National Technical University of Athens

Postgraduate Course:

Physics and Technological Applications

**PARAFERMIONIC BEHAVIOR
IN BOSE-EINSTEIN CONDENSATES**

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Abstract

The aim of this project is to explain the transition from the bosonic to the fermionic behavior of the Bose-Einstein Condensates that were created in an experiment in Rice University. In order to achieve that, we used as a theoritical tool the Theory of the Generalised Parafermionic Oscillators.

In the first chapter we define the Q numbers and the Generalised Parafermionic Oscillators' Theory. Afterwards, in the second and third chapter, we show that this theory is being successfully applied in various nuclear, molecular and superintergarble systems. Since we have been convinced that the theory works fine, in the forth chapter we derive the Gross-Pitaevskii equation which describes the Bose-Einstein Condensates (BECs) and we present the techniques used to create them. At this point we apply the theory of Parafermions in the BEC. Finally we conclude that the non linearity of the energy of the BEC signifies the divergence from the bosonic behavior. Last but not least we point out the similarities of the BECs with the Tonks-Giradeau gas.

Introduction

This project has been created for the purposes of the Master Class program "Physics and Technological Applications" of the "School of Applied Mathematical and Physical Sciences" of N.T.U.A. The first three chapters are based on work which had previously been done by Dr. Bonatsos and Dr. Daskaloyannis. The new part is actually chapter four where we apply the Generalised Parafermionic Oscillators' theory on BECs. The idea for this project was born when Dr Frantzeskakis has asked me " Why a collection of bosons exhibits a fermionic behavior?". This question had been set in mind till I attended a lecture of Dr. Bonatsos about parafermionic oscillators. Afterwards we began to aproach the issue as presented below. For any further information do not hesitate to contact me via email: amartinou@inp.demokritos.gr

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Chapter 1

Generalized deformed oscillators

In this chapter, the concept of the generalized deformed oscillator, of which the generalized parafermionic deformed oscillator is an extension, will be introduced. We shall start with the q-deformed oscillators and Q-deformed oscillators, which are special cases of the generalized deformed oscillator, introduced earlier. In order to do this, q-deformed numbers and Q-deformed numbers have to be introduced first.

1.1 q-numbers

The *q-number* corresponding to the ordinary number x is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{1.1}$$

where q is a parameter. The same definition holds if x is an operator. We remark that q-numbers remain invariant under the substitution $q \to q^{-1}$.

If q is real, q-numbers can easily be put in the form

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)}, \tag{1.2}$$

where $q = e^\tau$ and τ is real.

If q is a phase factor, q-numbers can be written as

$$[x] = \frac{\sin(\tau x)}{\sin(\tau)}, \tag{1.3}$$

where $q = e^{i\tau}$ and τ is real.

In both cases it is clear that in the limit $q \to 1$ (or, equivalently, $\tau \to 0$) q-numbers (or operators) tend to the ordinary numbers (or operators):

$$\lim_{q \to 1}[x] = x. \tag{1.4}$$

A few examples of q-numbers are given here:

$$[0] = 0, \qquad [1] = 1, \qquad [2] = q + q^{-1}, \qquad [3] = q^2 + 1 + q^{-2}. \tag{1.5}$$

Identities between q-numbers exist. They are, however, different from the familiar identities between usual numbers. As an exercise one can show (using the definition of q-numbers) that

$$[a][b + 1] - [b][a + 1] = [a - b]. \tag{1.6}$$

The q-factorial of an integer n is defined as

$$[n]! = [n][n - 1] \dots [2][1]. \tag{1.7}$$

The q-binomial coefficients are defined as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[m - n]![n]!}, \tag{1.8}$$

while the q-binomial expansion is given by

$$[a \pm b]^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} a^{m-k} (\pm b)^k. \tag{1.9}$$

In the limit $q \to 1$ we obviously have

$$[n]! \to n! \qquad \text{and} \qquad \begin{bmatrix} m \\ n \end{bmatrix} \to \binom{m}{n}, \tag{1.10}$$

where $n!$ and $\binom{m}{n}$ are the standard factorial and binomial coefficients respectively.

It should be noticed that two-parameter deformed numbers have also been introduced

$$[x]_{p,q} = \frac{q^x - p^{-x}}{q - p^{-1}}. \tag{1.11}$$

In the special case $p = q$ they reduce to the usual q-numbers.

1.2 q-deformed elementary functions

In addition to q-deformed numbers and operators, q-deformed elementary functions can be introduced. The q-exponential function is defined as

$$e_q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]!} x^n, \tag{1.12}$$

while the q-trigonometric functions are defined as

$$\sin_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n + 1]!}, \qquad \cos_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]!}. \tag{1.13}$$

It should also be noticed that q-deformed polynomials, such as q-deformed Hermite polynomials and q-deformed Laguerre polynomials also exist (see [1] for references).

The definitions of the q-exponential function and the q-trigonometric functions given above are not unique; for a different set of definitions, based on the Tsallis statistics, see [2].

1.3 Q-numbers

The definition of q-numbers given in sec. 2 is not the only possible one. We have already seen the two-parameter deformed numbers, defined in eq. (1.11). Furthermore, a different definition of quantum numbers (the Q-numbers) has been used in mathematics since the early ninetenth century, with rich literature existing on this subject [3, 4]. *Q-numbers* are defined as

$$[x]_Q = \frac{Q^x - 1}{Q - 1},$$ (1.14)

where x can be a number or an operator and Q is a deformation parameter. Q is a real number ($Q \neq 0, 1$). The notation $Q = e^T$, where T a real number, will be often used. The subscript Q will be used in this review in order to distinguish deformed numbers defined as in eq. (6.1) from these defined by eq. (2.1). It is clear that in the limit $Q \to 1$ (or, equivalently, $T \to 0$) Q-numbers become ordinary numbers, i.e. $[x]_Q \to x$.

A few examples of Q-numbers are given here:

$$[0]_Q = 0, \qquad [1]_Q = 1, \qquad [2]_Q = Q + 1, \qquad [3]_Q = Q^2 + Q + 1.$$ (1.15)

Q-numbers clearly do not remain invariant under the substitution $Q \to Q^{-1}$. One can easily prove that

$$[x]_Q = Q^{x-1} [x]_{1/Q}.$$ (1.16)

Q-numbers are connected to q-numbers through the relation [6]

$$[x] = q^{1-x}[x]_Q, \qquad \text{with} \qquad Q = q^2.$$ (1.17)

The definitions of Q-factorials and Q-binomial coefficients still look like the ones given in eqs. (1.7)–(1.8):

$$[n]_Q! = [n]_Q[n-1]_Q \ldots [1]_Q,$$ (1.18)

$$\begin{bmatrix} m \\ n \end{bmatrix}_Q = \frac{[m]_Q!}{[m-n]_Q![n]_Q!}.$$ (1.19)

As it can be easily seen from eq. (1.14) under the substitution $Q \to Q^{-1}$ one obtains

$$[n]_Q! = Q^{n(n-1)/2} [n]_{1/Q},$$ (1.20)

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_Q = Q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{1/Q}.$$ (1.21)

Q-factorials are connected to q-factorials by

$$[n]! = q^{-n(n-1)/2}[n]_Q!, \qquad \text{with} \qquad Q = q^2.$$ (1.22)

1.4 Q-deformed elementary functions

The definitions of Q-deformed elementary functions [3] look similar to these given in sec. 3. The *Q-deformed exponential function* is defined as

$$e_Q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]_Q!} x^n, \tag{1.23}$$

and satisfies the property

$$e_Q(x)e_{1/Q}(-x) = 1. \tag{1.24}$$

(Notice that $e_Q(x)e_Q(-x) \neq 1$.)

The *Q-deformed trigonometric functions* are defined as

$$\sin_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_Q!}, \tag{1.25}$$

$$\cos_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_Q!}. \tag{1.26}$$

One can easily show that

$$\sin_Q(x) = \frac{1}{2i} \left(e_Q(ix) - e_Q(-ix) \right), \tag{1.27}$$

$$\cos_Q(x) = \frac{1}{2} \left(e_Q(ix) + e_Q(-ix) \right). \tag{1.28}$$

Instead of the familiar identity $\sin^2(x) + \cos^2(x) = 1$ one has

$$\sin_Q(x) \sin_{1/Q}(x) + \cos_Q(x) \cos_{1/Q}(x) = 1. \tag{1.29}$$

The above defined Q-deformed functions are examples of Q-deformed hypergeometric functions[3, 5]. In addition Q-deformed polynomials, which are counterparts of the ordinary non deformed polynomials, can be defined, such as Q-deformed Hermite polynomials and Q-deformed Laguerre polynomials [3].

1.5   The q-deformed harmonic oscillator

The interest for possible applications of quantum algebras in physics has been triggered in 1989 by the introduction of the q-deformed harmonic oscillator [7, 8, 9], of which earlier equivalent versions existed [10, 11].

The q-deformed harmonic oscillator [7, 8, 9] is defined in terms of the creation and annihilation operators $a^\dagger$ and a and the number operator N, which satisfy the commutation relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{1.30}$$

$$aa^\dagger - q^{\mp 1}a^\dagger a = q^{\pm N}. \tag{1.31}$$

In addition the following conditions of hermitian conjugation hold (in the case of q being a real number or q being a root of unity)

$$(a^\dagger)^\dagger = a, \qquad N^\dagger = N. \tag{1.32}$$

Eq. (1.30) is the same as in ordinary quantum mechanics, while eq. (1.31) is modified by the presence of the deformation parameter q. For $q \to 1$ it is clear that eq. (1.32) goes to the usual boson commutation relation $[a, a^\dagger] = 1$. An immediate consequence of (1.32) is that

$$a^\dagger a = [N], \quad aa^\dagger = [N+1]. \tag{1.33}$$

Thus the number operator N is *not* equal to $a^\dagger a$, as in the ordinary case. The operators $a^\dagger$ and a are referred to as *q-deformed boson creation and annihilation operators* respectively.

The basis of the Fock space is defined by repeated action of the creation operator $a^\dagger$ on the vacuum state, which is annihilated by a:

$$a|0\rangle = 0, \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle. \tag{1.34}$$

The action of the operators on the basis is given by

$$N|n\rangle = n|n\rangle, \tag{1.35}$$

$$a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle, \tag{1.36}$$

$$a|n\rangle = \sqrt{[n]}|n-1\rangle. \tag{1.37}$$

We remark that these equations look very similar to the ones of the ordinary case, the only difference being that q-numbers appear under the square roots instead of usual numbers.

