\ \ \mathbb{P}-\text{a.e.},$$

for all $k\in\mathbb{N}$, and for a suitable sequence $\{f^{\mathcal{L}(Y^1|_{[0,s]})}\}_{k\in\mathbb{N}}\subseteq C_b(\mathbb{D}^d)$. Fix $k\in\mathbb{N}$, the above equality is equivalently written as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}f_k^{\mathcal{L}\left(Y^1|_{[0,s]}\right)}\left(Y^m|_{[0,s]}\right)=\mathbb{E}\left[f_k^{\mathcal{L}\left(Y^1|_{[0,s]}\right)}\left(Y^1|_{[0,s]}\right)\right]\ \ \mathbb{P}-\text{a.e.},$$

which in fact it is the strong law of large numbers for the independent and identically distributed random variables $\left\{f_k^{\mathcal{L}\left(Y^1|_{[0,s]}\right)}\left(Y^m|_{[0,s]}\right)\right\}_{m\in\mathbb{N}}$; recall Remark 4.1.1.(v). □

Theorem 4.1.4 (Propagation of chaos). *Assume* **(H1)-(H9)** *and let $i \leq N \in \mathbb{N}$. For the solution of the mean-field BSDE* (3.9)*, den. by* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$*, and the solution of the $i-$th McKean–Vlasov BSDE* (3.3)*, den. by* (Y^i, Z^i, U^i, M^i)*, we have*

$$\lim_{N\to\infty} \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i} = 0. \tag{4.10}$$

Proof. We work for N large enough such that $i \leq N$. The arguments presented in Theorem 4.1.3 can be followed almost verbatim in order to conclude. We provide the sketch of the proof for the convenience of the reader.

We derive (4.5) and define ψ^i as in Theorem 4.1.3. The upper bound of (4.8) worsens as follows

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T W^2_{2,\rho_{J_1^d}}\Big(L^N(\mathbf{Y}^N|_{[0,s]}), L^N(\widetilde{\mathbf{Y}}^N|_{[0,s]})\Big)\mathrm{d}\mathcal{E}(\hat{\beta}A^i)_s\right]\\
&\qquad\leq \Lambda_{\hat{\beta}}\,\frac{1}{N}\sum_{m=1}^N \|Y^{m,N} - Y^m\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N},A^{\overline{X}^m};\mathbb{R}^d)}\\
&\qquad\leq \Lambda_{\hat{\beta}}\,\frac{1}{N}\sum_{m=1}^N \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}.
\end{aligned}$$

Then, using the a priori estimates Lemma 2.9.1 for the system (4.5) in conjunction with the comments above, one gets for the $i-$element of the solution of the system

$$\begin{aligned}
&\| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}\\
&\qquad\leq (26 + 9\hat{\beta}\Phi)\|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T,A^i;\mathbb{R}^d)}\\
&+ \max\left\{2, \frac{\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M^{\Phi}_\star(\hat{\beta})\ \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}\\
&+\frac{2\Lambda_{\hat{\beta}}}{\hat{\beta}} M^{\Phi}_\star(\hat{\beta})\ \frac{1}{N}\sum_{i=1}^N \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}\\
&+ 2M^{\Phi}_\star(\hat{\beta})\frac{1}{\hat{\beta}}\ \mathbb{E}\left[\int_0^T W^2_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right), \mathcal{L}\left(Y^i|_{[0,s]}\right)\right)\mathrm{d}\mathcal{E}\left(\hat{\beta}A^i\right)_s\right].
\end{aligned}$$

In view of **(H9)** we get

$$\begin{aligned}
&\| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}\\
&\leq \frac{(26 + 9\hat{\beta}\Phi)\hat{\beta}}{2M^{\Phi}_\star(\hat{\beta})} C_{\star,\Phi,\hat{\beta}}\ \|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T,A^i;\mathbb{R}^d)}\\
&+ \Lambda_{\hat{\beta}} C_{\star,\Phi,\hat{\beta}}\ \frac{1}{N}\sum_{i=1}^N \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i}\\
&\qquad+ C_{\star,\Phi,\hat{\beta}}\ \mathbb{E}\left[\int_0^T W^2_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right), \mathcal{L}\left(Y^i|_{[0,s]}\right)\right)\mathrm{d}\mathcal{E}\left(\hat{\beta}A^i\right)_s\right],
\end{aligned}$$

for

$$C_{\star,\Phi,\hat{\beta}} := \frac{2M_\star^\Phi(\hat{\beta})}{1-\max\left\{2,\frac{\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\}M_\star^\Phi(\hat{\beta})}\frac{1}{\hat{\beta}}.$$

But the right-hand side of the last inequality vanishes as N increases to ∞. Indeed, the first term goes to zero from **(H2)**, the second term is the conclusion of Theorem 4.1.3 and for the third term we can follow exactly the same arguments as in the proof of Theorem 4.1.3. □

The following are the usual convergence in law results for the solutions of the mean-field systems to the solutions of the McKean–Vlasov BSDEs in the path-dependent case.

Corollary 4.1.5. *Assume* **(H1)-(H9)** *and let* $i \le N \in \mathbb{N}$. *For the solution of the mean-field BSDE* (3.9)*, den. by* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$*, and the solution of the* $i-th$ *McKean–Vlasov BSDE* (3.3)*, den. by* (Y^i, Z^i, U^i, M^i)*, we have*

(i) $\lim_{N\to\infty}\sup_{s\in[0,T]}\{W^2_{2,|\cdot|}(\mathcal{L}(Y^{i,N}_s),\mathcal{L}(Y^i_s))\} = 0,$

(ii) $\lim_{N\to\infty}\sup_{s\in[0,T]}\{W^2_{2,\rho_{\mathrm{J}_1^d}}(\mathcal{L}(Y^{i,N}|_{[0,s]}),\mathcal{L}(Y^i|_{[0,s]}))\} = 0,$

(iii) $\lim_{N\to\infty}\sup_{s\in[0,T]}\{W^2_{2,\rho_{\mathrm{J}_1^{id}}}\left(\mathcal{L}(Y^{1,N}|_{[0,s]},...,Y^{i,N}|_{[0,s]}),\mathcal{L}(Y^1|_{[0,s]},...,Y^i|_{[0,s]})\right)\} = 0.$

Proof. For every $s \in [0,T]$ by definition of the Wasserstein distance of order two we have that

$$\begin{aligned} W^2_{2,|\cdot|}(\mathcal{L}(Y^{i,N}_s),\mathcal{L}(Y^i_s)) &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}|x-z|^2\pi(\mathrm{d}x,\mathrm{d}z) = \mathbb{E}\left[|Y^{i,N}_s - Y^i_s|^2\right] \\ &\le \mathbb{E}\left[\sup_{0\le s\le T}\left\{|Y^{i,N}_s - Y^i_s|^2\right\}\right] = \|Y^{i,N}-Y^i\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N},A^{\overline{X}^i};\mathbb{R}^d)}, \end{aligned}$$

where we chose π to be the image measure on $\mathbb{R}^{2d}$ produced by the measurable function $(Y^{i,N}_t, Y^i_t) : \Omega \to \mathbb{R}^{2d}$. The right side of the above inequality is independent of s, hence from (4.10) we finished the proof of the first statement.

For the second, from (2.26), similarly we have

$$\begin{aligned} W^2_{2,\rho_{\mathrm{J}_1^d}}(\mathcal{L}(Y^{i,N}|_{[0,s]}),\mathcal{L}(Y^i|_{[0,s]})) &\le \int_{\mathbb{D}^d\times\mathbb{D}^d}\rho_{\mathrm{J}_1^d}(x,z)^2\pi(\mathrm{d}x,\mathrm{d}z) \\ \le \mathbb{E}\left[\sup_{0\le s\le T}\left\{|Y^{i,N}_s - Y^i_s|^2\right\}\right] &= \|Y^{i,N}-Y^i\|^2_{\mathcal{S}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N},A^{\overline{X}^i};\mathbb{R}^d)}. \end{aligned}$$

Finally for the last we have

$$\begin{aligned} &W^2_{2,\rho_{\mathrm{J}_1^{id}}}\left(\mathcal{L}(Y^{1,N}|_{[0,s]},...,Y^{i,N}|_{[0,s]}),\mathcal{L}(Y^1|_{[0,s]},...,Y^i|_{[0,s]})\right) \\ &\qquad\le \int_{\mathbb{D}^{id}\times\mathbb{D}^{id}}\rho_{\mathrm{J}_1^{id}}(x,z)^2\pi(\mathrm{d}x,\mathrm{d}z) \le \int_{\mathbb{D}^{id}\times\mathbb{D}^{id}}\sum_{m=1}^{i}\sup_{s\in[0,T]}|x_m-z_m|^2\pi(\mathrm{d}x,\mathrm{d}z), \end{aligned}$$

where we chose π to be the image measure produced by the measurable function $h : \Omega \to \mathbb{D}^{2id}$ with

$$h := (Y^{1,N}|_{[0,s]}, ..., Y^{i,N}|_{[0,s]}, Y^1|_{[0,s]}, ..., Y^i|_{[0,s]}).$$

By this we have

$$W^2_{2,\rho_{\mathrm{J}_1^{id}}}\left(\mathcal{L}(Y^{1,N}|_{[0,s]}, ..., Y^{i,N}|_{[0,s]}), \mathcal{L}(Y^1|_{[0,s]}, ..., Y^i|_{[0,s]})\right) \leq \sum_{m=1}^{i} \|Y^{m,N} - Y^m\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{F}^{1,\dots,N}, A^{\overline{X}^m}; \mathbb{R}^d)},$$

from which we get again from (4.10) the uniform convergence. $\square$

Remark 4.1.6 (Rates of convergence). *Let $i \in \{1, \dots, N\}$. From the proofs of Theorem 4.1.3 and Theorem 4.1.4 we can conclude that in order to get convergence rates from these methods one would have to control the quantities*

$$\int_0^T \mathbb{E}\left[W^2_{2,\rho_{\mathrm{J}_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right), \mathcal{L}\left(Y^1|_{[0,s]}\right)\right)\gamma_s\right] \mathrm{d}Q_s.$$

4.2 Propagation of chaos under the usual dependence

4.2.1 Main results

In the usual cases we are interested in the asymptotic behaviour of the mean-field system of BSDEs

$$\begin{aligned} Y^{i,N}_t = \xi^{i,N} &+ \int_t^T f\left(s, Y^{i,N}_s, Z^{i,N}_s c^i_s, \Gamma^{(\mathbb{F}^{1,\dots,N}, \overline{X}^i, \Theta)}(U^{i,N})_s, L^N(\mathbf{Y}^N_s)\right) \mathrm{d}C^{\overline{X}^i}_s \\ &- \int_t^T Z^{i,N}_s \,\mathrm{d}X^{i,\circ}_s - \int_t^T \int_{\mathbb{R}^d} U^{i,N}_s(x)\, \widetilde{\mu}^{(\mathbb{F}^{1,\dots,N}, X^{i,\natural})}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}M^{i,N}_s, \quad i = 1, ..., N. \end{aligned} \tag{3.1}$$

In other words, when the dependence in the generator is from the moment on time s and not the whole path up to that time. Naturally, we will need the corresponding "usual" McKean–Vlasov BSDE

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s c^{(\mathbb{G},\overline{X})}_s, \Gamma^{(,\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y_s)\right) \mathrm{d}C^{(\mathbb{G},\overline{X})}_s& \\ - \int_t^T Z_s \,\mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^d} U_s\, \widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}M_s.& \end{aligned} \tag{3.2}$$

To proceed we first need to make some reformulation of the **(H1)**-**(H9)** assumptions. **(H1)**-**(H3)** remain the same, as are **(H5)** and **(H6)**. Next, we give the modification of the rest.

(H4′) A generator $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y, z, u, \mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the map

$$t \longmapsto f(t, y, z, u, \mu) \text{ is } \mathcal{B}(\mathbb{R}_+) - \text{measurable}$$

and satisfies the following Lipschitz condition

$$|f(t,y,z,u,\mu) - f(t,y',z',u',\mu')|^2 \\ \leq\ r(t)\,|y-y'|^2 +\ \vartheta^o(t)\,|z-z'|^2 +\ \vartheta^\natural(t)\,|u-u'|^2 + \vartheta^*(t)\,W^2_{2,|\cdot|}\,(\mu,\mu')\,,$$

where $(r,\vartheta^o,\vartheta^\natural,\vartheta^*) : (\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)) \longrightarrow \left(\mathbb{R}^4_+,\mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(H7′) The martingale $\overline{X}^1$ has independent increments.

(H8′) For the same $\hat\beta$ as in **(H2)** we have $3\,\widetilde{M}^\Phi(\hat\beta) < 1$.

Remark 4.2.1. *(i) The independence of the increments of the martingale $\overline{X}^1$ is equivalent to its associated triplet being deterministic, see [40, Corollary 7.87] or [29, Theorem II.4.15]. As a result, recalling the notational simplification for which we argued in Remark 4.1.1.(ii) and which we will use hereinafter, the process A^1 is deterministic. Indeed, this is immediate by the way we have constructed C^i; see 2.9. In view of* **(H1)**, *in particular the fact that we assumed the sequence $\{\overline{X}^i\}_{i\in\mathbb{N}}$ to be identically distributed, we have that $A^1 = A^i$ for every $i \geq 2$, see Remark 3.2.5.*

(ii) Under the set of assumptions **(H1)**-**(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** *and* **(H8′)** *we can verify that Theorem 3.2.4 guarantees the existence of a unique solution of the mean-field BSDE* (3.1). *Additionally, Theorem 3.1.8 guarantees the existence of a unique solution of the McKean–Vlasov BSDE* (3.2). *We will use the same notation as in the previous section for the respective solutions,* i.e., *for fixed $N \in \mathbb{N}$, $(\boldsymbol{Y}^N,\boldsymbol{Z}^N,\boldsymbol{U}^N,\boldsymbol{M}^N)$ denotes the solution associated to the mean-field BSDE* (3.1). *Also, for $i \in \mathbb{N}$, we will call the $i-$th McKean–Vlasov BSDE* (3.2) *the one that corresponds to the standard data $\left(\overline{X}^i,\mathbb{F}^i,\Theta,\Gamma,T,\xi^i,f\right)$ under $\hat\beta$. Additionally, we will call the first N McKean–Vlasov BSDEs* (3.2) *those that correspond to the set of standard data $\left\{\left(\overline{X}^i,\mathbb{F}^i,\Theta,\Gamma,T,\xi^i,f\right)\right\}_{i\in\{1,\dots,N\}}$ under $\hat\beta$ with associated solution $(\widetilde{\boldsymbol{Y}}^N,\widetilde{\boldsymbol{Z}}^N,\widetilde{\boldsymbol{U}}^N,\widetilde{\boldsymbol{M}}^N)$.*

We are ready to proceed with the proofs of the desired properties of propagation. The method described in Section 4.1 in order to prove the backward propagation of chaos is transferred mutatis mutandis. To this end, we will provide the sketch of the proof as we did for Theorem 4.1.4. Before that we introduce the following notation. For

$$(Y,Z,U,M)\ \in\ \mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C^{(\mathbb{G},\overline{X})};\mathbb{R}^d)\ \times\ \mathbb{H}^2_\beta(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})\ \times\ \mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)\ \times\ \mathcal{H}^2_\beta(\mathbb{G},A,\overline{X}^{\perp_\mathbb{G}};\mathbb{R}^d)$$

we define (see (3.7))

$$\|\,(Y,Z,U,M)\,\|^2_{\star,\beta,\mathbb{G},\alpha,C^{(\mathbb{G},\overline{X})},\overline{X}} \\ := \|Y\|^2_{\mathcal{S}^2_{\hat\beta}(\mathbb{G},\alpha,C^{(\mathbb{G},\overline{X})};\mathbb{R}^d)} + \|Z\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\circ;\mathbb{R}^{d\times p})} + \|U\|^2_{\mathbb{H}^2_\beta(\mathbb{G},A,X^\natural;\mathbb{R}^d)} + \|M\|^2_{\mathcal{H}^2_\beta(\mathbb{G},A,\overline{X}^{\perp_\mathbb{G}};\mathbb{R}^d)}.$$

Theorem 4.2.2 (Propagation of chaos for the system). *Assume* **(H1)**-**(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** *and* **(H8′)**. *For the solution of the mean-field BSDE* (3.1), *den by $(\boldsymbol{Y}^N,\boldsymbol{Z}^N,\boldsymbol{U}^N,\boldsymbol{M}^N)$, and the* *solutions of the first N McKean–Vlasov BSDE* (3.2)*, den. by* $(\widetilde{\boldsymbol{Y}}^N, \widetilde{\boldsymbol{Z}}^N, \widetilde{\boldsymbol{U}}^N, \widetilde{\boldsymbol{M}}^N)$*, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \| \left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} = 0. \tag{4.11}$$

Proof. Let $N \in \mathbb{N}$. The analogous to (4.5) is now written as

$$\begin{aligned} Y^{i,N}_t - Y^i_t = \xi^{i,N} - \xi^i + \int_t^T & f\left(s, Y^{i,N}_s, Z^{i,N}_s c^i, \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta^i)}(U^{i,N})_s, L^N(\mathbf{Y}^N_s)\right) \\ & - f\left(s, Y^i_s, Z^i_s c^i, \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta^i)}(U^i)_s, \mathcal{L}(Y^i_s)\right) \mathrm{d}C^i_s \\ - \int_t^T \mathrm{d}\Big[\left(Z^{i,N} - Z^i\right)\cdot X^{i,\circ} & + \left(U^{i,N} - U^i\right) \star \widetilde{\mu}^{(\mathbb{F}^{1,\dots,N},X^{i,\natural})} + M^{i,N} - M^i\Big]_s, \end{aligned} \tag{4.12}$$

for $i \in \{1,\dots,N\}$. Let us, now, define $\psi := (\psi^1,\dots,\psi^N)$, where for $i \in \{1,...,N\}$ we define

$$\begin{aligned} \psi^i_t := f & \left(t, Y^{i,N}_t, Z^{i,N}_t c^i_t, \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta^i)}(U^{i,N})_t, L^N(\mathbf{Y}^N_t)\right) \\ & - f\left(t, Y^i_t, Z^i_t c^i_t, \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta^i)}(U^i)_t, \mathcal{L}(Y^i_t)\right). \end{aligned}$$

The properties of the generator, and using the triangular inequality of the Wasserstein distance, provide for every $i \in \{1,...,N\}$ that

$$\begin{aligned} \left\|\frac{\psi^i}{\alpha}\right\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N},A^i,C^i;\mathbb{R}^d)} & \leq \|\alpha\left(Y^{i,N} - Y^i\right)\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A^i,C^i;\mathbb{R}^d)} \\ & + \|Z^{i,N} - Z^i\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{F}^{1,\dots,N},A^i,X^{i,\circ};\mathbb{R}^{d\times p})} \\ + 2\,\|U^{i,N} - U^i\|^2_{\mathbb{H}^2_{\beta}(\mathbb{F}^{1,\dots,N},A^i,X^{i,\natural};\mathbb{R}^d)} & + \frac{2}{N}\sum_{m=1}^{N}\|\alpha\left(Y^{m,N} - Y^m\right)\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A^m,C^m;\mathbb{R}^d)} \\ & + 2\mathbb{E}\left[\int_0^T \alpha_s^2 \mathcal{E}(\hat{\beta}A^i)_{s-} W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y^i_s\right)\right) \mathrm{d}C^i_s\right]. \end{aligned}$$

From the a priori estimates Lemma 2.9.1, Lemma A.2.6 and by adding the above relations with respect to $i \in \{1,...,N\}$ we have

$$\begin{aligned} & \sum_{i=1}^{N} \| \left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} \\ & \leq \left(26 + \frac{2}{\hat{\beta}} + (9\hat{\beta} + 2)\Phi\right)\sum_{i=1}^{N}\|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T,A^i;\mathbb{R}^d)} \\ & \quad + 3\widetilde{M}^{\Phi}(\hat{\beta}) \sum_{i=1}^{N} \| \left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} \\ & \quad + 2\widetilde{M}^{\Phi}(\hat{\beta}) \sum_{i=1}^{N}\mathbb{E}\left[\int_0^T \alpha_s^2 \mathcal{E}\left(\hat{\beta}A^i\right)_{s-} W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y^i_s\right)\right) \mathrm{d}C^i_s\right]. \end{aligned}$$

So, from **(H8′)** we get

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} \| \left(Y^{i,N}-Y^{i}, Z^{i,N}-Z^{i}, U^{i,N}-U^{i}, M^{i,N}-M^{i}\right) \|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}\\
&\leq \frac{\left(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi\right)}{1-3\widetilde{M}^{\Phi}(\hat\beta)}\frac{1}{N}\sum_{i=1}^{N}\|\xi^{i,N}-\xi^{i}\|^2_{\mathbb{L}^2_{\hat\beta}(\mathcal{F}_T^{1,\dots,N},A^i;\mathbb{R}^d)}\\
&\quad+\frac{2\widetilde{M}^{\Phi}(\hat\beta)}{1-3\widetilde{M}^{\Phi}(\hat\beta)}\ \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^{N}\alpha_s^2\mathcal{E}\left(\hat\beta A^i\right)_{s-} W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^i_s\right)\right)\mathrm{d}C^i_s\right].
\end{aligned}$$

Let us now observe that we need to prove that the right-hand side of the above inequality vanishes as N increases to ∞. The result for the first summand follows from Assumption **(H2)** The rest of the proof is devoted to arguing about the validity of the desired claim. Essentially, we will use again the dominated convergence theorem in conjunction with the strong law of large numbers.

Using Remark 4.1.1.(v), we have for every $i,j\in\mathbb{N}$ that $\mathcal{L}(Y^i_s)=\mathcal{L}(Y^j_s)$. Thus by Remark 4.2.1.(i) and Tonelli's theorem we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^{N}\alpha_s^2\mathcal{E}\left(\hat\beta A^i\right)_{s-} W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^i_s\right)\right)\mathrm{d}C^i_s\right]\\
&\quad=\frac{1}{\hat\beta}\mathbb{E}\left[\int_0^T W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^1_s\right)\right)\mathrm{d}\mathcal{E}\left(\hat\beta A^1\right)_s\right]\\
&\quad=\frac{1}{\hat\beta}\int_0^T \mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^1_s\right)\right)\right]\mathrm{d}\mathcal{E}\left(\hat\beta A^1\right)_s.
\end{aligned}$$

We use once again the the triangle inequality for the Wasserstein distance as well as (2.25) and we have

$$\begin{aligned}
W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^1_s\right)\right)&\leq\ 2\,W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\delta_0\right)+\ 2\,W^2_{2,|\cdot|}\left(\delta_0,\mathcal{L}\left(Y^1_s\right)\right)\\
&\leq\ 2\,\frac{1}{N}\sum_{m=1}^{N}|Y^m_s|^2+\ 2\,\mathbb{E}\left[|Y^1_s|^2\right].
\end{aligned}\tag{4.13}$$

Moreover, from Remark 4.1.1.(v), for every $s\in[0,T]$ we have

$$\mathbb{E}\left[2\,\frac{1}{N}\sum_{m=1}^{N}|Y^m_s|^2+\ 2\,\mathbb{E}\left[|Y^1_s|^2\right]\right]=4\mathbb{E}[|Y^1_s|^2].$$

Furthermore, again from Tonelli's theorem we have

$$\int_0^T \mathbb{E}\left[|Y^1_s|^2\right]\,\mathrm{d}\mathcal{E}\left(\hat\beta A^1\right)_s=\hat\beta\,\|\alpha Y^1\|^2_{\mathbb{H}^2_{\hat\beta}(\mathbb{G},A^{\overline{X}^1},C^{\overline{X}^1};\mathbb{R}^d)}.$$

Hence, the Borel-measurable function $]0,T]\ni t\longmapsto 4\mathbb{E}\left[|Y^1_t|^2\right]$ is $\mathcal{E}\left(\hat\beta A^1\right)-$integrable and dominates the sequence $\left\{\mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right),\mathcal{L}\left(Y^1_s\right)\right)\right]\mathbb{1}_{]0,T]}(s)\right\}_{N\in\mathbb{N}}$. In order to apply the dominated convergence, we also need to show that

$$\lim_{N\to\infty} \mathbb{E}\left[W^2_{2,|\cdot|}\Big(L^N\big(\widetilde{\mathbf{Y}}^N_s\big), \mathcal{L}\big(Y^1_s\big)\Big)\right] = 0, \quad \text{for all } s \in (0,T].$$

To this end, we fix $s \in (0,T]$. Our new claim is that the sequence $\left\{W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y^1_s\right)\right)\right\}_{N\in\mathbb{N}}$ is uniformly integrable. In view of (4.13), for the above claim to hold it would suffice the sequence $\left\{\frac{1}{N}\sum_{m=1}^N |Y^m_s|^2\right\}_{N\in\mathbb{N}}$ to be uniformly integrable. So, to make the presentation easier, let us define the sequence of random variables $\{S_N\}_{N\in\mathbb{N}}$, where $S_N := \sum_{m=1}^N |Y^m_s|^2$. Also, we define the sequence of $\sigma-$algebras $\{\mathcal{G}_N\}_{N\in\mathbb{N}}$, where $\mathcal{G}_N := \sigma\left(S_N, S_{N+1}...\right)$. Arguing analogous to Remark 4.1.1.(v), the random variables $\left\{|Y^N_s|^2\right\}_{N\in\mathbb{N}}$ are integrable, independent and identically distributed. From the symmetry under permutation for the family $\left\{|Y^N_s|^2\right\}_{N\in\mathbb{N}}$ we can easily show that for every $N \in \mathbb{N}$

$$\mathbb{E}\left[|Y^1_s|^2 \Big| \mathcal{G}_N\right] = ... = \mathbb{E}\left[|Y^N_s|^2 \Big| \mathcal{G}_N\right].$$

By adding the terms of the above equalities, we get

$$\frac{1}{N} S_N = \mathbb{E}\left[|Y^1_s|^2 \Big| \mathcal{G}_N\right].$$

Hence, indeed the sequence $\left\{W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y^1_s\right)\right)\right\}_{N\in\mathbb{N}}$ is uniformly integrable. This fact makes it enough to show that

$$\lim_{N\to\infty} W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y^1_s\right)\right) = 0, \quad \text{for } \mathbb{P}-\text{a.e.}$$

We apply Villani [49, Definition 6.8 *(i)*], with $x_0 = 0$, Lemma 2.6.2 and Remark 4.1.1.(v) to get from the strong law of large numbers exactly what we want as at the end of the proof of Theorem 4.1.3. ☐

Theorem 4.2.3 (**Propagation of chaos**). *Assume* **(H1)**-**(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** *and* **(H8′)** *and let* $i \leq N \in \mathbb{N}$. *For the solution of the mean-field BSDE* (3.1), *den by* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *and the solution of the* $i-th$ *McKean–Vlasov BSDE* (3.2), *den. by* (Y^i, Z^i, U^i, M^i), *we have we have*

$$\lim_{N\to\infty} \left\| \left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right) \right\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} = 0. \tag{4.14}$$

Proof. We will give details only for the changes in the parts of Theorem 4.1.4 that might not be clear. So, from (2.25) we have arrived to the inequality

$$\alpha_s^2 \mathcal{E}\big(\hat{\beta}A^i\big)_{s-} W^2_{2,|\cdot|}\Big(L^N\big(\mathbf{Y}^N_s\big), L^N\big(\widetilde{\mathbf{Y}}^N_s\big)\Big) \leq \alpha_s^2 \mathcal{E}\big(\hat{\beta}A^i\big)_{s-} \frac{1}{N}\sum_{m=1}^N \left|Y^{m,N}_s - Y^m_s\right|^2.$$

Hence, by **(H7′)**, in particular Remark 4.2.1.(i), we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \alpha_s^2 \mathcal{E}\big(\hat\beta A^i\big)_{s-} W^2_{2,|\cdot|}\big(L^N\big(\mathbf{Y}^N_s\big), L^N\big(\widetilde{\mathbf{Y}}^N_s\big)\big)\,\mathrm{d}C^i_s\Big]\\
&\le \mathbb{E}\Big[\int_0^T \alpha_s^2 \mathcal{E}\big(\hat\beta A^i\big)_{s-} \frac{1}{N}\sum_{m=1}^N \big|Y^{m,N}_s - Y^m_s\big|^2\,\mathrm{d}C^i_s\Big]\\
&= \mathbb{E}\Big[\int_0^T \alpha_s^2 \mathcal{E}\big(\hat\beta A^m\big)_{s-} \frac{1}{N}\sum_{m=1}^N \big|Y^{m,N}_s - Y^m_s\big|^2\,\mathrm{d}C^m_s\Big]\\
&\le \frac{1}{N}\sum_{m=1}^N \|\big(Y^{m,N}-Y^m, Z^{m,N}-Z^m, U^{m,N}-U^m, M^{m,N}-M^m\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^m,\overline{X}^m}.
\end{aligned}$$

Then, as in the proof of Theorem 4.1.4 we end up in the inequality

$$\begin{aligned}
&\|\big(Y^{i,N}-Y^i, Z^{i,N}-Z^i, U^{i,N}-U^i, M^{i,N}-M^i\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}\\
&\le \Big(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi\Big)\|\xi^{i,N}-\xi^i\|^2_{\mathbb{L}^2_{\hat\beta}(\mathcal{F}^{1,\dots,N}_T,A^i;\mathbb{R}^d)}\\
&+ 2\widetilde{M}^\Phi(\hat\beta)\ \|\big(Y^{i,N}-Y^i, Z^{i,N}-Z^i, U^{i,N}-U^i, M^{i,N}-M^i\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}\\
&+ 2\widetilde{M}^\Phi(\hat\beta)\frac{1}{N}\sum_{m=1}^N \|\big(Y^{m,N}-Y^m, Z^{m,N}-Z^m, U^{m,N}-U^m, M^{m,N}-M^m\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^m,\overline{X}^m}\\
&+2\widetilde{M}^\Phi(\hat\beta)\ \mathbb{E}\Big[\int_0^T \alpha_s^2 \mathcal{E}\big(\hat\beta A^i\big)_{s-} W^2_{2,|\cdot|}\Big(L^N\Big(\widetilde{\mathbf{Y}}^N_s\Big),\mathcal{L}\big(Y^i_s\big)\Big)\,\mathrm{d}C^i_s\Big].
\end{aligned}$$

So, from **(H8′)** we get

$$\begin{aligned}
&\|\big(Y^{i,N}-Y^i, Z^{i,N}-Z^i, U^{i,N}-U^i, M^{i,N}-M^i\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}\\
&\le \frac{\Big(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi\Big)}{1-2\widetilde{M}^\Phi(\hat\beta)}\|\xi^{i,N}-\xi^i\|^2_{\mathbb{L}^2_{\hat\beta}(\mathcal{F}^{1,\dots,N}_T,A^i;\mathbb{R}^d)}\\
&+\frac{2\widetilde{M}^\Phi(\hat\beta)}{1-2\widetilde{M}^\Phi(\hat\beta)}\frac{1}{N}\sum_{m=1}^N \|\big(Y^{m,N}-Y^m, Z^{m,N}-Z^m, U^{m,N}-U^m, M^{m,N}-M^m\big)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^m,\overline{X}^m}\\
&+\frac{2\widetilde{M}^\Phi(\hat\beta)}{1-2\widetilde{M}^\Phi(\hat\beta)}\ \mathbb{E}\Big[\int_0^T \alpha_s^2 \mathcal{E}\big(\hat\beta A^i\big)_{s-} W^2_{2,|\cdot|}\Big(L^N\Big(\widetilde{\mathbf{Y}}^N_s\Big),\mathcal{L}\big(Y^i_s\big)\Big)\,\mathrm{d}C^i_s\Big].
\end{aligned}$$

From **(H7′)** and Tonelli's theorem our goal becomes to show that

$$\lim_{N\to\infty}\int_0^T \mathbb{E}\Big[W^2_{2,|\cdot|}\Big(L^N\Big(\widetilde{\mathbf{Y}}^N_s\Big),\mathcal{L}\big(Y^i_s\big)\Big)\Big]\,\mathrm{d}\mathcal{E}\big(\hat\beta A^i\big)_s = 0,$$

since the first term of the right-hand side goes to zero from **(H2)** and the second vanishes as N tends to ∞ by Theorem 4.2.2. The desired convergence can be derived by following the exact same arguments as at the end of the poof of Theorem 4.2.2. □

It is interesting to note that with our method the next result is a corollary of Theorem 4.2.2 while for example in [38] is a requirement in order to prove Theorem 4.2.3.

Corollary 4.2.4. *For the solution of the mean-field BSDE* (3.1)*, den by* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$*, we have for every* $t \in [0,T]$ *that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left\{ \mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t \right), \mathcal{L}(Y^1_t) \right) \right] \right\} = 0. \tag{4.15}$$

Proof. We remind that we denote by $(\widetilde{\mathbf{Y}}^N, \widetilde{\mathbf{Z}}^N, \widetilde{\mathbf{U}}^N, \widetilde{\mathbf{M}}^N)$ the solutions of the first N McKean–Vlasov BSDE (3.2). From the triangle inequality of the Wasserstein distance and (2.25) we have

$$\begin{aligned} W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t \right), \mathcal{L}(Y^1_t) \right) &\leq 2\, W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t \right), L^N\left(\widetilde{\mathbf{Y}}^N_t \right) \right) \\ &\quad + 2\, W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s \right), \mathcal{L}\left(Y^1_t \right) \right) \\ &\leq 2\, \frac{1}{N} \sum_{m=1}^N |Y^{m,N}_t - Y^m_t|^2 + 2\, W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_t \right), \mathcal{L}\left(Y^1_t \right) \right). \end{aligned}$$

Hence, from Theorem 4.2.2, for the first summand of the right hand side of the above inequality, and from the arguments presented in the proofs of Section 5.3.2, for the second summand, we are done. □

Now we give a version of the strong law of large numbers with respect to the $\mathbb{L}^2$ convergence for the solutions of the mean-field systems. Note however that for every $N \in \mathbb{N}$ and $t \in [0,T]$ the random variables $Y^{1,N}_t, \ldots, Y^{N,N}_t$ are neither independent nor exchangeable.