The Hamiltonian of the q-deformed harmonic oscillator is

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a), \tag{1.38}$$

and its eigenvalues in the basis given above are

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{1.39}$$

One can easily see that for q real the energy eigenvalues increase more rapidly than the ordinary case, in which the spectrum is equidistant, i.e. the spectrum gets "expanded". In contrast, for q being a phase factor ($q = e^{i\tau}$ with τ real) the eigenvalues of the energy increase less rapidly than the ordinary (equidistant) case, i.e. the spectrum is "compressed". In particular, for q real ($q = e^\tau$) the eigenvalues can be written as

$$E(n) = \frac{\hbar\omega}{2}\frac{\sinh\left(\tau\left(n + \frac{1}{2}\right)\right)}{\sinh\frac{\tau}{2}}, \tag{1.40}$$

while for q being a phase factor $(q = e^{i\tau})$ one has

$$E(n) = \frac{\hbar\omega}{2} \frac{\sin\left(\tau\left(n + \frac{1}{2}\right)\right)}{\sin\frac{\tau}{2}}. \tag{1.41}$$

In both cases in the limit $q \to 1$ $(\tau \to 0)$ the ordinary expression

$$E(n) = \hbar\omega\left(n + \frac{1}{2}\right) \tag{1.42}$$

is recovered.

In addition, the following commutation relation holds

$$[a, a^\dagger] = [N + 1] - [N]. \tag{10.14}$$

For q being a phase factor, this commutation relation takes the form

$$[a, a^\dagger] = \frac{\cos\frac{(2N+1)\tau}{2}}{\cos\frac{\tau}{2}}. \tag{1.43}$$

It is useful to notice that the q-deformed boson operators $a^\dagger$ and a can be expressed in terms of usual boson operators $\alpha^\dagger$ and α (satisfying $[\alpha, \alpha^\dagger] = 1$ and $N = \alpha^\dagger\alpha$) through the relations [12]

$$a = \sqrt{\frac{[N+1]}{N+1}}\alpha = \alpha\sqrt{\frac{[N]}{N}}, \qquad a^\dagger = \alpha^\dagger\sqrt{\frac{[N+1]}{N+1}} = \sqrt{\frac{[N]}{N}}\alpha^\dagger. \tag{1.44}$$

The square root factors in the last equation have been called *q-deforming functionals*.

For q being a primitive root of unity, i.e. $q = e^{2\pi i/k}$ $(k = 2, 3, \ldots)$, it is clear the the representation of eqs (10.5)–(10.8) becomes finite-dimensional and has dimension k, since only the vectors $|0>, |1>, \ldots, |k-1>$ can be present. This case has been related to the system of two anyons [13]. In what follows we are going to assume that q is *not* a primitive root of unity.

1.6   The Q-deformed harmonic oscillator

A different version of the deformed harmonic oscillator can be obtained by defining [12, 14, 15] the operators b, b^+ through the equations

$$a = q^{1/2}bq^{-N/2}, \qquad a^\dagger = q^{1/2}q^{-N/2}b^\dagger. \tag{1.45}$$

Eqs. (1.30) and (1.31) then give

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \tag{1.46}$$

$$bb^\dagger - q^2 b^\dagger b = 1. \tag{1.47}$$

This oscillator has been first introduced by Arik and Coon [10] and later considered also by Kuryshkin [11]. One then easily finds that

$$b^\dagger b = [N]_Q, \quad bb^\dagger = [N+1]_Q, \tag{1.48}$$

where $Q = q^2$ and Q-numbers are defined in (6.1). The basis is defined by

$$b|0\rangle = 0, \qquad |n\rangle = \frac{(b^\dagger)^n}{\sqrt{[n]_Q!}}|0\rangle, \tag{1.49}$$

while the action of the operators on the basis is given by

$$N|n\rangle = n|n\rangle, \tag{1.50}$$

$$b^\dagger|n\rangle = \sqrt{[n+1]_Q}|n+1\rangle, \tag{1.51}$$

$$b|n\rangle = \sqrt{[n]_Q}|n-1\rangle. \tag{1.52}$$

The Hamiltonian of the corresponding deformed harmonic oscillator has the form

$$H = \frac{\hbar\omega}{2}(bb^\dagger + b^\dagger b), \tag{1.53}$$

the eigenvalues of which are

$$E(n) = \frac{\hbar\omega}{2}([n]_Q + [n+1]_Q). \tag{1.54}$$

One can easily see that for $Q = e^T$, where $T > 0$ and real, the spectrum increases more rapidly than the ordinary (equidistant) spectrum, while for $Q = e^T$, with $T < 0$ and real, the spectrum is increasing less rapidly than the ordinary (equidistant) case.

From the above relations, it is clear that the following commutation relation holds

$$[b, b^\dagger] = Q^N. \tag{1.55}$$

1.7 The generalized deformed oscillator

In addition to the oscillators described in the last two sections, many kinds of deformed oscillators have been introduced in the literature (see [16] for a list). All of them can be accommodated within the common mathematical framework of the *generalized deformed oscillator* [17, 18], which is defined as the algebra generated by the operators $\{1, a, a^\dagger, N\}$ and the *structure function* $\Phi(x)$, satisfying the relations

$$[a, N] = a, \qquad [a^\dagger, N] = -a^\dagger, \tag{1.56}$$

$$a^\dagger a = \Phi(N) = [N], \qquad aa^\dagger = \Phi(N+1) = [N+1], \tag{1.57}$$

where $\Phi(x)$ is a positive analytic function with $\Phi(0) = 0$ and N is the number operator. From eq. (1.57) we conclude that

$$N = \Phi^{-1}(a^\dagger a), \tag{1.58}$$

and that the following commutation and anticommutation relations are obviously satisfied:

$$[a, a^\dagger] = [N + 1] - [N], \qquad \{a, a^\dagger\} = [N + 1] + [N]. \tag{1.59}$$

The *structure function* $\Phi(x)$ is characteristic to the deformation scheme. In Table 1 the structure functions corresponding to different deformed oscillators are given. They will be further discussed at the end of this section.

It can be proved that the generalized deformed algebras possess a Fock space of eigenvectors
$|0>, |1>, \ldots, |n>, \ldots$ of the number operator N

$$N|n> = n|n>, \quad <n|m> = \delta_{nm}, \tag{1.60}$$

if the *vacuum state* $|0>$ satisfies the following relation:

$$a|0> = 0. \tag{1.61}$$

These eigenvectors are generated by the formula:

$$|n> = \frac{1}{\sqrt{[n]!}} \left(a^\dagger\right)^n |0>, \tag{1.62}$$

where

$$[n]! = \prod_{k=1}^n [k] = \prod_{k=1}^n \Phi(k). \tag{1.63}$$

The generators $a^\dagger$ and a are the creation and annihilation operators of this deformed oscillator algebra:

$$a|n> = \sqrt{[n]}a|n-1>, \qquad a^\dagger|n> = \sqrt{[n+1]}a|n+1>. \tag{1.64}$$

These eigenvectors are also eigenvectors of the energy operator

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a), \tag{1.65}$$

corresponding to the eigenvalues

$$E(n) = \frac{\hbar\omega}{2}(\Phi(n) + \Phi(n+1)) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{1.66}$$

For

$$\Phi(n) = n \tag{1.67}$$

one obtains the results for the ordinary harmonic oscillator. For

$$\Phi(n) = \frac{q^n - q^{-n}}{q - q^{-1}} = [n] \tag{1.68}$$

one has the results for the q-deformed harmonic oscillator, while the choice

$$\Phi(n) = \frac{Q^n - 1}{Q - 1} = [n]_Q \tag{1.69}$$

leads to the results of the Q-deformed harmonic oscillator. Many more cases are shown in Table 1, on which the following comments apply:

i) Two-parameter deformed oscillators have been introduced [19, 20, 21], in analogy to the one-parameter deformed oscillators.

ii) Parafermionic oscillators [22] of order p represent particles of which the maximum number which can occupy the same state is p. Parabosonic oscillators [22] can also be introduced. The deformed oscillator realization of the parafermionic and parabosonic oscillator has been studied in [16], while in [23] an equivalent realization for the parabosonic oscillator has been introduced.

iii) q-deformed versions of the parafermionic and parabosonic oscillators have also been introduced [24, 25].

iv) q-deformed versions of the fermionic algebra [26] have also been introduced [27, 28, 29, 30, 31, 32], as well as q-deformed versions of generalized q-deformed fermionic algebras [33]. It has been proved, however, that q-deformed fermions are fully equivalent to the ordinary fermions [34, 35].

1.8 Generalized deformed parafermionic oscillators

It turns out that the generalized deformed $su_\Phi(2)$ algebras mentioned in the last section are related to generalized deformed parafermionic oscillators, which we will therefore describe here.

It has been proved [41] that any generalized deformed parafermionic algebra of order p can be written as a generalized oscillator (sec. 12) with structure function

$$\Phi(x) = x(\rho + 1 - x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots), \tag{1.70}$$

where λ, μ, ν, ρ, σ, $\ldots$ are real constants satisfying the conditions

$$\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \qquad x \in \{1, 2, \ldots, p\}. \tag{1.71}$$

Considering an $su_\Phi(2)$ algebra [36] with structure function

$$\Phi(J_0(J_0 + 1)) = A J_0(J_0 + 1) + B(J_0(J_0 + 1))^2 + C(J_0(J_0 + 1))^3, \tag{1.72}$$

and making the correspondence

$$J_+ \to A^\dagger, \qquad J_- \to A, \qquad J_0 \to N, \tag{1.73}$$

one finds that the $su_\Phi(2)$ algebra is equivalent to a generalized deformed parafermionic oscillator of the form

$$\Phi(N) = N(\rho + 1 - N)$$

$$[-(\rho^2(\rho+1)C+\rho B)+(\rho^3 C+(\rho-1)B)N+((\rho^2-\rho+1)C+B)N^2+(\rho-2)CN^3+CN^4], \quad (1.74)$$

if the condition

$$A + \rho(\rho+1)B + \rho^2(\rho+1)^2 C = 0 \tag{1.75}$$

holds. The condition of eq. (1.71) is always satisfied for $B > 0$ and $C > 0$.