Corollary 4.2.5. *For every* $t \in [0,T]$ *we have*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left\{ \mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N Y^{i,N}_t - \mathbb{E}[Y^1_t] \right|^2 \right] \right\} = 0. \tag{4.16}$$

Proof. From the triangle inequality for the Euclidean norm we get

$$\left| \frac{1}{N} \sum_{m=1}^N Y^{i,N}_t - \mathbb{E}[Y^1_t] \right| \leq \left| \frac{1}{N} \sum_{i=1}^N \left(Y^{i,N}_t - Y^i_t \right) \right| + \left| \frac{1}{N} \sum_{i=1}^N Y^i_t - \mathbb{E}[Y^1_t] \right|.$$

We are going to use the inequality

$$\left(\sum_{i=1}^N a_i \right)^2 \leq N \sum_{i=1}^N a_i^2,$$

for every set of real numbers $\{a_1, \ldots, a_N\}$. So, we have

$$\left| \frac{1}{N} \sum_{m=1}^N Y^{i,N}_t - \mathbb{E}[Y^1_t] \right|^2 \leq \frac{2}{N} \sum_{i=1}^N \left| Y^{i,N}_t - Y^i_t \right|^2 + \frac{2}{N^2} \left| \sum_{i=1}^N \left(Y^i_t - \mathbb{E}[Y^1_t] \right) \right|^2.$$

Hence, if we take expectations on the above inequality and use Remark 4.1.1(v) for the second term of its right-hand side we get

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\sum_{m=1}^{N} Y_t^{i,N} - \mathbb{E}[Y_t^1]\right|^2\right] &\leq \frac{2}{N}\mathbb{E}\left[\sum_{i=1}^{N}\left|Y_t^{i,N} - Y_t^i\right|^2\right] + \frac{2}{N^2}\mathbb{E}\left[\sum_{i=1}^{N}\left|Y_t^i - \mathbb{E}[Y_t^1]\right|^2\right]\\ &= \frac{2}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|Y_t^{i,N} - Y_t^i\right|^2\right] + \frac{2}{N}\mathbb{E}\left[\left|Y_t^1 - \mathbb{E}[Y_t^1]\right|^2\right]\\ &\leq \frac{2}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|Y_t^{i,N} - Y_t^i\right|^2\right] + \frac{2}{N}\mathbb{E}\left[\sup_{t\in[0,T]}\left\{\left|Y_t^1\right|^2\right\}\right].\end{aligned}$$

From Theorem 4.2.2 we can conclude. □

4.2.2 Rates of convergence

Assume **(H1)-(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)**, **(H8′)** and furthermore that exists a function $R : \mathbb{N} \longrightarrow \mathbb{R}_+$ with $\lim_{N\to\infty} R(N) = 0$ such that

$$\sup_{i\in\{1,\dots,N\}}\left\{\|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat\beta}(\mathcal{F}_T^{1,\dots,N}, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)}\right\} \leq R(N).$$

Let $i \in \{1,\dots,N\}$. From the proofs of Theorem 4.2.2 and Theorem 4.2.3 we can see that for the solution of the mean-field BSDE (3.1), den by $(\mathbf{Y}^N, \mathbf{Z}^N, \mathbf{U}^N, \mathbf{M}^N)$, and the solutions of the first N McKean–Vlasov BSDE (3.2), den. by $(\widetilde{\mathbf{Y}}^N, \widetilde{\mathbf{Z}}^N, \widetilde{\mathbf{U}}^N, \widetilde{\mathbf{M}}^N)$, we have the following inequalities

$$\begin{aligned}&\frac{1}{N}\sum_{m=1}^{N}\|\left(Y^{m,N} - Y^m, Z^{m,N} - Z^m, U^{m,N} - U^m, M^{m,N} - M^m\right)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^m,\overline{X}^m}\\ &\qquad\leq \frac{\left(26 + \frac{2}{\hat\beta} + (9\hat\beta + 2)\Phi\right)}{1 - 3\widetilde{M}^\Phi(\hat\beta)}R(N)\\ &\qquad+\frac{2\widetilde{M}^\Phi(\hat\beta)}{1 - 3\widetilde{M}^\Phi(\hat\beta)}\,\frac{1}{\hat\beta}\int_0^T \mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y_s^1\right)\right)\right]\mathrm{d}\mathcal{E}\left(\hat\beta A^1\right)_s\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}&\|\left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right)\|^2_{\star,\hat\beta,\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}\\ &\qquad\leq \frac{\left(26 + \frac{2}{\hat\beta} + (9\hat\beta + 2)\Phi\right)(2 - 5\widetilde{M}^\Phi(\hat\beta))}{(1 - 2\widetilde{M}^\Phi(\hat\beta))(1 - 3\widetilde{M}^\Phi(\hat\beta))}R(N)\\ &+\left(\frac{2\widetilde{M}^\Phi(\hat\beta)}{1 - 2\widetilde{M}^\Phi(\hat\beta)}\right)\left(\frac{1 - \widetilde{M}^\Phi(\hat\beta)}{1 - 3\widetilde{M}^\Phi(\hat\beta)}\right)\frac{1}{\hat\beta}\int_0^T \mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_s\right), \mathcal{L}\left(Y_s^1\right)\right)\right]\mathrm{d}\mathcal{E}\left(\hat\beta A^1\right)_s.\end{aligned}\tag{4.18}$$

So, in order to apply [20, Theorem 1] and get convergence rates we need to know the finiteness of the following quantity.

Definition 4.2.6. *For every real number* $q > 2$ *and deterministic* $T \in \mathbb{R}_+ \cup \{\infty\}$*, we define*

$$\Lambda_{q,T} := \frac{1}{\hat{\beta}} \int_0^T \left(\mathbb{E}\left[|Y_s^1|^q\right]\right)^{\frac{2}{q}} \mathrm{d}\mathcal{E}\left(\hat{\beta}A^1\right)_s.$$

Remark 4.2.7. *From Remark 4.1.1.(v) for all* $s \in [0,T]$ *we have* $\mathbb{E}[|Y_s^i|^q] = \mathbb{E}[|Y_s^j|^q]$*, for all* $i,j \in \mathbb{N}$*. Furthermore, obviously* $\|\alpha Y^1\|^2_{\mathbb{H}^2_{\hat{\beta}}(\mathbb{G},A^1,C^{\overline{X}^1};\mathbb{R}^d)} = \frac{1}{\hat{\beta}} \int_0^T \mathbb{E}[|Y_s^1|^2] \mathrm{d}\mathcal{E}\left(\hat{\beta}A^1\right)_s$
$\leq \Lambda_{q,T}$.

Theorem 4.2.8. *If* $\Lambda_{q,T} < \infty$ *for some* $q > 2$ *and deterministic* T*, then exists constant* $C_{d,q,2} > 0$ *depending on* $d, q, 2$ *such that*

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} \|\left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right)\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} \\
&\leq \frac{\left(26 + \frac{2}{\hat{\beta}} + (9\hat{\beta}+2)\Phi\right)}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})} R(N) + \frac{2\widetilde{M}^{\Phi}(\hat{\beta})}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})}\, \Lambda_{q,T}\, C_{d,q,2} \\
&\times \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}} & \text{, if } d < 4 \text{ and } q \neq 4 \\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}} & \text{, if } d = 4 \text{ and } q \neq 4 \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}} & \text{, if } d > 4. \end{cases}
\end{aligned}$$

Proof. It is immediate from (4.17) and [20, Theorem 1]. Note that in the present setting

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left\{|Y_s^i|^2\right\}\right] < \infty.$$

□

Identically, from (4.18) we also have the following.

Theorem 4.2.9. *If* $\Lambda_{q,T} < \infty$ *for some* $q > 2$ *and deterministic* T*, then exists constant* $C_{d,q,2} > 0$ *depending on* $d, q, 2$ *such that*

$$\begin{aligned}
&\|\left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right)\|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} \\
&\leq \frac{\left(26 + \frac{2}{\hat{\beta}} + (9\hat{\beta}+2)\Phi\right)(2 - 5\widetilde{M}^{\Phi}(\hat{\beta}))}{(1 - 2\widetilde{M}^{\Phi}(\hat{\beta}))(1 - 3\widetilde{M}^{\Phi}(\hat{\beta}))} R(N) \\
&+ \left(\frac{2\widetilde{M}^{\Phi}(\hat{\beta})}{1 - 2\widetilde{M}^{\Phi}(\hat{\beta})}\right)\left(\frac{1 - \widetilde{M}^{\Phi}(\hat{\beta})}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})}\right) \Lambda_{q,T}\, C_{d,q,2} \\
&\times \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}} & \text{, if } d < 4 \text{ and } q \neq 4 \\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}} & \text{, if } d = 4 \text{ and } q \neq 4 \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}} & \text{, if } d > 4. \end{cases}
\end{aligned}$$

The next result provides sufficient conditions for controlling the quantity of interest in (4.17) and (4.18), and thus to derive convrgence rates for the propagation of chaos results. These conditions are immediate to check, in contrast to the boundedness assumption of $\Lambda_{q,T}$. See also Remark 4.2.11 for further discussion in that direction.

Corollary 4.2.10. *If* $\sup_{t\in[0,T]}\left\{\mathbb{E}\left[|Y_t^1|^q\right]\right\} < \infty$ *for some* $q > 2$ *and deterministic* $T < \infty$*, then exists constant* $C_{d,q,2} > 0$ *depending on* $d, q, 2$ *such that for every* $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\left[W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t\right), \mathcal{L}(Y^1_t)\right)\right]\\
&\leq \frac{\left(26 + \frac{2}{\hat\beta} + (9\hat\beta + 2)\Phi\right)}{1 - 3\widetilde{M}^\Phi(\hat\beta)} R(N) + \left(\frac{4\widetilde{M}^\Phi(\hat\beta)}{1 - 3\widetilde{M}^\Phi(\hat\beta)}\,\Lambda_{q,T} + 2\left(\mathbb{E}\left[|Y^1_t|^q\right]\right)^{\frac{2}{q}}\right) C_{d,q,2}\\
&\quad\times \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}} & , \text{if } d < 4 \text{ and } q \neq 4\\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}} & , \text{if } d = 4 \text{ and } q \neq 4\\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}} & , \text{if } d > 4. \end{cases}
\end{aligned}$$

Proof. By definition, $\sup_{t\in[0,T]}\left\{\mathbb{E}\left[|Y_t^1|^q\right]\right\} < \infty$ and $T < \infty$ implies $\Lambda_{q,T} < \infty$.

From the triangle inequality of the Wasserstein distance and (2.25) we have

$$\begin{aligned}
W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t\right), \mathcal{L}(Y^1_t)\right) &\leq 2\, W^2_{2,|\cdot|}\left(L^N\left(\mathbf{Y}^N_t\right), L^N\left(\widetilde{\mathbf{Y}}^N_t\right)\right)\\
&\quad + 2\, W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_t\right), \mathcal{L}\left(Y^1_t\right)\right)\\
&\leq 2\,\frac{1}{N}\sum_{m=1}^N \left|Y^{m,N}_t - Y^m_t\right|^2 + 2\, W^2_{2,|\cdot|}\left(L^N\left(\widetilde{\mathbf{Y}}^N_t\right), \mathcal{L}\left(Y^1_t\right)\right).
\end{aligned}$$

Hence, from Theorem 4.2.8 and [20, Theorem 1] we are done. □

Remark 4.2.11. *For* $q > 2$ *and deterministic* $T < \infty$ *we have from Jensen's inequality that (note that from* **(H7′)** C^1 *is deterministic)*

$$|Y^1_t|^q \leq \left(4\,C^1_T + 4\right)^{\frac{q}{2}}\,\mathbb{E}\left[|\xi^1|^q + \left(\int_0^T \left|f\left(s, Y^1_s, Z^1_s c^1_s, \Gamma^{(\mathbb{F}^1,\overline{X}^1,\Theta^1)}(U^1_s)_s, \mathcal{L}(Y^1_s)\right)\right|^2 \mathrm{d}C^1_s\right)^{\frac{q}{2}} \middle| \mathcal{F}^1_t\right].$$

Hence, it seems we can satisfy the requirement $\Lambda_{q,T} < \infty$ *by an appropriate bounded condition on* f *and advanced integrability for* ξ^1*.*

Chapter 5

Stability of backward propagation of chaos

Backward propagation of chaos states that, under appropriate conditions, the solution of a mean-field system of backward stochastic differential equations (BSDEs) with N players (or particles) converges to the solutions of N independent and identically distributed McKean–Vlasov BSDEs, as N goes to infinity. Of course, every such phenomenon is associated with a set of data $\mathcal{D}$ that provides the basis of its mathematical description. In the theory of BSDEs the question of, if we assume that a sequence of data $\{\mathcal{D}^k\}_{k\in\mathbb{N}}$ converges to the data $\mathcal{D}^\infty$ then should the corresponding solutions converge, is called the stability problem for BSDEs. Naturally, the framework to tackle this kind of questions should determine a lot of technical details. For example, in what sense the data converge, how we measure the distance of the solutions and so on. Some notable works, among others, which provide such frameworks are Hu and Peng [24] in the special case of constant filtration, Briand et al. [6, 7] for Brownian drivers and more recently Papapantoleon et al. [44] where a very general framework is established.

At this chapter we will adopt the framework of [44] and enrich it as before to study the stability of the backward propagation of chaos. Let us be precise in what we mean by that. To start, set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, then assume a sequence of standard data

$$\Big\{\mathcal{D}^k := \Big(\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}, T^k, \Big\{\{\xi^{k,i,N}\}_{i\in\{1,\dots,N\}}\Big\}_{N\in\mathbb{N}}, \{\xi^{k,i}\}_{i\in\mathbb{N}}, \Theta^k, \Gamma, f^k\Big)\Big\}_{k\in\overline{\mathbb{N}}},$$

as defined in Chapter 4. For every $(k, N) \in \overline{\mathbb{N}} \times \mathbb{N}$ we have the following mean-field system of BSDEs

$$\begin{aligned} Y_t^{k,i,N} = \xi^{k,i,N} + \int_t^{T^k} f^k\Big(s, Y_s^{k,i,N}, Z_s^{k,i,N} c_s^k, \Gamma^{(\mathbb{F}^{k,(1,\dots,N)},\overline{X}^{k,i},\Theta^k)}(U^{k,i,N})_s, L^N(\mathbf{Y}_s^{k,N})\Big)\,\mathrm{d}C_s^k \\ - \int_t^{T^k} Z_s^{k,i,N}\,\mathrm{d}X_s^{k,i,\circ} - \int_t^{T^k}\int_{\mathbb{R}^n} U_s^{k,i,N}(x)\,\widetilde{\mu}^{(\mathbb{F}^{k,(1,\dots,N)},X^{k,i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} \mathrm{d}M_s^{k,i,N}, \\ i = 1, \dots, N, \end{aligned}$$

with unique solution $\mathbf{S}^{k,N} := \Big(\mathbf{Y}^{k,N}, \mathbf{Z}^{k,N}, \mathbf{U}^{k,N}, \mathbf{M}^{k,N}\Big)$, where $\mathbf{Y}^{k,N} := \Big(Y^{k,1,N}, ..., Y^{k,i,N}, ..., Y^{k,N,N}\Big)$ and so on. Furthermore, for $(k,i) \in \overline{\mathbb{N}} \times \mathbb{N}$ we also have the following McKean–Vlasov BSDE

$$Y_t^{k,i} = \xi^{k,i} + \int_t^{T^k} f^k\left(s, Y_s^{k,i}, Z_s^{k,i} c_s^k, \Gamma^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, \Theta^k)}(U^{k,i})_s, \mathcal{L}(Y_s^{k,i})\right) \mathrm{d}C_s^k \\ - \int_t^{T^k} Z_s^{k,i}\, \mathrm{d}X^{k,i,\circ} - \int_t^{T^k}\int_{\mathbb{R}^n} U_s^{k,i}\, \widetilde{\mu}^{(\mathbb{F}^{k,i}, X^{k,i,\natural})}(\mathrm{d}s, \mathrm{d}x) - \int_t^{T^k} \mathrm{d}M_s^{k,i},$$

where $\mathcal{L}(Y_s^{k,i}) := \mathrm{Law}(Y_s^{k,i})$. From Theorem 4.2.3 we know that for every $(k,i) \in \overline{\mathbb{N}} \times \mathbb{N}$ we have

$$\| \Big(Y^{k,i,N} - Y^{k,i}, Z^{k,i,N} - Z^{k,i}, U^{k,i,N} - U^{k,i}, M^{k,i,N} - M^{k,i}\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{k,(1,\dots,N)},\alpha^k, C^k, \overline{X}^{k,i}} \xrightarrow[N\to\infty]{|\cdot|} 0.$$

Our goal is to show that, for every $i \in \mathbb{N}$ we have

$$\Big(Y^{k,i,N}, Z^{k,i,N} \cdot X^{k,i,\circ} + U^{k,i,N} \star \widetilde{\mu}^{X^{k,i,\natural}}, M^{k,i,N}\Big) \\ \xrightarrow[(k,N)\to(\infty,\infty)]{} \Big(Y^{\infty,i}, Z^{\infty,i} \cdot X^{\infty,i,\circ} + U^{\infty,i} \star \widetilde{\mu}^{X^{\infty,i,\natural}}, 0\Big),$$

under some appropriate metric, which from now we will call the *stability or robustness property of backward propagation of chaos*. To the best of our knowledge this will be the first result of this kind.

In order to accommodate the reader, we provide Table 5.1 which provides a roadmap for our approach. Using this scheme we will describe the implications of the setting inherited from Chapter 4, *i.e.*, the set of Assumptions **(J1)-(J8)** presented in Section 5.1, as well as the forthcoming results we are intended to prove. The validity, for every $k \in \overline{\mathbb{N}}$, of the backward propagation of chaos is denoted by a solid, horizontal right arrow in Table 5.1. The set of Assumptions **(S1)-(S10)**, which is presented in Section 5.1 and amounts to the convergence of the sequence of standard data, is denoted by the solid, vertical down arrow in the first column of the table. Now, we claim that the framework we have already set allows for the validity, on the one hand, of the stability of McKean–Vlasov BSDEs and, on the other hand, of the uniform (over $k \in \overline{\mathbb{N}}$) backward propagation of chaos. The former is denoted by the dashed, vertical down arrow in the last column of the table. The latter was chosen not to be depicted in Table 5.1 in order to keep the scheme as simple as possible. The conjunction of the two aforementioned properties yields the convergence of the doubly indexed sequence $\{\mathcal{S}^{k,i,N}\}_{(k,N)\in\mathbb{N}\times\mathbb{N}}$ to $\mathcal{S}^{\infty,i}$, for every $i \in \mathbb{N}$; this convergence is denoted by the wiggly, diagonal arrow.

In order to prove the stability of backward propagation of chaos first we show in Section 5.3.2 that, based on results from Chapter 4, the propagation of chaos property is achieved uniformly with respect to the date $\mathcal{D}^k$. Next, by extending the stability results of [44] for the McKean–Vlasov BSDE we can immediately deduce what we want. At this point it is important to note that the stability of the mean-field systems of BSDEs was neither needed nor is implied from the stability of the propagation of chaos.

$\mathcal{D}^1$	$\mathcal{S}^{1,i,i}$	$\mathcal{S}^{1,i,i+1}$	$\mathcal{S}^{1,i,i+2}$	$\dots$	$\mathcal{S}^{1,i,N}$	$\xrightarrow{N\to\infty}$	$\mathcal{S}^{1,i}$
$\mathcal{D}^2$	$\mathcal{S}^{2,i,i}$	$\mathcal{S}^{2,i,i+1}$	$\mathcal{S}^{2,i,i+2}$	$\dots$	$\mathcal{S}^{2,i,N}$	$\xrightarrow{N\to\infty}$	$\mathcal{S}^{2,i}$
$\mathcal{D}^3$	$\mathcal{S}^{3,i,i}$	$\mathcal{S}^{3,i,i+1}$	$\mathcal{S}^{3,i,i+2}$	$\dots$	$\mathcal{S}^{3,i,N}$	$\xrightarrow{N\to\infty}$	$\mathcal{S}^{3,i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathcal{D}^k$	$\mathcal{S}^{k,i,i}$	$\mathcal{S}^{k,i,i+1}$	$\mathcal{S}^{k,i,i+2}$	$\dots$	$\mathcal{S}^{k,i,N}$	$\xrightarrow{N\to\infty}$	$\mathcal{S}^{k,i}$
$\downarrow$						$\searrow$	$\downarrow$
$\mathcal{D}^\infty$	$\mathcal{S}^{\infty,i,i}$	$\mathcal{S}^{\infty,i,i+1}$	$\mathcal{S}^{\infty,i,i+2}$	$\dots$	$\mathcal{S}^{\infty,i,N}$	$\xrightarrow{N\to\infty}$	$\mathcal{S}^{\infty,i}$

Table 5.1 The doubly-indexed scheme for (backward) propagation of chaos.

5.1 Setting

Let us fix for the remainder of the article a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a sequence of standard data

$$\mathcal{D}^k := \left(\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}, T^k, \left\{ \{\xi^{k,i,N}\}_{i\in\{1,\dots,N\}} \right\}_{N\in\mathbb{N}}, \{\xi^{k,i}\}_{i\in\mathbb{N}}, \Theta^k, \Gamma, f^k \right), \tag{5.1}$$

for all $k \in \overline{\mathbb{N}}$, under a universal $\hat{\beta} > 0$. We remind them for the convenience of the reader. So, for every $k \in \overline{\mathbb{N}}$:

(J1) The sequence $\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}$ are independent and identically distributed processes such that, for every $i \in \mathbb{N}$, $\overline{X}^{k,i} = (X^{k,i,\circ}, X^{k,i,\natural}) \in \mathcal{H}^2(\mathbb{F}^{k,i}; \mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^{k,i}; \mathbb{R}^n)$ with $M_{\mu^{X^{k,i,\natural}}}[\Delta X^{k,i,\circ} | \widetilde{\mathcal{P}}^{\mathbb{F}^{k,i}}] = 0$, where $\mathbb{F}^{k,i} := (\mathcal{F}^{k,i}_t)_{t\geq 0}$ is the usual augmentation of the natural filtration of $\overline{X}^{k,i}$ and $\mu^{X^{k,i,\natural}}$ is the random measure generated by the jumps of $X^{k,i,\natural}$.[1]

(J2) A deterministic time T^k and a sequence of identically distributed terminal conditions $\{\xi^{k,i}\}_{i\in\mathbb{N}}$ and a sequence of sets of terminal conditions $\left\{ \{\xi^{k,i,N}\}_{i\in\{1,\dots,N\}} \right\}_{N\in\mathbb{N}}$ such that under $\hat{\beta}$ it holds $\xi^{k,i}, \xi^{k,i,N} \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{k,i}_{T^k}, A^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, f^k)}; \mathbb{R}^d)$, $\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{k,(1,\dots,N)}_{T^k}, A^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, f^k)}; \mathbb{R}^d)$[2] respectively, for every $i \in \mathbb{N}$, where $\{A^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, f^k)}\}_{i\in\mathbb{N}}$ the ones defined in **(J5)**. Also, we have

$$\|\xi^{k,i,N} - \xi^{k,i}\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{k,(1,\dots,N)}_{T^k}, A^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, f^k)}; \mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0$$

for every $i \in \mathbb{N}$, and

$$\frac{1}{N} \sum_{i=1}^{N} \|\xi^{k,i,N} - \xi^{k,i}\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{k,(1,\dots,N)}_{T^k}, A^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, f^k)}; \mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0.$$

(J3) Functions Θ^k, $\Gamma^{k,i} \triangleq \Gamma^{(\mathbb{F}^{k,i}, \overline{X}^{k,i}, \Theta^k)}$ for every $i \in \mathbb{N}$ as in Definition 2.5.1, with Θ^k deterministic.

[1] Since for every $i \in \mathbb{N}$ the filtration $\mathbb{F}^{k,i}$ is associated to $\overline{X}^{k,i}$, we will make use of $C^{(\mathbb{F}^{k,i}, \overline{X}^{k,i})}$, resp. $c^{(\mathbb{F}^{k,i}, \overline{X}^{k,i})}$, as defined in (2.9), resp. (2.11). Moreover, we will use the kernels $K^{(\mathbb{F}^{k,i}, \overline{X}^{k,i})}$ as determined by (2.10).

[2] see Remark 4.1.1 *(i)*

(J4) A generator $f^k : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ such that for any $(y,z,u,\mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the map

$$t \longmapsto f^k(t,y,z,u,\mu) \text{ is } \mathcal{B}(\mathbb{R}_+) - \text{measurable}$$

and satisfies the following Lipschitz condition

$$\begin{aligned}&|f^k(t,y,z,u,\mu) - f^k(t,y',z',u',\mu')|^2\\ &\qquad\le r^k(t)\,|y-y'|^2 + \vartheta^{k,\circ}(t)\,|z-z'|^2 + \vartheta^{k,\natural}(t)\,|u-u'|^2 + \vartheta^{k,*}(t)\,W^2_{2,|\cdot|}(\mu,\mu'),\end{aligned}$$

where $(r^k, \vartheta^{k,\circ}, \vartheta^{k,\natural}, \vartheta^{k,*}) : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(J5) Define $(\alpha^k)^2 := \max\{\sqrt{r^k}, \vartheta^{k,\circ}, \vartheta^{k,\natural}, \sqrt{\vartheta^{k,*}}\}$. For the $\mathbb{F}^{k,i}$-predictable and càdlàg processes

$$A_t^{(\mathbb{F}^{k,i},\overline{X}^{k,i},f^k)} := \int_0^t (\alpha^k)^2_s \mathrm{d}C_s^{(\mathbb{F}^{k,i},\overline{X}^{k,i})} \tag{5.2}$$

there exists $\Phi^k > 0$ such that $\Delta A_t^{(\mathbb{F}^{k,i},\overline{X}^{k,i},f^k)}(\omega) \le \Phi^k$, $\mathbb{P}\otimes C^{(\mathbb{F}^{k,i},\overline{X}^{k,i})}$ – a.e, for every $i \in \mathbb{N}$.

(J6) We have

$$\mathbb{E}\left[\int_0^{T^k} \mathcal{E}\left(\hat{\beta} A^{(\mathbb{F}^{k,i},\overline{X}^{k,i},f^k)}\right)_{s-} \frac{|f^k(s,0,0,0,\delta_0)|^2}{(\alpha^k)^2_s} \mathrm{d}C_s^{(\mathbb{F}^{k,i},\overline{X}^{k,i})}\right] < \infty, \text{ for every } i \in \mathbb{N}, \tag{5.3}$$

where δ_0 is the Dirac measure on the domain of the last argument concentrated at 0, the neutral element of the addition.

(J7) The martingale $\overline{X}^{k,1}$ has independent increments.

(J8) We have $3\,\widetilde{M}^{\Phi^k}(\hat{\beta}) < 1$.

Remark 5.1.1. *(i) Above, we denoted with $\mathbb{F}^{k,(1,\dots,N)}$ the filtration $\mathbb{F}^{k,(1,\dots,N)} := \bigvee_{i=1}^N \mathbb{F}^{k,i}$. From [52, Theorem 1] we have that $\mathbb{F}^{k,(1,..,N)}$ satisfies the usual conditions. For $i \in \{1,\dots,N\}$ and $N \in \mathbb{N}$, a direct consequence of the independence of filtrations is that every $\mathbb{F}^{k,i}-$martingale, remains martingale under $\mathbb{F}^{k,(1,\dots,N)}$,* i.e., *the filtration $\mathbb{F}^{k,i}$ is immersed in the filtration $\mathbb{F}^{k,(1,\dots,N)}$. In particular, $X^{k,i,\natural} \in \mathcal{H}^{2,d}(\mathbb{F}^{k,(1,\dots,N)};\mathbb{R}^n)$; see Corollary A.2.7. Additionally, from the assumption $M_{\mu^{X^{k,i,\natural}}}[\Delta X^{k,i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^{k,i}}] = 0$ one can deduce that it is also true $M_{\mu^{X^{k,i,\natural}}}[\Delta X^{k,i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^{k,(1,\dots,N)}}] = 0$; one can follow the exact same arguments as in Lemma A.2.9.*

(ii) The independence of the increments of the martingale $\overline{X}^{k,i}$ is equivalent to its associated triplet being deterministic, see [40, Corollary 7.87] or [29, Theorem II.4.15]. As a result, recalling the notational simplification for which we argued in Remark 4.1.1.(ii) and which we will use hereinafter, the process $A^{k,i}$ is deterministic. Indeed, this is immediate by the way we have constructed $C^{k,i}$; see 2.9. In view of **(J1)**, *in particular the fact that we assumed the sequence $\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}$ to be identically distributed, we have that $A^{k,1} = A^{k,i}$ for every $i \ge 2$, see Remark 3.2.5. So, moving forward we drop the dependence from i in the notation of $C^{k,i}, A^{k,i}$, i.e. we will instead use C^k, A^k.*

(iii) *We clarify that the stochastic processes associated to the data $\mathcal{D}^k$ will be assumed that are stopped at time T^k.*

We proceed with a few remarks about the notation involving filtrations, which will help in further simplifying the notation. In the following we assume that a set of standard data is fixed with associated index k, for $k \in \overline{\mathbb{N}}$. Let us start with a subtle point, namely the distinction between the use of the filtration associated to the $i-$th player and of that associated to the set of the first N players; recall **(J1)** and Remark 5.1.1 for the respective notation. Given the set of standard data and because of the immersion of the aforementioned smaller filtration into the larger one, see Remark 5.1.1.(i), there is no harm in placing ourselves under the framework of any of the two when there are no measurability conflicts, *e.g.*, see Remark 2.2.1 and Section A.2.2. If, however, there are sound reasons for a clarification, then this will be promptly provided. Therefore, we will omit the symbol of a filtration, *e.g.*, the predictable quadratic covariation will be simply denoted by $\langle\cdot\rangle$, since the respective measurability will be easily concluded by the argument process. Having this simplification in mind, we will denote the predictable $\sigma-$algebra associated to the filtration $\mathbb{F}^{k,i}$ simply by $\mathcal{P}^{k,i}$, instead of $\mathcal{P}^{\mathbb{F}^{k,i}}$, and analogous simplification will be used for $\widetilde{\mathcal{P}}^{k,i}$. Moreover, we simplify the notation associated to (integer-valued, random) measures. The integer-valued measure $\mu^{X^{k,\natural}}$ will be denoted by $\mu^{k,\natural}$, its compensator will be denoted by $\nu^{k,\natural}$ and the compensated integer-valued measure will be denoted by $\widetilde{\mu}^{k,\natural}$.

Finally, we conclude the discussion about the notational simplifications with the comments related to the spaces introduced in Section 2.4. Given the set of standard data $\mathcal{D}^k$, for $k \in \overline{\mathbb{N}}$, a label i of a player and the total number of players N, the main symbol of a space will be kept and the superscripts will be modified as follows: the indices k, i, N (if all necessary) will be affixed, succeeding the number 2 and preceding any of the symbols $\circ, \natural, \perp$, if they are required; the dimensions of the state spaces will be omitted only when no confusion arises. For example, $\mathcal{H}^2_\beta(\mathbb{F}^{k,i}, A^k, \overline{X}^{\perp_{\mathbb{F}^{k,i}}}; \mathbb{R}^d)$ will be simply denoted by $\mathcal{H}^{2,k,i,\perp}_\beta(\mathbb{R}^d)$. The rule for the norms makes the indices k, i, N perform as subscripts which succeed the symbols $\star$ (if any) and β and precede the value t, *e.g.*, $\|\cdot\|_{\star,\beta,k,i}$, $|||\cdot|||_{k,i,t}$. In the case that the considered filtration is $\mathbb{F}^{k,(1,\dots,N)}$, instead of the index which denotes the label of the player it will be used $(1,\dots,N)$, *e.g.*, $\mathcal{H}^2_\beta(\mathbb{F}^{k,(1,\dots,N)}, A^k, \overline{X}^{\perp_{\mathbb{F}^{k,i}}}; \mathbb{R}^d)$ will be simply denoted by $\mathcal{H}^{2,k,(1,\dots,N),\perp}_\beta(\mathbb{R}^d)$.

Now, following [44], we are going to complement the above assumptions with the ones needed for the convergence of the data $\{\mathcal{D}^k\}_{k\in\mathbb{N}}$ and the convergence of the Lebesgue-Stieltjes integrals associated to the generators of the BSDEs.

(S1) For every $i \in \mathbb{N}$, the martingale $X^{\infty,i,\circ}$ is continuous and the martingale $X^{\infty,i,\natural}$ is quasi-left-continuous.

(S2) For every $i \in \mathbb{N}$, we have

$$\overline{X}^{k,i} \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{p+n}),\mathbb{P})} \overline{X}^{\infty,i} \text{ and } \overline{X}^{k,i}_\infty \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^{p+n})} \overline{X}^{\infty,i}_\infty.$$

(S3) For every $i \in \mathbb{N}$, the pair $\overline{X}^{\infty,i}$ possesses the $\mathbb{F}^{\infty,i}-$predictable representation property.

(S4) For every $i \in \mathbb{N}$,

$$\left\|\xi^{k,i,N} - \xi^{k,i}\right\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)} \xrightarrow[(k,N)\to(\infty,\infty)]{|\cdot|} 0.$$

Moreover,

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\xi^{k,i,N}-\xi^{k,i}\right\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)} \xrightarrow[(k,N)\to(\infty,\infty)]{|\cdot|} 0.$$

(S5) For every $i\in\mathbb{N}$, we have

$$\xi^{k,i} \xrightarrow[k\to\infty]{\mathbb{L}_2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} \xi^{\infty,i}.$$

(S6) For every $i\in\mathbb{N}$, the sequence of random variables

$$\left\{\int_0^\infty \frac{|f^k(s,0,0,0,\delta_0)|^2}{(\alpha^k)^2_s}\mathrm{d}C^k_s\right\}_{k\in\overline{\mathbb{N}}},$$

is uniformly integrable, where δ_0 is the Dirac measure on the domain of the last argument concentrated at 0, the neutral element of the addition.

(S7) There exists $\overline{A}\in\mathbb{R}_+$ such that, the sequence of random variables $\{A^k_\infty\}_{k\in\overline{\mathbb{N}}}$ is bounded from $\overline{A}$ (see Remark 5.1.1 (i)).

(S8) The generators $\{f^k\}_{k\in\overline{\mathbb{N}}}$ posses additionally the following properties:

(i) For every $k\in\overline{\mathbb{N}}, i\in\mathbb{N}, a\in\mathbb{D}(\mathbb{R}^d,|\cdot|), Z\in D^{\circ,d\times p}, U\in D^{\natural}, \mu\in\mathbb{D}(\mathcal{P}_2(\mathbb{R}^d),W_{2,|\cdot|})^3$, it holds that

$$\left(f^k(t,a_t,Z_t,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t,\mu_t)\right)_{t\in\mathbb{R}_+}\in\mathbb{D}(\mathbb{R}^d,|\cdot|).$$

(ii) For every $i\in\mathbb{N}, Z\in D^{\circ,d\times p}, U\in D^{\natural}$ and a sequence of $\mathbb{R}^d-$valued stochastic processes $\{a^k\}_{k\in\overline{\mathbb{N}}}$ such that $\mathbb{E}\left[\sup_{t\in\mathbb{R}_+}\left\{|a^k_t|^2\right\}\right]<\infty$ for every $k\in\overline{\mathbb{N}}$, if $a^k\xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^d,|\cdot|)}a^\infty$, $\mathbb{P}-a.e.$, then

$$\left(f^k(t,a^k_t,Z_t,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t,\mathcal{L}(a^k_t))\right)_{t\in\mathbb{R}_+} \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^d,|\cdot|)}\left(f^\infty(t,a^\infty_t,Z_t,\Gamma^{(\mathbb{F}^{\infty,i},\overline{X}^{\infty,i},\Theta^\infty)}(U)_t,\mathcal{L}(a^\infty_t))\right)_{t\in\mathbb{R}_+}\ \mathbb{P}-a.e.$$

Furthermore, if $\sup_{k\in\overline{\mathbb{N}}}\left\{\|a^k(\omega)\|_\infty\right\}_{k\in\overline{\mathbb{N}}}<\infty$, $\mathbb{P}-a.e.$, then

$$\sup_{k\in\overline{\mathbb{N}}}\left\{\left\|\left(f^k(t,a^k_t,Z_t,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t,\mathcal{L}(a^k_t))\right)_{t\in\mathbb{R}_+}\right\|_\infty\right\}_{k\in\overline{\mathbb{N}}}<\infty,\ \mathbb{P}-a.e.$$

(S9) (i) The sequence $\{\Phi^k\}_{k\in\overline{\mathbb{N}}}$ satisfies $\Phi^k\xrightarrow[k\to\infty]{|\cdot|}\Phi^\infty:=0$.

(ii) $\widetilde{M}^0(\hat{\beta})\triangleq\frac{2\sqrt{\frac{2}{\hat{\beta}}+9}\sqrt{\frac{2}{\hat{\beta}}+17}+\frac{4}{\hat{\beta}}+35}{\hat{\beta}}<\frac{1}{4}$.

[3]We denote with $\mathbb{D}(\mathcal{P}_2(\mathbb{R}^d),W_{2,|\cdot|})$ the Skorokhod space of cadlag functions with values in $(\mathcal{P}_2(\mathbb{R}^d),W_{2,|\cdot|})$.

(S10) The stopping time T^∞ is finite and $T^k \xrightarrow[k\to\infty]{|\cdot|} T^\infty$.

A couple of remarks regarding the set of Assumptions **(S1)**-**(S10)** are in order. In view of **(J1)**, the following are true for every $i\in\mathbb{N}$. In **(S1)** we have assumed that the process $\overline{X}^{\infty,i}$ is quasi-left-continuous. Hence, the filtration $\mathbb{F}^{\infty,i}$ is also quasi-left-continuous, as the one generated by $\overline{X}^{\infty,i}$. Actually, $\overline{X}^{\infty,i}$ having independent increments renders equivalent the property of being quasi-left-continuous and the property of having no fixed time discontinuity, see Wang [50, Section 1]. Next, the alerted reader may observe that we do not (explicitly) mention any assumptions for the weak convergence of the filtrations $\{\mathbb{F}^{k,i}\}_{k\in\overline{\mathbb{N}}}$, *i.e.*, we do not mention the analogous to [44, Assumption (S4)]. This apparent omission can be justified because combining **(J1)** and **(S2)** yields that

$$\mathbb{F}^{k,i} \xrightarrow[k\to\infty]{\mathrm{w}} \mathbb{F}^{\infty,i}; \tag{5.4}$$

see [14, Proposition 2].

Having justified the validity of (5.4) for every $i\in\mathbb{N}$, we may proceed to another apparent omission, namely the lack of assumptions on the convergence of the sequences $\{C^k_{\cdot}\}_{k\in\overline{\mathbb{N}}}$ and $\{C^k_\infty\}_{k\in\overline{\mathbb{N}}}$, *i.e.*, we do not mention the analogous to [44, Assumption (S9)]. Recalling the note before Remark 2.2.1, the function C^k is defined as the trace of $\langle\overline{X}^k\rangle$, for every $k\in\overline{\mathbb{N}}$. The combination of **(S2)** and (5.4) yields, via [41, Corollary 12], the convergence

$$\langle\overline{X}^{k,i}\rangle \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^{p+n})} \langle\overline{X}^{\infty,i}\rangle,$$

which in particular implies,

$$C^k = \mathrm{Tr}\Big[\langle\overline{X}^{k,i}\rangle\Big] \xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^{p})} \mathrm{Tr}\Big[\langle\overline{X}^{\infty,i}\rangle\Big] = C^\infty.$$

The continuity of C^∞, because of the quasi-left-continuity of $\overline{X}^{\infty,i}$ for every $i\in\mathbb{N}$, allows the above convergence to hold under the locally uniform convergence. In view of **(S10)**, we have

$$C^k_\infty = C^k_{T^k} \xrightarrow[k\to\infty]{|\cdot|} C^\infty_{T^\infty} = C^\infty_\infty,$$

because the limit time horizon T^∞ is finite.

At this point we present an equivalent form of Assumption **(S4)**. More precisely, in view of **(J2)**, **(S4)** is equivalent with

$$\sup_{k\in\mathbb{N}}\Big\{\|\xi^{k,i,N}-\xi^{k,i}\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)}\Big\} \xrightarrow[N\to\infty]{|\cdot|} 0$$

and

$$\sup_{k\in\mathbb{N}}\left\{\frac{1}{N}\sum_{i=1}^N\|\xi^{k,i,N}-\xi^{k,i}\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)}\right\} \xrightarrow[N\to\infty]{|\cdot|} 0.$$

Finally, we turn our attention to Assumption **(S9)**.(ii); we remark that the left-hand side of the inequality corresponds to the summand of $\widetilde{M}^\Phi(\hat{\beta})$ which is independent of Φ, *i.e.*, it coincides with $\widetilde{M}^0(\hat{\beta})$.

Combining the two parts of Assumption **(S9)**, we get that finally for all $k \in \mathbb{N}$ it holds

$$\widetilde{M}^{\Phi^k}(\hat{\beta}) < \frac{1}{4}. \tag{5.5}$$

In other words, **(S9)** strengthens **(J8)**.

5.2 Stability properties

In this section we will present the main result of the current work, namely the stability of backward propagation of chaos. We have already described in the introduction the approach we intend to follow, which seemingly passes through the stability of McKean–Vlasov BSDEs. At this point, it is noteworthy to underline that under the specific framework the stability of mean-field BSDEs cannot be derived for any $N \in \mathbb{N}$ and for this reason there is no down arrow appearing between $\mathcal{S}^{k,i,N}$ and $\mathcal{S}^{\infty,i,N}$ in the second but last row of the Table 5.1. The lack of the stability property in the aforementioned case is mainly because we have not imposed any condition on the convergence of the respective terminal conditions. Once such a condition is imposed, *i.e.*, for $N \in \mathbb{N}$

$$\xi^{k,i,N} \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} \xi^{\infty,i,N},$$

accompanied with the suitable extension of the continuity property of the generators, then the stability of mean-field BSDEs for N players can be obtained. These conditions appear later as **(S5′)** and **(S8′)**.

The complementation of **(J1)**-**(J8)** with the set of Assumptions **(S1)**-**(S10)** leads, in particular, to the desired stability property of McKean–Vlasov BSDEs associated to the $i-$player. The next theorem serves as the precise statement of the above.

Theorem 5.2.1 (**Stability of McKean–Vlasov BSDEs**). *Given the sequence of data $\{\mathcal{D}^k\}_{k\in\overline{\mathbb{N}}}$ as described in* (5.1)*, which satisfy* **(J1)**-**(J8)** *for every $k \in \overline{\mathbb{N}}$, and assuming that* ***(S1)**–**(S10)*** *are in force, then the following are true for every $i \in \mathbb{N}$:*

$$\Big(Y^{k,i}, Z^{k,i}\cdot X^{k,i,\circ} + U^{k,i}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i}\Big) \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^2)} \Big(Y^{\infty,i}, Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}, 0\Big), \tag{5.6}$$

$$\begin{aligned}&\Big([Y^{k,i}], [Z^{k,i}\cdot X^{k,i,\circ} + U^{k,i}\star\widetilde{\mu}^{k,i,\natural}], [M^{k,i}], [Y^{k,i}, X^{k,i,\circ}], [Y^{k,i}, X^{k,i,\natural}], [Y^{k,i}, M^{k,i}]\Big)\\ &\qquad \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^1)} \Big([Y^{\infty,i}], [Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}], 0, [Y^{\infty,i}, X^{\infty,i,\circ}], [Y^{\infty,i}, X^{\infty,1,\natural}], 0\Big)\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}&\Big(\langle Y^{k,i}\rangle, \langle Z^{k,i}\cdot X^{k,i,\circ} + U^{k,i}\star\widetilde{\mu}^{k,i,\natural}\rangle, \langle M^{k,i}\rangle, \langle Y^{k,i}, X^{k,i,\circ}\rangle, \langle Y^{k,i}, X^{k,i,\natural}\rangle, \langle Y^{k,i}, M^{k,i}\rangle\Big)\\ &\qquad \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^1)} \Big(\langle Y^{\infty,i}\rangle, \langle Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}\rangle, 0, \langle Y^{\infty,i}, X^{\infty,i,\circ}\rangle, \langle Y^{\infty,i}, X^{\infty,i,\natural}\rangle, 0\Big),\end{aligned} \tag{5.8}$$

where in (5.6) *the state space is $\mathbb{R}^{d\times 3}$, while in the other two the state space is $\mathbb{R}^{d\times(4d+p+n)}$.*

Proof. For notational simplicity we will treat the case $i = 1$. We will follow the same strategy as in [44], which is described schematically in Table 5.2. In other words, we will apply the Moore-Osgood theorem on the doubly-indexed sequence $\left\{\mathcal{S}^{k,1,(q)}\right\}_{(k,q)\in\mathbb{N}\times\mathbb{N}}$ of the Picard approximation scheme, where $\mathcal{S}^{k,1,(q)}$ denotes the representation obtained at the $q-$th step of the Picard iteration under the data $\mathcal{D}^k$.

$\mathcal{D}^1$	$\mathcal{S}^{1,1,(0)}$	$\mathcal{S}^{1,1,(1)}$	$\mathcal{S}^{1,1,(2)}$	$\dots$	$\mathcal{S}^{1,1,(q)}$	$\xrightarrow{q\to\infty}$	$\mathcal{S}^{1,1}$
$\mathcal{D}^2$	$\mathcal{S}^{2,1,(0)}$	$\mathcal{S}^{2,1,(1)}$	$\mathcal{S}^{2,1,(2)}$	$\dots$	$\mathcal{S}^{2,1,(q)}$	$\xrightarrow{q\to\infty}$	$\mathcal{S}^{2,1}$
$\mathcal{D}^3$	$\mathcal{S}^{3,1,(0)}$	$\mathcal{S}^{3,1,(1)}$	$\mathcal{S}^{3,1,(2)}$	$\dots$	$\mathcal{S}^{3,1,(q)}$	$\xrightarrow{q\to\infty}$	$\mathcal{S}^{3,1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathcal{D}^k$	$\mathcal{S}^{k,1,(0)}$	$\mathcal{S}^{k,1,(1)}$	$\mathcal{S}^{k,1,(2)}$	$\dots$	$\mathcal{S}^{k,1,(q)}$	$\xrightarrow{q\to\infty}$	$\mathcal{S}^{k,1}$
$\downarrow$	⇣	⇣	⇣		⇣		⇣
$\mathcal{D}^\infty$	$\mathcal{S}^{\infty,1,(0)}$	$\mathcal{S}^{\infty,(1)}$	$\mathcal{S}^{\infty,1,(2)}$	$\dots$	$\mathcal{S}^{\infty,1,(q)}$	$\xrightarrow{q\to\infty}$	$\mathcal{S}^{\infty,1}$

Table 5.2 The doubly-indexed Picard scheme for McKean–Vlasov BSDEs.

From Proposition 5.3.1 we get the uniform (over $k \in \overline{\mathbb{N}}$) horizontal convergence; in Table 5.2 this corresponds to the horizontal, right arrows denoting the convergence of the Picard schemes. In order for the Moore–Osgood theorem to be applied, it is sufficient to prove that the convergence indicated by the dashed, vertical down arrows are indeed true for every $q \in \mathbb{N}$. To avoid the redundant repeating of arguments and results presented in [44] we will give the modifications only at the needed points, which are presented in Lemma 5.3.2. $\square$

It follows the main theorem of this chapter.

Theorem 5.2.2 (Stability for backward propagation of chaos). *Given the sequence of data* $\{\mathcal{D}^k\}_{k\in\overline{\mathbb{N}}}$ *as described in* (5.1), *which satisfy* **(J1)**-**(J8)** *for every* $k \in \overline{\mathbb{N}}$, *and assuming that* ***(S1)**–**(S10)*** *are in force, then the following are true for every* $i \in \mathbb{N}$*:*

$$\Big(Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i,N}\Big) \xrightarrow[(k,N)\to(\infty,\infty)]{(\mathrm{J}_1,\mathbb{L}^2)} \Big(Y^{\infty,i}, Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}, 0\Big), \tag{5.9}$$

$$\begin{aligned}\Big([Y^{k,i,N}], [Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}], [M^{k,i,N}],\\ [Y^{k,i,N}, X^{k,i,\circ}], [Y^{k,i,N}, X^{k,i,\natural}], [Y^{k,i,N}, M^{k,i,N}]\Big)\\ \xrightarrow[(k,N)\to(\infty,\infty)]{(\mathrm{J}_1,\mathbb{L}^1)} \Big([Y^{\infty,i}], [Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}], 0, [Y^{\infty,i}, X^{\infty,i,\circ}], [Y^{\infty,i}, X^{\infty,1,\natural}], 0\Big)\end{aligned} \tag{5.10}$$

and

$$\Big(\langle Y^{k,i,N}\rangle, \langle Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}\rangle, \langle M^{k,i,N}\rangle,$$
$$\langle Y^{k,i,N}, X^{k,i,\circ}\rangle, \langle Y^{k,i,N}, X^{k,i,\natural}\rangle, \langle Y^{k,i,N}, M^{k,i,N}\rangle\Big) \tag{5.11}$$
$$\xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^1)} \Big(\langle Y^{\infty,i}\rangle, \langle Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}\rangle, 0, \langle Y^{\infty,i}, X^{\infty,i,\circ}\rangle, \langle Y^{\infty,i}, X^{\infty,i,\natural}\rangle, 0\Big),$$

where in (5.9) *the state space is* $\mathbb{R}^{d\times 3}$*, while in the other two the state space is* $\mathbb{R}^{d\times(4d+p+n)}$.

Proof. For notational simplicity let us consider the case $i=1$. The main argument will be (again) an application of the Moore–Osgood theorem for a doubly-indexed sequence, namely $\{\mathcal{S}^{k,1,N}\}_{(k,N)\in\mathbb{N}\times\mathbb{N}}$; see [22, Chapter VI, Sections 336–338]. As it was described at an earlier point, we first need to prove the uniform (over k) backward propagation of chaos, which is provided by Theorem 5.3.7. The existence of the iterated limit

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathcal{S}^{k,1,N} = \mathcal{S}^{\infty,1}$$

is guaranteed by Theorem 4.2.3 and Theorem 5.2.1. Then with the help of Corollary 5.3.8 we can conclude with the same arguments as the ones in [44, Sections 3.3.2, 3.3.3]. □

Corollary 5.2.3. *For every* $(k,i,N)\in\overline{\mathbb{N}}\times\mathbb{N}\times\mathbb{N}$ *let us define*

$$\theta^{k,i,N} := \mathrm{Law}\Big(\big(Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i,N}\big)\Big)$$

and

$$\theta^{\infty,i} := \mathrm{Law}\Big(\big(Y^{\infty,i}, Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}, 0\big)\Big)$$

Then, for every $i\in\mathbb{N}$ *we have*

$$W^2_{2,\rho_{\mathrm{J}_1^{d\times 3}}}\big(\theta^{k,i,N},\theta^{\infty,i}\big) \xrightarrow[(k,N)\to(\infty,\infty)]{|\cdot|} 0. \tag{5.12}$$

Proof. For notational simplicity, we define for $(k,i,N)\in\overline{\mathbb{N}}\times\mathbb{N}\times\mathbb{N}$

$$Q^{k,i,N} := (Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i,N})$$

and

$$Q^{k,i} := (Y^{k,i}, Z^{k,i}\cdot X^{k,i,\circ} + U^{k,i}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i}).$$

From the definition of the Wasserstein distance we have

$$W^2_{2,\rho_{\mathrm{J}_1^{d\times 3}}}\big(\theta^{k,i,N},\theta^{\infty,i}\big) \le \int_{\mathbb{D}^{d\times 3}\times\mathbb{D}^{d\times 3}} \rho^2_{\mathrm{J}_1^{d\times 3}}(x,z)\pi(\mathrm{d}x,\mathrm{d}z) \le \mathbb{E}\Big[\rho^2_{\mathrm{J}_1^{d\times 3}}\big(Q^{k,i,N}, Q^{\infty,i}\big)\Big],$$

where we chose π to be the image measure on $\mathbb{D}^{d\times 3}\times\mathbb{D}^{d\times 3}$ produced by the measurable function

$$(Q^{k,i,N},Q^{\infty,i}):\Omega\longrightarrow\mathbb{D}^{d\times 3}\times\mathbb{D}^{d\times 3}.$$

Hence the result follows from Theorem 5.2.2. □

As it was pointed out earlier, from the previous theorem one cannot deduce the stability of mean-field BSDE systems under the framework **(S1)**-**(S10)**, for any $N\in\mathbb{N}$, but only the following

$$\lim_{N\to\infty}\limsup_{k\to\infty}\|\mathcal{S}^{k,1,N}-\mathcal{S}^{\infty,i,N}\|_{\mathcal{S}^2}=0,$$

which can be interpreted as *asymptotically* obtaining the stability of mean-field BSDE systems. We conclude this section with a theorem which deals with the stability of the mean-field BSDE systems. To this end, let us fix the number of players $N\in\mathbb{N}$ until the end of this section. We need to modify **(S5)** and **(S8)** as follows:

(S5′) For every $i\in\{1,...,N\}$, we have

$$\xi^{k,i,N}\xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)}\xi^{\infty,i,N}.$$

(S8′) The generators $\{f^k\}_{k\in\overline{\mathbb{N}}}$ posses the following properties:

(i) For every $k\in\overline{\mathbb{N}}$, $i\in\{1,...,N\}, a\in\mathbb{D}(\mathbb{R}^d), Z\in D^{\circ,d\times p}, U\in D^{\natural}, \mu\in\mathbb{D}(\mathcal{P}_2(\mathbb{R}^d),W_{2,|\cdot|})$, it holds that

$$f^k(\cdot,a_\cdot,Z_\cdot,\Gamma^{k,(1,\dots,N)}(U)_\cdot,\mu_\cdot)\in\mathbb{D}(\mathbb{R}^d).$$

(ii) For every $Z\in D^{\circ,d\times p}$, $U\in D^{\natural}$ and a sequence of $\mathbb{R}^{d\times N}-$valued càdlàg stochastic processes $\{\mathbf{a}^k\}_{k\in\overline{\mathbb{N}}}$, where $\mathbf{a}^k:=(a^{k,1},...,a^{k,N})$ such that $\mathbb{E}\Big[\sup_{t\in\mathbb{R}_+}|\mathbf{a}^k_t|^2\Big]<\infty$ for every $k\in\overline{\mathbb{N}}$, if $\mathbf{a}^k\xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^{d\times N})}\mathbf{a}^\infty$, $\mathbb{P}-a.e.$, then, for every $i\in\{1,\dots,N\}$, it holds $\mathbb{P}-a.e.$

$$f^k\Big(\cdot,a^{k,i}_\cdot,Z_\cdot,\Gamma^{k,(1,\dots,N)}(U)_\cdot,L^N(\mathbf{a}^k_\cdot)\Big)\xrightarrow[k\to\infty]{\mathrm{J}_1(\mathbb{R}^d)}f^\infty\Big(\cdot,a^{\infty,i}_\cdot,Z_\cdot,\Gamma^{(\mathbb{F}^{\infty,(1,\dots,N)},\overline{X}^{\infty,i},\Theta^\infty)}(U)_\cdot,L^N(\mathbf{a}^\infty_\cdot)\Big).$$

Furthermore, if $\sup_{k\in\overline{\mathbb{N}}}\Big\{\|a^{k,i}(\omega)\|_\infty\Big\}_{k\in\overline{\mathbb{N}}}<\infty$, $\mathbb{P}-a.e.$, then

$$\sup_{k\in\overline{\mathbb{N}}}\left\{\left\|\Big(f^k(t,a^{k,i}_t,Z_t,\Gamma^{k,(1,\dots,N)}(U)_t,L^N(\mathbf{a}^k_t))\Big)_{t\in\mathbb{R}_+}\right\|_\infty\right\}_{k\in\overline{\mathbb{N}}}<\infty,\ \mathbb{P}-a.e.$$

Remark 5.2.4. *(i) The filtration $\mathbb{F}^{k,(1,\dots,N)}$ can be seen as the usual augmentation of the natural filtration of the square integrable martingale $\overline{X}^{k,(1,\dots,N)}:=(X^{k,1,\circ},X^{k,1,\natural},...,X^{k,N,\circ},X^{k,N,\natural})$. Then, $\overline{X}^{k,(1,\dots,N)}$*

has independent increments with respect to $\mathbb{F}^{k,(1,\dots,N)}$ *and so from [14, Proposition 2] we get as before that*

$$\mathbb{F}^{k,(1,\dots,N)} \xrightarrow[k\to\infty]{\mathrm{w}} \mathbb{F}^{\infty,(1,\dots,N)},$$

i.e. the filtrations converge weakly.

(ii) *From (i) above,* ***(S1)*** *and [50, Section 1] we have that for every* $N \in \mathbb{N}$ *the filtrations* $\mathbb{F}^{\infty,(1,\dots,N)}$ *are quasi-left-continuous.*

The next theorem provides the stability of mean-field BSDEs for N players.

Theorem 5.2.5 (**Stability of mean-field systems of BSDEs**). *Assume* ***(S1)**–**(S4)***,(S5′),***(S6)***, ***(S7)***, (S8′), ***(S9)***,***(S10)*** *and fix* $N \in \mathbb{N}$. *Then, we have for every* $i \in \{1,\dots,N\}$

$$\Big(Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}, M^{k,i,N}\Big) \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^2)} \Big(Y^{\infty,i,N}, Z^{\infty,i,N}\cdot X^{\infty,i,\circ} + U^{\infty,i,N}\star\widetilde{\mu}^{\infty,i,\natural}, 0\Big), \tag{5.13}$$

$$\begin{aligned}\Big([Y^{k,i,N}], [Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}], [M^{k,i,N}],\\ [Y^{k,i,N}, X^{k,i,\circ}], [Y^{k,i,N}, X^{k,i,\natural}], [Y^{k,i,N}, M^{k,i,N}]\Big)\\ \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^1)} \Big([Y^{\infty,i,N}], [Z^{\infty,i,N}\cdot X^{\infty,i,\circ} + U^{\infty,i,N}\star\widetilde{\mu}^{\infty,i,\natural}], 0,\\ [Y^{\infty,i,N}, X^{\infty,i,\circ}], [Y^{\infty,i,N}, X^{\infty,i,\natural}], 0\Big)\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\Big(\langle Y^{k,i,N}\rangle, \langle Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}\rangle, \langle M^{k,i,N}\rangle,\\ \langle Y^{k,i,N}, X^{k,i,\circ}\rangle, \langle Y^{k,i,N}, X^{k,i,\natural}\rangle, \langle Y^{k,i,N}, M^{k,i,N}\rangle\Big)\\ \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{L}^1)} \Big(\langle Y^{\infty,i,N}\rangle, \langle Z^{\infty,i,N}\cdot X^{\infty,i,\circ} + U^{\infty,i,N}\star\widetilde{\mu}^{\infty,i,\natural}\rangle, 0,\\ \langle Y^{\infty,i,N}, X^{\infty,i,\circ}\rangle, \langle Y^{\infty,i,N}, X^{\infty,i,\natural}\rangle, 0\Big),\end{aligned} \tag{5.15}$$

where in (5.13) *the state space is* $\mathbb{R}^{d\times 3}$, *while in the other two the state space is* $\mathbb{R}^{d\times(4d+p+n)}$.

Proof. The approach we need to follow is completely analogous to the one followed in Theorem 5.2.1. Then, the combination of the results presented in Section 5.3.3 complete the missing details. □

5.3 Proofs of the theorems presented in Section 5.2

In the following subsections they will be presented the auxiliary results required for the proof of Theorems 5.2.1, 5.2.2 and 5.2.5. For the convenience of the reader, each next subsection is devoted to a specific theorem.

At this point, it is required to extend the convention we have made about the role of the indices k, i, N by introducing another integer, namely q, which may also take the value 0 and whose role is to denote the step of the Picard iteration. The newly introduced index will always succeed the other indices and it will be placed within parentheses. For example, $\mathcal{S}^{k,i,N,(q)}$ denotes the Picard representation obtained at the $q-$th iteration under the data $\mathcal{D}^k$ for the $i-$th player participating in a game of N players. The interpretation is obvious in the case fewer indices appear.

5.3.1 Auxiliary results for Theorem 5.2.1: Stability of McKean–Vlasov BSDes

For this subsection the framework of Theorem 5.2.1 is adopted. The next proposition is the analogous to [7, Corollary 10] and [44, Proposition 3.2] for the McKean–Vlasov case.

Proposition 5.3.1 (**Uniform *a priori* McKean–Vlasov BSDE estimates**). *Let $i \in \mathbb{N}$. For every $k \in \overline{\mathbb{N}}$ we associate to the standard data $\mathcal{D}^k$ the sequence of Picard iterations $\{\mathcal{S}^{k,i,(q)}\}_{q\in\mathbb{N}\cup\{0\}}$, where $\mathcal{S}^{k,i,(0)}$ is the zero element of $\mathcal{S}^{2,k}_{\hat\beta} \times \mathbb{H}^{2,k,\circ}_{\hat\beta} \times \mathbb{H}^{2,k,\natural}_{\hat\beta} \times \mathcal{H}^{2,k,\perp_{\mathbb{F}^{k,i}}}_{\hat\beta}$. There exists $k_{\star,0}$, independent of i, such that*[4]

$$\lim_{q\to\infty} \sup_{k\geq k_{\star,0}} \left\{ \|\mathcal{S}^{k,i,(q)} - \mathcal{S}^{k,i}\|^2_{\star,\hat\beta,k,i} \right\} = 0.$$

Additionally, we have $\sup_{k\geq k_{\star,0}} \left\{ \|\mathcal{S}^{k,i}\|^2_{\star,\hat\beta,k,i} \right\} < \infty$.

Proof. Let us fix $i \in \mathbb{N}$. From **(S9)**.(ii) we can pick $k_{\star,0}$ such that for every $k \geq k_{\star,0}$ we have $4\widetilde{M}^{\Phi^k}(\hat\beta) < 1$. Then, from the triangle inequality we have for every $k \in \overline{\mathbb{N}}$ and $q \in \mathbb{N}$ that

$$\begin{aligned}
\|\mathcal{S}^{k,i,(q)} - \mathcal{S}^{k,i}\|^2_{\star,\hat\beta,k,i} &\leq \sum_{m=0}^{\infty} 2^{m+1} \left\| \mathcal{S}^{k,i,(m+1+q)} - \mathcal{S}^{k,i,(m+q)} \right\|^2_{\star,\hat\beta,k,i} \\
&\leq \sum_{m=0}^{\infty} 2^{m+1} \left(2\widetilde{M}^{\Phi^k}(\hat\beta)\right)^{m+q} \left\| \mathcal{S}^{k,i,(1)} - \mathcal{S}^{k,i,(0)} \right\|^2_{\star,\hat\beta,k,i} \\
&= 2\left(2\widetilde{M}^{\Phi^k}(\hat\beta)\right)^{q} \left\| \mathcal{S}^{k,i,(1)} \right\|^2_{\star,\hat\beta,k,i} \sum_{m=0}^{\infty} \left(4\widetilde{M}^{\Phi^k}(\hat\beta)\right)^{m} \\
&= 2\frac{\left(2\widetilde{M}^{\Phi^k}(\hat\beta)\right)^{q}}{1 - 4\widetilde{M}^{\Phi^k}(\hat\beta)} \|\mathcal{S}^{k,i,(1)}\|^2_{\star,\hat\beta,k,i}.
\end{aligned}$$

[4]For the notation of the norm recall the convention we made on p. 85.

The right-hand side is finite. Indeed, we have from Lemma 2.9.1

$$\begin{aligned}\|\mathcal{S}^{k,i,(1)}\|^2_{\star,\hat\beta,k,i} &\leq \Big(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi^k\Big)\big\|\xi^{k,i}\big\|^2_{\mathbb{L}^{2,k,i}_{\hat\beta}(\mathbb{R}^d)}+\widetilde{M}^{\Phi^k}(\hat\beta)\Big\|\frac{f^k(\cdot,0,0,0,\delta_0)}{\alpha^k_\cdot}\Big\|^2_{\mathbb{H}^{2,k,i}_{\hat\beta}(\mathbb{R}^d)}\\ &\overset{\textbf{(S7)}}{\leq} e^{\hat\beta\overline{A}}\Big(26+\frac{2}{\hat\beta}+(9\hat\beta+2)\Phi^k\Big)\big\|\xi^{k,i}\big\|^2_{\mathbb{L}^{2,k,i}_{0}(\mathbb{R}^d)}+e^{\hat\beta\overline{A}}\widetilde{M}^{\Phi^k}(\hat\beta)\Big\|\frac{f^k(\cdot,0,0,0,\delta_0)}{\alpha^k_\cdot}\Big\|^2_{\mathbb{H}^{2,k,i}_{0}(\mathbb{R}^d)},\end{aligned}$$

which in particular implies

$$\sup_{k\geq k_{\star,0}}\|\mathcal{S}^{k,i,(1)}\|^2_{\star,\hat\beta,k,i}<\infty$$

in view of **(S5)**, **(S6)** and **(S9)**. □

Notation 1. *In order to simplify the notation, we define the following for $i\in\mathbb{N}$, $t\in\mathbb{R}_+\cup\{\infty\}$, $k\in\overline{\mathbb{N}}$ and $q\in\mathbb{N}\cup\{0\}$:*

- $L^{k,i,(q)}_t:=\displaystyle\int_0^t f^k\Big(s,Y^{k,i,(q)}_s,Z^{k,i,(q)}_s c^k_s,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i,(q)})_s,\mathcal{L}(Y^{k,i,(q)}_s)\Big)\,\mathrm{d}C^k_s,$

- $L^{k,i}_t:=\displaystyle\int_0^t f^k\Big(s,Y^{k,i}_s,Z^{k,i}_s c^k_s,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i})_s,\mathcal{L}(Y^{k,i}_s)\Big)\,\mathrm{d}C^k_s$ *and*

- $G^{k,i,(q)}:=\displaystyle\int_0^\infty \frac{\Big|f^k\Big(s,Y^{k,i,(q)}_s,Z^{k,i,(q)}_s c^k_s,\Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i,(q)})_s,\mathcal{L}(Y^{k,i,(q)}_s)\Big)\Big|^2}{(\alpha^k)^2}\,\mathrm{d}C^k_s.$

By replicating exactly the arguments presented in [44, Sections 3.3.1, 3.3.2 and 3.3.3] we arrive to the conclusion that the convergences (5.6), (5.7) and (5.8) are equivalent to the combination of Proposition 5.3.1 and the next result.

Lemma 5.3.2. *For every $q\in\mathbb{N}\cup\{0\}$ we have*

(i) $L^{k,i,(q)}_\infty\xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)}L^{\infty,i,(q)}_\infty$,

(ii) $L^{k,i,(q)}_\cdot\xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^d),\mathbb{L}^2)}L^{\infty,i,(q)}_\cdot$,

(iii) the sequence $\big\{G^{k,i,(q)}\big\}_{k\in\overline{\mathbb{N}}}$ is uniformly integrable.

Proof. The above results are proved by induction. For the first step of the inductions we have nothing to add to that of [44, Section 3.4] except that in the proof of [44, Lemma 3.12] when we apply the Lipschitz condition of the generators we would get for every $k \in \overline{\mathbb{N}}$ that the following hold $\mathbb{P}-a.e.$[5]

$$G^{k,i,(1)} = \int_0^\infty \frac{\left|f^k\Big(s, Y_s^{k,i,(1)}, Z_s^{k,i,(1)}c_s^k, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i,(1)})_s, \mathcal{L}(Y_s^{k,i,(1)})\Big)\right|^2}{(\alpha^k)^2}\,\mathrm{d}C_s^k$$
$$\leq \int_0^\infty (a_s^k)^2|Y_s^{k,i,(1)}|^2 + \|Z_s^{k,i,(1)}c_s^k\|^2 + 2\left|\!\left|\!\left|U_s^{k,i,(1)}\right|\!\right|\!\right|_{k,i,s}^2\,\mathrm{d}C_s^k$$
$$+ \int_0^\infty (a_s^k)^2\mathbb{E}\Big[|Y_s^{k,i,(1)}|^2\Big] + \frac{\left|f^k\left(s,0,0,0,\delta_0\right)\right|^2}{(\alpha^k)^2}\,\mathrm{d}C_s^k$$
$$\leq 2\int_0^\infty |Y_s^{k,i,(1)}|^2\,\mathrm{d}A_s^k + 2\mathbb{E}\bigg[\int_0^\infty |Y_s^{k,i,(1)}|^2\,\mathrm{d}A_s^k\bigg]$$
$$+ 2\int_0^\infty \mathrm{dTr}\Big[\Big\langle Z^{k,i,(1)}\cdot X^{k,i,\circ} + U^{k,i,(1)} \star \widetilde{\mu}^{k,i,\natural}\Big\rangle_s\Big] + 2G^{k,i,(0)}$$
$$\leq 4\overline{A}\sup_{s\in\mathbb{R}_+}\Big|\mathbb{E}\Big[\xi^{k,i}\Big|\mathcal{F}_s^{k,i}\Big]\Big|^2 + 4\overline{A}\sup_{s\in\mathbb{R}_+}\bigg|\mathbb{E}\bigg[\int_s^\infty f^k(t,0,0,0,\delta_0)\mathrm{d}C_t^k\bigg|\mathcal{F}_s^{k,i}\bigg]\bigg|^2 + 2\,\mathrm{Tr}\Big[\langle\widetilde{M}^{k,i,(0)}\rangle_\infty\Big]$$
$$+ 2\,G^{k,i,(0)} + 4\overline{A}\,\mathbb{E}\bigg[\sup_{s\in\mathbb{R}_+}\Big|\mathbb{E}\Big[\xi^{k,i}\Big|\mathcal{F}_s^{k,i}\Big]\Big|^2 + \sup_{s\in\mathbb{R}_+}\bigg|\mathbb{E}\bigg[\int_s^\infty f^k(t,0,0,0,\delta_0)\mathrm{d}C_t^k\bigg|\mathcal{F}_s^{k,i}\bigg]\bigg|^2\bigg],$$

where we have introduced the notation $\widetilde{M}_\cdot^{k,i,(0)}$ for the martingale

$$\widetilde{M}_\cdot^{k,i,(0)} := Y_\cdot^{k,i,(1)} + \int_0^\cdot f^k\left(s, Y_s^{k,i,(0)}, Z_s^{k,i,(0)}c_s^k, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i,(0)})_s, \mathcal{L}(Y_s^{k,i,(0)})\right)\mathrm{d}C_s^k$$
$$= Z^{k,i,(1)}\cdot X_\cdot^{k,i,\circ} + U^{k,i,(1)} \star \widetilde{\mu}_\cdot^{k,i,\natural}.$$

Hence, to show that $\{G^{k,i,1}\}_{k\in\overline{\mathbb{N}}}$ is uniformly integrable we will show that each of the summands in the last inequality belongs to a uniformly integrable sequence. We begin with the sequence

$$\Big\{\mathrm{Tr}\Big[\langle\widetilde{M}^{k,i,(0)}\rangle_\infty\Big]\Big\}_{k\in\overline{\mathbb{N}}}.$$

From the induction hypothesis we have

$$L_\infty^{k,i,(0)} \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} L_\infty^{\infty,i,(0)},$$

and because for every $k \in \overline{\mathbb{N}}$ we have $Y_\infty^{k,i,(1)} = \xi^{k,i}$, from **(S5)** we get

$$\widetilde{M}_\infty^{k,i,(0)} \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} \widetilde{M}_\infty^{\infty,i,(0)}.$$

Then, from [43, Theorem 2.16] we have the convergence

$$\mathrm{Tr}\Big[\langle\widetilde{M}^{k,i,(0)}\rangle_\infty\Big] \xrightarrow[k\to\infty]{\mathbb{L}^1(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} \mathrm{Tr}\Big[\langle\widetilde{M}^{k,i,(0)}\rangle_\infty\Big],$$

[5]We use the inequalities $\frac{r^k}{(\alpha^k)^2}, \frac{\vartheta^{k,*}}{(\alpha^k)^2} \leq (\alpha^k)^2$ and $\frac{\vartheta^{k,\circ}}{(\alpha^k)^2}, \frac{\vartheta^{k,\natural}}{(\alpha^k)^2} \leq 1$.

from which the wanted uniform integrability property follows. For the rest of the terms our main tool would be [44, Lemma A.17]. So, from the aforementioned lemma would be enough the sequences

$$\{|\xi^{k,i}|^2\}_{k\in\overline{\mathbb{N}}} \text{ and } \{G^{k,i,(0)}\}_{k\in\overline{\mathbb{N}}}$$

to be uniformly integrable, which holds from **(S5)** and **(S6)**. The last claim is true because

$$\begin{aligned}
&\sup_{s\in\mathbb{R}_+}\left|\mathbb{E}\left[\int_s^\infty f^k(t,0,0,0,\delta_0)\mathrm{d}C^k_t\middle|\mathcal{F}^{k,i}_s\right]\right|^2 \\
&\quad= \sup_{s\in\mathbb{R}_+}\left|\mathbb{E}\left[\int_0^\infty f^k(t,0,0,0,\delta_0)\mathrm{d}C^k_t\middle|\mathcal{F}^{k,i}_s\right] - \int_0^s f^k(t,0,0,0,\delta_0)\mathrm{d}C^k_t\right|^2 \\
&\quad\le 2\sup_{s\in\mathbb{R}_+}\left|\mathbb{E}\left[\int_0^\infty f^k(t,0,0,0,\delta_0)\mathrm{d}C^k_t\middle|\mathcal{F}^{k,i}_s\right]\right|^2 + 2\sup_{s\in\mathbb{R}_+}\left|\int_0^s f^k(t,0,0,0,\delta_0)\mathrm{d}C^k_t\right|^2,
\end{aligned}$$

and from Jensen's and Cauchy-Schwarz inequalities in conjunction with **(S7)** we get it.