In the special case of $C = 0$ one finds that the $\mathrm{su}_\Phi(2)$ algebra with structure function

$$\Phi(J_0(J_0+1)) = A J_0(J_0+1) + B(J_0(J_0+1))^2 \tag{1.76}$$

is equivalent to a generalized deformed parafermionic oscillator characterized by

$$\Phi(N) = BN(\rho+1-N)(-\rho+(\rho-1)N+N^2), \tag{1.77}$$

if the condition

$$A + \rho(\rho+1)B = 0 \tag{1.78}$$

is satisfied. The condition of eq. (18.2) is satisfied for $B > 0$.

Including higher powers of $J_0(J_0+1)$ in eq. (1.72) results in higher powers of N in eq. (1.74) and higher powers of $\rho(\rho+1)$ in eq. (1.75). If, however, one sets $B = 0$ in eq. (1.76), then eq. (1.77) vanishes, indicating that no parafermionic oscillator equivalent to the usual su(2) rotator can be constructed.

It turns out that several other mathematical structures, like the finite W algebras $\bar{\mathrm{W}}_0$ [38] and $\mathrm{W}_3^{(2)}$ (see subsec. 36.5) can be put into the generalized deformed parafermionic oscillator form. The same is true for several physical systems, such as the isotropic oscillator and the Kepler problem in a 2-dim curved space with constant curvature [39, 40], and the Fokas–Lagerstrom [42], Smorodinsky–Winternitz [43], and Holt [44] potentials. Further details can be found in [37].

A detailed discussion of the representation theory of several deformed oscillator algebras can be found in [45, 46].

Table 1.1: Structure functions of special deformation schemes

	$\Phi(x)$	Reference
i	x	harmonic oscillator, bosonic algebra
ii	$\frac{q^x - q^{-x}}{q - q^{-1}}$	q-deformed harmonic oscillator [7, 8]
iii	$\frac{q^x - 1}{q - 1}$	Arik–Coon, Kuryshkin, or Q-deformed oscillator [10, 11]
iv	$\frac{q^x - p^{-x}}{q - p^{-1}}$	2-parameter deformed oscillator [19, 20, 21]
v	$x(\rho + 1 - x)$	parafermionic oscillator [22]
vi	$\frac{\sinh(\tau x)\sinh(\tau(p+1-x))}{\sinh^2(\tau)}$	q-deformed parafermionic oscillator [24, 25]
vii	$x\cos^2(\pi x/2) + (x + \rho - 1)\sin^2(\pi x/2)$	parabosonic oscillator [22]
viii	$\frac{\sinh(\tau x)}{\sinh(\tau)}\frac{\cosh(\tau(x+2N_0-1))}{\cosh(\tau)}\cos^2(\pi x/2) + $ $+\frac{\sinh(\tau(x+2N_0-1))}{\sinh(\tau)}\frac{\cosh(\tau x)}{\cosh(\tau)}\sin^2(\pi x/2)$	q-deformed parabosonic oscillator [24, 25]
ix	$\sin^2 \pi x/2$	fermionic algebra [26]
x	$q^{x-1}\sin^2 \pi x/2$	q-deformed fermionic algebra [27, 28, 29, 30, 31, 32]
xi	$\frac{1 - (-q)^x}{1 + q}$	generalized q-deformed fermionic algebra [33]
xii	x^n	[17]
xiii	$\frac{sn(\tau x)}{sn(\tau)}$	[17]

Chapter 2

Generalized parafermionic oscillators in nuclear and molecular systems

In this chapter some cases in which generalized parafermionic oscillators appear in nuclear and molecular systems will be studied. It will be seen that in nuclear shells the order of the parafermions is related to the size of the shell.

2.1 The generalized parafermionic oscillator

The structure function of the generalized deformed parafermionic algebra of order p is [47]

$$\Phi(N) = N(\rho + 1 - N)(\lambda + \mu N + \nu N^2 + \rho N^3 + \ldots), \tag{2.1}$$

where λ, μ, ν, ρ are real constants satisfying the conditions $\lambda + \mu N + \nu N^2 + \rho N^3 + \ldots > 0$ if $N \in \{1, 2, \ldots, p\}$.

Using these generalized deformed parafermions as building blocks, one can consider the Bose-like Hamiltonian

$$H = \frac{1}{2}\{a, a^\dagger\} = \frac{1}{2}(aa^\dagger + a^\dagger a), \tag{2.2}$$

possessing the energy eigenvalues

$$E(N) = \frac{1}{2}(\Phi(N) + \Phi(N+1)), \qquad = 0, 1, 2, \ldots \tag{2.3}$$

Using Eq. (2.1), and keeping terms up to N^4, the energy eigenvalues are written as

$$E(N) = (\lambda+\mu+\nu)\rho+((2\lambda+2\mu+3\nu)\rho-\mu-\nu)N+((2\mu+3\nu)\rho-2\lambda-\mu-3\nu)N^2+(2\nu\rho-2\mu-2\nu)N^3-2\nu N^4. \tag{2.4}$$

One can also consider the Fermi-like Hamiltonian

$$H = \frac{1}{2}[a^\dagger, a] = \frac{1}{2}(a^\dagger a - aa^\dagger), \tag{2.5}$$

possessing the energy eigenvalues

$$E(N) = \frac{1}{2}(\Phi(N) - \Phi(N+1)), \qquad N = 0, 1, 2, \ldots \tag{2.6}$$

Using again Eq. (2.1), and keeping terms up to [4], the energy eigenvalues are written as

$$E(N) = -(\lambda + \mu + \nu)\rho - ((2\mu + 3\nu)\rho - 2\lambda - \mu - \nu)N - (3\nu\rho - 3\mu - 3\nu)n^2 + 4\nu N^3. \quad (2.7)$$

2.2 The Morse potential

The energy eigenvalues of the Morse potential [48]

$$V(x) = e^{-2ax} - 2e^{-ax}, \quad (2.8)$$

are known to have the form [49]

$$E(N) = \left(N + \frac{1}{2}\right) - \alpha\left(N + \frac{1}{2}\right)^2, \quad (2.9)$$

where a, α are constants.

Trying to describe the Morse oscillator in terms of a Fermi-like oscillator built with generalized deformed parafermions, one should equate the coefficients of the various powers of n in Eq. (2.9) with the coefficients in Eq. (2.7). There is no need the equate the constant terms, since they just correspond to a shift of the energy scale. Solving the relevant simple system of linear equations one obtains the conditions

$$\lambda = 1 - \frac{2}{3}(\rho + 1)\alpha, \qquad \mu = -\frac{2}{3}\alpha, \qquad \nu = 0, \quad (2.10)$$

which are equivalent to the result obtained in Ref. [47].

Alternatively, trying to describe the Morse oscillator in terms of a Bose-like oscillator built with generalized deformed parafermions, one should equate the spectrum of Eq. (2.9) with the spectrum of Eq. (2.2). Following the same procedure one obtains the conditions

$$\lambda = \alpha, \qquad \mu = \nu = 0, \qquad \rho = \frac{1}{\alpha} - 1. \quad (2.11)$$

Comparing the two results we see the following.

1) In the Fermi-like case the coefficients λ, μ, ν of the generalized deformed parafermionic oscillator are determined in terms of the parameter α appearing in the spectrum and the order of the parafermions ρ. In other words, for each Morse oscillator with a specific value of α we can have many different Fermi-like Hamiltonians built with generalized deformed parafermionic oscillators of different orders.

2) In the Bose-like case we have an extra equation in the linear system. As a result, the coefficients λ, μ, ν of the generalized deformed parafermionic oscillator and the order of the parafermions, ρ, are determined in terms of the parameter α appearing in the spectrum. Since ρ has to be a positive integer, only values of α leading to such values of ρ can be considered. In other words, for a series of Morse oscillators corresponding to specific values of α we obtain a corresponding series of Bose-like Hamiltonians built with generalized deformed parafermionic oscillators of specific order.

A qualitative discussion of the latter case is in order. For Morse oscillators deviating little from the harmonic oscillator behavior, the parameter α obtains small values, thus leading through Eq. (2.11) to large values of the order ρ, corresponding to near-bosonic behavior. In the limiting case of $\alpha \to 0$ one obtains $\rho \to \infty$, i.e., pure harmonic behavior. On the contrary, large deviations from the harmonic behavior correspond to larger values of α, leading to smaller values of ρ, i.e., to behaviors closer to the fermionic one.

2.3 Pairing in a single-j nuclear shell

n the usual formulation of the theory of pairing in a single-j shell [50], fermion pairs of angular momentum $J = 0$ are created by the pair creation operators

$$S^+ = \frac{1}{\sqrt{\Omega}} \sum_{m>0} (-1)^{j+m} a_{jm}^+ a_{j-m}^+,$$

(2.12)

where a_{jm}^+ are fermion creation operators and $2\Omega = 2j + 1$ is the degeneracy of the shell. In addition, pairs of nonzero angular momentum are created by the $\Omega - 1$ operators

$$B_J^+ = \sum_{m>0} (-1)^{j+m} (jmj - m|J0) a_{jm}^+ a_{j-m}^+,$$

(2.13)

where $(jmj - m|J0)$ are the usual Clebsch Gordan coefficients. The fermion number operator is defined as

$$N_F = \sum_m a_{jm}^+ a_{jm} = \sum_{m>0} (a_{jm}^+ a_{jm} + a_{j-m}^+ a_{j-m}).$$

(2.14)

The $J = 0$ pair creation and annihilation operators satisfy the commutation relation

$$[S, S^+] = 1 - \frac{N_F}{\Omega},$$

(2.15)

while the pairing Hamiltonian is

$$H = -G\Omega S^+ S.$$

(2.16)

The seniority V_F is defined as the number of fermions not coupled to $J = 0$. If only pairs of $J = 0$ are present (i.e. $V_F = 0$), the eigenvalues of the Hamiltonian are

$$E(N_F, V_F = 0) = -G\Omega \left(\frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right).$$

(2.17)

For non-zero seniority the eigenvalues of the Hamoltonian are

$$E(N_F, V_F) = -\frac{G}{4}(N_F - V_F)(2\Omega - N_F - V_F + 2).$$

(2.18)

We denote the operators N_F, V_F and their eigenvalues by the same symbol for simplicity.