Next, we proceed with the proof that the $q-$th step of the induction is valid. Following [44, Sections 3.5.1, 3.5.2, 3.5.3, 3.5.4, 3.5.5] we can conclude with exactly the same line of thoughts, noting only that in [44, Proposition 3.19] we utilize our assumption **(S8)**. □

5.3.2 Auxiliary results for Theorem 5.2.2: Stability of backward propagation of chaos

For this subsection the framework of Theorem 5.2.2 is adopted. We remark that the adopted framework is identical to that imposed for Theorem 5.2.1. Hence, we can use the results presented in Section 5.3.1.

Lemma 5.3.3. *Let $i \in \mathbb{N}$. The random variables $\left\{\sup_{s\in\mathbb{R}_+}|Y^{k,i}_s|^2\right\}_{k\in\mathbb{N}}$ are uniformly integrable.*

Proof. Let $i \in \mathbb{N}$. Initially, we will show that the random variables $\left\{\sup_{s\in\mathbb{R}_+}|Y^{k,i,(q)}_s|^2\right\}_{k\in\mathbb{N}}$ are uniformly integrable, for every $q \in \mathbb{N}$. This will be achieved by induction. The first step is obvious, we remind that from Remark 5.1.1.(iii) for every $k \in \mathbb{N}$ we have $C^k_t = C^k_{T^k}$ for $t \in [T^k,\infty)$. From Picard's scheme we know that for every $q \in \mathbb{N}$ and $t \in \mathbb{R}_+$

$$\begin{aligned}
Y^{k,i,(q)}_t &= \mathbb{E}\left[\xi^{k,i} + \int_t^\infty f^k\left(s, Y^{k,i,(q-1)}_s, Z^{k,i,(q-1)}_s c^k_s, \Gamma^{k,i}(U^{k,i,(q-1)}_s)_s, \mathcal{L}(Y^{k,i,(q-1)}_s)\right)\mathrm{d}C^k_s\middle|\mathcal{F}^{k,i}_t\right] \\
&= \mathbb{E}\left[\xi^{k,i} + \int_0^\infty f^k\left(s, Y^{k,i,(q-1)}_s, Z^{k,i,(q-1)}_s c^k_s, \Gamma^{k,i}(U^{k,i,(q-1)}_s)_s, \mathcal{L}(Y^{k,i,(q-1)}_s)\right)\mathrm{d}C^k_s\middle|\mathcal{F}^{k,i}_t\right] \\
&\qquad - \int_0^t f^k\left(s, Y^{k,i,(q-1)}_s, Z^{k,i,(q-1)}_s c^k_s, \Gamma^{k,i}(U^{k,i,(q-1)}_s)_s, \mathcal{L}(Y^{k,i,(q-1)}_s)\right)\mathrm{d}C^k_s.
\end{aligned}$$

Hence we get,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}_+}|Y_s^{k,i,(q)}|^2\\
&\quad\le 4\sup_{t\in\mathbb{R}_+}\Big|\mathbb{E}[\xi^{k,i}|\mathcal{F}_t^{k,i}]\Big|^2\\
&\qquad+4\sup_{t\in\mathbb{R}_+}\bigg|\mathbb{E}\bigg[\int_0^\infty f^k\left(s,Y_s^{k,i,(q-1)},Z_s^{k,i,(q-1)}c_s^k,\Gamma^{k,i}(U_s^{k,i,(q-1)})_s,\mathcal{L}(Y_s^{k,i,(q-1)})\right)\,\mathrm{d}C_s^k\bigg|\mathcal{F}_t^{k,i}\bigg]\bigg|^2 \qquad (5.16)\\
&\qquad\qquad+2\sup_{t\in\mathbb{R}_+}\bigg|\int_0^t f^k\left(s,Y_s^{k,i,(q-1)},Z_s^{k,i,(q-1)}c_s^k,\Gamma^{k,i}(U_s^{k,i,(q-1)})_s,\mathcal{L}(Y_s^{k,i,(q-1)})\right)\,\mathrm{d}C_s^k\bigg|^2.
\end{aligned}$$

Now, from the Cauchy–Schwarz inequality, and recalling **(S7)**, *i.e.*, $\sup_{k\in\mathbb{N}}\left\{A_\infty^k\right\}\le\overline{A}\in\mathbb{R}_+$, we have for every $t\in\mathbb{R}_+\cup\{\infty\}$

$$\begin{aligned}
&\bigg|\int_0^t f^k\left(s,Y_s^{k,i,(q-1)},Z_s^{k,i,(q-1)}c_s^k,\Gamma^{k,i}(U_s^{k,i,(q-1)})_s,\mathcal{L}(Y_s^{k,i,(q-1)})\right)\,\mathrm{d}C_s^k\bigg|^2\\
&\qquad\le\overline{A}\int_0^\infty\frac{\Big|f^k\left(s,Y_s^{k,i,(q-1)},Z_s^{k,i,(q-1)}c_s^k,\Gamma^{k,i}(U_s^{k,i,(q-1)})_s,\mathcal{L}(Y_s^{k,i,(q-1)})\right)\Big|^2}{(\alpha_s^k)^2}\,\mathrm{d}C_s^k.
\end{aligned}$$

From Lemma 5.3.2.(iii), **(S5)** and [44, Lemma A.17] the right side of inequality (5.16) consists of elements of sequences of uniformly integrable random variables. Hence, our first claim is proved.

So, for every $q\in\mathbb{N}$ and $\varepsilon>0$ there exists $\delta(\varepsilon,q)>0$ such that for every $S\in\mathcal{F}$ with $\mathbb{P}(S)\le\delta(\varepsilon,q)$ we have

$$\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i,(q)}|^2\mathbb{1}_S\Big]\le\frac{\varepsilon}{4}.$$

From Proposition 5.3.1 there exists $k_{\star,0}$ such that we can choose $q(\varepsilon)\in\mathbb{N}$ large enough such that

$$\sup_{k\ge k_{\star,0}}\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i,(q(\varepsilon))}-Y_s^{k,i}|^2\Big]\le\frac{\varepsilon}{4}.$$

Then, we get that for every $S\in\mathcal{F}$ with $\mathbb{P}(S)\le\delta(\varepsilon,q(\varepsilon))$ we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i}|^2\mathbb{1}_S\Big]&=\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i}-Y_s^{k,i,(q(\varepsilon))}+Y_s^{k,i,(q(\varepsilon))}|^2\mathbb{1}_S\Big]\\
&\le 2\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i}-Y_s^{k,i,(q(\varepsilon))}|^2\Big]+2\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i,(q(\varepsilon))}|^2\mathbb{1}_S\Big]\le\varepsilon,
\end{aligned}$$

for every $k\in\mathbb{N}$. From the above and the fact that $\sup_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{s\in\mathbb{R}_+}|Y_s^{k,i}|^2\Big]<\infty$, recall Proposition 5.3.1, we are done. □

Lemma 5.3.4. *Let $\widetilde{\mathbf{Y}}^{k,N}$ be the vector of the Y-component of the solutions of the first N McKean–Vlasov BSDEs under the data $\mathcal{D}^k$, for $k\in\overline{\mathbb{N}}$ and $N\in\mathbb{N}$. Then,*

$$\lim_{N\to\infty}\sup_{k\in\overline{\mathbb{N}},s\in\mathbb{R}_+}\mathbb{E}\Big[W_{2,|\cdot|}^2\Big(L^N\Big(\widetilde{\mathbf{Y}}_s^{k,N}\Big),\mathcal{L}\Big(Y_s^{k,1}\Big)\Big)\Big]=0.$$

Proof. For every $m \in \mathbb{N}$ define

$$M_2(Y_s^{k,1}, m) := \int_{B_m} |x|^2 \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x),$$

where $B_m := (-2^m, 2^m]^d \setminus (-2^{m-1}, 2^{m-1}]^d$ and also $B_0 := (-1,1]^d$. Let $\ell \in \mathbb{N}$, then for every $k \in \mathbb{N}$ and $s \in \mathbb{R}_+$ we have

$$\begin{aligned}\sum_{m=\ell}^{\infty} M_2(Y_s^{k,1}, m) &= \int_{\bigcup_{m=\ell}^{\infty} B_m} |x|^2 \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x) = \int_{\mathbb{R}^d} |x|^2 \mathbb{1}_{[2^{\ell-1},\infty)}(x) \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x) \\ &\leq \int_{\mathbb{R}^d} |x|^2 \mathbb{1}_{[2^{2(\ell-1)},\infty)}(|x|^2) \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x) = \mathbb{E}\Big[\big|Y_s^{k,1}\big|^2 \mathbb{1}_{\{|Y_s^{k,1}|^2 \geq 2^{2(\ell-1)}\}}\Big] \\ &\leq \mathbb{E}\Big[\sup_{s\in\mathbb{R}_+} \big\{|Y_s^{k,1}|^2\big\} \mathbb{1}_{\{\sup_{s\in\mathbb{R}_+} |Y_s^{k,1}|^2 \geq 2^{2(\ell-1)}\}}\Big]. \end{aligned} \tag{5.17}$$

From Lemma 5.3.3, the random variables $\big\{\sup_{s\in\mathbb{R}_+} |Y_s^{k,1}|^2\big\}_{k\in\mathbb{N}}$ are uniformly integrable. Hence, using this information on inequality (5.17), for every positive number ε_1, we can choose $\ell_1(\varepsilon_1) \in \mathbb{N}$, which is universal with respect to $k \in \mathbb{N}$ and $s \in \mathbb{R}_+$, such that

$$\sum_{m=\ell_1(\varepsilon_1)+1}^{\infty} M_2(Y_s^{k,1}, m) < \varepsilon_1. \tag{5.18}$$

Also, for every positive ε_2 there exists $\ell_2(\varepsilon_2) \in \mathbb{N}$ such that

$$\sum_{l=\ell_2(\varepsilon_2)+1}^{\infty} 2^{-2l} < \varepsilon_2. \tag{5.19}$$

Let $\mathcal{R}_0 := \sup_{k\in\mathbb{N}} \mathbb{E}\big[\sup_{s\in\mathbb{R}_+} |Y_s^{k,1}|^2\big] < \infty$. Then, trivially, for every $k \in \mathbb{N}$ and $s \in \mathbb{R}_+$ we have $M_2(Y_s^{k,1}, m) \leq \mathcal{R}_0$. From [20, Lemma 6 and Inequality (4) on p. 716] we get

$$\mathbb{E}\Big[W_{2,|\cdot|}^2\Big(L^N\big(\widetilde{\mathbf{Y}}_s^{k,N}\big), \mathcal{L}\big(Y_s^{k,1}\big)\Big)\Big] \leq C_{d,2} \sum_{m=0}^{\infty} 2^{2m} \sum_{l=0}^{\infty} 2^{-2l} \min\Big\{2\mathcal{L}\big(Y_s^{k,1}\big)(B_m), 2^{\frac{dl}{2}}\Big(\frac{1}{N}\mathcal{L}\big(Y_s^{k,1}\big)(B_m)\Big)^{\frac{1}{2}}\Big\}.$$

Trivially, for every $k \in \mathbb{N}$ and $s \in \mathbb{R}_+$, we have for $m = 0$

$$\mathcal{L}\big(Y_s^{k,1}\big)(B_0) \leq 1$$

and for $m \in \mathbb{N}$

$$\mathcal{L}\big(Y_s^{k,1}\big)(B_m) = \int_{B_m} 1 \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x) \leq \int_{B_m} 2^{-2(m-1)} |x|^2 \,\mathrm{d}\mathcal{L}\big(Y_s^{k,1}\big)(x) = 2^{-2(m-1)} M_2(Y_s^{k,1}, m). \tag{5.20}$$

Let us fix, now, $\varepsilon > 0$ and define

$$\varepsilon_1 := \frac{\varepsilon}{33 C_{d,2}}. \tag{5.21}$$

To the defined ε_1 it corresponds a natural number $\ell_1(\varepsilon_1)$ such that (5.18) holds. Next, to

$$\varepsilon_2 := \frac{\varepsilon}{\Big(6 + 24\ell_1(\varepsilon_1)\mathcal{R}_0\Big)C_{d,2}} \tag{5.22}$$

we can associate a natural number $\ell_2(\varepsilon_2)$ such that (5.19) holds. Finally, we define

$$N(\varepsilon) := \left\lfloor \frac{9C_{d,2}^2 2^{d\ell_2(\varepsilon_2)+2}\Big(2^{\ell_1(\varepsilon_1)+1}\ell_1(\varepsilon_1)\mathcal{R}_0{}^{\frac{1}{2}} + 1\Big)^2}{\varepsilon^2} \right\rfloor + 1.^6 \tag{5.23}$$

Collecting the information presented above, we can deduce that for the fixed $\varepsilon > 0$ we have (we omit the dependencies for the sake of a readable notation), for every $N \geq N(\varepsilon)$, universal with respect to $k \in \mathbb{N}$ and $s \in \mathbb{R}_+$

$$\begin{aligned}
&\mathbb{E}\left[W_{2,|\cdot|}^2\Big(L^N\big(\widetilde{\mathbf{Y}}_s^{k,N}\big), \mathcal{L}\big(Y_s^{k,1}\big)\Big)\right] \\
&\overset{(5.20)}{\leq} C_{d,2}\sum_{l=0}^{\infty} 2^{-2l}\min\Big\{2, 2^{\frac{dl}{2}}N^{-\frac{1}{2}}\Big\} \\
&\qquad + C_{d,2}\sum_{m=1}^{\infty} 2^{2m}\sum_{l=0}^{\infty} 2^{-2l}\min\left\{2^{-2(m-1)+1}M_2\Big(Y_s^{k,1}, m\Big), 2^{\frac{dl}{2}}\left(\frac{2^{-2(m-1)}}{N}M_2\Big(Y_s^{k,1}, m\Big)\right)^{\frac{1}{2}}\right\} \\
&\leq C_{d,2}\Big[2\sum_{l=\ell_2+1}^{\infty} 2^{-2l} + \frac{4}{3}2^{\frac{d\ell_2}{2}}N^{-\frac{1}{2}}\Big] + 11C_{d,2}\sum_{m=\ell_1+1}^{\infty} M_2\Big(Y_s^{k,1}, m\Big) \\
&\qquad + C_{d,2}\sum_{m=1}^{\ell_1}\left[8\mathcal{R}_0\sum_{l=\ell_2+1}^{\infty} 2^{-2l} + 2^{\ell_1}\frac{4}{3}2^{\frac{d\ell_2}{2}+1}\left(\frac{\mathcal{R}_0}{N}\right)^{\frac{1}{2}}\right] \\
&\leq C_{d,2}\frac{4}{3}2^{\frac{d\ell_2}{2}}N^{-\frac{1}{2}} + 2C_{d,2}\varepsilon_2 + 11C_{d,2}\varepsilon_1 + 8C_{d,2}\ell_1\mathcal{R}_0\varepsilon_2 + C_{d,2}\ell_1\frac{4}{3}2^{\frac{d\ell_2}{2}+1+\ell_1}\left(\frac{\mathcal{R}_0}{N}\right)^{\frac{1}{2}} \\
&\leq 11C_{d,2}\varepsilon_1 + \Big(2 + 8\ell_1\mathcal{R}_0\Big)C_{d,2}\varepsilon_2 + \frac{C_{d,2}2^{\frac{d\ell_2}{2}+1}\Big(2^{\ell_1+1}\ell_1\mathcal{R}_0{}^{\frac{1}{2}} + 1\Big)}{N^{\frac{1}{2}}} \\
&\leq \varepsilon,
\end{aligned}$$

where the last inequality is the outcome of (5.21), (5.22) and (5.23). □

Remark 5.3.5. *The approach we used in the last proof was inspired by the approach used in [20]. There, the authors used advanced inegrability assumptions in order to control the tails of the distributions and get rates of convergence for the different cases. In our setting we had to work with sharp square integrability, this made possible by noticing that we can bound $\mathcal{L}\Big(Y_s^{k,1}\Big)(B_m)$ from the quantities $2^{-2(m-1)}M_2(Y_s^{k,1}, m)$, thus make them controllable. Then we proceeded in a similar fashion as the authors of [20] did, while ensuring that our bounds are universal with respect to k by the virtue of Lemma 5.3.3.*

[6] We denote with $\lfloor x \rfloor$ the integer part of the real number x.

Theorem 5.3.6 (**Uniform propagation of chaos for the system**). *Under the adopted framework it holds*

$$\lim_{N\to\infty}\sup_{k\in\mathbb{N}}\left\{\frac{1}{N}\sum_{i=1}^{N}\Big\|\Big(Y^{k,i,N}-Y^{k,i},Z^{k,i,N}-Z^{k,i},U^{k,i,N}-U^{k,i},M^{k,i,N}-M^{k,i}\Big)\Big\|^2_{\star,\hat{\beta},k,(1,\dots,N)}\right\}=0.$$

Proof. From the proof of Theorem 4.2.2 we see that for every $k,N\in\mathbb{N}$ we get that

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}&\Big\|\Big(Y^{k,i,N}-Y^{k,i},Z^{k,i,N}-Z^{k,i},U^{k,i,N}-U^{k,i},M^{k,i,N}-M^{k,i}\Big)\Big\|^2_{\star,\hat{\beta},k,(1,\dots,N)}\\&\le\frac{\Big(26+\frac{2}{\hat{\beta}}+(9\hat{\beta}+2)\Phi^k\Big)}{1-3\widetilde{M}^{\Phi^k}(\hat{\beta})}\frac{1}{N}\sum_{i=1}^{N}\big\|\xi^{k,i,N}-\xi^{k,i}\big\|^2_{\mathbb{L}^{2,k,(1,\dots,N)}_{\hat{\beta}}(\mathbb{R}^d)}\\&\qquad+\frac{2\widetilde{M}^{\Phi^k}(\hat{\beta})}{1-3\widetilde{M}^{\Phi^k}(\hat{\beta})}\frac{1}{\hat{\beta}}\int_0^\infty\mathbb{E}\Big[W^2_{2,|\cdot|}\Big(L^N\big(\widetilde{\mathbf{Y}}^{k,N}_s\big),\mathcal{L}\big(Y^{k,1}_s\big)\Big)\Big]\,\mathrm{d}\mathcal{E}\big(\hat{\beta}A^k\big)_s,\end{aligned}$$

where we denoted by $\widetilde{\mathbf{Y}}^{k,N}$ the vector of the Y-component of the solutions of the first N McKean–Vlasov BSDEs under the data $\mathcal{D}^k$. The first summand of the right-hand side of the above inequality converges in view of **(S4)** and **(S9)**. The convergence of the second summand comes from Lemma 5.3.4 in conjunction with **(S7)**, which implies $\sup_{k\in\mathbb{N}}\Big\{\mathcal{E}\big(\hat{\beta}A^k\big)_\infty\Big\}\le e^{\hat{\beta}\overline{A}}$. Hence, these combined with **(S4)** and **(S9)** complete the proof.

□

Theorem 5.3.7 (**Uniform propagation of chaos**). *For every $i\in\mathbb{N}$, under the adopted framework it holds*

$$\lim_{N\to\infty}\sup_{k\in\mathbb{N}}\Big\|\Big(Y^{k,i,N}-Y^{k,i},Z^{k,i,N}-Z^{k,i},U^{k,i,N}-U^{k,i},M^{k,i,N}-M^{k,i}\Big)\Big\|^2_{\star,\hat{\beta},k,(1,\dots,N)}=0.$$

Proof. Let $i\in\mathbb{N}$. From the proof of Theorem 4.2.3 we get that

$$\begin{aligned}&\Big\|\Big(Y^{k,i,N}-Y^{k,i},Z^{k,i,N}-Z^{k,i},U^{k,i,N}-U^{k,i},M^{k,i,N}-M^{k,i}\Big)\Big\|^2_{\star,\hat{\beta},k,(1,\dots,N)}\\&\le\frac{\Big(26+\frac{2}{\hat{\beta}}+(9\hat{\beta}+2)\Phi^k\Big)}{1-2\widetilde{M}^{\Phi^k}(\hat{\beta})}\big\|\xi^{k,i,N}-\xi^{k,i}\big\|^2_{\mathbb{L}^{2,k,(1,\dots,N)}_{\hat{\beta}}(\mathbb{R}^d)}\\&\quad+\frac{2\widetilde{M}^{\Phi^k}(\hat{\beta})}{1-2\widetilde{M}^{\Phi^k}(\hat{\beta})}\frac{1}{N}\sum_{i=1}^{N}\Big\|\Big(Y^{k,i,N}-Y^{k,i},Z^{k,i,N}-Z^{k,i},U^{k,i,N}-U^{k,i},M^{k,i,N}-M^{k,i}\Big)\Big\|^2_{\star,\hat{\beta},k,(1,\dots,N)}\\&\quad+\frac{2\widetilde{M}^{\Phi^k}(\hat{\beta})}{1-2\widetilde{M}^{\Phi^k}(\hat{\beta})}\frac{1}{\hat{\beta}}\int_0^\infty\mathbb{E}\Big[W^2_{2,|\cdot|}\Big(L^N\big(\widetilde{\mathbf{Y}}^{k,N}_s\big),\mathcal{L}\big(Y^{k,1}_s\big)\Big)\Big]\,\mathrm{d}\mathcal{E}\big(\hat{\beta}A^k\big)_s.\end{aligned}$$

From **(S4)**, **(S9)**, Theorem 5.3.6 and Lemma 5.3.4 we conclude. □

Corollary 5.3.8. *For every $k\in\overline{\mathbb{N}}$ and $i,N\in\mathbb{N}$ we define*

$$L^{k,i,N}_\cdot:=\int_0^\cdot f^k\left(s,Y^{k,i,N}_s,Z^{k,i,N}_s c^k_s,\Gamma^{k,(1,\dots,N)}(U^{k,i,N})_s,L^N(\mathbf{Y}^{k,N}_s)\right)\,\mathrm{d}C^k_s.$$

Then, the following are true:[7]

$$\sup_{k\in\mathbb{N}} \mathbb{E}\Big[\Big|\mathrm{Var}\Big[L^{k,i,N} - L^{k,i}\Big]_\infty\Big|^2\Big] \xrightarrow[N\to\infty]{|\cdot|} 0, \tag{5.24}$$

where the total variation is calculated per coordinate,

$$\sup_{k\in\mathbb{N}} \mathbb{E}\Big[\Big\|[L^{k,i,N}]_\cdot - [L^{k,i}]_\cdot\Big\|_\infty\Big] \xrightarrow[N\to\infty]{|\cdot|} 0, \tag{5.25}$$

$$[L^{k,i,N}] \xrightarrow[(k,N)\to(\infty,\infty)]{\left(\mathrm{J}_1(\mathbb{R}^{d\times d}),\mathbb{L}^1\right)} 0 \tag{5.26}$$

and

$$Z^{k,i,N}\cdot X^{k,i,\circ}_\infty + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}_\infty + M^{k,i,N}_\infty \xrightarrow[(k,N)\to(\infty,\infty)]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} Z^{\infty,i}\cdot X^{\infty,i,\circ}_\infty + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}_\infty. \tag{5.27}$$

Finally, the sequence of random variables $\left\{\left|Z^{k,i,N}\cdot X^{k,i,\circ}_\infty + U^{k,i,N}\star\widetilde{\mu}^{k,i,\natural}_\infty + M^{k,i,N}_\infty\right|^2\right\}_{k\in\overline{\mathbb{N}},N\in\mathbb{N}}$ *is uniformly integrable.*

Proof. For (5.24), we start by reminding the fact that

$$\frac{r^k}{(\alpha^k)^2} \le (\alpha^k)^2, \vartheta^{k,\circ} \le (\alpha^k)^2, \vartheta^{k,\natural} \le (\alpha^k)^2 \text{ and } \frac{\vartheta^{k,*}}{(\alpha^k)^2} \le (\alpha^k)^2.$$

Define for every $s\in\mathbb{R}_+$

$$\delta f^{k,N}_s := f^k\left(s, Y^{k,i,N}_s, Z^{k,i,N}_s c^k_s, \Gamma^{k,i}(U^{k,i,N})_s, L^N(\mathbf{Y}^{k,N}_s)\right) - f^k\left(s, Y^{k,i}_s, Z^{k,i}_s c^k_s, \Gamma^{k,i}(U^{k,i})_s, \mathcal{L}(Y^{k,i}_s)\right).$$

From Cauchy-Schwarz inequality, **(H4)** and **(S7)** we get for every $k\in\mathbb{N}$ that

$$\begin{aligned}
\mathbb{E}\Big[\Big|\mathrm{Var}\Big[L^{k,i,N}_\cdot - L^{k,i}_\cdot\Big]\Big|^2\Big] &= \mathbb{E}\Big[\Big|\mathrm{Var}\Big[\int_0^\cdot \delta f^{k,N}_s \mathrm{d}C^k_s\Big]\Big|^2\Big] \le \overline{A}\,\mathbb{E}\left[\int_0^\infty \mathcal{E}\big(\hat\beta A^k\big)_{s-}\frac{|\delta f^{k,N}_s|^2}{(a^k_s)^2}\mathrm{d}C^k_s\right]\\
&\le \overline{A}\Bigg(\|\alpha^k(Y^{k,i,N}-Y^{k,i})\|^2_{\mathbb{H}^{2,k,(1,\dots,N)}_{\hat\beta}(\mathbb{R}^d)} + \|Z^{k,i,N}-Z^{k,i}\|^2_{\mathbb{H}^{2,k,(1,\dots,N),\circ}_{\hat\beta}(\mathbb{R}^{d\times p})}\\
&\qquad + \|U^{k,i,N}-U^{k,i}\|^2_{\mathbb{H}^{2,k,(1,\dots,N),\natural}_{\hat\beta}(\mathbb{R}^d)} + \frac{1}{\hat\beta}\,\mathbb{E}\left[\int_0^\infty W^2_{2,|\cdot|}(L^N(\mathbf{Y}^{k,N}_s),\mathcal{L}(Y^{k,i}_s))\mathrm{d}\mathcal{E}\big(\hat\beta A^k\big)_s\right]\Bigg).
\end{aligned}$$

Now, in view of the previous results, we only need to take care of the convergence of the last summand of the right-hand side of the last inequality. From the triangle inequality of the Wasserstein distance and Tonelli's theorem we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty W^2_{2,|\cdot|}(L^N(\mathbf{Y}^{k,N}_s),\mathcal{L}(Y^{k,i}_s))\mathrm{d}\mathcal{E}\big(\hat\beta A^k\big)_s\right]\\
&\qquad\le 2\mathbb{E}\left[\int_0^\infty W^2_{2,|\cdot|}(L^N(\mathbf{Y}^{k,N}_s),L^N(\widetilde{\mathbf{Y}}^{k,N}_s))\mathrm{d}\mathcal{E}\big(\hat\beta A^k\big)_s\right]\\
&\qquad\qquad + 2\int_0^\infty \mathbb{E}\Big[W^2_{2,|\cdot|}\Big(L^N\Big(\widetilde{\mathbf{Y}}^{k,N}_s\Big),\mathcal{L}\Big(Y^{k,1}_s\Big)\Big)\Big]\,\mathrm{d}\mathcal{E}\big(\hat\beta A^k\big)_s{}^{8}.
\end{aligned}$$

[7]The reader may also recall 1.

On the right-hand side of the above inequality, the convergence of the first summand is concluded from Theorem 5.3.6 in conjunction with **(S7)**, once we observe that by (2.25)

$$W^2_{2,|\cdot|}(L^N(\mathbf{Y}^{k,N}_s), L^N(\widetilde{\mathbf{Y}}^{k,N}_s)) \leq \frac{1}{N}\sum_{i=1}^{N}|Y^{k,i,N}_s - Y^{k,i}_s|^2.$$

The convergence of the second summand is guaranteed by Lemma 5.3.4 in conjunction with **(S7)**.

For (5.25) we argue as follows. For $j \in \{1,\dots,d\}$, we will adjoin the upper index $[j]$ in order to denote the $j-$element of a $d-$dimensional vector.

$$\begin{aligned}\mathrm{Tr}\Big[[L^{k,i,N} - L^{k,i}]_\infty\Big] &= \sum_{j=1}^{d}\sum_{t\geq 0}\Big(\Delta(L^{k,i,N,[j]} - L^{k,i,[j]})\Big)^2 \leq \sum_{j=1}^{d}\Big(\sum_{t\geq 0}\Delta|L^{k,i,N,[j]} - L^{k,i,[j]}|\Big)^2\\ &\leq \Big|\mathrm{Var}[L^{k,i,N} - L^{k,i}]_\infty\Big|^2.\end{aligned}$$

Now, the pathwise identity

$$[L^{k,i,N}] - [L^{k,i}] = [L^{k,i,N} - L^{k,i}] + 2[L^{k,i,N} - L^{k,i}, L^{k,i}]$$

implies for every $t \in \mathbb{R}_+$ (we understand the following elementwise)

$$\begin{aligned}\mathrm{Var}\Big([L^{k,i,N}]_t - [L^{k,i}]_t\Big) &= \mathrm{Var}\Big([L^{k,i,N} - L^{k,i}]_t + 2[L^{k,i,N} - L^{k,i}, L^{k,i}]_t\Big)\\ &\leq \mathrm{Var}\Big([L^{k,i,N} - L^{k,i}]_\infty\Big) + 2\mathrm{Var}\Big([L^{k,i,N} - L^{k,i}, L^{k,i}]_\infty\Big)\end{aligned}$$

Then, the desired convergence (5.25) is derived by the convergence (5.24), the Kunita–Watanabe inequality, see [21, Theorem 6.33], and the $\mathbb{L}^1-$boundedness of $\{[L^{k,i}]_\infty\}_{k\in\overline{\mathbb{N}}}$.

The convergence (5.26) is immediate from the convergence (5.25) if the next limit holds

$$[L^{k,i}] \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^{d\times d}),\mathbb{L}^1)} 0.$$

But this follows by following for the McKean–Vlasov case the exact same reasoning as in the analogous results of [44, Section 3.3.2], *i.e.*,

$$[L^{k,i,(q)}] \xrightarrow[q\to\infty]{(\|\cdot\|_\infty,\mathbb{L}^1)} [L^{k,i}] \quad \text{and} \quad [L^{k,i,(q)}] \xrightarrow[(k,q)\to(\infty,\infty)]{(\mathrm{J}_1(\mathbb{R}^{d\times d}),\mathbb{L}^1)} 0.$$

Now, we intend to prove convergence (5.27). To this end, we have from **(S5)**, Lemma 5.3.2.(i) and Proposition 5.3.1 by an application of the Moore–Osgood theorem that

$$Z^{k,i}\cdot X^{k,i,\circ}_\infty + U^{k,i}\star\widetilde{\mu}^{k,i,\natural}_\infty + M^{k,i}_\infty \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} Z^{\infty,i}\cdot X^{\infty,i,\circ}_\infty + U^{\infty,i}\star\widetilde{\mu}^{\infty,i,\natural}_\infty. \tag{5.28}$$

Applying again the Moore–Osgood theorem, we conclude from (5.28) and Theorem 5.3.7.

[8] Note that $\mathcal{L}(Y^{k,1}_s) = \mathcal{L}(Y^{k,i}_s)$ for every $s \in \mathbb{R}_+$ and $k \in \overline{\mathbb{N}}, i \in \mathbb{N}$.

The desired uniform integrability property is immediate from Vitalli's theorem, see [5, Theorem 4.5.4] which can be adapted for the $\mathbb{L}^2-$case. $\square$

5.3.3 Auxiliary results for Theorem 5.2.5: Stability of mean-field BSDE systems

For this subsection the framework of Theorem 5.2.5 is adopted.

For the convenience of the reader we present the following table which corresponds to the stability of mean-field BSDEs with N players. The outline of the approach is exactly the same as in the case of the stability of McKean–Vlasov BSDEs and therefore we avoid the repetition by omitting the respective description.

$\mathcal{D}^1$	$\mathbf{S}^{1,N,(0)}$	$\mathbf{S}^{1,N,(1)}$	$\mathbf{S}^{1,N,(2)}$	$\dots$	$\mathbf{S}^{1,N,(q)}$	$\xrightarrow{q\to\infty}$	$\mathbf{S}^{1,N}$
$\mathcal{D}^2$	$\mathbf{S}^{2,N,(0)}$	$\mathbf{S}^{2,N,(1)}$	$\mathbf{S}^{2,N,(2)}$	$\dots$	$\mathbf{S}^{2,N,(q)}$	$\xrightarrow{q\to\infty}$	$\mathbf{S}^{2,N}$
$\mathcal{D}^3$	$\mathbf{S}^{3,N,(0)}$	$\mathbf{S}^{3,N,(1)}$	$\mathbf{S}^{3,N,(2)}$	$\dots$	$\mathbf{S}^{3,N,(q)}$	$\xrightarrow{q\to\infty}$	$\mathbf{S}^{3,N}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathcal{D}^k$	$\mathbf{S}^{k,N,(0)}$	$\mathbf{S}^{k,N,(1)}$	$\mathbf{S}^{k,N,(2)}$	$\dots$	$\mathbf{S}^{k,N,(q)}$	$\xrightarrow{q\to\infty}$	$\mathbf{S}^{k,N}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$		$\wr\downarrow$
$\mathcal{D}^\infty$	$\mathbf{S}^{\infty,N,(0)}$	$\mathbf{S}^{\infty,N,(1)}$	$\mathbf{S}^{\infty,N,(2)}$	$\dots$	$\mathbf{S}^{\infty,N,(q)}$	$\xrightarrow{q\to\infty}$	$\mathbf{S}^{\infty,N}$

Table 5.3 The doubly-indexed Picard scheme for mean-field systems of BSDEs.

Proposition 5.3.9 (Uniform *a priori* mean-field BSDE system estimates). *For $k \in \overline{\mathbb{N}}$ and $N \in \mathbb{N}$ we associate to the standard data $\mathcal{D}^k$ the sequence of Picard iterations $\{\boldsymbol{S}^{k,N,(q)}\}_{q\in\mathbb{N}\cup\{0\}}$, where $\boldsymbol{S}^{k,N,(0)}$ is the zero element of*

$$\prod_{i=1}^{N} \mathcal{S}^{2,k,(1,\dots,N)}_{\hat{\beta}} \times \prod_{i=1}^{N} \mathbb{H}^{2,k,(1,\dots,N),\circ}_{\hat{\beta}} \times \prod_{i=1}^{N} \mathbb{H}^{2,k,(1,\dots,N),\natural}_{\hat{\beta}} \times \prod_{i=1}^{N} \mathcal{H}^{2,k,(1,\dots,N),\perp}_{\hat{\beta}}.$$

There exists $k_{\star,0}$ such that

$$\lim_{q\to\infty} \sup_{k\geq k_{\star,0}} \left\| \boldsymbol{S}^{k,N,(q)} - \boldsymbol{S}^{k,N} \right\|^2_{\star,k,(1,\dots,N),\hat{\beta}} = 0.$$

Additionally, we also have $\sup_{k\geq k_{\star,0}} \left\| \boldsymbol{S}^{k,N} \right\|^2_{\star,k,(1,\dots,N),\hat{\beta}} < \infty$.

Proof. As the reader can confirm, one can follow, *mutatis mutandis*, the same arguments as in the proof of Proposition 5.3.1, which deals with the stability of McKean–Vlaosv BSDEs. $\square$

To prove Theorem 5.2.5 one follows the same strategy as for the proof of Theorem 5.2.1, but working now for the $N-$system of BSDEs. Hence, we need to take care of the appropriate modifications.

Notation 2. *For every $i \in \{1, ..., N\}$, $t \in \mathbb{R}_+ \cup \{\infty\}$, $k \in \overline{\mathbb{N}}$ and $q \in \mathbb{N} \cup \{0\}$ we define:*

- $L_t^{k,i,N,(q)} := \int_0^t f^k\Big(s, Y_s^{k,i,N,(q)}, Z_s^{k,i,N,(q)} c_s^k, \Gamma^{k,(1,\dots,N)}(U^{k,i,N,(q)})_s, L^N(\boldsymbol{Y}_s^{k,N,(q)})\Big)\,\mathrm{d}C_s^k,$

- $L_t^{k,i,N} := \int_0^t f^k\Big(s, Y_s^{k,i,N}, Z_s^{k,i,N} c_s^k, \Gamma^{k,(1,\dots,N)}(U^{k,i,N})_s, L^N(\boldsymbol{Y}_s^{k,N})\Big)\,\mathrm{d}C_s^k,$ *and*

- $G^{k,i,N,(q)} := \int_0^\infty \frac{\Big|f^k\Big(s, Y_s^{k,i,N,(q)}, Z_s^{k,i,N,(q)} c_s^k, \Gamma^{k,(1,\dots,N)}(U^{k,i,N,(q)})_s, L^N(\boldsymbol{Y}_s^{k,N,(q)})\Big)\Big|^2}{(\alpha^k)^2}\,\mathrm{d}C_s^k.$

Then, as before, from [44, Sections 3.3.1, 3.3.2 and 3.3.3] we get that the convergences of Theorem 5.2.5 are equivalent to the next result.

Lemma 5.3.10. *For every $i \in \{1, ..., N\}$ and $q \in \mathbb{N} \cup \{0\}$ we have*

(i) $L_\infty^{k,i,N,(q)} \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} L_\infty^{\infty,i,N,(q)},$

(ii) $L_\cdot^{k,i,N,(q)} \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^d),\mathbb{L}^2)} L_\cdot^{\infty,i,N,(q)},$

(iii) the sequence $\left\{G^{k,i,N,(q)}\right\}_{k\in\overline{\mathbb{N}}}$ *is uniformly integrable.*

The above result is proved by induction with exactly the same arguments as Lemma 5.3.2, but note that in each step of the induction we treat all $i \in \{1, ..., N\}$ simultaneously. As an example we will provide the computations needed in the proof of [44, Lemma 3.12.], as we did at Lemma 5.3.2 above. So, for $i \in \{1, ..., N\}$ and $k \in \overline{\mathbb{N}}$ we have $\mathbb{P}-a.s.$

$$
\begin{aligned}
G^{k,i,N,(1)} &= \int_0^\infty \frac{\Big|f^k\Big(s, Y_s^{k,i,N,(1)}, Z_s^{k,i,N,(1)} c_s^k, \Gamma^{k,(1,\dots,N)}(U^{k,i,N,(1)})_s, L^N(\mathbf{Y}_s^{k,N,(1)})\Big)\Big|^2}{(\alpha^k)^2}\,\mathrm{d}C_s^k \\
&\leq \int_0^\infty (a_s^k)^2 |Y_s^{k,i,N,(1)}|^2 + \|Z_s^{k,i,N,(1)} c_s^k\|^2 + 2\Big(\big|\!\big|\!\big|U_s^{k,i,N,(1)}(\cdot)\big|\!\big|\!\big|_s^{(\mathbb{F}^{k,(1,\dots,N)},\overline{X}^{k,i})}\Big)^2\,\mathrm{d}C_s^k \\
&\qquad + \int_0^\infty (a_s^k)^2 \frac{1}{N}\sum_{m=1}^N |Y_s^{k,m,N,(1)}|^2 + \frac{\big|f^k(s,0,0,0,\delta_0)\big|^2}{(\alpha^k)^2}\,\mathrm{d}C_s^k \\
&\leq 2\int_0^\infty |Y_s^{k,i,N,(1)}|^2 + \frac{1}{N}\sum_{m=1}^N |Y_s^{k,m,N,(1)}|^2\,\mathrm{d}A_s^k \\
&\qquad + 2\int_0^\infty \mathrm{d}\mathrm{Tr}\Big[\Big\langle Z^{k,i,N,(1)}\cdot X^{k,i,\circ} + U^{k,i,N,(1)} \star \widetilde{\mu}^{k,i,\natural}\Big\rangle_s\Big] + 2G^{k,i,N,(0)} \\
&\leq 4\overline{A}\sup_{s\in\mathbb{R}_+}\left\{\left|\mathbb{E}\left[\xi^{k,i,N}\middle|\mathcal{F}_s^{k,(1,\dots,N)}\right]\right|^2\right\} + \frac{4\overline{A}}{N}\sum_{m=1}^N \sup_{s\in\mathbb{R}_+}\left\{\left|\mathbb{E}\left[\xi^{k,m,N}\middle|\mathcal{F}_s^{k,(1,\dots,N)}\right]\right|^2\right\} \\
&\qquad + 8\overline{A}\sup_{s\in\mathbb{R}_+}\left\{\left|\mathbb{E}\left[\int_s^\infty f^k(t,0,0,0,\delta_0)\,\mathrm{d}C_t^k\middle|\mathcal{F}_s^{k,(1,\dots,N)}\right]\right|^2\right\} + 2\,\mathrm{Tr}\Big[\langle \widetilde{M}^{k,i,N,(0)}\rangle_\infty\Big] + 2\,G^{k,i,N,(0)},
\end{aligned}
$$

with $\widetilde{M}^{k,i,N,(0)}_{\cdot}$ defined to be the martingale

$$Y^{k,i,N,(1)}_{\cdot} + \int_0^{\cdot} f^k\left(s, Y^{k,i,N,(0)}_s, Z^{k,i,N,(0)}_s c^k_s, \Gamma^{k,(1,\ldots,N)}(U^{k,i,N,(0)})_s, L^N(\mathbf{Y}^{k,N,(0)}_s)\right) \mathrm{d}C^k_s.$$

Hence, by the information provided from the previous step, for every $i \in \{1, ..., N\}$, we can conclude as before in the McKean–Vlasov case.

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Appendix A

A.1 Remainder of the proof of Proposition 2.9.2

We define for, every $\gamma \in (0,\beta)$ the functions

$$g_1(\gamma) := \frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma},$$
$$g_2(\gamma) := \frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}.$$

We have

$$\begin{aligned} g_1(\gamma) &= \frac{9}{\beta} + \left(\frac{1}{\gamma}+\Phi\right)\left(8 + 9\frac{\beta\gamma}{\beta-\gamma}\left(\frac{1}{\gamma}+\Phi\right)\right) \\ &= \frac{9}{\beta} + \left(\frac{1}{\gamma}+\Phi\right)\left(8 + 9\frac{\left(\frac{1}{\gamma}+\Phi\right)}{\left(\frac{1}{\gamma}+\Phi\right)-\left(\frac{1}{\beta}+\Phi\right)}\right). \end{aligned}$$

Note that $\gamma \in (0,\beta) \iff \frac{1}{\gamma}+\Phi \in \left(\frac{1}{\beta}+\Phi, \infty\right)$. So, we let $\frac{1}{\gamma}+\Phi := \lambda\left(\frac{1}{\beta}+\Phi\right)$, for $\lambda \in (1,\infty)$ and we only need to find the minimun of the function

$$\tilde{g}_1(\lambda) := \frac{9}{\beta} + \left(\frac{1}{\beta}+\Phi\right)\left(8\lambda + 9\frac{\lambda^2}{\lambda-1}\right), \quad \lambda \in (1,\infty).$$

Trivially, we have that

$$\lim_{\lambda\to 1^+}\left(8\lambda + 9\frac{\lambda^2}{\lambda-1}\right) = \lim_{\lambda\to\infty}\left(8\lambda + 9\frac{\lambda^2}{\lambda-1}\right) = \infty.$$

Hence, we will calculate the critical points of the function

$$h(\lambda) := 8\lambda + 9\frac{\lambda^2}{\lambda-1}, \quad \lambda \in (1,\infty).$$

$h'(\lambda) = 0 \iff 8 + 9\frac{\lambda^2-2\lambda}{(\lambda-1)^2} = 0 \iff 8(\lambda-1)^2 + 9(\lambda^2 - 2\lambda) = 0 \iff 17(\lambda-1)^2 = 9 \overset{\lambda>1}{\Longrightarrow} \lambda = \frac{3}{\sqrt{17}} + 1.$
Because

$$h\left(\frac{3}{\sqrt{17}} + 1\right) = \left(8 + \left(3 + \sqrt{17}\right)^2\right)\frac{3}{\sqrt{17}} + 8 = \left(8 + 9 + 17 + 6\sqrt{17}\right)\frac{3}{\sqrt{17}} + 8 = 6\sqrt{17} + 26,$$

we can conclude that

$$M_\star^\Phi(\beta) = \frac{9}{\beta} + \left(\frac{1}{\beta} + \Phi\right)\left(6\sqrt{17} + 26\right) = \frac{6\sqrt{17} + 35}{\beta} + \left(6\sqrt{17} + 26\right)\Phi. \tag{A.1}$$

Similarly, we have

$$\begin{aligned} g_2(\gamma) &= \frac{9}{\beta} + 8\left(\frac{1}{\gamma} + \Phi\right) + 2\frac{\left(\frac{1}{\beta} + \Phi\right) - \Phi}{\left(\frac{1}{\gamma} + \Phi\right) - \left(\frac{1}{\beta} + \Phi\right)}\left(\frac{1}{\gamma} + \Phi\right)^2 + 9\frac{\left(\frac{1}{\gamma} + \Phi\right)^2}{\left(\frac{1}{\gamma} + \Phi\right) - \left(\frac{1}{\beta} + \Phi\right)} \\ &= \frac{9}{\beta} + 8\left(\frac{1}{\gamma} + \Phi\right) + \frac{\left(\frac{1}{\gamma} + \Phi\right)^2}{\left(\frac{1}{\gamma} + \Phi\right) - \left(\frac{1}{\beta} + \Phi\right)}\left(\frac{2}{\beta} + 9\right). \end{aligned}$$

Same as before, we let $\frac{1}{\gamma} + \Phi := \lambda\left(\frac{1}{\beta} + \Phi\right)$, for $\lambda \in (1, \infty)$ and we only need to find the minimun of the function

$$\widetilde{g}_2(\lambda) := \frac{9}{\beta} + 8\lambda\left(\frac{1}{\beta} + \Phi\right) + \frac{\lambda^2}{\lambda - 1}\left(\frac{1}{\beta} + \Phi\right)\left(\frac{2}{\beta} + 9\right), \quad \lambda \in (1, \infty).$$

We have

$$\widetilde{g}_2(\lambda) = \frac{9}{\beta} + \left(\frac{1}{\beta} + \Phi\right)\left(8\lambda + \frac{\lambda^2}{\lambda - 1}\left(\frac{2}{\beta} + 9\right)\right)$$

and

$$\begin{aligned} \widetilde{g}_2'(\lambda) = 0 &\iff 8 + \left(\frac{2}{\beta} + 9\right)\frac{\lambda^2 - 2\lambda}{(\lambda - 1)^2} = 0 \\ &\iff \left(\frac{2}{\beta} + 17\right)(\lambda - 1)^2 = \frac{2}{\beta} + 9 \\ &\overset{\lambda>1}{\Longrightarrow} \lambda = \frac{\sqrt{\frac{2}{\beta} + 9}}{\sqrt{\frac{2}{\beta} + 17}} + 1. \end{aligned}$$

Finally, we have

$$
\begin{aligned}
M^{\Phi}(\beta) &= \frac{9}{\beta} + \left(\frac{1}{\beta}+\Phi\right)\left(\left(8+\left(\sqrt{\frac{2}{\beta}+9}+\sqrt{\frac{2}{\beta}+17}\right)^2\right)\left(\frac{\sqrt{\frac{2}{\beta}+9}}{\sqrt{\frac{2}{\beta}+17}}\right)+8\right)\\
&= \frac{9}{\beta} + \left(\frac{1}{\beta}+\Phi\right)\left(\left(2\left(\frac{2}{\beta}+17\right)+2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17}\right)\left(\frac{\sqrt{\frac{2}{\beta}+9}}{\sqrt{\frac{2}{\beta}+17}}\right)+8\right)\\
&= \frac{9}{\beta} + \left(\frac{1}{\beta}+\Phi\right)\left(2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17}+\frac{4}{\beta}+26\right)\\
&= \frac{2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17}+\frac{4}{\beta}+35}{\beta} + \left(2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17}+\frac{4}{\beta}+26\right)\Phi. \qquad \text{(A.2)}
\end{aligned}
$$

A.2 Auxiliary results

A.2.1 Construction of a space satisfying (H1).

In this subsection we will show that there exists a space satisfying **(H1)**. Since we discuss about a sequence of independent identically distributed processes, one naturally expects to construct a countable product space, den. by $(\Omega, \mathcal{G}, \mathbb{P})$, based on a prototype probability space $(\Omega^1, \mathcal{G}^1, \mathbb{P}^1)$. On this prototype probability space we will construct the pair of martingales $\overline{X}^1$ which satisfies the desired properties. As one may expect, this is not a condition that is trivially satisfied. Hence, we are led to consider specific cases, which nevertheless demonstrate the generality of the framework we are using. We remind that a Lévy processes is square–integrable if and only if for the corresponding Lévy measure ν we have up to evanescence that

$$\int_{\mathbb{R}^d} \mathbb{1}_{[1,\infty)}(|x|)|x|^2 \mathrm{d}\nu(x) < \infty.$$

Example A.2.1. *Let $(\Omega^1, \mathcal{G}^1, \mathbb{P}^1)$ be the probability space that supports two independent square–integrable Lévy processes, say $(X^{1,\circ}, X^{1,\natural})$,which are martingales with respect to their natural filtrations. We further assume that $X^{1,\natural}$ is purely discontinuous. The independence of the Lévy processes implies the property of having no common jumps; see [47, Proposition 5.3]. Hence the desired condition $M_{\mu^{X^{1,\natural}}}[\Delta X^{1,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^1}] = 0$ is trivially satisfied.*

For completeness, we mention that [47, Proposition 5.3] refers to Lévy processes with no Gaussian part, but this property remains valid to the case we describe.

Example A.2.2. *Let $(\Omega^1, \mathcal{G}^1, \mathbb{P}^1)$ be a probability space that supports a p-dimensional, purely discontinuous square–integrable* Lévy *process $X^{1,\natural}$, for the construction see [4, Theorem 4.6.17]. By taking product if necessary, we assume that our probability space supports also a sequence of independent random variables, $\{h^k\}_{k\in\mathbb{N}} \subseteq \mathbb{L}^2(\mathcal{G}^1;\mathbb{R}^n)$, such that the $\sigma-$algebras $\bigvee_{t\in\mathbb{R}_+}\sigma\left(X_t^{1,\circ}\right)$ and $\bigvee_{k=1}^{\infty}\sigma\left(h^k\right)$ are independent and*

$$\mathbb{E}[h^k] = 0, \forall k \in \mathbb{N}, \quad \sum_{k=1}^{\infty}\mathbb{E}\left[|h^k|^2\right] < \infty. \qquad \text{(A.3)}$$

Furthermore, let $\{t_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ *be a family of deterministic times indexed in increasing order. We define*

$$X^{1,\circ}(\omega^1,t) := \sum_{k=1}^{\infty} h^k(\omega^1)\mathbb{1}_{[t_k,\infty)}(t).$$

Then we have that $(X^{1,\circ}, X^{1,\natural}) \in \mathcal{H}^2(\mathbb{F}^1;\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^1;\mathbb{R}^n)$*, we remind that* $\mathbb{F}^1$ *is the usual augmentation of the natural filtration of the pair. The martingale property for* $X^{1,\circ}$ *comes from* (A.3). *Finally, because* Lévy *processes are quasi-left-continuous, they jump only at totally inaccessible times. Hence* $X^{1,\circ}$ *and* $X^{1,\natural}$ *have no common jumps and again the condition* $M_{\mu^{X^{1,\natural}}}[\Delta X^{1,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^1}] = 0$ *is trivially satisfied.*

Example A.2.3. *Two independent random walks defined on the same grid. Then, if we denote as in example A.2.2 by* $\{h^k\}_{k\in\mathbb{N}}$ *the jumps of* $X^{1,o}$*,* $\{\widetilde{h}^k\}_{k\in\mathbb{N}}$ *the jumps of* $X^{1,\natural}$ *and* $\{t_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ *the grid, by reducing the general case to the one described from a single deterministic time we have the desired property if the jumps are 0 on average. That is because* $\mathcal{F}^1_{t_k-} = \left(\bigvee_{\{m\in\mathbb{N}:t_m<t_k\}} \sigma\left(h^m, \widetilde{h}^m\right)\right) \bigvee \mathcal{N}^1$*, here* $\mathcal{N}^1$ *is the* $\sigma-$*algebra generated from the null sets under* $\mathbb{P}^1$*. Note that* $X^{1,\natural} \in \mathcal{H}^{2,d}(\mathbb{F}^1;\mathbb{R}^n)$ *due to the fact that has finite variation and [21, 6.23 Theorem 3)].*

In the above examples key feature was the concept of independence. We now provide an example which illustrates that independence is not necessary.

Example A.2.4. *Let* $(\Omega^1,\mathcal{G}^1,\mathbb{P}^1)$ *be a probability space that supports* $h^1 \in \mathbb{L}^2(\mathcal{G}^1;\mathbb{R}^p)$ *and* $h^2 \in \mathbb{L}^2(\mathcal{G}^1;\mathbb{R}^n)$ *such that*

$$\mathbb{E}\Big[h^1\Big|\sigma(h^2)\Big] = 0 \quad \text{and} \quad \mathbb{E}\Big[h^2\Big|\sigma(h^1)\Big] = 0. \tag{A.4}$$

The relation that is expressed through (A.4) *is a generalization of independence, when the random variables have zero expectation. Let* $t_1, t_2 \in \mathbb{R}_+$*, we define*

$$X^{1,\circ}(\omega^1,t) := \begin{cases} 0, & if\, t < t_1 \\ h^1(\omega^1), & if\, t \geq t_1 \end{cases} \quad \text{and} \quad X^{1,\natural}(\omega^1,t) := \begin{cases} 0, & if\, t < t_2 \\ h^2(\omega^1), & if\, t \geq t_2. \end{cases}$$

Then we have that $(X^{1,\circ}, X^{1,\natural}) \in \mathcal{H}^2(\mathbb{F}^1;\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^1;\mathbb{R}^n)$*, we remind that* $\mathbb{F}^1$ *is the usual augmentation of the natural filtration of the pair. The martingale property comes from* (A.4). *To see that* $X^{1,\natural} \in \mathcal{H}^{2,d}(\mathbb{F}^1;\mathbb{R}^n)$ *note that* $X^{1,\natural}$ *has finite variation and use [21, 6.23 Theorem 3)]. Finally, from* (A.4) *and [12, Lemma 13.3.15 (ii)] we have* $M_{\mu^{X^{\natural,1}}}[\Delta X^{1,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^1}] = 0$.

In view of the presented examples, we may assume a product space $\Omega^1 \times \mathbb{R}_+$ where $(\Omega^1,\mathcal{G}^1,\mathbb{P}^1)$ is a probability space. Let $\mathbb{F}^1$ be the usual augmentation of the natural filtration of a pair $\overline{X}^1 := (X^{1,\circ}, X^{1,\natural}) \in \mathcal{H}^2(\mathbb{F}^1;\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^1;\mathbb{R}^n)$ (defined on the product space $\Omega^1 \times \mathbb{R}_+$) , with $M_{\mu^{X^{1,\natural}}}[\Delta X^{1,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^1}] = 0$, where $\mu^{X^{1,\natural}}$ is the random measure generated by the jumps of $X^{1,\natural}$. Additionally, let a random variable $\xi^1 \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^1_T;\mathbb{R}^d)$ for a deterministic time T, which will be assumed fixed from now on.

Then, let $\{(\Omega^i,\mathcal{G}^i,\mathbb{F}^i,\mathbb{P}^i)\}_{i\in\mathbb{N}}$ be copies of the stochastic base $(\Omega^1,\mathcal{G}^1,\mathbb{F}^1,\mathbb{P}^1)$, $\{\overline{X}^i := (X^{i,\circ}, X^{i,\natural})\}_{i\in\mathbb{N}}$ the corresponding copies of $(X^{1,\circ}, X^{1,\natural})$ and $\{\xi^i\}_{i\in\mathbb{N}}$ the corresponding copies of ξ^1. We define the product probability space $\left(\prod_{i=1}^{\infty}\Omega^i, \bigotimes_{i=1}^{\infty}\mathcal{G}^i, \bigotimes_{i=1}^{\infty}\mathbb{P}^i\right)$. We denote by $\widehat{\mathbb{F}}^i$ the augmented natural filtration of the pair

$\overline{X}^i$ in the product space $(\prod_{i=1}^{\infty}\Omega^i)\times\mathbb{R}_+$ under the probability measure $\mathbb{P}:=\bigotimes_{i=1}^{\infty}\mathbb{P}^i$. Because the pair $\overline{X}^i$ depends only on ω^i we have

$$\widehat{\mathbb{F}}^i=\Big(\mathbb{F}^i\times\prod_{m\in\mathbb{N}\setminus\{i\}}^{\infty}\Omega^m\Big)\bigvee\mathcal{N},\tag{A.5}$$

where $\mathcal{N}$ is the σ–algebra generated from the subsets of the null sets under the measure $\mathbb{P}$. Using the methods of corollary A.2.7 we get that $\overline{X}^i\in\mathcal{H}^2(\widehat{\mathbb{F}}^i;\mathbb{R}^p)\times\mathcal{H}^{2,d}(\widehat{\mathbb{F}}^i;\mathbb{R}^n)$. To prove that $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\widehat{\mathbb{F}}^i}]=0$ we work as in the end of the proof of lemma A.2.9. So, let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of disjoint $\mathbb{F}^i-$stopping times that exhausts the jumps of $X^{i,\natural}$ and also satisfies the assumptions of [12, Lemma 13.3.15 (*ii*)]; it is known that such a sequence always exists for every $\mathbb{F}^i-$adapted, càdlàg process, *e.g.*, see [29, Definition I.1.30, Proposition I.1.32]. Of course, the aforementioned stopping times when viewed in the product space depend only on ω^i. Moreover, $X^{i,\circ}$ is also an $\mathbb{F}^i-$martingale. Hence, $\Delta X^{i,\circ}_{\tau_k}$ will be measurable with respect to $\mathcal{F}^i_\infty$, for every $k\in\mathbb{N}$. If we denote by $\widehat{\mathcal{F}}^i_{\tau_k-}$ the $\sigma-$algebra of events occurring strictly before the stopping time τ_k produced under the filtration $\widehat{\mathbb{F}}^i$ and with $\mathcal{F}^i_{\tau_k-}$ the respective $\sigma-$algebra under the filtration $\mathbb{F}^i$, then from (A.5) we have

$$\widehat{\mathcal{F}}^i_{\tau_k-}=\Big(\mathcal{F}^i_{\tau_k-}\times\prod_{m\in\mathbb{N}\setminus\{i\}}^{\infty}\Omega^m\Big)\bigvee\mathcal{N}$$

and

$$\sigma\Big(\Delta X^{i,\natural}_{\tau_k};\prod_{m=1}^{\infty}\Omega^m\Big)=\sigma(\Delta X^{i,\natural}_{\tau_k};\Omega^i)\times\prod_{m\in\mathbb{N}\setminus\{i\}}^{\infty}\Omega^m.$$

Then, because $\mathcal{N}$ is independent from any other sub σ–algebra of $\overline{\bigotimes_{i=1}^{\infty}\mathcal{G}^i}$, where we denoted by $\overline{\bigotimes_{i=1}^{\infty}\mathcal{G}^i}$ the completion under the measure $\mathbb{P}$, we get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\Delta X^{i,\circ}_{\tau_k}\Big|\widehat{\mathcal{F}}^i_{\tau_k-}\bigvee\sigma\Big(\Delta X^{i,\natural}_{\tau_k};\prod_{m=1}^{\infty}\Omega^m\Big)\Big]\\
&=\mathbb{E}^{\mathbb{P}}\Big[\Delta X^{i,\circ}_{\tau_k}\Big|\Big(\Big(\mathcal{F}^i_{\tau_k-}\bigvee\sigma(\Delta X^{i,\natural}_{\tau_k};\Omega^i)\Big)\times\prod_{m\in\mathbb{N}\setminus\{i\}}^{\infty}\Omega^m\Big)\bigvee\mathcal{N}\Big]\\
&=\mathbb{E}^{\mathbb{P}}\left[\Delta X^{i,\circ}_{\tau_k}\middle|\Big(\mathcal{F}^i_{\tau_k-}\bigvee\sigma(\Delta X^{i,\natural}_{\tau_k};\Omega^i)\Big)\times\prod_{m\in\mathbb{N}\setminus\{i\}}^{\infty}\Omega^m\right]\\
&=\mathbb{E}^{\mathbb{P}^i}\left[\Delta X^{i,\circ}_{\tau_k}\middle|\mathcal{F}^i_{\tau_k-}\bigvee\sigma(\Delta X^{i,\natural}_{\tau_k};\Omega^i)\right](\omega^i)\\
&=0,
\end{aligned}$$

where we used [51, Section 9.7, Property (k) on p. 88] in the second equality and [12, Lemma 13.3.15 (*ii*)] in the last equality.

Lastly, note from (A.5) that the sequence $\{\widehat{\mathbb{F}}^i\}_{i\in\mathbb{N}}$ consists of independent filtrations of $\Big(\prod_{i=1}^{\infty}\Omega^i,\overline{\bigotimes_{i=1}^{\infty}\mathcal{G}^i},\mathbb{P}\Big)$, where we abused notation and denoted the extended measure again with $\mathbb{P}$.

Next, for every $i \in \mathbb{N}$ define the bimeasurable bijections
$g^i : \left(\prod_{i=1}^{\infty} \Omega^i, \overline{\bigotimes_{i=1}^{\infty} \mathcal{G}^i}\right) \longrightarrow \left(\prod_{i=1}^{\infty} \Omega^i, \overline{\bigotimes_{i=1}^{\infty} \mathcal{G}^i}\right)$ by

$$g^i((\omega^1, \omega^2, \dots, \omega^{i-1}, \omega^i, \omega^{i+1}, \dots)) := (\omega^i, \omega^2, \dots, \omega^{i-1}, \omega^1, \omega^{i+1}, \dots),$$

i.e., the function g^i switches the places of ω^i and ω^1. It is easy to check the following properties of the sequence $\{g^i\}_{i\in\mathbb{N}}$:

(i) For every $i \in \mathbb{N}$, we have $g^i \circ g^i = \mathrm{Id}_{\prod_{i=1}^{\infty} \Omega^i}$, where $\mathrm{Id}_{\prod_{i=1}^{\infty} \Omega^i}$ is the identity function.

(ii) For every $i \in \mathbb{N}$ and for every $t \in \mathbb{R}_+$, we have $g^i\left(\widehat{\mathcal{F}}_t^1\right)^{-1} = \widehat{\mathcal{F}}_t^i$,

(iii) For every $i \in \mathbb{N}$ and for every $A \in \overline{\bigotimes_{i=1}^{\infty} \mathcal{G}^i}$, we have $\mathbb{P}(A) = \mathbb{P}(g^i(A)^{-1})$.

(iv) For every $i \in \mathbb{N}$, we have $\overline{X}^i := \overline{X}^1 \circ (g^i, \mathrm{Id}_{\mathbb{R}_+})$ and $\xi^i := \xi^1 \circ g^i$.

Now, for every $i \in \mathbb{N}$ we have that

$$\mathcal{P}^{\widehat{\mathbb{F}}^i} = \sigma\left(\left\{A_t \times (t, \infty) : t \in \mathbb{R}_+, A_t \in \widehat{\mathcal{F}}_t^i\right\} \cup \left\{A_0 \times \{0\} : A_0 \in \widehat{\mathcal{F}}_0^i\right\}\right).$$

So, from (A.5) and (iii) we have that

$$Z \in \mathcal{P}^{\widehat{\mathbb{F}}^i} \iff Z \circ (g^i, \mathrm{Id}_{\mathbb{R}_+}) \in \mathcal{P}^{\widehat{\mathbb{F}}^1}$$

and

$$M \in \mathcal{H}^2(\widehat{\mathbb{F}}^i; \mathbb{R}^d) \iff M \circ (g^i, \mathrm{Id}_{\mathbb{R}_+}) \in \mathcal{H}^2(\widehat{\mathbb{F}}^1; \mathbb{R}^d).$$

From the above properties one can show that

$$\langle X^{i,\circ}\rangle^{\widehat{\mathbb{F}}^i} = \langle X^{1,\circ}\rangle^{\widehat{\mathbb{F}}^1} \circ (g^i, \mathrm{Id}_{\mathbb{R}_+}) \quad \text{and} \quad |I|^2 * \nu^{(\widehat{\mathbb{F}}^i, X^{i,\natural})} = |I|^2 * \nu^{(\widehat{\mathbb{F}}^1, X^{1,\natural})} \circ (g^i, \mathrm{Id}_{\mathbb{R}_+}).$$

Hence, from (2.9) we have $C^{(\widehat{\mathbb{F}}^i, \overline{X}^i)} = C^{(\widehat{\mathbb{F}}^1, \overline{X}^1)} \circ (g^i, \mathrm{Id}_{\mathbb{R}_+})$ and $b^i = b^1 \circ (g^i, \mathrm{Id}_{\mathbb{R}_+})$.

Assuming **(H3)** – **(H9)**, note that $\mathcal{E}\left(\hat{\beta} A^{\overline{X}^i}\right)_T = \mathcal{E}\left(\hat{\beta} A^{\overline{X}^1}\right) \circ (g^i, T)$, from the existence and uniqueness theorem 3.1.3 and theorem 3.1.8 due to symmetry we have that for all $i \in \mathbb{N}$

$$Y^i = Y^1 \circ (g^i, \mathrm{Id}_{\mathbb{R}_+}).$$

A.2.2 Technicalities

In this section we will present some useful technical lemmata and their proofs. To this end, let us fix $\mathbb{G}$ and $\mathbb{H}$ be filtrations on the probability space $(\Omega, \mathcal{G}^\circ, \mathbb{P})$ such that $\mathbb{G}$ is immersed in $\mathbb{H}$ and both satisfy the usual conditions.