For the case of nonzero seniority, one observes that Eq. (2.18) can be written as

$$E(N_F, V_F) = G\Omega \left(\frac{V_F}{2} + \frac{V_F}{2\Omega} - \frac{V_F^2}{4\Omega} \right) - G\Omega \left(\frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right),$$

(2.19)

i.e. it can be separated into two parts, formally identical to each other.

Let us first consider the case in which only $J = 0$ pairs are present. Equating the spectrum of Eq. (2.17) to Eq. (2.4), one finds

$$\lambda = -\frac{1}{8}, \qquad \rho = 2\Omega + 2 = 2j + 3. \tag{2.20}$$

Therefore, fermion pairs with $J = 0$ in a single-j shell can be described as parafermions of order $\rho = 2j + 3$ through a Bose-like Hamiltonian. Since j is half-integer, ρ is an integer number.

Considering now the pairs with non-zero angular momentum, we see that the first term in Eq. (2.19), when equated to Eq. (2.4), gives

$$\lambda = \frac{1}{8}, \qquad \rho = 2\Omega + 2 = 2j + 3. \tag{2.21}$$

Thus Eq. (2.19) can be described in terms of two parafermionic oscillators of order ρ.

2.4   Fermion monopole and quadrupole pairing model

In the fermion monopole and quadrupole pairing model with SO(8) symmetry [51], fermions occupy orbitals with angular momenta $j = k + \frac{3}{2}$, $k + \frac{1}{2}$, $k - \frac{1}{2}$, $k - \frac{3}{2}$, where k is the pseudo-angular momentum, which is an integer number. The total number of single-particle states is $2\Omega = 4(2k + 1)$.

Within this space one defines monopole and quadrupole pair creation operators

$$S^\dagger = \frac{1}{2} \sum_{j,m} (-1)^{j-m} a^\dagger_{j,m} a^\dagger_{j,-m}, \tag{2.22}$$

$$D^\dagger_\mu = \sum_{j,j'} (-1)^{j+k+\frac{3}{2}} \sqrt{(2j+1)(2j'+1)} \left\{ \begin{matrix} j & j' & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{matrix} \right\} [a^\dagger_j a^\dagger_{j'}]^{(2)}_\mu, \tag{2.23}$$

where $a^\dagger_{j,m}$ is the creation operator of a fermion with orbital angular momentum j and projection m, $[a^\dagger_j a^\dagger_{j'}]^{(J)}_\mu$ is the creation operator for a fermion pair with orbital angular momentum J and projection μ, and the symbol in curly brackets is a 6-j symbol.

In addition, multipole operators of rank $l = 0$, 1, 2, 3 are defined as

$$P^{(l)}_\mu = \sum_{j,j'} (-1)^{l+j+k+\frac{3}{2}} \sqrt{(2j+1)(2j'+1)} \left\{ \begin{matrix} j & j' & l \\ \frac{3}{2} & \frac{3}{2} & k \end{matrix} \right\} [a^\dagger_j \tilde{a}_{j'}]^{(l)}_\mu, \tag{2.24}$$

where $\tilde{a}_{j,m} = (-1)^{j+m} a_{j,-m}$.

The pair creation operators defined above, together with the corresponding annihilation operators, and the multipole operators defined above, close the algebra SO(8), which contains three chains of subalgebras involving SO(3), the SO(5) (vibrational limit), SO(6) (γ-unstable rotor limit), and SO(7) (transitional limit) chains.

The SO(7) limit has attracted attention, since it is an exactly soluble symmetry limit, which in addition is a transitional limit betweem vibrators and γ-unstable rotors. The relevant chain reards

$$SO(8) \supset SO(7) \supset SO(5) \otimes U(1) \supset SO(3) \otimes U(1). \tag{2.25}$$

In the SO(7) limit, the allowed irreducible representations of SO(7) are $(\Lambda, 0, 0)$, with $\Lambda = 0, 1, \ldots, \frac{\Omega}{2}$. The allowed values of N (the total number of pairs) for a given Λ are $N = \frac{\Omega}{2} - \Lambda, \frac{\Omega}{2} - \Lambda + 1, \ldots, \frac{\Omega}{2} + \Lambda$. As a consequence
i) the irrep $\Lambda = \frac{\Omega}{2}$ occurs for all numbers of pairs $N = 0, 1, 2, \ldots, \Omega$,
ii) the irrep $\Lambda = \frac{\Omega}{2} - 1$ occurs for the numbers of pairs $N = 1, 2, \ldots, \Omega - 1$, etc
iii) the irrep $\Lambda = 0$ occurs only at midshell, $N = \frac{\Omega}{2}$.

The additional quantum numbers needed in the SO(7) chain are the SO(5) quantum numbers τ and n_Δ, and the SO(3) quantum numbers J and M. The quantum number τ represents the number of quadrupole pairs not coupled to angular momentum zero, while the quantum number n_Δ represents the number of quadrupole triplets not coupled to angular momentum zero.

The most general Hamiltonian in this symmetry limit is

$$H_7 = c_8 C_8 + c_7 C_7 + c_5 C_5 + c_3 C_3 + c_1 C_1, \tag{2.26}$$

where c_i are free coefficients, while C_i are the Casimir operators of the relevant algebras in i dimensions. For example, C_7 stands for the second order Casimir operator of SO(7).

The energy eigenvalues in the basis described above are

$$E_7(N, \Lambda, \tau, n_\Delta, J) = c_8 \frac{\Omega}{2}\left(\frac{\Omega}{2} + 6\right) + c_7 \Lambda(\Lambda+5) + c_5 \tau(\tau+3) + c_3 J(J+1) + c_1\left(\frac{\Omega}{2} - N\right). \tag{2.27}$$

For the ground state one has $\tau = 0$, $n_\Delta =$, $J = 0$, while

$$\Lambda_0 = \frac{\Omega}{2} - N, \qquad N \leq \frac{\Omega}{2}, \tag{2.28}$$

$$\Lambda_0 = N - \frac{\Omega}{2} = \frac{\Omega}{2} - \bar{N}, \qquad N \geq \frac{\Omega}{2}, \tag{2.29}$$

where $\bar{N} = \Omega - N$ is the number of hole pairs. Therefore the excitation energy in a given nucleus is

$$E_7^*(N, \Lambda, \tau, n_\Delta, J) = E_7(N, \Lambda, \tau, n_\Delta, J) - E_7(N, \Lambda_0, \tau = 0, n_\Delta = 0, J = 0) \tag{2.30}$$

$$= c_7(\Lambda - \Lambda_0)(\Lambda + \Lambda_0 + 5) + c_5 \tau(\tau + 3) + c_3 J(J + 1). \tag{2.31}$$

Using Eq. (2.28) the excitation energy becomes

$$E_7^*(N, \Lambda, \tau, n_\Delta, J) = \tag{2.32}$$

$$c_7\left(\Lambda - \frac{\Omega}{2} + N\right)\left(\Lambda + \frac{\Omega}{2} - N + 5\right) + c_5 \tau(\tau + 3) + c_3 J(J + 1), \quad N \leq \frac{\Omega}{2}, \tag{2.33}$$

$$E_7^*(N, \Lambda, \tau, n_\Delta, J) = \tag{2.34}$$

$$c_7\left(\Lambda - \frac{\Omega}{2} + \bar{N}\right)\left(\Lambda + \frac{\Omega}{2} - \bar{N} + 5\right) + c_5 \tau(\tau + 3) + c_3 J(J + 1), \quad N \geq \frac{\Omega}{2}. \tag{2.35}$$

We now focus attention on the states with $\tau = 0$, $J = 0$, i.e., on the states with no pairs not coupled to angular momentum zero. (In other words, all pairs are coupled to angular momentum zero.) Equating, as usually, the present spectrum to Eq. (2.4), we obtain

$$\lambda = 1, \qquad \rho = \Omega + 5 = 2(2k + 1) + 5. \tag{2.36}$$

This result is valid for any Λ, since Λ cancels out during the calculation.

We see that, in this shell model space, fermion pairs coupled to angular momentum zero behave as parafermions of order $\rho = 2(2k + 1) + 5$.

Within a given nucleus (given N, Ω), this result regards $\tau = 0$, $J = 0$ states lying in different SO(7) irreps (characterized by different values of Λ).

2.5   Nuclear shells in j-j coupling

In a single-j nuclear shell one can define [52] pair creation operators

$$A_M^{(J)\dagger} = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm'|JM) a_{jm}^{\dagger} a_{jm'}^{\dagger}, \tag{2.37}$$

and multipole operators

$$B_M^{(J)} = \frac{1}{\sqrt{2J+1}} \sum_{mm'} (jmj - m'|JM)(-1)^{j-m'} a_{jm}^{\dagger} a_{jm'}, \tag{2.38}$$

where $a_{jm}^{\dagger}$ (a_{jm}) are fermion creation (annihilation) operators and $(jmjm'|JM)$ are the usual Clebsch–Gordan coefficients.

These operators, together with the corresponding fermion pair annihilation operators, close the algebra SO($2(2j+1)$). The multipole operators alone close the algebra U($2j+1$). Defining the size of the shell as $2\Omega = 2j + 1$, the relevant chain reads SO(4Ω)$\supset$U(2Ω).

A simple two-body Hamiltonian can be the second order Casimir operator of U(2Ω). Because of the Pauli principle, only completely antisymmetric irreps $[1^N]$ of U(2Ω) will occur, with $N \leq 2\Omega$ [53]. The eigenvalues of the second order Casimir operator in these irreps are [54, 55]

$$C_2 = N(2\Omega - N + 1). \tag{2.39}$$

Equating the energy (i.e., the eigenvalues of the second order Casimir operator) to Eq. (2.4) we obtain

$$\lambda = 1, \qquad \rho = 2\Omega + 1. \tag{2.40}$$

Therefore one can try to represent fermion pairs in a single-j nuclear shell in terms of parafermions of order $\rho = 2\Omega + 1 = 2j + 2$.

In each and every case we have studied we found that the order parameter ρ is proportional to the size of the shell:

$$\rho \sim \Omega \tag{2.41}$$

Chapter 3

Generalized parafermionic oscillators in superintegrable and mathematical systems

In this chapter, the appearance of generalized parafermionic oscillators in some superintegrable systems, as well as within other constructions of mathematical physics, will be studied.

3.1 Isotropic harmonic oscillator in a 2-dim curved space

The algebra of the isotropic harmonic oscillator in a 2-dim curved space with constant curvature λ for finite representations can be put in the form [71]

$$\Phi(N) = 4N(\rho + 1 - N)\left(\lambda(\rho + 1 - N) + \sqrt{\omega^2 + \lambda^2/4}\right)\left(\lambda N + \sqrt{\omega^2 + \lambda^2/4}\right), \qquad (3.1)$$

the relevant energy eigenvalues being

$$E_\rho = \sqrt{\omega^2 + \frac{\lambda^2}{4}}(\rho + 1) + \frac{\lambda}{2}(\rho + 1)^2, \qquad (3.2)$$

where ω is the angular frequency of the oscillator. It is clear that the condition of eq. (1.71) is satisfied without any further restrictions.

3.2 The Kepler problem in a 2-dim curved space

The algebra of the Kepler problem in a 2-dim curved space with constant curvature λ for finite representations can be put in the form [71]

$$\Phi(N) = N(\rho + 1 - N)\left(\frac{4\mu^2}{(\rho + 1)^2} + \lambda\frac{(\rho + 1 - 2N)^2}{4}\right), \qquad (3.3)$$

the corresponding energy eigenvalues being

$$E_\rho = -\frac{2\mu^2}{(\rho+1)^2} + \lambda \frac{\rho(\rho+2)}{8}, \tag{3.4}$$

where μ is the coefficient of the $-1/r$ term in the Hamiltonian. It is clear that the restrictions of eq. (1.71) are satisfied automatically.

3.3   The Fokas–Lagerstrom potential

The Fokas–Lagerstrom potential [64] is described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{x^2}{2} + \frac{y^2}{18}. \tag{3.5}$$

It is therefore an anisotropic oscillator with ratio of frequencies 3:1. For finite representations it can be seen [71] that the relevant algebra can be put in the form

$$\Phi(N) = 16N(\rho+1-N)\left(\rho+\frac{2}{3}-N\right)\left(\rho+\frac{4}{3}-N\right) \tag{3.6}$$

for energy eigenvalues $E_p = p + 1$, or in the form

$$\Phi(N) = 16N(\rho+1-N)\left(\rho+\frac{2}{3}-N\right)\left(\rho+\frac{1}{3}-N\right) \tag{3.7}$$

for eigenvalues $E_\rho = \rho + 2/3$, or in the form

$$\Phi(N) = 16N(\rho+1-N)\left(\rho+\frac{5}{3}-N\right)\left(\rho+\frac{4}{3}-N\right) \tag{3.8}$$

for energies $E_\rho = \rho + 4/3$. In all cases it is clear that the restrictions of eq. (1.71) are satisfied.

3.4   The Smorodinsky–Winternitz potential

The Smorodinsky–Winternitz potential [65] is described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + k(x^2 + y^2) + \frac{c}{x^2}, \tag{3.9}$$

i.e. it is a generalization of the isotropic harmonic oscillator in two dimensions. For finite representations it can be seen [71] that the relevant algebra takes the form

$$\Phi(N) = 1024k^2N(\rho+1-N)\left(N+\frac{1}{2}\right)\left(\rho+1+\frac{\sqrt{1+8c}}{2}-N\right) \tag{3.10}$$

for $c \geq -1/8$ and energy eigenvalues

$$E_\rho = \sqrt{8k}\left(\rho + \frac{5}{4} + \frac{\sqrt{1+8c}}{4}\right), \qquad p = 1, 2, \dots. \tag{3.11}$$

In the special case of $-1/8 \leq c \leq 3/8$ and energy eigenvalues

$$E_\rho = \sqrt{8k}\left(\rho + \frac{5}{4} - \frac{\sqrt{1+8c}}{4}\right), \qquad \rho = 1, 2, \dots \tag{3.12}$$

the relevant algebra is

$$\Phi(N) = 1024k^2 N(\rho + 1 - N)\left(N + \frac{1}{2}\right)\left(\rho + 1 - \frac{\sqrt{1+8c}}{2} - N\right). \tag{3.13}$$

In both cases the restrictions of eq. (1.71) are satisfied.

3.5 The Holt potential

The Holt potential [66]

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (x^2 + 4y^2) + \frac{\delta}{x^2} \tag{3.14}$$

is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. The relevant algebra can be put [71] in the form of an oscillator with

$$\Phi(N) = 2^{23/2} N(\rho + 1 - N)\left(\rho + 1 + \frac{\sqrt{1+8\delta}}{2} - N\right), \tag{3.15}$$

where $(1 + 8\delta) \geq 0$, the relevant energies being given by

$$E_\rho = \sqrt{8}\left(\rho + 1 + \frac{\sqrt{1+8\delta}}{4}\right). \tag{3.16}$$

In this case it is clear that the condition of eq. (1.71) is always satisfied without any further restrictions.

In the special case $-\frac{1}{8} \leq \delta \leq \frac{3}{8}$ one obtains [71]

$$\Phi(N) = 2^{23/2} N(\rho + 1 - N)\left(\rho + 1 - \frac{\sqrt{1+8\delta}}{2} - N\right), \tag{3.17}$$

the relevant energies being

$$E_\rho = \sqrt{8}\left(\rho + 1 - \frac{\sqrt{1+8\delta}}{4}\right). \tag{3.18}$$

The condition of eq. (1.71) is again satisfied without any further restrictions within the given range of δ values.

The deformed oscillator commutation relations in these cases take the form

$$[N, A^\dagger] = A^\dagger, \qquad [N, A] = -A, \tag{3.19}$$

$$[A, A^\dagger] = 2^{23/2} \left(3N^2 - N \left(4p + 1 \pm \sqrt{1 + 8\delta} \right) + p^2 \pm \frac{1}{2} p \sqrt{1 + 8\delta} \right). \tag{3.20}$$

It can easily be seen that they are the same as the $W_3^{(2)}$ commutation relations [67, 68, 69] with the identifications

$$F = \sigma A^\dagger, \qquad E = \rho A, \qquad C = f(p), \qquad H = -2N + k(\rho), \tag{3.21}$$

where

$$\rho\sigma = 2^{-19/2}/3, \qquad k(\rho) = \frac{1}{3} \left(4\rho + 1 \pm \sqrt{1 + 8\delta} \right), \tag{3.22}$$

$$f(p) = \frac{2}{9} \left(14\rho^2 + 4\rho \pm (7\rho + 1)\sqrt{1 + 8\delta} + 1 + 4\delta \right). \tag{3.23}$$

It is thus shown that the Holt potential possesses the $W_3^{(2)}$ symmetry.

The results of this section are summarized in Table 2.

3.6   Two identical particles in two dimensions

Let us consider the system of two identical particles in two dimensions. For identical particles observables of the system have to be invariant under exchange of particle indices. A set of appropriate observables in this case is [58]

$$u = (x_1)^2 + (x_2)^2, \qquad v = (x_1)^2 - (x_2)^2, \qquad w = 2x_1x_2, \tag{3.24}$$

$$U = (p_1)^2 + (p_2)^2, \qquad V = (p_1)^2 - (p_2)^2, \qquad W = 2p_1p_2, \tag{3.25}$$

$$C_1 = \frac{1}{4}(x_1p_1 + p_1x_1), \qquad C_2 = \frac{1}{4}(x_2p_2 + p_2x_2), \qquad M = x_1p_2 + x_2p_1, \tag{3.26}$$

where the indices 1 and 2 indicate the two particles. These observables are known to close an sp(4,R) algebra. A representation of this algebra can be constructed [11,23] using one arbitrary constant η and three matrices Q, R, and S satisfying the commutation relations

$$[S, Q] = -2iR, \qquad [S, R] = 2iQ, \qquad [Q, R] = -8iS(\eta - 2S^2). \tag{3.27}$$

The explicit expressions of the generators of sp(4,R) in terms of η, S, Q, R are given in [58] and need not be repeated here. Defining the operators

$$X = Q - iR, \qquad Y = Q + iR, \qquad S_0 = \frac{S}{2}, \tag{3.28}$$

one can see that the commutators of eq. (16) take the form

$$[S_0, X] = X, \qquad [S_0, Y] = -Y, \qquad [X, Y] = 32S_0(\eta - 8(S_0)^2), \tag{3.29}$$

which is a deformed version of su(2).

Using the same procedure as above, we set the deformation function in the form:

$$\Phi(N) = N(\rho + 1 - N)64(\rho + (1 - \rho)N - N^2), \qquad (3.30)$$

if the condition

$$\eta = 4\rho(\rho + 1) \qquad (3.31)$$

holds. However, the condition of eq. (1.71) is violated in this case.

3.7 The finite W algebra $\bar{\mathrm{W}}_0$

The finite W algebra $\bar{\mathrm{W}}_0$ [61]

$$[U_0, L_0^\pm] = \pm L_0^\pm, \qquad (3.32)$$

$$[L_0^+, L_0^-] = (-k(k-1) - 2(k+1)h)U_0 + 2(U_0)^3, \qquad (3.33)$$

is equivalent to a generalized deformed parafermionic algebra with

$$\Phi(N) = N(\rho + 1 - N)\frac{1}{2}(-\rho + (\rho - 1)N + N^2), \qquad (3.34)$$

provided that the condition

$$k(k-1) + 2(k+1)h = \rho(\rho+1) \qquad (3.35)$$

holds. One can easily check that the condition of eq. (1.71) is satisfied without any further restriction.

3.8 The finite W algebra $\mathrm{W}_3^{(2)}$

The finite W algebra $\mathrm{W}_3^{(2)}$ [67, 68, 69] is characterized by the commutation relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H^2 + C, \qquad (3.36)$$

$$[C, E] = [C, F] = [C, H] = 0. \qquad (3.37)$$

Defining $\tilde{H} = H/2$ these can be put in the form

$$[\tilde{H}, E] = E, \qquad [\tilde{H}, F] = -F, \qquad [E, F] = 4\tilde{H}^2 + C, \qquad (3.38)$$

$$[C, E] = [C, F] = [C, \tilde{H}] = 0. \qquad (3.39)$$

This algebra is equivalent to a parafermionic oscillator with

$$\Phi(N) = \frac{2}{3}N(\rho + 1 - N)(2\rho - 1 + 2N), \qquad (3.40)$$

provided that the condition

$$C = -\frac{2}{3}\rho(2\rho + 1) \qquad (3.41)$$

holds. One can easily see that the condition of eq. (1.71) is satisfied without any further restriction.