Remark A.2.5. *Special cases of the below results are included in [18]. Although these are sufficient for our purposes, we present here the more general results for completeness.*

Lemma A.2.6. *Given* $U \in \widetilde{\mathcal{P}}^{\mathbb{G}}_{+}$ *and* $g \in (\mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+))_+$, *for every pair* $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$, *for* $C^{(\mathbb{H},\overline{X})}$ *as defined via* (2.9) *and for* $K^{(\mathbb{G},\overline{X})}$, *as well as for* $K^{(\mathbb{H},\overline{X})}$, *as defined via* (2.10), *we have that*

(i) *for* $\mathbb{P} \otimes C^{(\mathbb{H},\overline{X})} -$ *a.e.* $(\omega, t) \in \Omega \times \mathbb{R}_+$

$$\int_{\mathbb{R}^n} U(\omega,t,x)\, K^{(\mathbb{G},\overline{X})}(\omega,t,\mathrm{d}x) = \int_{\mathbb{R}^n} U(\omega,t,x)\, K^{(\mathbb{H},\overline{X})}(\omega,t,\mathrm{d}x).$$

(ii)

$$\int_{\mathbb{R}_+\times\mathbb{R}^n} g(\omega,t)\, U(\omega,t,x)\, \nu^{(\mathbb{G},X^\natural)}(\omega,\mathrm{d}t,\mathrm{d}x) = \int_{\mathbb{R}_+\times\mathbb{R}^n} g(\omega,t)\, U(\omega,t,x)\, \nu^{(\mathbb{H},X^\natural)}(\omega,\mathrm{d}t,\mathrm{d}x),\ \mathbb{P}-a.e.$$

(iii) $U * \nu^{(\mathbb{G},X^\natural)} = U * \nu^{(\mathbb{H},X^\natural)}$, *up to evanescence.*

(iv) $\widehat{U}^{(\mathbb{G},X^\natural)} = \widehat{U}^{(\mathbb{H},X^\natural)}$, *up to evanescence; see* (2.6) *for their definition.*

(v) $\zeta^{(\mathbb{G},X^\natural)} = \zeta^{(\mathbb{H},X^\natural)}$, *up to evanescence; see* (2.7) *for their definition.*

Proof. Let us fix $\overline{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^2(\mathbb{G};\mathbb{R}^n)$. We remind the reader that the immersion of the filtrations implies that $C^{(\mathbb{G},\overline{X})} = C^{(\mathbb{H},\overline{X})}$; see remark 2.2.1. Therefore, we may simplify the notation and simply write C for $C^{(\mathbb{H},\overline{X})}$. We proceed to prove our claims:

(i) We consider $\{B_m\}_{m\in\mathbb{N}} \subseteq \widetilde{\mathcal{P}}^{\mathbb{G}}$ to be a partition of $\Omega \times \mathbb{R}_+ \times \mathbb{R}^n$ that makes $M_{\mu^{X^\natural}}$ $\sigma-$integrable with respect to $\widetilde{\mathcal{P}}^{\mathbb{G}}$. Then, we define the sequence $\{A_m\}_{m\in\mathbb{N}}$ as $A_m := (\bigcup_{k=1}^m B_k) \bigcap \{|U| \leq m\}$; of course $\mathbb{1}_{A_m} \nearrow 1$ for every (ω,t,x). For every $m \in \mathbb{N}$, we have that $(U\mathbb{1}_{A_m}) * \mu^{X^\natural} - (U\mathbb{1}_{A_m}) * \nu^{(\mathbb{G},X^\natural)}$ is a $\mathbb{G}-$martingale of finite variation, while $(U\mathbb{1}_{A_m}) * \mu^{X^\natural} - (U\mathbb{1}_{A_m}) * \nu^{(\mathbb{H},X^\natural)}$ is an $\mathbb{H}-$martingale of finite variation. In view of the immersion hypothesis, *i.e.*, every $\mathbb{G}-$martingale is also an $\mathbb{H}-$martingale, $(U\mathbb{1}_{A_m}) * \nu^{(\mathbb{G},X^\natural)} - (U\mathbb{1}_{A_m}) * \nu^{(\mathbb{H},X^\natural)}$ is a predictable, $\mathbb{H}-$martingale of finite variation starting at 0. In other words, it is 0 up to indistinguishability for every $m \in \mathbb{N}$, which equivalently reads

$$(U\mathbb{1}_{A_m}) * \nu^{(\mathbb{G},X^\natural)} = (U\mathbb{1}_{A_m}) * \nu^{(\mathbb{H},X^\natural)} \tag{A.6}$$

up to indistinguishability for every $m \in \mathbb{N}$. By (2.10) we get

$$\int_{\mathbb{R}_+} \mathbb{1}_{[0,s]}(t) \int_{\mathbb{R}^n} U(\omega,t,x)\mathbb{1}_{A_m}(\omega,t,x)\, K^{(\mathbb{G},\overline{X})}(\omega,t,\mathrm{d}x)\mathrm{d}C_t$$
$$= \int_{\mathbb{R}_+} \mathbb{1}_{[0,s]}(t) \int_{\mathbb{R}^n} U(\omega,t,x)\mathbb{1}_{A_m}(\omega,t,x)\, K^{(\mathbb{H},\overline{X})}(\omega,t,\mathrm{d}x)\mathrm{d}C_t,$$

up to evanescence, for every $s \in \mathbb{Q}_+$ and $m \in \mathbb{N}$. Recalling that $\{[0,s]\}_{s\in\mathbb{Q}_+}$ is a π-system whose $\lambda-$system produces $\mathcal{B}(\mathbb{R}_+)$, by an application of Dynkin's lemma we can replace $[0,s]$ in the above equality with any set $D \in \mathcal{B}(\mathbb{R}_+)$. By the monotone convergence theorem, in respect to the sequence $\{\mathbb{1}_{A_m}\}_{m\in\mathbb{N}}$, we get what we want.

(ii) Immediate from (i) and the disintegration formula (2.10).

(iii) Immediate from (A.6) by means of monotone convergence.

(iv) Immediate from (A.6) because from [21, 5.27 Theorem, 2) and 11.11 Theorem] we have by monotone convergence

$$\widehat{U}^{(\mathbb{G},X^{\natural})} = \lim_{m\to\infty} \Delta\left((U\mathbb{1}_{A_m}) * \nu^{(\mathbb{G},X^{\natural})}\right)$$

and

$$\widehat{U}^{(\mathbb{H},X^{\natural})} = \lim_{m\to\infty} \Delta\left((U\mathbb{1}_{A_m}) * \nu^{(\mathbb{H},X^{\natural})}\right).$$

(v) This is a direct consequence of (iv) for $U = 1$.

□

Corollary A.2.7. *If $Z \in \mathbb{H}^2(\mathbb{G}, X^\circ; \mathbb{R}^{d\times p})$, then $(Z\cdot X^\circ)^{\mathbb{G}} = (Z\cdot X^\circ)^{\mathbb{H}}$, up to evanescence. Moreover, if $U \in G_2(\mathbb{G}, \mu^{X^\natural})$, then $U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} = U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}$, up to evanescence. In particular, for $\mathbb{R}^n \ni x \xrightarrow{\mathrm{Id}} x \in \mathbb{R}^n$ one gets $X^\natural = \mathrm{Id} \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} = \mathrm{Id} \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}$.*

Proof. For the Itō stochastic integral, it is immediate from the definition of the integrals, see [29, Definition I.2.1] and the fact that $C^{(\mathbb{G},\overline{X})} = C^{(\mathbb{H},\overline{X})}$.

For the stochastic integral with respect to the integer–valued measure $\mu^{X^\natural}$, let $U \in G_2(\mathbb{G}, \mu^{X^\natural})$. Then, we have

$$\begin{aligned}
\|U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\|^2_{\mathcal{H}^2(\mathbb{G};\mathbb{R}^d)} &= \mathbb{E}\left[\left|U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right|^2_\infty\right] \\
&= \|U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)} \\
&\overset{(2.1)}{=} \left\|\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},c)}\right\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)} + \left\|\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},d)}\right\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)}.
\end{aligned}$$

Note that we denoted with

$$\left(\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},c)}, \left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},d)}\right)$$

the unique pair of $\mathcal{H}^{2,c}(\mathbb{H};\mathbb{R}^d) \times \mathcal{H}^{2,d}(\mathbb{H};\mathbb{R}^d)$ such that

$$U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} = \left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},c)} + \left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},d)}.$$

From [21, 6.23 Theorem] we have

$$\|U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\|^2_{\mathcal{H}^2(\mathbb{G};\mathbb{R}^d)} = \mathbb{E}\left[\sum_{t>0}\left|\Delta\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)_t\right|^2\right] = \left\|\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},d)}\right\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)}.$$

Hence,

$$\left\|\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\right)^{(\mathbb{H},c)}\right\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)} = 0,$$

and $U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} \in \mathcal{H}^{2,d}(\mathbb{H};\mathbb{R}^d)$.

Next, from lemma A.2.6.(iv)

$$\begin{aligned}\Delta\Big(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)}\Big)_t &= U(\omega,t,\Delta X^\natural_t)\mathbb{1}_{\{\Delta X^\natural \neq 0\}} - \widehat{U}^{(\mathbb{G},X^\natural)}_t \\ &= U(\omega,t,\Delta X^\natural_t)\mathbb{1}_{\{\Delta X^\natural \neq 0\}} - \widehat{U}^{(\mathbb{H},X^\natural)}_t \\ &= \Delta\Big(U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}\Big)_t.\end{aligned}$$

From the above, because $U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} - U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)} \in \mathcal{H}^{2,d}(\mathbb{H};\mathbb{R}^d)$, again from [21, 6.23 Theorem] we conclude that

$$\|U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} - U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}\|^2_{\mathcal{H}^2(\mathbb{H};\mathbb{R}^d)} = \mathbb{E}\left[\sum_{t>0}\left|\Delta\left(U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} - U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}\right)_t\right|^2\right] = 0,$$

and $U \star \widetilde{\mu}^{(\mathbb{G},X^\natural)} = U \star \widetilde{\mu}^{(\mathbb{H},X^\natural)}$, up to indistinguishability.

For $U = \mathrm{Id}$ we get the last claim. □

A.2.3 Conservation of solutions under immersion of filtrations

In this subsection we will identify the solutions of the McKean–Vlasov BSDE (3.2) when we fix all the elements of the standard data except for the filtrations. lemma A.2.9 makes precise the previous sentence.

We remind the reader the necessary notation and terminology. Let us fix $N \in \mathbb{N}$ and assume **(H1)**-**(H9)**. For each $i \in \{1,\dots,N\}$, the McKean–Vlasov BSDE (3.2) associated to the standard data $\left(\overline{X}^i, \mathbb{F}^i, \Theta, \Gamma, T, \xi^i, f\right)$ under $\hat{\beta}$ admits, by theorem 3.1.3, a unique solution, which will be denoted by (Y^i, Z^i, U^i, M^i). For later reference, we will say that $(\widetilde{\mathbf{Y}}^N, \widetilde{\mathbf{Z}}^N, \widetilde{\mathbf{U}}^N, \widetilde{\mathbf{M}}^N)$ is the solution of the first N McKean–Vlasov BSDEs, where we define

$$\widetilde{\mathbf{Y}}^N := (Y^1,\dots,Y^n),\ \widetilde{\mathbf{Z}}^N := (Z^1,\dots,Z^n),\ \widetilde{\mathbf{U}}^N := (U^1,\dots,U^n)$$
$$\text{and } \widetilde{\mathbf{M}}^N := (M^1,\dots,M^n).$$

Remark A.2.8. *Let $i \in \{1,...,N\}$. Under* **(H1)**, **(H3)** *and for $U \in \mathbb{H}^2(\mathbb{F}^i, X^{i,\natural}; \mathbb{R}^d)$ we have from lemma A.2.6 that*

$$\Gamma^{(\mathbb{F}^i,\overline{X}^i,\Theta^i)}(U) = \Gamma^{(\mathbb{F}^{1,\dots,N},\overline{X}^i,\Theta^i)}(U), \quad \mathbb{P}\otimes C^{\overline{X}^i} - a.e.$$

Lemma A.2.9 (**Conservation of solutions**). *Assume* **(H1)**-**(H9)** *and fix $N \in \mathbb{N}$ and $i \in \{1,\dots,N\}$. The unique solution of the McKean–Vlasov BSDE* (3.2) *associated to the standard data $\left(\overline{X}^i, \mathbb{F}^i, \Theta, \Gamma, T, \xi^i, f\right)$ under $\hat{\beta}$, is also the unique solution of the McKean–Vlasov BSDE* (3.2) *associated to the standard data $\left(\overline{X}^i, \mathbb{F}^{1,\dots,N}, \Theta, \Gamma, T, \xi^i, f\right)$ under $\hat{\beta}$.*

Proof. Let us fix $N \in \mathbb{N}$ and $i \in \{1,\dots,N\}$. We denote by (Y^i, Z^i, U^i, M^i) the solution of the McKean–Vlasov BSDE (3.2) associated to the standard data $\left(\overline{X}^i, \mathbb{F}^i, \Theta, \Gamma, T, \xi^i, f\right)$ under $\hat{\beta}$. From lemma A.2.6, corollary A.2.7, remark A.2.8 and theorem 3.1.3 we deduce that it will be enough to show that $M^i \in$

$\mathcal{H}^2(\overline{X}^{i^{\perp_{\mathbb{F}^{1,\dots,N}}}})$. From proposition 2.1.1 we will need to check that

$$\langle X^{i,\circ}, M^i\rangle^{\mathbb{F}^{1,\dots,N}} = 0 \text{ and } M_{\mu^{X^{i,\natural}}}[\Delta M^i|\widetilde{\mathcal{P}}^{\mathbb{F}^{1,\dots,N}}] = 0. \tag{A.7}$$

We remind the reader that $M^i \in \mathcal{H}^2(\overline{X}^{i^{\perp_{\mathbb{F}^i}}})$, *i.e.*, $\langle X^{i,\circ}, M^i\rangle^{\mathbb{F}^i} = 0$ and $M_{\mu^{X^{i,\natural}}}[\Delta M^i|\widetilde{\mathcal{P}}^{\mathbb{F}^i}] = 0$.

For the first requirement in (A.7), we have that $X^{i,\circ}M^i$ remains an $\mathbb{F}^{1,\dots,N}-$martingale, since $\mathbb{F}^i$ is immersed in $\mathbb{F}^{1,\dots,N}$. Hence, $\langle X^{i,\circ}, M^i\rangle^{\mathbb{F}^{1,\dots,N}} = 0$.

For the second requirement in (A.7), we are going to use [12, Lemma 13.3.15 (*ii*)]. The martingale $X^{i,\natural}$ is adapted under the filtration $\mathbb{F}^i$. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of disjoint $\mathbb{F}^i-$stopping times that exhausts the jumps of $X^{i,\natural}$ and also satisfies the assumptions of [12, Lemma 13.3.15 (*ii*)]; it is known that such a sequence always exists for every $\mathbb{F}^i-$adapted, càdlàg process, *e.g.*, see [29, Definition I.1.30, Proposition I.1.32]. Moreover, M^i is also an $\mathbb{F}^i-$martingale. Hence, $\Delta M^i_{\tau_k}$ will be measurable with respect to $\mathcal{F}^i_\infty$, for every $k \in \mathbb{N}$. If we denote by $\mathcal{F}^{1,\dots,N}_{\tau_k-}$ the $\sigma-$algebra of events occuring strictly before the stopping time τ_k produced under the filtration $\mathbb{F}^{1,\dots,N}$ and with $\mathcal{F}^i_{\tau_k-}$ the respective $\sigma-$algebra under the filtration $\mathbb{F}^i$, then we have

$$\begin{aligned}
&\mathcal{F}^{1,\dots,N}_{\tau_k-} \subseteq \mathcal{F}^i_{\tau_k-} \bigvee \left(\bigvee_{m\in\{1,\dots,N\}\setminus\{i\}} \mathcal{F}^m_\infty \right),\\
&\sigma(\Delta X^{i,\natural}_{\tau_k}) \subseteq \mathcal{F}^i_\infty.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\mathbb{E}\Big[\Delta M^i_{\tau_k}\Big|\mathcal{F}^{1,\dots,N}_{\tau_k-} \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\Big[\Delta M^i_{\tau_k}\Big|\mathcal{F}^i_{\tau_k-} \bigvee \Big(\bigvee_{m\in\{1,\dots,N\}\setminus\{i\}} \mathcal{F}^m_\infty\Big) \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\Big|\mathcal{F}^{1,\dots,N}_{\tau_k-} \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\Big[\Delta M^i_{\tau_k}\Big|\mathcal{F}^i_{\tau_k-} \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\Big|\mathcal{F}^{1,\dots,N}_{\tau_k-} \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\\
&= \mathbb{E}\Big[M_{\mu^{X^{i,\natural}}}\Big[\Delta M^i\Big|\widetilde{\mathcal{P}}^{\mathbb{F}^i}\Big](\tau_k, \Delta X^{i,\natural}_{\tau_k})\Big|\mathcal{F}^{1,\dots,N}_{\tau_k-} \bigvee \sigma(\Delta X^{i,\natural}_{\tau_k})\Big]\\
&= 0,
\end{aligned}$$

where we used the tower property in the first equality, [51, Section 9.7, Property (k) on p. 88] in the second equality, [12, Lemma 13.3.15 (*ii*)] in the second to last equality and we concluded in view of the known information $M_{\mu^{X^{i,\natural}}}[\Delta M^i|\widetilde{\mathcal{P}}^{\mathbb{F}^i}] = 0$. □

Lemma A.2.10. *Let* $i \in \{1,\dots,N\}$. *The process* $W^2_{2,\rho_{J^d_1}}\Big(L^N\Big(\widetilde{\boldsymbol{Y}}^N|_{[0,\cdot]}\Big), \mathcal{L}(Y^i|_{[0,\cdot]})\Big)$ *is càdlàg and adapted to the filtration* $\mathbb{F}^{1,\dots,N}$.

Proof. First of from remark 2.6.4 because $\rho_{J^d_1} \leq 1$ the Wasserstein distance of order 2 as a function $W_{2,\rho_{J^d_1}} : \mathcal{P}(\mathbb{D}^d) \times \mathcal{P}(\mathbb{D}^d) \to \mathbb{R}_+$ is continuous, if we supply $\mathcal{P}(\mathbb{D}^d)$ with the weak topology $\mathcal{T}$, as it metrizes it.

Alternatively, more generally one can use remark 2.6.3 to claim the measurability of the Wasserstein distance with respect to $\mathcal{B}_{\mathcal{T}}(\mathcal{P}(\mathbb{D}^d))$.

Then from (2.22) for the random measure $L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right)$ we have $\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right)\right)^{-1}(S) \in \mathcal{F}_s^{1,\dots,N}$, with $S = \left(I^f\right)^{-1}(A)$ for some A open set in the usual topology of $\mathbb{R}$ and $f \in C_b(\mathbb{D}^d)$. To see this note that

$$\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right)\right)^{-1}(S) = \left(\frac{1}{N}\sum_{m=1}^{N} Y^m|_{[0,s]}\right)^{-1}(A).$$

Hence, from (2.27), $L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right)$ is an $\left(\mathcal{F}_s^{1,\dots,N}/\mathcal{B}_{\mathcal{T}}(\mathcal{P}(\mathbb{D}^d))\right)$ −measurable function. Because $\mathcal{L}\left(Y^i|_{[0,s]}\right)$ is constant with respect to ω, by composition of the functions we get that $W^2_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s]}\right), \mathcal{L}\left(Y^i|_{[0,s]}\right)\right)$ is adapted. To show that it is càdlàg choose $t \in [0,\infty)$ and a decreasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $s_j \searrow t$. For every $j \in \mathbb{N}$ we have from the triangle inequality, (2.25) and (2.26) that

$$\begin{aligned}
&\left|W_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s_j]}\right), \mathcal{L}\left(Y^i|_{[0,s_j]}\right)\right) - W_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,t]}\right), \mathcal{L}\left(Y^i|_{[0,t]}\right)\right)\right| \\
&\quad\le W_{2,\rho_{J_1^d}}\left(L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,s_j]}\right), L^N\left(\widetilde{\mathbf{Y}}^N|_{[0,t]}\right)\right) + W_{2,\rho_{J_1^d}}\left(\mathcal{L}\left(Y^i|_{[0,s_j]}\right), \mathcal{L}\left(Y^i|_{[0,t]}\right)\right) \\
&\quad\le \sqrt{\frac{1}{N}\sum_{m=1}^{N}\sup_{z\in[t,s_j]}\{|Y_z^m - Y_t^m|\}^2} + \sqrt{\mathbb{E}\Big[\sup_{z\in[t,s_j]}\{|Y_z^i - Y_t^i|\}^2\Big]}.
\end{aligned} \tag{A.8}$$

The first term in the above inequality goes to zero from the right continuity of the $\{Y^m\}_{m\in\{1,\dots,N\}}$. For the second term, because $\mathbb{E}\left[\sup_{z\in[0,T]}\{|Y_z^i|\}^2\right] < \infty$, by dominated convergence again from the right continuity of Y^i the term goes to 0. To complete the proof note that the term $\sqrt{\frac{1}{N}\sum_{m=1}^{N}\sup_{z\in[t,s_j]}\{|Y_z^m - Y_t^m|\}^2}$ which depends from ω is non-increasing with respect to time. So, the convergence holds independent of the choice we do for the sequence $\{s_j\}_{j\in\mathbb{N}}$.

Similarly, for a $t \in (0,\infty)$ and an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $s_j \nearrow t$ we can carry out the exact same argument as above with the only difference being that in the inequalities we replace t with $t-$ and then use the existence of left limits for the $\{Y^m\}_{m\in\{1,\dots,N\}}$. □

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Detailed abstract in Greek

Η οπισθόδρομη διάδοση του χάους αναφέρεται στο φαινόμενο όπου η συμπεριφορά αλληλεπιδραστικών παραγόντων (ή σωματιδίων), που περιγράφεται από ένα σύστημα οπισθόδρομων στοχαστικών διαφορικών εξισώσεων (BSDEs), μοιάζει προοδευτικά με αυτήν σαν να ήταν ανεξάρτητοι, ενώ ο αριθμός των παραγόντων τείνει άπειρο. Η παρούσα διατριβή στοχεύει να μελετήσει την οπισθόδρομη (ή προς τα πίσω) διάδοση του χάους σε ένα περιβάλλον όσο το δυνατόν γενικότερο και να εισαγάγει την έννοια της ευστάθειας της οπισθόδρομης διάδοσης του χάους. Εδώ η ευστάθεια νοείται ως η ιδιότητα συνέχειας της οπισθόδρομης διάδοσης του χάους σε σχέση με τα σύνολα δεδομένων.

Η αλληλεπίδραση μεταξύ των διαφορετικών παραγόντων εκφράζεται μέσω του εμπειρικού τους μέτρου. Για να προσδιορίσουμε την ασυμπτωτική συμπεριφορά των mean-field συστημάτων από BSDEs θα χρησιμοποιήσουμε τη McKean–Vlasov BSDE. Θεωρούμε δύο περιπτώσεις αντίστροφης διάδοσης του χάους, όταν έχουμε εξάρτηση μονοπατιού στον γεννήτορα και όταν έχουμε τη συνήθη στιγμιαία εξάρτηση. Έτσι, ξεκινάμε με την καθιέρωση της ύπαρξης και της μοναδικότητας για τις λύσεις του συστήματος mean-field και της McKean–Vlasov BSDE, κάτω από κατάλληλα πλαίσια. Στη συνέχεια, εισάγουμε έναν νέο τρόπο απόδειξης της οπισθόδρομης διάδοσης του χάους που επιτρέπει ασύμμετρες τερματικές συνθήκες για τα συστήματα μέσου πεδίου και γενικά τετραγωνικά ολοκληρώσιμα martingales με ανεξάρτητες προσαυξήσεις ως οδηγούς. Επιπλέον, δείχνουμε επίσης ότι οι γνωστοί ρυθμοί σύγκλισης για την αντίστροφη διάδοση του χάους επεκτείνονται και στο γενικό μας περιβάλλον. Τέλος, εισάγουμε την έννοια της ευστάθειας της οπισθόδρομης διάδοσης του χάους σε σχέση με τα σύνολα δεδομένων, και αποδεικνύουμε την εγκυρότητά της εντός φυσιολογικού πλαισίου, για τη περίπτωση της συνήθους εξάρτησης. Πρώτα καθιερώνουμε την ομοιόμορφη σύγκλιση των συστημάτων μέσου πεδίου με τα BSDE McKean–Vlasov σε σχέση με τα σύνολα δεδομένων και στη συνέχεια επεκτείνουμε φυσιολογικά τη γνωστή ευστάθεια των BSDE για να συμπεριλάβουμε και BSDE McKean–Vlasov. Η σύνθεση τους μας δίνει το αποτέλεσμα ευστάθειας. Επειδή το πλαίσιο μας ενσωματώνει τόσο συνεχείς όσο και ασυνεχείς περιπτώσεις, επιτρέπει την ανάπτυξη αριθμητικών σχημάτων για την αντίστροφη διάδοση του χάους υπό προσεγγίσεις τύπου $\mathbb{L}^2$.

Επιπρόσθετα της έρευνας της αντίστροφης διάδοσης του χάους, στο Κεφάλαιο 1 παρουσιάζουμε έναν νέο τρόπο για την απόδειξη μερικών από τα θεμελιώδη θεωρήματα του στοχαστικού λογισμού. Κάποια από τα πλεονεκτήματα της προσέγγισής μας είναι ότι απαιτεί ελάχιστες προϋποθέσεις, αποφεύγει οποιαδήποτε άμεση αναφορά σε χωρητικότητες, εργάζεται άμεσα με τη προβλέψιμη τομή ενώ η μετρήσιμη τομή είναι ένα άμεσο συμπέρασμα και έτσι αποφεύγεται η διπλή εργασία που κρύβεται στο παρασκήνιο στις συνήθεις αποδείξεις. Τελευταίο αλλά εξίσου σημαντικό, η επιλεκτική (αντιστ. προβλέψιμη) τομή απορρέει από ένα διαισθητικό επιχείρημα προσέγγισης που βασίζεται στη διχοτόμηση προβλέψιμων και ολοκληρωτικά απρόσιτων χρόνων που διασαφηνίζει περαιτέρω τη σχέση μεταξύ αυτών των εννοιών. Το κεφάλαιο κλείνει με μια ενδιαφέρουσα σύντομη απόδειξη ενός θεωρήματος αποσύνθεσης μέτρων, που παρέχεται για λόγους πληρότητας.

Κεφάλαιο 1

Το κεφάλαιο έχει παιδαγωγικό σκοπό, προσπαθεί να καταστήσει πιο κατανοητά τα βασικά θεωρήματα τομών της στοχαστικής ανάλυσης. Για να το πετύχει αυτό χρησιμοποιεί μία καινούρια μέθοδο απόδειξης τους όπου γίνεται χρήση της "έσωτερικής" περιγραφής των σ-αλγεβρών που δίνεται από το τελεστή Souslin. Μετά τη παρουσίαση των θερημάτων τομών δίνεται μια άμεση εφαρμογή τους με τα θεωρήματα προβολών. Το κεφάλαιο κλείνει με μία ενδιαφέρουσα απόδειξη ενός χρήσιμου θεωρήματος για την αποσύνθεση των μέτρων.

Έτσι, παρακάτω αποδεικνύουμε απευθείας το θεώρημα της προβλέψιμης τομής. Δείχνουμε ότι το μόνο που πραγματικά χρειάζεται κάποιος από την αναλυτική θεωρία συνόλων είναι το σχεδόν τετριμμένο Λεμμα 1.1.2, το οποίο συνδέει τις κλάσεις Σουσλιν με τις $\sigma-$άλγεβρες. Επιπλέον, η προσέγγισή μας επιτρέπει χωρίς πολύ κόπο να αποφύγουμε οποιαδήποτε άμεση αναφορά σε χωρητικότητες και να βασιστούμε μόνο στις άμεσες ιδιότητες του $\mathbb{P}^*$, βλέπε Τηεορεμ 1.1.3 και Τηεορεμ 1.1.5. Στη συνέχεια παρατηρούμε ότι η μετρήσιμη προβολή και τομή είναι άμεσα επακόλουθα της προβλέψιμης τομής, βλέπε Τηεορεμ 1.1.6. Η τελική μας παρατήρηση είναι ότι η επιλεκτική τομή έπεται επίσης άμεσα από την προβλέψιμη τομή, εφόσον χρησιμοποιηθεί ένα κατάλληλο αποτέλεσμα προσέγγισης επιλεκτικών συνόλων από προβλέψιμα, βλέπε Λεμμα 1.1.11.

Ορισμός 1.1. *Έστω E ένα αυθαίρετο μη κενό σύνολο και $\emptyset \subset \mathcal{E} \subseteq 2^E$.*
Μια συνάρτηση $A_{\{n_1,\ldots,n_k\}} : \bigcup_{k=1}^{\infty} \mathbb{N}^k \longrightarrow \mathcal{E} \cup \{E\}$ λέγεται ότι είναι ένα σχήμα Souslin με τιμές στο $\mathcal{E}$.
Ο τελεστής Souslin αντιστοιχίζει σε ένα σχήμα Souslin $A_{\{n_1,\ldots,n_k\}}$ το σύνολο $A := \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}$.
Η συλλογή όλων αυτών των συνόλων συμβολίζεται με $\mathcal{S}(\mathcal{E})$ και ονομάζεται κλάση Souslin του $\mathcal{E}$.
Τέλος, ένα σχήμα Souslin $A_{\{n_1,\ldots,n_k\}}$ ονομάζεται μονότονο όταν $A_{n_1,\ldots,n_k,n_{k+1}} \subseteq A_{n_1,\ldots,n_k}$ για οποιαδήποτε $n \in \mathbb{N}^{\mathbb{N}}$ και $k \in \mathbb{N}$.

Το παρακάτω απλό θεώρημα μας δίνει την "έσωτερική" περιγραφή, η απόδειξη του είναι σχεδόν τετριμμένη.

Θεώρημα 1.2. *Έστω E ένα αυθαίρετο μη κενό σύνολο και $\emptyset \subset \mathcal{E} \subseteq 2^E$. Τα παρακάτω είναι αληθή.*

i. *Η $\mathcal{S}(\mathcal{E})$ είναι κλειστή ως προς τις αριθμήσιμες ενώσεις και τομές.*

ii. *Αν για κάθε $D \in \mathcal{E} \cup \{E\}$ έχουμε $D^c \in \mathcal{S}(\mathcal{E})$, τότε $\sigma(\mathcal{E}) \subseteq \mathcal{S}(\mathcal{E})$.*

iii. *Έστω $A_{\{n_1,\ldots,n_k\}}$ ένα μονότονο σχήμα Souslin με τιμές στο $\mathcal{E}$ και $m^* \in \mathbb{N}^{\mathbb{N}}$. Αν για κάθε $k \in \mathbb{N}$ ορίσουμε ως $S_k := \bigcup_{n_1=1}^{m_1^*} \ldots \bigcup_{n_k=1}^{m_k^*} A_{n_1,\ldots,n_k}$, τότε $\bigcap_{k=1}^{\infty} S_k \subseteq \bigcup_{n\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}$.*

Τα επόμενα είναι τα κύρια θερωρήματα μας.

Θεώρημα 1.5 (Προβλέψιμης τομής). *Για κάθε προβλέψιμο σύνολο P στη $\mathcal{P}$ και $\epsilon > 0$ υπάρχει ένας προβλέψιμος χρόνος $\rho^{P,\epsilon}$ τέτοιος ώστε $[\![\rho^{P,\epsilon}]\!] \subseteq P$ και $\mathbb{P}^*(\pi_\Omega(P)) - \mathbb{P}\left(\{\rho^{P,\epsilon} < \infty\}\right) \leq \epsilon$.*

Θεώρημα 1.6 (Μετρήσιμης τομής και προβολής). *Για κάθε σύνολο $S \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ υπάρχει μία $\mathcal{F}-$μετρήσιμη συνάρτηση $\tau_S : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ και ένα σύνολο $A^S \in \mathcal{F}$ τέτοια ώστε $[\![\tau_S]\!] \subseteq S$, $\{\tau_S < \infty\} \subseteq \pi_\Omega(S) \subseteq A^S$ και $\mathbb{P}\left(A^S\right) = \mathbb{P}(\{\tau_S < \infty\}) = \mathbb{P}^*(\pi_\Omega(S))$.*

Για να ολοκληρώσουμε τη παρουσιάση μας με το θεώρημα επιλεκτικής τομής θα χρειαστούμε το επόμενο λήμμα που περιγράφει τη "προσέγγιση" των επιλεκτικών συνόλων από τα προβλέψιμα.

Λήμμα 1.7. *Για κάθε σύνολο $O \in \mathcal{O}$ υπάρχει ένα σύνολο $P \in \mathcal{P}$ τέτοιο ώστε το $O \setminus P$ να είναι λεπτό σύνολο και το $P \setminus O$ να είναι ολοκληρωτικά μη προσβάσιμο λεπτό σύνολο.*

Θεώρημα 1.8 (Επιλεκτικής τομής). *Για κάθε σύνολο $O \in \mathcal{O}$ και $\epsilon > 0$ υπάρχει ένας επιλεκτικός χρόνος $\tau^{O,\epsilon}$ τέτοιος ώστε $[\![\tau^{O,\epsilon}]\!] \subseteq O$ και $\mathbb{P}^*(\pi_\Omega(O)) - \mathbb{P}\left(\{\tau^{O,\epsilon} < \infty\}\right) \leq \epsilon$.*

Κλείνοντας, παρουσιάζουμε το θεώρημα αποσύνθυσης μέτρων το οποίο αναφέραμε παραπάνω. Πριν από αυτό είναι απαραίτητοι οι παρακάτω ορισμοί.

Ορισμός 1.9. *Έστω $(E, \mathcal{E})$ ένας μετρήσιμος χώρος. Λέμε ότι η $\mathcal{E}$* είναι αριθμήσιμα παραγόμενη *εάν και μόνο εάν υπάρχει μια αριθμήσιμη οικογένεια $\mathcal{C} \subseteq \mathcal{P}(E)$ τέτοια ώστε $\mathcal{E} = \sigma(\mathcal{C})$. Επιπλέον, λέμε ότι η $\mathcal{C}$* διαχωρίζει *το E εάν και μόνο εάν για κάθε ζεύγος x, y διακριτών σημείων του E υπάρχει $A \in \mathcal{C}$ έτσι ώστε $x \in A$ και $y \in A^c$. Εάν υπάρχει μια οικογένεια $\mathcal{C}$ που δημιουργεί το $\mathcal{E}$ και διαχωρίζει το E, λέμε ότι ο $(E, \mathcal{E})$ είναι διαχωρίσιμος και αριθμήσιμα παραγόμενος.*

Ορισμός 1.10. *Έστω $(\Omega, \mathcal{F})$ και $(E, \mathcal{E})$ μετρήσιμοι χώροι. Μια συνάρτηση $\mu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ θα ονομάζεται* πεπερασμένο τυχαίο μέτρο *εάν και μόνο εάν τα ακόλουθα είναι αληθή:*

i . Για κάθε $\omega \in \Omega$ το $\mu(\omega, \cdot)$ είναι ένα πεπερασμένο μέτρο στο $(E, \mathcal{E})$.

ii . Για κάθε $S \in \mathcal{E}$ η συνάρτηση $\mu(\cdot, S)$ είναι $\mathcal{F}-$μετρήσιμη.

Επιπλέον, το μ θα ονομάζεται τυχαίο μέτρο *εάν και μόνο εάν $\mu = \sum_{n=1}^{\infty} \mu_n$ και το μ_n είναι πεπερασμένο τυχαίο μέτρο για κάθε $n \in \mathbb{N}$.*

Θεώρημα 1.11. *Έστω $(\Omega, \mathcal{F}, \mathbb{P})$ ένας χώρος πιθανότητας και $(E, \mathcal{E})$ ενας διαχωρίσιμος και αριθμήσιμα παραγόμενος μετρήσιμος χώρος. Επιπλέον, έστω $m : \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathbb{R}_+$ ένα πεπερασμένο μέτρο στο $(\Omega \times E, \mathcal{F} \otimes \mathcal{E})$. Ορίζουμε το πεπερασμένο μέτρο $m_1 : \mathcal{F} \longrightarrow \mathbb{R}_+$ με $m_1(S) := m(S \times E)$. Εάν $m_1 \ll \mathbb{P}$, τότε υπάρχει ένα πεπερασμένο τυχαίο μέτρο $\mu : \Omega \times \mathcal{E} \longrightarrow \mathbb{R}_+$ τέτοιο ώστε για κάθε σύνολο $S \in \mathcal{F} \otimes \mathcal{E}$ να έχουμε ότι*

$$m(S) = \int_\Omega \int_E \mathbb{1}_S(\omega, x) \mu(\omega, \mathrm{d}x) \mathbb{P}(\mathrm{d}\omega).$$

Επιλέον, το πεπερασμένο τυχαίο μέτρο μ είναι μοναδικό $\mathbb{P}-$σ.π.