3.9   The quadratic Hahn algebra QH(3)

For the quadratic Hahn algebra QH(3) [60]

$$[K_1, K_2] = K_3, \tag{3.42}$$

$$[K_2, K_3] = A_2 K_2^2 + C_1 K_1 + D K_2 + G_1, \tag{3.43}$$

$$[K_3, K_1] = A_2(K_1 K_2 + K_2 K_1) + C_2 K_2 + D K_1 + G_2, \tag{3.44}$$

one obtains the parafermionic oscillator with

$$\Phi(N) = N(\rho + 1 - N)A_2^2(\rho + (1 - \rho)N - N^2), \tag{3.45}$$

if the condition

$$\rho(\rho + 1)A_2^2 + 2A_2 G_1 + C_2 = 0 \tag{3.46}$$

holds. Again, eq. (1.71) is violated in this case.

Table 3.1: Structure functions of deformed oscillators. For conditions of validity and further explanations in the case of the various generalized deformed parafermionic oscillators see the corresponding subsection in the text.

	$\Phi(N)$	Reference
i	N	harmonic oscillator
ii	$\frac{q^N - q^{-N}}{q - q^{-1}} = [N]_q$	q-deformed harmonic oscillator [7, 8, 9]
iii	$N(\rho + 1 - N)$	parafermionic oscillator [72]
iv	$[N]_q[\rho + 1 - N]_q$	q-deformed parafermionic oscillator [73]
v	$N(\rho + 1 - N)(\lambda + \mu N + \nu N^2 + \rho N^3 + \sigma N^4 + \ldots)$	generalized deformed parafermionic oscillator [62]
vi	$N(\rho + 1 - N)[-(\rho^2(\rho + 1)C + \rho B) + (\rho^3 C + (\rho - 1)B)N$ $+((\rho^2 - \rho + 1)C + B)N^2 + (\rho - 2)CN^3 + CN^4]$	3-term $su_\Phi(2)$ algebra (eq. 54)
vii	$BN(\rho + 1 - N)(-\rho + (\rho - 1)N + N^2)$	2-term $su_\Phi(2)$ algebra (eq. 58)
viii	$N(\rho + 1 - N)\frac{1}{2}(-\rho + (\rho - 1)N + N^2)$	finite W algebra $\bar{W}_0$ [61]
ix	$4N(\rho + 1 - N)\left(\lambda(\rho + 1 - N) + \sqrt{\omega^2 + \lambda^2/4}\right)$ $\left(\lambda N + \sqrt{\omega^2 + \lambda^2/4}\right)$	isotropic oscillator in 2-dim curved space [59, 71]
x	$N(\rho + 1 - N)\left(\frac{4\mu^2}{(\rho+1)^2} + \lambda\frac{(\rho+1-2N)^2}{4}\right)$	Kepler system in 2-dim curved space [59, 71]
xi	$16N(\rho + 1 - N)\left(\rho + \frac{2}{3} - N\right)\left(\rho + \frac{4}{3} - N\right)$ or $16N(\rho + 1 - N)\left(\rho + \frac{2}{3} - N\right)\left(\rho + \frac{1}{3} - N\right)$ or $16N(p + 1 - N)\left(p + \frac{5}{3} - N\right)\left(\rho + \frac{4}{3} - N\right)$	Fokas–Lagerstrom potential [64, 71]
xii	$1024k^2 N(\rho + 1 - N)\left(N + \frac{1}{2}\right)\left(\rho + 1 \pm \frac{\sqrt{1+8c}}{2} - N\right)$	Smorodinsky-Winternitz potential [65, 71]
xiii	$\frac{2}{3}N(\rho + 1 - N)(2\rho - 1 + 2N)$	finite W algebra $W_3^{(2)}$ [67, 68, 69]
xiv	$2^{23/2}N(\rho + 1 - N)\left(\rho + 1 \pm \frac{\sqrt{1+8\delta}}{2} - N\right)$	Holt potential [66, 71]

Chapter 4

Bose-Einstein Condensates

Bose-Einstein Condensates are particles that have been created in vitro by a collection of atoms. In order to create a BEC, the collection of atoms has to be cooled down at a temperature of approximately zero Kelvin degrees and be trapped at a very small area of space. At this temperature the atoms are almost immobilised. Therefor, due the Pauli Principle, the atoms occupy all the space given to them. In the figure (4.1) we show the probability density functions of the atoms during the cooling and the trapping. The resulting particle is the BEC.

4.1 Experimental techniques

There have already been performed many experiments that created BECs up today. Although they are not exactly the same they have some things in common:
a) for the cooling of the gas are used LASER beams with opposite directions or the evaporation method in a magnetic trap potential.
b) for the confinment of the gas magneto-optical traps are oftenly being used.

Furthermore, through a strong magnetic field one can tune the kind of the interaction between the atoms (attractive or repulsive). The scattering lenght of each atom is a_o. If the atoms are muttually attracted then $a_o < 0$ and vice versa. Figure (4.2) is a plot of the scattering lenght as a function of the external strong magnetic field.

4.1.1 Cooling procedure by LASER beams

If a LASER beam with an adequate frequency collides with an atom, then the atom will absorb and emit photons constantly. The emitting photons are spherical waves with no preferable direction. Obviously the total momentum of the system must be conserved. The momentum is conserved if the atom gradually moves to the direction of the beam.

Suppose that we choose LASER beams with less frequency than the one the atom can absorb, in the Laboratory System [84]. We can set these two opposite directing LASER beams to collide with the atom gas. The results will be the above:
a) If te atom moves in an opposite direction of that of the LASER beam, then due to Doppler effect the frequency of the beam, in the System of the atom, will have the right frequency for
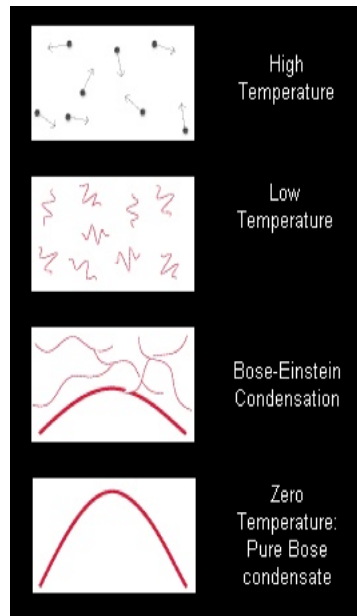
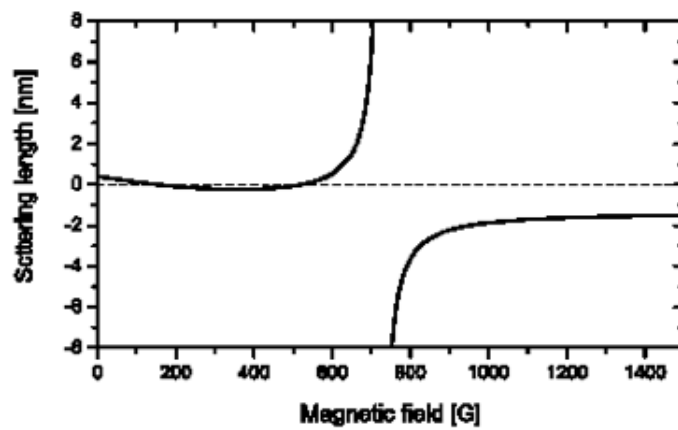Figure 4.1: Probability density functions of the atoms during the cooling and the trapping

Figure 4.2: The scattering lenght of the atoms as a function of the external magnetic field [74]
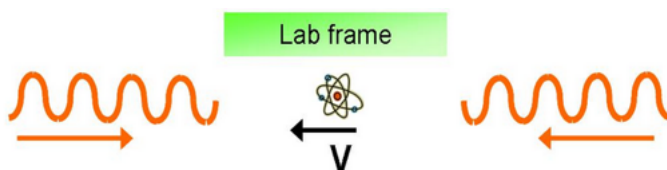
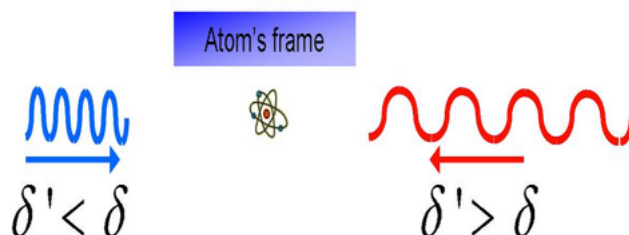Figure 4.3: Collision of two opposite directing LASER beams in the Laboratory System

Figure 4.4: Collision of two opposite directing LASER beam in the System of the atom

the absorbion. So the atom will move to the direction of the beam.
b) On the other hand, if the atom moves into the same direction of that of the beam, then due to the Doppler effect the frequency will be even greater, in the System of the atom, than the one appropriate for the absorbion.

4.1.2 Cooling procedure through evaporation in an magnetic trap

A magnetic field can interact with an magnetic dipole such as an atom is. The magnetic potential of this interaction serves as a trap where only the atoms with the energy less than the maximum of the potential of the trap remain inside. So in order to have only zero temperature atoms one can gradually reduce the maximum of the potential. This is called evaporating cooling because the atoms with the highest energy escape the trap and only the coolest atoms remain inside [84].
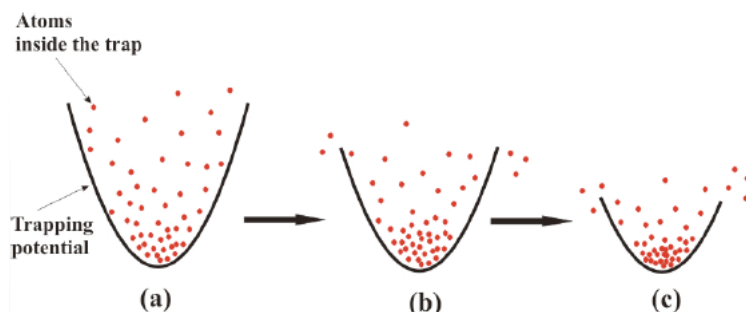

Figure 4.5: The atoms with high energy escape the trap potential and only the cool atoms remain inside.
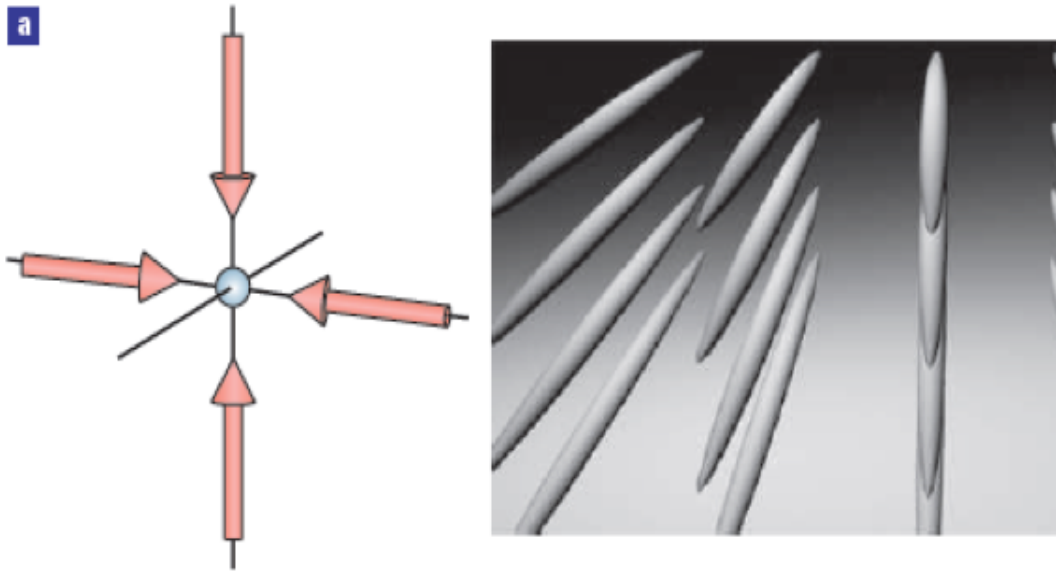
Figure 4.6: Opposite directing LASER beams create a cigar shape trap, with cylindrical symmetry.

4.1.3   Optical trap

The trapping of the atoms in an optical trap is based on a force called Gradient Force [85]. This force is the result of the interaction between the electric field $\vec{E}$ of the LASER beam and the electrical dipole momentum $\vec{p}$ of the atoms. It appears that the electrical dipole momentum is:

$$\vec{p} = a_e \vec{E}, \tag{4.1}$$

where a_e is parameter related to the size of the atom. The energy of the interaction is:

$$U = -\frac{1}{2} < \vec{p} \cdot \vec{E} > . \tag{4.2}$$

The gradient force is:

$$F = \frac{a_e}{2} \nabla < E^2 > . \tag{4.3}$$

The above force is leading the atoms to the minimum of the energy of the interaction. This minimum is actually the maximum of the electric field. Thus one can combine LASER beams to create several shapes of traps.

4.2   The experiment in Rice University

A precise description of the experiment and it's results can be found at [74] .In brief ,the experiment proceeded as follows: A number of $_3^7$Li atoms were cooled down by evaporation in a magnetic potential at practically zero Kelvin.During the cooling some atoms escaped and

a number "N_{tot}" of cool atoms remained in the trap.Through a strong magnetic field the remaining cool atoms were tuned to attract one another.Afterwards they were transferred into a magneto-optical trap so as to be almost immobilized.By using laser beams, the experimentalists forced them to occupy the same space.Because of Heisenberg's principle, the least their momentum became, the biggest space area each atom occupied, so this is how their wave functions looked like during the cooling and trapping procedure:

The resulting condensates are particles called Bose Einstein Condensates (BECs) and each of them consists of N number of atoms. Every atom of each BEC is under the influence of two types of potentials:

a) The potential of the magneto-optical trap

$$V_{ext}(\mathbf{r}) \tag{4.4}$$

b) The mean field potential caused by the other atoms.

$$V(\mathbf{r} - \mathbf{r}') \tag{4.5}$$

More extensicely the experimental procedure was the following: The potential of the optical trap was an anisotropic oscillator of cylindrical symmetry. The radial frequency was $\omega_r = 800Hz$ while the axial was $\omega_z = 70Hz$. The result was an one dimentional confinment of the gas in a cigar-shape potential. The $^{7}_{3}$Li atoms have a total angular momentum F and its projection on the z axis is m_F. The atoms with $(F, m_F) = (2, 2)$ are magnetically trapable, while those with $(F, m_F) = (1, 1)$ are not. Therefor the 98% of the atoms are being trasmitted in the the state (2,2). After the the cooling and the trapping, the scattering lenght was set to be $a_0 \approx -3a_B$, where a_B is the Bohr radius. At the end, four BECs were created of $N \approx 5000$ atoms each.

The $^{7}_{3}$Li atoms and the BEC are expected to have a boson-like behavior because they made of an even number of fermions [75].We know that ideal fermions or bosons can be only the elementary particles.The combinations of them them are neither ideal fermions nor ideal bosons, but something in the middle.If we match the ideal fermions at the color black and the ideal bosons at the color white, then the non elementary particles are shades of grey, from dark grey to light grey.This can be translated in the language of parafermions: a parafermion is any kind of particle, elementary or not, characterized by a parameter ρ which is called the "order of the parafermion".If the parameter $\rho = 1$ then the particle is an ideal fermion and if the parameter $\rho \rightarrow \infty$ then the particle is an ideal boson.So when we say that a particle has a boson-like behavior we mean that the order of the parafermion is $\rho \gg 1$.

4.3 Derivation of the Gross-Pitaevskii equation

According to the Many Body Theory the Hamiltonian of N interacting bosons is:

$$\hat{H} = \int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}))\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\Psi}^{\dagger}(\mathbf{r}')\hat{\Psi}^{\dagger}(\mathbf{r})V(\mathbf{r} - \mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}) \tag{4.6}$$
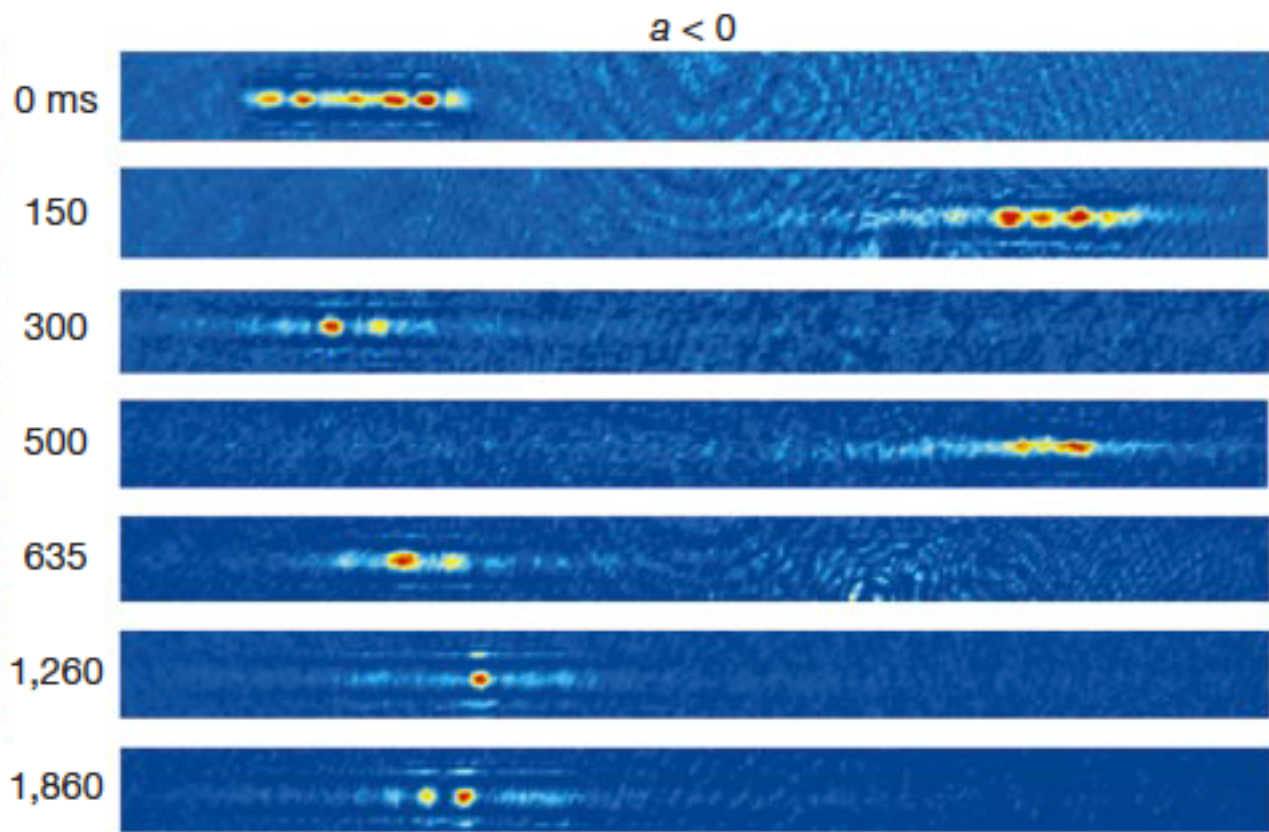
Figure 4.7: Four BECs consisting of $^{7}_{3}$Li atoms with scattering lenght $a_0 \approx -3a_B$. Among the BECs there is a distance which indicates a fermionic behaviour [74].

where $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^{\dagger}(\mathbf{r})$ are the bosonic field operators which destroy and create one atom at position $\mathbf{r}$.