Κεφάλαιο 2

Σε αυτό το κεφάλαιο παρουσιάζουμε τον συμβολισμό που χρησιμοποιούμε στη συνέχεια ενώ κάνουμε μια ανασκόπηση κλασικών αποτλεσμάτων της στοχαστικής ανάλυσης, καθώς επίσης και ορισμένα βασικά στοιχεία των μετρικών Wasserstein και του χώρου Skorokhod. Ακόμα εισάγουμε τη συνάρτηση Γ, βασικό συστατικό του πλαισίου μας. Στο τέλος του κεφαλαίου δίνουμε το βασικό μας εργαλείο για την εκτίμηση νορμών, τις *εκ των προτέρων εκτιμήσεις*.

Παρακάτω ακολουθεί ο ορισμός της συνάρτησης Γ και η Lipschitz ιδιότητα της.

Ορισμός 2.1. *Έστω ζεύγος* $\overline{X} := (X^{\circ}, X^{\natural}) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$, $C^{(\mathbb{G},\overline{X})}$ *ορισμένη όπως στη σχέση* (2.9) *και* $K^{(\mathbb{G},\overline{X})}$ *που ικανοποιεί τη* (2.10). *Επιπλέον, έστω* Θ *μία* $\widetilde{\mathcal{P}}^{\mathbb{G}}-$*μετρήσιμη συνάρτηση με τιμές στο* $\mathbb{R}$, *τέτοια ώστε* $|\Theta| \leq |I|$, *για* $|I|(x) := |x| + \mathbb{1}_{\{0\}}(x)$. *Ορίζουμε τη διαδικασία* $\Gamma^{(\mathbb{G},\overline{X},\Theta)} : \mathbb{H}^2(\mathbb{G}, X^{\natural};\mathbb{R}^d) \longrightarrow \mathcal{P}^{\mathbb{G}}(\mathbb{R}^d)$ *τέτοια ώστε, για κάθε* $s \in \mathbb{R}_+$, *να ισχύει ότι*

$$\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s(\omega) := \int_{\mathbb{R}^n} \Big(U(\omega,s,x) - \widehat{U}_s^{(\mathbb{G},X^{\natural})}(\omega)\Big)\Big(\Theta(\omega,s,x) - \widehat{\Theta}_s^{(\mathbb{G},X^{\natural})}(\omega)\Big)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)$$
$$+(1-\zeta_s^{(\mathbb{G},X^{\natural})}(\omega))\Delta C_s^{(\mathbb{G},\overline{X})}(\omega)\int_{\mathbb{R}^n} U(\omega,s,x)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x)\int_{\mathbb{R}^n}\Theta(\omega,s,x)\, K_s^{(\mathbb{G},\overline{X})}(\omega,\mathrm{d}x).$$

Παρατήρηση 2.2. *Η επιλογή της* Γ *βασίστηκε και εμπνεύστηκε από τις εφαρμογές. Ο αναγνώστης μπορεί να θυμηθεί, για παράδειγμα, τη σύνδεση μεταξύ BSDE και μερικών ολοκληρο-διαφορικών εξισώσεων και την ειδική δομή που απαιτείται για τον γεννήτορα, βλέπε π.χ. Barles et al. [3] ή Delong [17, Section 4.2]. Επιπλέον, μπορεί κανείς εύκολα να επαληθεύσει ότι η* Γ *είναι ίση με*

$$\frac{\mathrm{d}\langle U \star \widetilde{\mu}^{(\mathbb{G},X^{\natural})}, \Theta \star \widetilde{\mu}^{(\mathbb{G},X^{\natural})}\rangle^{\mathbb{G}}}{\mathrm{d}C^{(\mathbb{G},\overline{X})}}.$$

Λήμμα 2.3. *Έστω ζεύγος* $\overline{X} \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p)\times\mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ *και* $\Theta \in \widetilde{\mathcal{P}}^{\mathbb{G}}$ *με τιμές στο* $\mathbb{R}$ *τέτοια ώστε* $|\Theta| \leq |I|$. *Τότε, για κάθε* $U^1, U^2 \in \mathbb{H}^2(\mathbb{G}, X^{\natural};\mathbb{R}^d)$, *έχουμε ότι*

$$|\Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^1)_t(\omega) - \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U^2)_t(\omega)|^2 \leq 2\left(\left|\!\left|\!\left| U_t^1(\omega;\cdot) - U_t^2(\omega;\cdot)\right|\!\right|\!\right|_t^{(\mathbb{G},\overline{X})}(\omega)\right)^2,\ \mathbb{P}\otimes C^{(\mathbb{G},\overline{X})} - \sigma.\pi.$$

Κλείνουμε, τη περίληψη του κεφαλαίου 2 με μία αναφορά στις *εκ των προτέρων εκτιμήσεις*.

Λήμμα 2.4. *Έστω ότι μας δίναεται ένα* d*-διάστατο semimartingale* y *της μορφής*

$$y_t = \xi + \int_t^T f_s\,\mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s,$$

όπου T *ένας χρόνος διακοπής,* $\xi \in \mathbb{L}^2(\mathcal{G}_T;\mathbb{R}^d)$, f *μία* d*-διάσταστη επιλεκτική διαδικασία, και* $\eta \in \mathcal{H}^2(\mathbb{R}^d)$. *Επιπροσθέτως, υποθέτουμε ότι υπάρχει* $\Phi \geq 0$ *τέτοια ώστε* $\Delta A \leq \Phi, \mathbb{P}\otimes C-$*σχεδόν παντού. Τέλος, υποθέτουμε ότι υπάρχει* $\beta \in (0,\infty)$ *τέτοια ώστε*

$$\|\xi\|_{\mathbb{L}^2_\beta(\mathcal{G}_T;\mathbb{R}^d)} + \left\|\frac{f}{\alpha}\right\|_{\mathbb{H}^2_\beta(\mathbb{R}^d)} < \infty.$$

Τότε, για κάθε $(\gamma,\delta) \in (0,\beta]^2$ *με* $\gamma \neq \delta$ *έχουμε ότι*

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} \leq \frac{2(1+\delta\Phi)}{\delta}\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 2\Lambda^{\gamma,\delta,\Phi}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)},$$
$$\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} \leq 8\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 8\frac{1+\gamma\Phi}{\gamma}\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}$$

και

$$\|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq 9(2+\delta\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + 9\Big(\frac{1}{\gamma\vee\delta} + \delta\Lambda^{\gamma,\delta,\Phi}\Big)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)},$$

όπου

$$\Lambda^{\gamma,\delta,\Phi} := \frac{(1+\gamma\Phi)^2}{\gamma|\delta-\gamma|}.$$

Ενώνοντας τα κομμάτια έχουμε ότι

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq \Big(18 + \frac{2}{\delta} + (9\delta+2)\Phi\Big)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + \Big(\frac{9}{\gamma\vee\delta} + (9\delta+2)\Lambda^{\gamma,\delta,\Phi}\Big)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)},$$

$$\|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq (26+9\delta\Phi)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathbb{R}^d)} + \Big(\frac{8}{\gamma} + 8\Phi + \frac{9}{\gamma\vee\delta} + 9\delta\Lambda^{\gamma,\delta,\Phi}\Big)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}$$

και

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta(\mathbb{R}^d)} + \|y\|^2_{\mathcal{S}^2_\delta(\mathbb{R}^d)} + \|\eta\|^2_{\mathcal{H}^2_\delta(\mathbb{R}^d)} \leq \Big(26 + \frac{2}{\delta} + (9\delta+2)\Phi\Big)\|\xi\|^2_{\mathbb{L}^2_\delta(\mathcal{G}_T;\mathbb{R}^d)} + \Big(\frac{8}{\gamma} + 8\Phi + \frac{9}{\gamma\vee\delta} + (9\delta+2)\Lambda^{\gamma,\delta,\Phi}\Big)\left\|\frac{f}{\alpha}\right\|^2_{\mathbb{H}^2_{\gamma\vee\delta}(\mathbb{R}^d)}.$$

Έστω $\mathcal{C}_\beta := \{(\gamma,\delta)\in(0,\beta]^2 : \gamma<\delta\}$. Ορίζουμε

$$M^\Phi_\star(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta}\left\{\frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\delta}{\delta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma}\right\}$$

και

$$\widetilde{M}^\Phi(\beta) := \inf_{(\gamma,\delta)\in\mathcal{C}_\beta}\left\{\frac{9}{\delta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\delta}{\delta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\}.$$

Για να ολοκληρώσουμε την ανάλυση μας δίνουμε ασυμπτωτικά φράγματα για τους συντελεστές $M^\Phi_\star(\beta)$ και $\widetilde{M}^\Phi(\beta)$ σε σχέση με τη σταθερά Φ.

Πρόταση 2.5. *Για $\Phi\geq 0$ και $\beta\in(0,\infty)$ έχουμε ότι*

$$\begin{aligned} M^\Phi_\star(\beta) &= \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + 9\,\frac{\beta}{\beta-\gamma}\frac{(1+\gamma\Phi)^2}{\gamma}\right\} \\ &= \frac{6\sqrt{17}+35}{\beta} + \Big(6\sqrt{17}+26\Big)\Phi \end{aligned}$$

και

$$\begin{aligned} \widetilde{M}^\Phi(\beta) &= \min_{\gamma\in(0,\beta)}\left\{\frac{9}{\beta} + 8\frac{(1+\gamma\Phi)}{\gamma} + \frac{2+9\beta}{\beta-\gamma}\,\frac{(1+\gamma\Phi)^2}{\gamma}\right\} \\ &= \frac{2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17} + \frac{4}{\beta} + 35}{\beta} + \left(2\sqrt{\frac{2}{\beta}+9}\sqrt{\frac{2}{\beta}+17} + \frac{4}{\beta} + 26\right)\Phi. \end{aligned}$$

Έτσι, παίρνουμε ότι

$$\lim_{\beta\to\infty} M^{\Phi}_{\star}(\beta) = \lim_{\beta\to\infty} \widetilde{M}^{\Phi}(\beta) = \left(6\sqrt{17}+26\right)\Phi.$$

Κεφάλαιο 3

Στο κεφάλαιο 3 παρουσιάζουμε τα θεωρήματα ύπραξης και μοναδικότητας για τη McKean–Vlasov BSDE και το mean-field σύστημα από BSDEs. Πριν από κάθε αποτέλεσμα θα εισάγουμε το πλαίσιο υπό το οποίο το αποδεικνύουμε.

McKean–Vlasov BSDE

Έστω η στοχαστική βάση $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ που ικανοποιεί τις συνήθεις υποθέσεις και υποστηρίζει τα ακόλουθα:

(Φ1) Ένα ζεύγος martingales $\overline{X} := (X^{\circ}, X^{\natural}) \in \mathcal{H}^2(\mathbb{G};\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G};\mathbb{R}^n)$ που ικανοποιεί $M_{\mu^{X^\natural}}[\Delta X^{\circ}|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$, όπου $\mu^{X^\natural}$ είναι το τυχαίο μέτρο που παράγεται από τα άλματα της $X^{\natural}$.[1]

(Φ2) Ένα $\mathbb{G}-$χρόνος διακοπής T και τερματική συνθήκη $\xi \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{G}_T, A^{(\mathbb{G},\overline{X},f)}; \mathbb{R}^d)$, για κάποιο $\hat{\beta} > 0$ και $A^{(\mathbb{G},\overline{X},f)}$ όπως ορίζεται στη **(Φ5)** παρακάτω.

(Φ3) Συναρτήσεις Θ, Γ όπως ορίζονται δεφινιτιον 2.5.1, όπου τα δεδομένα για τον ορισμό τους είναι το ζεύγος $(\mathbb{G}, \overline{X})$, η διαδικασία $C^{(\mathbb{G},\overline{X})}$ και οι πυρήνες $K^{(\mathbb{G},\overline{X})}$.

(Φ4) Ένας γεννήτορας $f : \Omega \times \mathbb{R}_+ \times \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y, z, u, \mu) \in \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d)$, η συνάρτηση

$$(\omega, t) \longmapsto f(\omega, t, y, z, u, \mu) \text{ είναι } \mathbb{G} - \text{προοδευτικά μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$\begin{aligned} &|f(\omega,t,y,z,u,\mu) - f(\omega,t,y',z',u',\mu')|^2 \\ &\quad\leq\; r(\omega,t)\,\rho^2_{J^d_1}(y,y') + \;\vartheta^o(\omega,t)\,|z-z'|^2 + \;\vartheta^\natural(\omega,t)\,|u-u'|^2 + \vartheta^*(\omega,t)\,W^2_{2,\rho_{J^d_1}}(\mu,\mu')\,, \end{aligned}$$

όπου $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : \left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \;\longrightarrow\; \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(Φ5) Ορίζουμε $\alpha^2 := \max\{\sqrt{r}, \vartheta^o, \vartheta^\natural, \sqrt{\vartheta^*}\}$. Για τη $\mathbb{G}$-προβλέψιμη και càdlàg διαδικασία

$$A^{(\mathbb{G},\overline{X},f)}_{\cdot} := \int_0^{\cdot} \alpha_s^2 \,\mathrm{d}C^{(\mathbb{G},\overline{X})}_s$$

υπάρχει $\Phi > 0$ τέτοιο ώστε $\Delta A^{(\mathbb{G},\overline{X},f)} \leq \Phi$, $\mathbb{P}\otimes C^{(\mathbb{G},\overline{X})} -$ σ.π.

(Φ6) Για το ίδιο $\hat{\beta}$ όπως στη **(Φ2)** υπάρχει $\Lambda_{\hat{\beta}} > 0$ τέτοιο ώστε $\mathcal{E}\left(\hat{\beta}A^{(\mathbb{G},\overline{X},f)}\right)_T \leq \Lambda_{\hat{\beta}}$ $\mathbb{P}-$σ.π.

[1] Αφού η διύλιση $\mathbb{G}$ και το ζεύγος $\overline{X}$ μας δίνονται, θα κάνουμε χρήση της $C^{(\mathbb{G},\overline{X})}$, αντιστ. $c^{(\mathbb{G},\overline{X})}$, όπως ορίζεται από τη (2.9), αντιστ. (2.11). Επιπλέον, θα χρησιμοποιήσουμε το πυρήνα $K^{(\mathbb{G},\overline{X})}$ όπως ορίζεται από τη (2.10).

(Φ7) Για το ίδιο $\hat{\beta}$ όπως στη **(Φ2)** έχουμε

$$\mathbb{E}\left[\int_0^T \mathcal{E}\left(\hat{\beta} A^{(\mathbb{G},\overline{X},f)}\right)_{s-} \frac{|f(s,0,0,0,\delta_0)|^2}{\alpha_s^2} \mathrm{d}C_s^{(\mathbb{G},\overline{X})}\right] < \infty,$$

όπου δ_0 το μέτρο Dirac συγκεντρωμένο στο ουδέτερο στοιχείο 0 της πρόσθεσης.

Θεώρημα 3.1. *Έστω $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ standard δεδομένα υπό τη σταθερά $\hat{\beta}$ για τη McKean–Vlasov BSDE* (3.3) *υπό την εξάρτηση μονοπατιού. Εάν*

$$\max\left\{2, \frac{2\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M_\star^\Phi(\hat{\beta}) < 1,$$

τότε η McKean–Vlasov BSDE

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y|_{[0,s]}, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s \,\mathrm{d}X^\circ - \int_t^T \int_{\mathbb{R}^n} U_s \,\widetilde{\mu}^{(\mathbb{G},X^\natural)}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s \end{aligned} \tag{3.3}$$

έχει μοναδική λύση

$$(Y,Z,U,M) \in \mathcal{S}^2(\mathbb{G};\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{G},X^\circ;\mathbb{R}^{d\times p}) \times \mathbb{H}^2(\mathbb{G},X^\natural;\mathbb{R}^d) \times \mathcal{H}^2(\mathbb{G},\overline{X}^{\perp_{\mathbb{G}}};\mathbb{R}^d).$$ [2]

Για να δώσουμε το θεώρημα ύπαρξης και μοναδικότητας για τη McKean–Vlasov BSDE υπό τη συνήθη εξάρτηση χρειάζεται να αντικαταστήσουμε τις **(Φ4)** και **(Φ6)** με τις παρακάτω

(Φ4′) Ένας γεννήτορας $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y,z,u,\mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, η συνάρτηση

$$(\omega,t) \longmapsto f(\omega,t,y,z,u,\mu) \text{ είναι } \mathbb{G} - \text{προοδευτικά μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$\begin{aligned} &|f(\omega,t,y,z,u,\mu) - f(\omega,t,y',z',u',\mu')|^2 \\ &\qquad \leq r(\omega,t)\,|y-y'|^2 + \vartheta^o(\omega,t)\,|z-z'|^2 + \vartheta^\natural(\omega,t)\,|u-u'|^2 + \vartheta^*(\omega,t)\,W_{2,|\cdot|}^2(\mu,\mu'), \end{aligned}$$

όπου $(r,\vartheta^o,\vartheta^\natural,\vartheta^*) : \left(\Omega\times\mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}_+^4, \mathcal{B}\left(\mathbb{R}_+^4\right)\right)$.

(Φ6″) Η διαδικασία $A^{(\mathbb{G},\overline{X},f)}$ είναι ντετερμινιστική.

[2] Υπενθυμίζεται στον αναγνώστη η παρατήρηση 2.4.1 και το γεγονός πως υπό τη **(Φ5)** οι $\hat{\beta}-$νόρμες είναι ισοδύναμες με τις αντίστοιχες τους.

Θεώρημα 3.2. *Έστω* $\left(\mathbb{G}, \overline{X}, T, \xi, \Theta, \Gamma, f\right)$ *που ικανοποιούν τις* **(Φ1)**-**(Φ3)**,**(Φ4′)**,**(Φ5)**,**(Φ6″)** *και* **(Φ7)** *υπό τη σταθερά* $\hat{\beta}$. *Έστω, επίσης ότι, ο* T *στη* **(Φ2)** *είναι ντετερμινιστική. Εάν*

$$2\widetilde{M}^{\Phi}(\hat{\beta}) < 1,$$

τότε η McKean–Vlasov BSDE

$$\begin{aligned} Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s c_s^{(\mathbb{G},\overline{X})}, \Gamma^{(\mathbb{G},\overline{X},\Theta)}(U)_s, \mathcal{L}(Y_s)\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X})} \\ - \int_t^T Z_s \,\mathrm{d}X^{\circ} - \int_t^T \int_{\mathbb{R}^n} U_s\, \widetilde{\mu}^{(\mathbb{G},X^{\natural})}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}M_s. \end{aligned} \tag{3.2}$$

έχει μοναδική λύση

$$(Y, Z, U, M) \in \mathbb{S}^2_{\hat{\beta}}(\mathbb{G}, \alpha, C^{(\mathbb{G},\overline{X})}; \mathbb{R}^d) \times \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, X^{\circ}; \mathbb{R}^{d\times p}) \times \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, X^{\natural}; \mathbb{R}^d) \times \mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, \overline{X}^{\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Mean-field σύστημα από BSDEs

Τώρα, έστω μια στοχαστική βάση $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ που ικανοποιεί τις συνήθεις υποθέσεις και υποστηρίζει τα ακόλουθα:

(Γ1) N ζεύγη martingales $\{\overline{X}^i := (X^{i,\circ}, X^{i,\natural})\}_{i\in\{1,\dots,N\}} \in \left(\mathcal{H}^2(\mathbb{G}; \mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{G}; \mathbb{R}^n)\right)^N$ που ικανοποιούν $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{G}}] = 0$, για $i \in \{1, \dots, N\}$, όπου $\mu^{X^{i,\natural}}$ είναι το τυχαίο μέτρο που παράγεται από τα άλματα της $X^{i,\natural}$.[3]

(Γ2) Ένας $\mathbb{G}$ χρόνος διακοπής T και τερματικές συνθήκες $\{\xi^{i,N}\}_{i\in\{1,\dots,N\}} \in \prod_{i=1}^N \mathbb{L}^2_{\hat{\beta}}(\mathcal{G}_T, A^{(\mathbb{G},\overline{X}^i,f)}; \mathbb{R}^d)$, για κάποιο $\hat{\beta} > 0$ και $\{A^{(\mathbb{G},\overline{X}^i,f)}\}_{i\in\{1,\dots,N\}}$ όπως ορίζονται στη **(Γ5)** παρακάτω.

(Γ3) Συναρτήσεις $\{\Theta^i\}_{i\in\{1,\dots,N\}}, \Gamma$ όπως ορίζονται στη 2.5.1, όπου για $i \in \{1, \dots, N\}$ τα δεδομένα για τους ορισμούς είναι τα ζεύγη $(\mathbb{G}, \overline{X}^i)$, οι διαδικασίες $C^{(\mathbb{G},\overline{X}^i)}$ και οι πυρήνες $K^{(\mathbb{G},\overline{X}^i)}$.

(Γ4) Ένας γεννήτορας $f : \Omega \times \mathbb{R}_+ \times \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y, z, u, \mu) \in \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d)$, η συνάρτηση

$$(\omega, t) \longmapsto f(\omega, t, y, z, u, \mu) \text{ είναι } \mathbb{G} - \text{προοδευτικά μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$\begin{aligned} &|f(\omega, t, y, z, u, \mu) - f(\omega, t, y', z', u', \mu')|^2 \\ &\quad\le r(\omega, t)\, \rho^2_{J_1^d}(y, y') + \vartheta^o(\omega, t)\, |z - z'|^2 + \vartheta^{\natural}(\omega, t)\, |u - u'|^2 + \vartheta^*(\omega, t)\, W^2_{2,\rho_{J_1^d}}(\mu, \mu'), \end{aligned}$$

όπου $(r, \vartheta^o, \vartheta^{\natural}, \vartheta^*) : \left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

[3] Αφού η διύλιση $\mathbb{G}$ και τα ζεύγη $\overline{X}^i$ μας δίνονται, για $i \in \{1, \dots, N\}$, θα κάνουμε χρήση των $C^{(\mathbb{G},\overline{X}^i)}$, αντιστ. $c^{(\mathbb{G},\overline{X}^i)}$, όπως ορίζονται από τη (2.9), αντιστ. (2.11). Επιπλέον, θα χρησιμοποιήσουμε τους πυρήνες $K^{(\mathbb{G},\overline{X}^i)}$ όπως ορίζονται από τη (2.10).

(Γ5) Ορίζουμε $\alpha^2 := \max\{\sqrt{r}, \vartheta^o, \vartheta^\natural, \sqrt{\vartheta^*}\}$. Για τις $\mathbb{G}$-προβλέψιμες και càdlàg διαδικασίες

$$A_t^{(\mathbb{G},\overline{X}^i,f)} := \int_0^t \alpha_s^2 \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)}$$

υπάρχει $\Phi > 0$ τέτοιο ώστε $\Delta A_t^{(\mathbb{G},\overline{X}^i,f)}(\omega) \leq \Phi$, $\mathbb{P} \otimes C^{(\mathbb{G},\overline{X}^i)} -$ σ.π., $\quad i \in \{1,...,N\}$.

(Γ6) Για το ίδιο $\hat{\beta}$ όπως στη **(Γ2)** υπάρχει $\Lambda_{\hat{\beta}} > 0$ τέτοιο ώστε $\max_{i\in\{1,\dots,N\}} \left\{ \mathcal{E}\left(\hat{\beta} A^{(\mathbb{G},\overline{X}^i,f)}\right)_T \right\} \leq \Lambda_{\hat{\beta}}$.

(Γ7) Για το ίδιο $\hat{\beta}$ όπως στη **(Γ2)** έχουμε

$$\mathbb{E}\left[\int_0^T \mathcal{E}\left(\hat{\beta} A^{(\mathbb{G},\overline{X}^i,f)}\right)_{s-} \frac{|f(s,0,0,0,\delta_0)|^2}{\alpha_s^2} \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)}\right] < \infty, \quad i \in \{1,...,N\},$$

όπου δ_0 το μέτρο Dirac συγκεντρωμένο στο ουδέτερο στοιχείο 0 της πρόσθεσης.

Θεώρημα 3.3. *Έστω* $\left(\mathbb{G}, \{\overline{X}^i\}_{i\in\{1,\dots,N\}}, T, \left\{\xi^{i,N}\right\}_{i\in\{1,\dots,N\}}, \{\Theta^i\}_{i\in\{1,\dots,N\}}, \Gamma, f\right)$ *standard δεδομένα υπό τη σταθερά* $\hat{\beta}$ *για τη mean-field BSDE* (3.9) *υπό την εξάρτηση μονοπατιού. Εάν*

$$\max\left\{2, \frac{2\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M_\star^\Phi(\hat{\beta}) < 1,$$

τότε το σύστημα των $N-$*BSDEs*

$$\begin{aligned} Y_t^{i,N} = \xi^{i,N} &+ \int_t^T f\left(s, Y^{i,N}|_{[0,s]}, Z_s^{i,N} c_s^{(\mathbb{G},\overline{X}^i)}, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(U^{i,N})_s, L^N(\boldsymbol{\Psi}^N|_{[0,s]})\right) \mathrm{d}C_s^{(\mathbb{G},\overline{X}^i)} \\ &- \int_t^T Z_s^{i,N}\, \mathrm{d}X_s^{i,\circ} - \int_t^T \int_{\mathbb{R}^n} U_s^{i,N}(x)\, \widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M_s^{i,N}, \quad i = 1,...,N, \end{aligned} \tag{3.9}$$

έχει μοναδικά λύση τη $N-$*τετράδα* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *τέτοια ώστε*

$$\boldsymbol{Y}^N := (Y^{1,N}, \dots, Y^{N,N}) \in \prod_{i=1}^N \mathcal{S}_{\hat{\beta}}^2(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}; \mathbb{R}^d),$$

$$\boldsymbol{Z}^N := (Z^{1,N}, \dots, Z^{N,N}) \in \prod_{i=1}^N \mathbb{H}_{\hat{\beta}}^2(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ}; \mathbb{R}^{d\times p}),$$

$$\boldsymbol{U}^N := (U^{1,N}, \dots, U^{N,N}) \in \prod_{i=1}^N \mathbb{H}_{\hat{\beta}}^2(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural}; \mathbb{R}^d)$$

και

$$\boldsymbol{M}^N := (M^{1,N}, \dots, M^{N,N}) \in \prod_{i=1}^N \mathcal{H}_{\hat{\beta}}^2(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i\perp_{\mathbb{G}}}; \mathbb{R}^d).$$

Όπως πριν, για να δώσουμε το θεώρημα ύπαρξης και μοναδικότητας για τη mean-field σύστημα από BSDEs υπό τη συνήθη εξάρτηση χρειάζεται να αντικαταστήσουμε τις **(Γ4)** και **(Γ6)** με τις παρακάτω

(Γ4′) Ένας γεννήτορας $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y, z, u, \mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, η συνάρτηση

$$(\omega, t) \longmapsto f(\omega, t, y, z, u, \mu) \text{ είναι } \mathbb{G} - \text{προοδευτικά μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$\begin{aligned}|f(\omega, t, y, z, u, \mu) - f(\omega, t, y', z', u', \mu')|^2 \\ \leq\ r(\omega, t)\, |y - y'|^2 + \vartheta^o(\omega, t)\, |z - z'|^2 + \vartheta^\natural(\omega, t)\, |u - u'|^2 + \vartheta^*(\omega, t)\, W^2_{2,|\cdot|}(\mu, \mu'),\end{aligned}$$

όπου $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : \left(\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{G}}\right) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(Γ6′) Για $i, j \in \{1, ..., N\}$ έχουμε ότι $A^{(\mathbb{G},\overline{X}^i,f)} = A^{(\mathbb{G},\overline{X}^j,f)}$.[4]

Θεώρημα 3.4. *Έστω* $\left(\mathbb{G}, \{\overline{X}^i\}_{i\in\{1,...,N\}}, T, \{\xi^i\}_{i\in\{1,...,N\}}, \{\Theta^i\}_{i\in\{1,...,N\}}, \Gamma, f\right)$ *που ικανοποιούν* **(Γ1)**-**(Γ3)**, **(Γ4′)**, **(Γ5)**, **(Γ6′)** *και* **(Γ7)**. *Εάν*

$$2\widetilde{M}^{\Phi}(\hat{\beta}) < 1,$$

τότε το σύστημα των $N-$*BSDEs*

$$\begin{aligned}Y^{i,N}_t = \xi^{i,N} + \int_t^T f\left(s, Y^{i,N}_s, Z^{i,N}_s c^{(\mathbb{G},\overline{X}^i)}_s, \Gamma^{(\mathbb{G},\overline{X}^i,\Theta^i)}(U^{i,N})_s, L^N(\boldsymbol{\Psi}^N_s)\right) \mathrm{d}C^{(\mathbb{G},\overline{X}^i)}_s \\ - \int_t^T Z^{i,N}_s \,\mathrm{d}X^{i,\circ}_s - \int_t^T \int_{\mathbb{R}^n} U^{i,N}_s(x)\, \widetilde{\mu}^{(\mathbb{G},X^{i,\natural})}(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}M^{i,N}_s, \quad i = 1, ..., N,\end{aligned} \tag{3.1}$$

έχει μοναδική λύση $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$ *τέτοια ώστε*

$$\boldsymbol{Y}^N := (Y^{1,N}, \dots, Y^{N,N}) \in \prod_{i=1}^N \mathcal{S}^2_{\hat{\beta}}(\mathbb{G}, \alpha, C^{(\mathbb{G},\overline{X}^i)}; \mathbb{R}^d),$$[5]

$$\boldsymbol{Z}^N := (Z^{1,N} \dots, Z^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\circ}; \mathbb{R}^{d\times p}),$$

$$\boldsymbol{U}^N := (U^{1,N}, \dots, U^{N,N}) \in \prod_{i=1}^N \mathbb{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, X^{i,\natural}; \mathbb{R}^d)$$

και

$$\boldsymbol{M}^N := (M^{1,N}, \dots, M^{N,N}) \in \prod_{i=1}^N \mathcal{H}^2_{\hat{\beta}}(\mathbb{G}, A^{(\mathbb{G},\overline{X}^i,f)}, \overline{X}^{i^{\perp_{\mathbb{G}}}}; \mathbb{R}^d).$$

[4] Η ισότητα ερμηνεύεται up to evanescence. Επιπλέον, από τον ορισμό των $A^{(\mathbb{G},\overline{X}^i,f)}$, για $i \in \{1, \dots, N\}$, αυτή η συνθήκη είναι ισοδύναμη με τις ισότητες $C^{(\mathbb{G},\overline{X}^i)} = C^{(\mathbb{G},\overline{X}^j)}$ για $i, j \in \{1, \dots, N\}$. Επιλέξαμε να τη παρουσιάσουμε έτσι γιατί μας διευκολύνει στις πράξεις.

[5] Οι $\mathcal{S}^2-$χώροι έχουν εισαχθεί πριν από τον ορισμό 3.1.5.

Κεφάλαιο 4

Στο παρών κεφάλαιο θα μελετήσουμε τη οπισθόδρομη διάδοση του χάους.

Η έννοια της διάδοσης του χάους άρχισε να τραβάει πολύ την προσοχή όταν σε μια σειρά διαλέξεων οι Lasry και Lyons [35–37] τη χρησιμοποίησαν για να απλοποιήσουν τη μελέτη των παιχνιδιών μέσου πεδίου. Εισήγαγαν ιδέες από τη στατιστική φυσική στη μελέτη των ισορροπιών κατά Nash για στοχαστικά διαφορικά παιχνίδια με συμμετρικές αλληλεπιδράσεις μαζί με τον Malhamé και τον Caines στο [25, 26]. Γενικά, τα προβλήματα με μεγάλο αριθμό παικτών είναι εμφανώς δύσκολο να ελεγχθούν. Ωστόσο, όπως μας έχει δείξει η στατιστική φυσική, κάτω από τις κατάλληλες υποθέσεις (η πιο σημαντική είναι η συμμετρία) μπορεί κανείς να μελετήσει την ασυμπτωτική συμπεριφορά ενός συστήματος καθώς ο αριθμός των παικτών αυξάνεται στο άπειρο πολύ πιο εύκολα. Φυσικά δεν υπάρχει ένας μόνο τρόπος για να εκφραστεί μαθηματικά η ιδέα ότι οι παίκτες, ή τα σωματίδια στη στατιστική φυσική, αλληλεπιδρούν μεταξύ τους, πρέπει να γίνει μια επιλογή. Στη θεωρία πιθανοτήτων, ένα από τα πρώτα βαθιά θεωρήματα ήταν ο ισχυρός νόμος των μεγάλων αριθμών. Ο νόμος δίνει ένα σύνολο συνθηκών υπό τις οποίες η τυχαιότητα ασυμπτωτικά καταρρέει στον ντετερμινισμό. Αυτή η δυνατότητα είναι ιδανική για τη μελέτη συστημάτων με μεγάλο αριθμό πρακτόρων, καθώς επιτρέπει τον υπολογισμό των απλοποιήσεων. Ίσως εξαιτίας αυτού, η αλληλεπίδραση που έχει μελετηθεί εκτενώς στη βιβλιογραφία της στατιστικής φυσικής είναι αυτή που προκύπτει από το εμπειρικό μέτρο των καταστάσεων των σωματιδίων. Ως εκ τούτου, μια αλληλεπίδραση που περιλαμβάνει το εμπειρικό μέτρο των καταστάσεων των συμμετεχόντων ονομάζεται αλληλεπίδραση mean-field.

Η σύγχρονη αντίληψη για τη διάδοση του χάους ξεκίνησε τη δεκαετία του ΄50. Ο M. Kac στη διαδικασία διερεύνησης των προσεγγίσεων συστημάτων σωματιδίων για ορισμένες μη τοπικές μερικές διαφορικές εξισώσεις (PDE) που προκύπτουν στη θερμοδυναμική, βλέπε [30], έκανε μια σημαντική παρατήρηση σχετικά με ένα χαρακτηριστικό μεγάλων συστημάτων. Ας υποθέσουμε ότι η συμπεριφορά των σωματιδίων είναι συμμετρική και αλληλεπιδρούν με αδύναμο τρόπο ώστε το μέγεθός τους να μειώνεται αντιστρόφως ανάλογα με το μέγεθος του συστήματος. Ίσως λόγω ακυρώσεων των συνεισφορών διαφορετικών σωματιδίων. Στη συνέχεια, εάν οι αρχικές θέσεις τους είναι χαοτικές, εδώ νοούνται ως ανεξάρτητες και ισόνομες, αυτή η αρχική κατάσταση του συστήματος θα μπορούσε να φανεί ασυμπτωτικά ότι διαδίδεται (ή εξαπλώνεται) στα άλλα σημεία του χρόνου, όταν το μέγεθός του αυξάνεται στο άπειρο. Αυτή η ιδέα της διάδοσης έχει χρησιμοποιηθεί από τότε σε διάφορα θέματα με πολλές εφαρμογές, μερικές πρόσφατες βρίσκονται στο Jabin and Wang [27, 28], Malrieu [39].

Σε αυτή την εργασία, συνδυάζοντας τις παραπάνω ιδέες θέλουμε να μελετήσουμε την αντίστροφη διάδοση του χάους για λύσεις συστημάτων μέσου πεδίου οπισθοδρομικών στοχαστικών διαφορικών εξισώσεων (BSDEs) κάτω από μια γενική ρύθμιση. Η οπισθόδρομη διάδοση εννοείται ότι έχει χαοτική συμπεριφορά στις τερματικές συνθήκες, αντί να έχει στις αρχικές συνθήκες.

Αν και η διάδοση του χάους έχει μελετηθεί εκτενώς για τις (Εμπρός) Στοχαστικές Διαφορικές Εξισώσεις (SDEs), π.χ., δείτε την ανασκόπηση Chaintron and Diez [11], για την οπισθόδρομη διάδοση του χάους έχουν δημοσιευθεί μέχρι στιγμής μόνο λίγες εργασίες Hu et al. [23], Briand and Hibon [8], Djehiche et al. [19], Buckdahn et al. [9], Laurière and Tangpi [38]. Καμία από αυτές τις εργασίες δεν λειτουργεί σε ένα περιβάλλον τόσο γενικό όσο της εργασίας που παρουσιάζεται εδώ. Πιο συγκεκριμένα, για τη διάδοση της ιδιότητας χάους, το τρέχον πλαίσιο επιτρέπει τετραγωνικά–ολοκληρώσιμα martingales με ανεξάρτητες προσαυξήσεις ως ολοκληρωτές των στοχαστικών ολοκληρωμάτων, càdlàg προβλέψιμες αύξουσες διαδικασίες ως ολοκληρωτές των Lebesgue–Stieltjes ολοκληρωμάτων καθώς επίσης και εξάρτηση στον γεννήτορα από τα αρχικά τμήματα των μονοπατιών της λύσης $\mathbf{Y}$. Στην πραγματικότητα, τα αποτελέσματα των ςηαπτερ 3 και σεςτιον 4.1 ισχύουν επίσης χωρίς αλλαγή στην περίπτωση που η εξάρτηση στον γεννήτορα προέρχεται από το $x|_{[0,s-]}$ αντί για το $x|_{[0,s]}$· δείτε τη σημείωση που εισάγεται στο (2.28). Ομοίως στο τμήμα 4.2 μπορούμε να αντικαταστήσουμε το Y_s με το Y_{s-}.

Για ευκολία της παρουσίασης, ο γεννήτορας f υποτίθεται ότι είναι ντετερμινιστικός, αλλά μπορεί επίσης να υποτεθεί, όπως συνηθίζεται, ότι εργαζόμαστε σε έχαν χώρο γινόμενο από αντίγραφα ενός στοχαστικού γεννήτορα και δεδομένων ενός πρωτότυπου χώρου πιθανοτήτων, υπό τις προφανείς τροποποιήσεις στις αποδείξεις.

Στο τμήμα 4.2.2 παρέχουμε ρυθμούς σύγκλισης για τη συνήθη εξάρτηση. Τα θεωρήματα γενικεύουν εκείνα που βρίσκονται στο [38] για το Brownian-πλαίσιο. Αν και, στην εργασία τους οι Lauriere και Tangpi αποδεικνύουν επίσης ότι οι απαιτήσεις των θεωρημάτων, δηλαδή η προηγμένη ολοκληρωσιμότητα των λύσεων, μπορούν να ικανοποιηθούν υπό μια πρόσθετη ειδική συνθήκη γραμμικής ανάπτυξης για τον γεννήτορα. Στην παρούσα εργασία δεν μελετάμε υπό ποιες συνθήκες για τον γεννήτορα μπορούν να επιτευχθούν αυτές οι απαιτήσεις ολοκληρωσιμότητας, αλλά μπορούμε να πούμε ότι επιτυγχάνονται τετριμμένα όταν ο γεννήτορας είναι φραγμένος, βλέπε παρατήρηση 4.2.11. Προφανώς, στα προηγούμενα αποτελέσματα, πρέπει να υποθέσει κανείς επίσης προηγμένη ενσωμάτωση για το τερματικό συνθήκες. Εναλλακτικά, εάν κάποιος θέλει να διατηρήσει τετραγωνικές συνθήκες ολοκληρωσιμότητας για τα δεδομένα, τότε χρειάζεται να εξειδικεύσει την εξάρτηση της f από τα μέτρα πιθανότητας σε έναν τύπο που το επιτρέπει, όπως στο [9] ή στο [38, Πρόταση 2.12.].

Οπισθόδρομη διάδοση του χάους υπό την εξάρτηση μονοπατιού

Πρώτα εισάγουμε το πλαίσιο για τη οπισθόδρομη διάδοση του χάους υπό την εξάρτηση μονοπατιού.

Έστω πλήρης χώρος πιθανότητας $(\Omega, \mathcal{G}, \mathbb{P})$ που υποστηρίζει τα ακόλουθα:

(H1) Μια ακολουθία από ανεξάρτητα και ισόνομα ζεύγη $\{\overline{X}^i\}_{i\in\mathbb{N}}$ τέτοια ώστε, για $i \in \mathbb{N}$, $\overline{X}^i = (X^{i,\circ}, X^{i,\natural}) \in \mathcal{H}^2(\mathbb{F}^i;\mathbb{R}^p) \times \mathcal{H}^{2,d}(\mathbb{F}^i;\mathbb{R}^n)$ με $M_{\mu^{X^{i,\natural}}}[\Delta X^{i,\circ}|\widetilde{\mathcal{P}}^{\mathbb{F}^i}] = 0$, όπου $\mathbb{F}^i := (\mathcal{F}^i_t)_{t\geq 0}$ είναι η συνήθης επέκταση της διύλισης που παράγεται από τη $\overline{X}^i$ και $\mu^{X^{i,\natural}}$ το τυχαίο μέτρο που παράγεται από τα άλματα της $X^{i,\natural}$.[6]

(H2) Ένας ντετερμινιστικός χρόνος T και μία ακολουθία από ανεξάρτητες, ισόνομες τερματικές συνθήκες $\{\xi^i\}_{i\in\mathbb{N}}$ καθώς επίσης μία ακολουθία συνόλων τερματικών συνθηκών $\left\{\{\xi^{i,N}\}_{i\in\{1,\dots,N\}}\right\}_{N\in\mathbb{N}}$ τέτοια ώστε, για κάποιο $\hat{\beta} > 0$ να ισχύει $\xi^i, \xi^{i,N} \in \mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^i_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)$, $\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)$[7] αντίστοιχα για κάθε $i \in \mathbb{N}$, όπου $\{A^{(\mathbb{F}^i,\overline{X}^i,f)}\}_{i\in\mathbb{N}}$ όπως ορίζονται στη **(H5)**.

Επιπλέον, υποθέτουμε ότι $\|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0$, για κάθε $i \in \mathbb{N}$, και $\frac{1}{N}\sum_{i=1}^N \|\xi^{i,N} - \xi^i\|^2_{\mathbb{L}^2_{\hat{\beta}}(\mathcal{F}^{1,\dots,N}_T, A^{(\mathbb{F}^i,\overline{X}^i,f)};\mathbb{R}^d)} \xrightarrow[N\to\infty]{|\cdot|} 0$.

(H3) Συναρτήσεις Θ, Γ όπως ορίζονται δεφινιτιον 2.5.1, με Θ ντετερμινιστικό και για κάθε $i \in \mathbb{N}$ τα δεδομένα για τους ορισμούς είναι τα ζεύγη $(\mathbb{F}^i, \overline{X}^i)$, οι διαδικασίες $C^{(\mathbb{F}^i,\overline{X}^i)}$ και οι πυρήνες $K^{(\mathbb{F}^i,\overline{X}^i)}$. Υπογραμμίζεται ότι $\Theta \in \widetilde{\mathcal{P}}^{\mathbb{F}^i}$, για κάθε $i \in \mathbb{N}$.

[6] Αφού για κάθε $i \in \mathbb{N}$ η διύλιση $\mathbb{F}^i$ σχετίζεται με το ζεύγος $\overline{X}^i$, θα κάνουμε χρήση $C^{(\mathbb{F}^i,\overline{X}^i)}$, αντιστ. $c^{(\mathbb{F}^i,\overline{X}^i)}$, όπως ορίζεται στη (2.9), αντιστ. (2.11). Επιπλέον, θα κάνουμε χρήση των πυρήνων $K^{(\mathbb{F}^i,\overline{X}^i)}$ όπως ορίζονται στη (2.10).

[7] δες παρατήρηση 4.1.1 *(ι)*

(H4) Ένας γεννήτορας $f : \mathbb{R}_+ \times \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y, z, u, \mu) \in \mathbb{D}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}(\mathbb{D}^d)$, η συνάρτηση

$$t \longmapsto f(t, y, z, u, \mu) \text{ είναι } \mathcal{B}(\mathbb{R}_+) - \text{μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$|f(t, y, z, u, \mu) - f(t, y', z', u', \mu')|^2 \\ \leq r(t)\, \rho^2_{J_1^d}(y, y') + \vartheta^o(t)\, |z - z'|^2 + \vartheta^\natural(t)\, |u - u'|^2 + \vartheta^*(t)\, W^2_{2,\rho_{J_1^d}}(\mu, \mu'),$$

όπου $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \longrightarrow \left(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\right)$.

(H5) Ορίζουμε $\alpha^2 := \max\{\sqrt{r}, \vartheta^o, \vartheta^\natural, \sqrt{\vartheta^*}\}$. Για τις $\mathbb{F}^i$-προβλέψιμες και càdlàg διαδικασίες

$$A_t^{(\mathbb{F}^i, \overline{X}^i, f)} := \int_0^t \alpha_s^2 \mathrm{d}C_s^{(\mathbb{F}^i, \overline{X}^i)} \tag{6.1}$$

υπάρχει $\Phi > 0$ τέτοιο ώστε $\Delta A_t^{(\mathbb{F}^i, \overline{X}^i, f)}(\omega) \leq \Phi$, $\mathbb{P} \otimes C^{(\mathbb{F}^i, \overline{X}^i)}$ – α.ε, για κάθε $i \in \mathbb{N}$.

(H6) Για το ίδιο $\hat{\beta}$ όπως στη **(H2)** έχουμε

$$\mathbb{E}\left[\int_0^T \mathcal{E}\left(\hat{\beta} A^{(\mathbb{F}^i, \overline{X}^i, f)}\right)_{s-} \frac{|f(s, 0, 0, 0, \delta_0)|^2}{\alpha_s^2} \mathrm{d}C_s^{(\mathbb{F}^i, \overline{X}^i)}\right] < \infty, \quad i \in \mathbb{N}, \tag{6.2}$$

όπου δ_0 το μέτρο Dirac συγκεντρωμένο στο ουδέτερο στοιχείο 0 της πρόσθεσης.

(H7) Υπάρχει μία μη-φθίνουσα, δεξιά συνεχής Q, μία Borel-μετρήσιμη συνάρτηση γ και μία οικογένεια $\{b^i\}_{i\in\mathbb{N}}$, με $b^i \in \mathcal{P}_+^{\mathbb{F}^i}$ για κάθε $i \in \mathbb{N}$, τέτοια ώστε

$$\mathcal{E}\left(\hat{\beta} A^{(\mathbb{F}^i, \overline{X}^i, f)}\right)_{\cdot} = 1 + \int_0^{\cdot} b_s^i \, \mathrm{d}Q_s, \quad i \in \mathbb{N}$$

και

$$\sup_{i\in\mathbb{N}}\{b^i\} \leq \gamma, \quad Q - a.e.$$

(H8) Για το ίδιο $\hat{\beta}$ όπως στη **(H2)** και γ όπως στη **(H7)** υπάρχει ένα $\Lambda_{\hat{\beta}} > 0$ τέτοιο ώστε $1 + \int_0^T \gamma_s \, \mathrm{d}Q_s = \Lambda_{\hat{\beta}}$.

(H9) Για το ίδιο $\hat{\beta}$ όπως στη **(H2)** έχουμε $\max\left\{2, \frac{3\Lambda_{\hat{\beta}}}{\hat{\beta}}\right\} M_\star^\Phi(\hat{\beta}) < 1$.

Τα κύρια αποτελέσματα είναι τα ακόλουθα.

Θεώρημα 4.1 (**Διάδοση του χάους για το σύστημα**). *Έστω ότι ισχύουν οι* **(H1)**-**(H9)**. *Για τη λύση της mean-field BSDE* (3.9), *συμβ. με* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *και τις λύσεις των πρώτων N McKean–Vlasov BSDE* (3.3), *συμβ. με* $(\widetilde{\boldsymbol{Y}}^N, \widetilde{\boldsymbol{Z}}^N, \widetilde{\boldsymbol{U}}^N, \widetilde{\boldsymbol{M}}^N)$, *έχουμε*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \| \left(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\right) \|^2_{\star, \hat{\beta}, \mathbb{F}^{1,\ldots,N}, A^i, \overline{X}^i} = 0.$$

Θεώρημα 4.2 (**Διάδοση του χάους**). *Έστω ότι ισχύουν οι* **(H1)-(H9)** *και έστω* $i \le N \in \mathbb{N}$. *Για τη λύση της mean-field BSDE* (3.9), *συμβ. με* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *και τη λύση της* $i-$*th McKean–Vlasov BSDE* (3.3), *συμβ. με* (Y^i, Z^i, U^i, M^i), *έχουμε*

$$\lim_{N\to\infty} \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i \Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},A^i,\overline{X}^i} = 0.$$

Οπισθόδρομη διάδοση του χάους υπό τη συνήθη εξάρτηση

Πριν δώσουμε τα κύρια αποτελέσματα αυτού του τμήματος θα τροποποιήσουμε ορισμένες από τις **(H1)-(H9)** ως εξής:

(H4′) Ένας γεννήτορας $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ τέτοιος ώστε για κάθε $(y, z, u, \mu) \in \mathbb{R}^d \times \mathbb{R}^{d\times p} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, η συνάρτηση

$$t \longmapsto f(t, y, z, u, \mu) \text{ είναι } \mathcal{B}(\mathbb{R}_+) - \text{μετρήσιμη}$$

και ικανοποιεί τις παρακάτω Lipschitz συνθήκες

$$|f(t, y, z, u, \mu) - f(t, y', z', u', \mu')|^2 \\ \le r(t)\, |y - y'|^2 + \vartheta^o(t)\, |z - z'|^2 + \vartheta^\natural(t)\, |u - u'|^2 + \vartheta^*(t)\, W^2_{2,|\cdot|}(\mu, \mu'),$$

όπου $(r, \vartheta^o, \vartheta^\natural, \vartheta^*) : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \longrightarrow \Big(\mathbb{R}^4_+, \mathcal{B}\left(\mathbb{R}^4_+\right)\Big)$.

(H7′) Το martingale $\overline{X}^1$ έχει ανεξάρτητες προσαυξήσεις.

(H8′) Για το ίδιο $\hat{\beta}$ όπως στη **(H2)** έχουμε $3\, \widetilde{M}^\Phi(\hat{\beta}) < 1$.

Θεώρημα 4.3 (**Διάδοση του χάους για το σύστημα**). *Έστω ότι ισχύουν οι* **(H1)-(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** *και* **(H8′)**. *Για τη λύσης της mean-field BSDE* (3.1), *συμβ. με* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *και τις λύσεις των πρώτων* N *McKean–Vlasov BSDE* (3.2), *συμβ. με* $(\widetilde{\boldsymbol{Y}}^N, \widetilde{\boldsymbol{Z}}^N, \widetilde{\boldsymbol{U}}^N, \widetilde{\boldsymbol{M}}^N)$, *έχουμε*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i \Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} = 0.$$

Θεώρημα 4.4 (**Διάδοση του χάους**). *Έστω ότι ισχύουν οι* **(H1)-(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** *και* **(H8′)** *και έστω ότι* $i \le N \in \mathbb{N}$. *Για τη λύση της mean-field BSDE* (3.1), *συμβ. με* $(\boldsymbol{Y}^N, \boldsymbol{Z}^N, \boldsymbol{U}^N, \boldsymbol{M}^N)$, *και τη λύση της* $i-$*th McKean–Vlasov BSDE* (3.2), *συμβ. με* (Y^i, Z^i, U^i, M^i), *έχουμε*

$$\lim_{N\to\infty} \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i \Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i} = 0.$$

Παρακάτω δίνουμε τους αντίστοιχους ρυθμούς σύγκλισης.

Θεώρημα 4.5. *Εάν* $\Lambda_{q,T} < \infty$ *για κάποιο* $q > 2$ *και ντετερμινιστικό* T, *τότε υπάρχει σταθερά* $C_{d,q,2} > 0$ *εξαρτώμενη από τις* $d, q, 2$ *τέτοια ώστε*

$$\frac{1}{N}\sum_{i=1}^{N} \| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}$$
$$\leq \frac{\Big(26 + \frac{2}{\hat{\beta}} + (9\hat{\beta}+2)\Phi\Big)}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})} R(N) + \frac{2\widetilde{M}^{\Phi}(\hat{\beta})}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})}\, \Lambda_{q,T}\, C_{d,q,2}$$
$$\times \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}} & , \text{αν } d < 4 \text{ και } q \neq 4 \\ N^{-\frac{1}{2}} \log(1+N) + N^{-\frac{q-2}{q}} & , \text{αν } d = 4 \text{ και } q \neq 4 \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}} & , \text{αν } d > 4. \end{cases}$$

Θεώρημα 4.6. *Εάν* $\Lambda_{q,T} < \infty$ *για κάποιο* $q > 2$ *και ντετερμινιστικό* T, *τότε υπάρχει σταθερά* $C_{d,q,2} > 0$ *εξαρτώμενη από τις* $d, q, 2$ *τέτοια ώστε*

$$\| \Big(Y^{i,N} - Y^i, Z^{i,N} - Z^i, U^{i,N} - U^i, M^{i,N} - M^i\Big) \|^2_{\star,\hat{\beta},\mathbb{F}^{1,\dots,N},\alpha,C^i,\overline{X}^i}$$
$$\leq \frac{\Big(26 + \frac{2}{\hat{\beta}} + (9\hat{\beta}+2)\Phi\Big)(2 - 5\widetilde{M}^{\Phi}(\hat{\beta}))}{(1 - 2\widetilde{M}^{\Phi}(\hat{\beta}))(1 - 3\widetilde{M}^{\Phi}(\hat{\beta}))} R(N)$$
$$+ \left(\frac{2\widetilde{M}^{\Phi}(\hat{\beta})}{1 - 2\widetilde{M}^{\Phi}(\hat{\beta})}\right)\left(\frac{1 - \widetilde{M}^{\Phi}(\hat{\beta})}{1 - 3\widetilde{M}^{\Phi}(\hat{\beta})}\right)\, \Lambda_{q,T}\, C_{d,q,2}$$
$$\times \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}} & , \text{αν } d < 4 \text{ και } q \neq 4 \\ N^{-\frac{1}{2}} \log(1+N) + N^{-\frac{q-2}{q}} & , \text{αν } d = 4 \text{ και } q \neq 4 \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}} & , \text{αν } d > 4. \end{cases}$$

Κεφάλαιο 5

Η οπισθόδρομη διάδοση του χάους δηλώνει ότι, κάτω από κατάλληλες συνθήκες, η λύση του συστήματος mean-field των οπισθοδρομικών στοχαστικών διαφορικών εξισώσεων (BSDEs) με N παίκτες (ή σωματίδια) συγκλίνει στις λύσεις των N ανεξάρτητων και ισόνομων McKean– Vlasov BSDEs, καθώς το N πηγαίνει στο άπειρο. Φυσικά, κάθε τέτοιο φαινόμενο σχετίζεται με ένα σύνολο δεδομένων $\mathcal{D}$ που παρέχει τη βάση της μαθηματικής περιγραφής του. Στη θεωρία των BSDE, το ερώτημα αν υποθέσουμε ότι μια ακολουθία δεδομένων $\{\mathcal{D}^k\}_{k\in\mathbb{N}}$ συγκλίνει στα δεδομένα $\mathcal{D}^\infty$ τότε εάν οι αντίστοιχες λύσεις συγκλίνουν, ονομάζεται πρόβλημα ευστάθειας για BSDE. Φυσικά, το πλαίσιο για την αντιμετώπιση αυτού του είδους ζητημάτων θα πρέπει να καθορίσει πολλές τεχνικές λεπτομέρειες. Για παράδειγμα, με ποια έννοια συγκλίνουν τα δεδομένα, πώς μετράμε την απόσταση των λύσεων και ούτω καθεξής. Μερικά αξιοσημείωτα έργα, μεταξύ άλλων, που παρέχουν τέτοια πλαίσια είναι το Hu and Peng [24] στην ειδική περίπτωση συνεχούς διύλισης, Briand et al. [6, 7] για Brownian οδηγούς και πιο πρόσφατα Papapantoleon et al. [44] όπου καθιερώθηκε ένα πολύ γενικό πλαίσιο .

Θα υιοθετήσουμε το πλαίσιο του [44] και θα το εμπλουτίσουμε για να μελετήσουμε την ευστάθεια της οπισθόδρομης διάδοσης του χάους. Ας είμαστε ακριβείς σε αυτό που εννοούμε. Για να ξεκινήσουμε, ορίζουμε

$\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ και, στη συνέχεια, υποθέτουμε μια ακολουθία δεδομένων

$$\Big\{\mathcal{D}^k := \Big(\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}, T^k, \Big\{\{\xi^{k,i,N}\}_{i\in\{1,\dots,N\}}\Big\}_{N\in\mathbb{N}}, \{\xi^{k,i}\}_{i\in\mathbb{N}}, \Theta^k, \Gamma, f^k\Big)\Big\}_{k\in\overline{\mathbb{N}}}.$$

Για κάθε $(k, N) \in \overline{\mathbb{N}} \times \mathbb{N}$ έχουμε το ακόλουθο mean-field σύστημα από BSDE

$$\begin{aligned} Y_t^{k,i,N} = \xi^{k,i,N} + \int_t^{T^k} f^k\Big(s, Y_s^{k,i,N}, Z_s^{k,i,N} c_s^k, \Gamma^{(\mathbb{F}^{k,(1,\dots,N)},\overline{X}^{k,i},\Theta^k)}(U^{k,i,N})_s, L^N(\mathbf{\Psi}_s^{k,N})\Big)\,\mathrm{d}C_s^k \\ - \int_t^{T^k} Z_s^{k,i,N}\,\mathrm{d}X_s^{k,i,\circ} - \int_t^{T^k}\int_{\mathbb{R}^n} U_s^{k,i,N}(x)\,\widetilde{\mu}^{(\mathbb{F}^{k,(1,\dots,N)},X^{k,i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k}\mathrm{d}M_s^{k,i,N}, \\ i = 1, \dots, N, \end{aligned}$$

με μοναδική λύση $\mathbf{S}^{k,N} := \Big(\mathbf{Y}^{k,N}, \mathbf{Z}^{k,N}, \mathbf{U}^{k,N}, \mathbf{M}^{k,N}\Big)$, όπου $\mathbf{Y}^{k,N} := \Big(Y^{k,1,N}, \dots, Y^{k,i,N}, \dots, Y^{k,N,N}\Big)$ και ούτω καθεξής. Επιπλέον, για $(k, i) \in \overline{\mathbb{N}} \times \mathbb{N}$ έχουμε επίσης την ακόλουθη McKean–Vlasov BSDE

$$\begin{aligned} Y_t^{k,i} = \xi^{k,i} + \int_t^{T^k} f^k\Big(s, Y_s^{k,i}, Z_s^{k,i} c_s^k, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U^{k,i})_s, \mathcal{L}(Y_s^{k,i})\Big)\,\mathrm{d}C_s^k \\ - \int_t^{T^k} Z_s^{k,i}\,\mathrm{d}X^{k,i,\circ} - \int_t^{T^k}\int_{\mathbb{R}^n} U_s^{k,i}\,\widetilde{\mu}^{(\mathbb{F}^{k,i},X^{k,i,\natural})}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k}\mathrm{d}M_s^{k,i}, \end{aligned}$$

όπου $\mathcal{L}(Y_s^{k,i}) := \mathrm{Law}(Y_s^{k,i})$. Από το θεώρημα 4.4 γνωρίζουμε ότι για κάθε $(k, i) \in \overline{\mathbb{N}} \times \mathbb{N}$ έχουμε

$$\|\Big(Y^{k,i,N} - Y^{k,i}, Z^{k,i,N} - Z^{k,i}, U^{k,i,N} - U^{k,i}, M^{k,i,N} - M^{k,i}\Big)\|^2_{\star,\hat{\beta},\mathbb{F}^{k,(1,\dots,N)},\alpha^k,C^k,\overline{X}^{k,i}} \xrightarrow[N\to\infty]{|\cdot|} 0.$$

Στόχος μας είναι να δείξουμε ότι, για κάθε $i \in \mathbb{N}$ έχουμε τη σύγκλιση

$$\Big(Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N}\star\widetilde{\mu}^{X^{k,i,\natural}}, M^{k,i,N}\Big) \xrightarrow[(k,N)\to(\infty,\infty)]{} \Big(Y^{\infty,i}, Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i}\star\widetilde{\mu}^{X^{\infty,i,\natural}}, 0\Big),$$

κάτω από κάποια κατάλληλη μετρική, την οποία από τώρα θα ονομάζουμε ιδιότητα ευστάθειας ή ευρωστίας της οπισθόδρομης διάδοσης του χάους. Από όσο γνωρίζουμε αυτό θα είναι το πρώτο αποτέλεσμα αυτού του είδους, δείτε ωστόσο Del Moral and Tugaut [15].

Προκειμένου να αποδείξουμε την ευστάθεια της οπισθόδρομης διάδοσης του χάους, πρώτα δείχνουμε ότι η διάδοση της ιδιότητας χάους επιτυγχάνεται ομοιόμορφα σε σχέση με τα δεδομένα $\mathcal{D}^k$. Στη συνέχεια, επεκτείνοντας τα αποτελέσματα ευστάθειας του [44] για τη McKean–Vlasov BSDE μπορούμε αμέσως να συμπεράνουμε αυτό που θέλουμε. Σε αυτό το σημείο είναι σημαντικό να σημειωθεί ότι η ευστάθεια των συστημάτων mean-field δεν χρειαζόταν ούτε συνεπάγεται από την ευστάθεια της διάδοσης του χάους. Ωστόσο, για πληρότητα, παρέχουμε επίσης ένα υσχυρότερο πλαίσιο που τη περιλαμβάνει, επεκτείνοντας και πάλι φυσιολογικά τα αποτελέσματα του [44].

Είμαστε έτοιμοι να εισάγουμε τις υποθέσεις κάτω από τις οποίες θα εργαστούμε.

Για το υπόλοιπο του κεφαλαίου σταθεροποιούμε ένα χώρο πιθανότητας $(\Omega, \mathcal{G}, \mathbb{P})$ και μία ακολουθία δεδομένων, $\mathcal{D}^k := \Big(\{\overline{X}^{k,i}\}_{i\in\mathbb{N}}, T^k, \Big\{\{\xi^{k,i,N}\}_{i\in\{1,\dots,N\}}\Big\}_{N\in\mathbb{N}}, \{\xi^{k,i}\}_{i\in\mathbb{N}}, \Theta^k, \Gamma, f^k\Big)$ για κάθε $k \in \overline{\mathbb{N}}$, υπό μία παγκόσμια σταθερά $\hat{\beta} > 0$, έτσι ώστε για κάθε $k \in \overline{\mathbb{N}}$ να ικανοποιούνται οι **(H1)**-**(H3)**, **(H4′)**, **(H5)**, **(H6)**, **(H7′)** και **(H8′)**.

Τώρα, ακολουθώντας το [44], θα συμπληρώσουμε στα παραπάνω τις υποθέσεις που χρειάζονται για τη σύγκλιση των δεδομένων $\{\mathcal{D}^k\}_{k\in\mathbb{N}}$ καθώς και για τη σύγκλιση των Lebesgue-Stieltjes ολοκληρωμάτων που σχετίζονται με τους γεννήτορες των BSDEs.

(Σ1) Για κάθε $i \in \mathbb{N}$, η διαδικασία $X^{\infty,i,\circ}$ είναι συνεχής και η $X^{\infty,i,\natural}$ δεν έχει άλματα σε ντετερμινιστικούς χρόνους (σχεδόν παντού).

(Σ2) Για κάθε $i \in \mathbb{N}$, έχουμε

$$\overline{X}^{k,i} \xrightarrow[k\to\infty]{(\Theta_1(\mathbb{R}^{p+n}),\mathbb{P})} \overline{X}^{\infty,i} \quad \text{και} \quad \overline{X}^{k,i}_\infty \xrightarrow[k\to\infty]{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^{p+n})} \overline{X}^{\infty,i}_\infty.$$

(Σ3) Για κάθε $i \in \mathbb{N}$, το ζεύγος $\overline{X}^{\infty,i}$ ικανοποιεί τη $\mathbb{F}^{\infty,i}-$predictable representation ιδιότητα.

(Σ4) Για κάθε $i \in \mathbb{N}$,

$$\left\|\xi^{k,i,N} - \xi^{k,i}\right\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)} \xrightarrow[(k,N)\to(\infty,\infty)]{|\cdot|} 0.$$

Επιπλέον,

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\xi^{k,i,N} - \xi^{k,i}\right\|^2_{\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P};\mathbb{R}^d)} \xrightarrow[(k,N)\to(\infty,\infty)]{|\cdot|} 0.$$

(Σ5) Για κάθε $i \in \mathbb{N}$, έχουμε

$$\xi^{k,i} \xrightarrow[k\to\infty]{\mathbb{L}_2(\Omega,\mathcal{G},\mathbb{P},\mathbb{R}^d)} \xi^{\infty,i}.$$

(Σ6) Για κάθε $i \in \mathbb{N}$, η ακολουθία πραγματικών αριθμών

$$\left\{\int_0^\infty \frac{|f^k(s,0,0,0,\delta_0)|^2}{(\alpha^k)^2_s}\mathrm{d}C^k_s\right\}_{k\in\overline{\mathbb{N}}},$$

είναι φραγμένη, όπου δ_0 το μέτρο Dirac συγκεντρωμένο στο ουδέτερο στοιχείο 0 της πρόσθεσης.

(Σ7) Υπάρχει $\overline{A} \in \mathbb{R}_+$ τέτοια ώστε, η ακολουθία πραγματικών αριθμών $\{A^k_\infty\}_{k\in\overline{\mathbb{N}}}$ είναι φραγμένη $\overline{A}$ (δες παρατήρηση 5.1.1 (i)).

(Σ8) Οι γεννήτορες $\{f^k\}_{k\in\overline{\mathbb{N}}}$ έχουν επιπλέον τις ακόλουθες ιδιότητες:

(i) Για κάθε $k \in \overline{\mathbb{N}}, i \in \mathbb{N}, a \in \mathbb{D}(\mathbb{R}^d, |\cdot|), Z \in D^{\circ,d\times p}, U \in D^\natural, \mu \in \mathbb{D}(\mathcal{P}_2(\mathbb{R}^d), W_{2,|\cdot|})^8$, ισχύει ότι

$$\left(f^k(t, a_t, Z_t, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t, \mu_t)\right)_{t\in\mathbb{R}_+} \in \mathbb{D}(\mathbb{R}^d, |\cdot|).$$

[8] Συμβολίζουμε με $\mathbb{D}(\mathcal{P}_2(\mathbb{R}^d), W_{2,|\cdot|})$ το Skorokhod χώρο των cadlag συναρτήσεων με τιμές στο $(\mathcal{P}_2(\mathbb{R}^d), W_{2,|\cdot|})$.

(ii) Για κάθε $i \in \mathbb{N}, Z \in D^{\circ,d\times p}, U \in D^{\natural}$ και ακολυθία στοχαστικών διαδικασιώνε με τιμές στο $\mathbb{R}^d$ $\{a^k\}_{k\in\overline{\mathbb{N}}}$ τέτοια ώστε $\mathbb{E}\left[\sup_{t\in\mathbb{R}_+}\left\{|a_t^k|^2\right\}\right] < \infty$ για κάθε $k \in \overline{\mathbb{N}}$, εάν $a^k \xrightarrow[k\to\infty]{J_1(\mathbb{R}^d,|\cdot|)} a^\infty$, $\mathbb{P}-$σ.π., τότε

$$\Big(f^k(t, a_t^k, Z_t, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t, \mathcal{L}(a_t^k))\Big)_{t\in\mathbb{R}_+} \xrightarrow[k\to\infty]{J_1(\mathbb{R}^d,|\cdot|)} \Big(f^\infty(t, a_t^\infty, Z_t, \Gamma^{(\mathbb{F}^{\infty,i},\overline{X}^{\infty,i},\Theta^\infty)}(U)_t, \mathcal{L}(a_t^\infty))\Big)_{t\in\mathbb{R}_+} \ \mathbb{P}-\text{σ.π.}$$

Επιπλέον, εάν $\sup_{k\in\overline{\mathbb{N}}}\left\{\|a^k(\omega)\|_\infty\right\}_{k\in\overline{\mathbb{N}}} < \infty$, $\mathbb{P}-$σ.π., τότε

$$\sup_{k\in\overline{\mathbb{N}}}\left\{\left\|\Big(f^k(t, a_t^k, Z_t, \Gamma^{(\mathbb{F}^{k,i},\overline{X}^{k,i},\Theta^k)}(U)_t, \mathcal{L}(a_t^k))\Big)_{t\in\mathbb{R}_+}\right\|_\infty\right\}_{k\in\overline{\mathbb{N}}} < \infty, \ \mathbb{P}-\text{σ.π.}$$

(Σ9) (i) Η ακολουθία $\{\Phi^k\}_{k\in\overline{\mathbb{N}}}$ ικανοποιεί $\Phi^k \xrightarrow[k\to\infty]{|\cdot|} \Phi^\infty := 0$.

(ii) $\widetilde{M}^0(\hat{\beta}) \triangleq \frac{2\sqrt{\frac{2}{\hat{\beta}}+9}\sqrt{\frac{2}{\hat{\beta}}+17+\frac{4}{\hat{\beta}}+35}}{\hat{\beta}} < \frac{1}{4}$.

(Σ10) Ο χρόνος διακοπής T^∞ είναι πεπερασμένος και $T^k \xrightarrow[k\to\infty]{|\cdot|} T^\infty$.

Για να ολοκληρώσουμε δίνουμε το θεώρημα περί της ευστάθειας της οπισθόδρομης διάδοσης του χάους.

Θεώρημα 5.1 (**Ευστάθεια της οπισθόδρομης διάδοσης του χάους**). *Για κάθε $i \in \mathbb{N}$ έχουμε*

$$\Big(Y^{k,i,N}, Z^{k,i,N}\cdot X^{k,i,\circ} + U^{k,i,N} \star \widetilde{\mu}^{X^{k,i,\natural}}, M^{k,i,N}\Big) \xrightarrow[(k,N)\to(\infty,\infty)]{\left(J_1(\mathbb{R}^{d\times 3}),\mathbb{L}^2\right)} \Big(Y^{\infty,i}, Z^{\infty,i}\cdot X^{\infty,i,\circ} + U^{\infty,i} \star \widetilde{\mu}^{X^{\infty,i,\natural}}, 0\Big).$$