The field operator can be expressed as [76]:

$$\hat{\Psi}(\mathbf{r}) = \sum_p \Psi_p(\mathbf{r})\hat{a}_p = \sum_p \frac{1}{\sqrt{\nu}}e^{ipx/\hbar}\hat{a}_p \tag{4.7}$$

It is noteworthy that the exponentials are an orthogonal set of functions, the $\hat{a}_p$ serve as coefficients and the $\frac{1}{\sqrt{\nu}}$ is the normalization constant since ν is the volume.The coefficients are the annihilation operators of Fock space. The action of the annihilation operator over a Fock space vector destroys one atom from the 'p' state:

$$\hat{a}_p|n_0, n_1, ..., n_p, ... >= \sqrt{n_p}|n_0, n_1, ..., n_p - 1, ... > \tag{4.8}$$

Therefor the creation operator creates an atom at the 'p' state:

$$\hat{a}_p^{\dagger}|n_0, n_1, ..., n_p, ... >= \sqrt{n_p + 1}|n_0, n_1, ..., n_p + 1, ... > \tag{4.9}$$

The annihilation and creation operators' commutators are:

$$[\hat{a}_p, \hat{a}_l^{\dagger}] = \delta_{pl} \tag{4.10}$$

$$[\hat{a}_p, \hat{a}_l] = 0 \tag{4.11}$$

$$[\hat{a}_p^{\dagger}, \hat{a}_l^{\dagger}] = 0 \tag{4.12}$$

The bosonic field operators' commutators are:

$$[\hat{\Psi}(\mathbf{r}, t), \hat{\Psi}^{\dagger}(\mathbf{r}', t)] = \delta^3(\mathbf{r} - \mathbf{r}') \tag{4.13}$$

$$[\hat{\Psi}(\mathbf{r}, t), \hat{\Psi}(\mathbf{r}', t)] = [\hat{\Psi}^{\dagger}(\mathbf{r}, t), \hat{\Psi}^{\dagger}(\mathbf{r}', t)] = 0 \tag{4.14}$$

In order to have a Bose-Einstein Condensate almost every atom must occupy the ground state.So the number of atoms of the ground state is $n_0 = N_0 \gg 1$, while the limit $lim_{N \to \infty}\frac{N_0}{N}$ has to remain finite[?].Since $N_0 \gg 1$ then $N_0 1 \approx N_0 \implies \sqrt{N_0 1} \approx \sqrt{N_0}$.Now let's get back at the equations which define the action of the annihilation and the creation operators and write them for the ground state, p=0 ,which is occupied by the majority of atoms $N_0 \gg 1$.One can realize that the operators serve as simple constants: $a_0 \approx a^{\dagger} \approx \sqrt{N_0}$ and $a_0 a_0^{\dagger} \approx N_0$[76].This means that in the case of a uniform gas of bosons the field operator can be expressed as[77]:

$$\hat{\Psi}(\mathbf{r}) = \Psi_{p=0}\hat{a}_{p=0} + \sum_{p>0} \Psi_p(\mathbf{r})\hat{a}_p = \sqrt{\frac{N_0}{\nu}} + \hat{\Psi}'(\mathbf{r}) \tag{4.15}$$

If the gas of bosons is not uniform then:

$$\hat{\Psi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \hat{\Psi}'(\mathbf{r}, t) \tag{4.16}$$

where $\Phi(\mathbf{r}, t)$ is the field operator ,which became a simple function, for the bosons at the p=0 state and $\hat{\Psi}'(\mathbf{r}, t)$ is the field operator for the bosons at the $p > 0$ states.The function $\Phi(\mathbf{r}, t)$ actually is the wavefunction of the BEC.The probability density function of the problem is $|\Phi(\mathbf{r}, t)|^2 = \frac{dN_0}{d\nu}$, so it represents the concentration of atoms around position $\mathbf{r}$.The strict definition of the wavefunction of the BEC is $\Phi(\mathbf{r}, t) = < \hat{\Psi}(\mathbf{r}, t) >$. About the next term, $\hat{\Psi}'(\mathbf{r})$, Bogoliubov assumed it can be considered as a small perturbation at the system since most of the bosons are at the ground state.

4.3.1   The mean field potential and the coupling constant

As we have already mentioned the mean field potential is the result of the interaction of the atoms. The simplest model we can use to describe their interaction is the " two body collisions" model, where we suppose that only two atoms collide at a time. This is a small range potential:

$$V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') \tag{4.17}$$

where g is a negative constant for attractive interactions and positive for repulsive, called "pairing constant".

   In order to be this model adequate for our experiment, the gas of bosons must be a dilute gas, otherwise "three body collisions" take part too.In that case the GP equation is no longer valid.One can argue that " We can not have a dilute gas in a BEC since the laser beams are pressing the atoms to occupy the same space.In a dilute gas an atom should have plenty of space to move!". This is not true. In a dilute gas , each atom has plenty of space to move , comparing to the scattering length "a" of the atoms, $r \gg a$.If the scattering length has a small value then the gas can still be dilute, although it is a BEC[?].

   By setting $\mathbf{r} - \mathbf{r}' = \tilde{\mathbf{r}}$, $\mathbf{q} = \mathbf{k} - \mathbf{k}'$,$\mathbf{p} = \hbar\mathbf{q}$ we can write the Fourier Transformation of the potential :

$$V(\mathbf{p}) = \int_{-\infty}^{+\infty} V(\tilde{\mathbf{r}})e^{-i\frac{\mathbf{p}\tilde{\mathbf{r}}}{\hbar}}d\tilde{\mathbf{r}} = \int_{-\infty}^{+\infty} g\delta(\tilde{\mathbf{r}})e^{-i\frac{\mathbf{p}\tilde{\mathbf{r}}}{\hbar}}d\tilde{\mathbf{r}} = g = constant \tag{4.18}$$

The pairing constant will prove to be a very important parameter at the end of this project.We can relate this constant to the scattering length.We will use the collisions theory, where the colliding particle is represented by flat waves[78].In this theory, we suppose that one particle is immobilized (target) and the other (projectile) is moving towards the first one. After the collision, the projectile leaves as a spherical wave and its wave function is:

$$\Psi_{sph}(\tilde{\mathbf{r}}, t) = \frac{f(\theta, \phi)}{\tilde{r}}e^{i\mathbf{k}\tilde{\mathbf{r}}}e^{-i\omega t} \tag{4.19}$$

   The function $f(\theta, \phi)$ is the "scattering length".By using a first order perturbation theory one can prove that [78]:

$$f(\theta, \phi) = -\frac{\acute{\mu}}{2\pi\hbar^2}\int_{-\infty}^{+\infty} e^{-i\mathbf{k}\tilde{\mathbf{r}}}V(\tilde{\mathbf{r}})d^3\tilde{r} = -\frac{\acute{\mu}}{2\pi\hbar^2}g \tag{4.20}$$

where $\acute{\mu}$ is the reduced mass of the two atoms ,of mass m each.We set that [76] $a_0 = \pm|f(\theta, \phi)|$ and finally we get the relation between the scattering length and the coupling constant :

$$a_0 = \frac{mg}{4\pi\hbar^2} \Rightarrow g = \frac{4\pi a_0 \hbar^2}{m} \tag{4.21}$$

It is very crucial to note that if the coupling constant has a small value then the scattering length is small enough to let us characterize the gas dilute. The smallest the g is the less the atoms interact . This is what I call "bosonic behavior" and by the end of this project we will have proved it.In other words, bosonic interaction means no interaction at all!

4.3.2 Analysis of the bosonic field operator into s-waves

The order of the kinetic energy of each atom is $\frac{p^2}{2m} \sim k_B T$. Therefor the momentum is $p \sim \sqrt{2mk_BT}$. The scattering theory states that only the atoms which satisfy the below inequality will be scattered [78]:

$$\sqrt{l(l+1)\hbar^2} < pr_0 \tag{4.22}$$

If $\sqrt{l(l+1)\hbar^2} \simeq l\hbar$, then the angular momentum must be

$$l < \frac{pr_0}{\hbar} \simeq \frac{r_0}{\hbar}\sqrt{2mk_BT} \tag{4.23}$$

By taking under consideration that the temperature is $T \ll 1$ we conclude that only atoms with $l \ll 1$ will take part in the scattering procedure. So it is inevitable to choose l=0. The colliding atoms are those with $l = 0$ (s-waves) and are at states with $p > 0$.

We are going to edit the bosonic field operator, because the primitive form is not helpful. It is useful to analyze the states with $p > 0$ in s-waves which travel in opposite directions [77]:

$$\Psi(\mathbf{r}, t) = \Phi(\mathbf{r})e^{-i\mu t} + \Psi_k e^{i(\mathbf{kr}-(\mu+\omega)t)/\hbar} + \Psi_{-k}e^{-i(\mathbf{kr}-(\mu-\omega)t)/\hbar} \tag{4.24}$$

where μ is the chemical potential and Ψ_k the amplitudes.The frequencies $\mu + \omega$ and $\mu - \omega$ are choosen so as the energy of the system is conserved.

4.3.3 The Heisenberg equation

It is high time we derive the GP equation. There will be a lot of math, so if anyone not interested may skip this section.The below equation of the many body theory is the Heisenberg equation and is the analogous of the Schrödinger equation of quantum mechanics:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r}, t) = [\hat{\Psi}(\mathbf{r}, t), \hat{H}] \tag{4.25}$$

Calculation

$$[\hat{\Psi}(\mathbf{r},t),\hat{H}] = [\hat{\Psi}(\mathbf{r},t),\int(\hat{\Psi}^\dagger(\mathbf{r})(-\frac{\hbar^2}{2m}\nabla^2)\hat{\Psi}(\mathbf{r})+\hat{\Psi}^\dagger(\mathbf{r})V_{ext}(\mathbf{r})\hat{\Psi}(\mathbf{r}))d\mathbf{r}$$

$$+\frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})] =$$

$$\int([\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})(-\frac{\hbar^2}{2m}\nabla^2)\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})V_{ext}(\mathbf{r})\hat{\Psi}(\mathbf{r})])d\mathbf{r}$$

$$+\frac{1}{2}\int[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r}',t)\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})]d\mathbf{r}d\mathbf{r}' =$$

$$\int(\hat{\Psi}^\dagger(\mathbf{r})[\hat{\Psi}(\mathbf{r},t),(-\frac{\hbar^2}{2m}\nabla^2)\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})](-\frac{\hbar^2}{2m}\nabla^2\hat{\Psi}(\mathbf{r}))+$$

$$\hat{\Psi}^\dagger(\mathbf{r})[\hat{\Psi}(\mathbf{r},t),V_{ext}\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})]V_{ext}\hat{\Psi}(\mathbf{r}))d\mathbf{r}+$$

$$\frac{1}{2}\int(\hat{\Psi}^\dagger(\mathbf{r}')[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r}')]\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}'))d\mathbf{r}d\mathbf{r}'$$

$$=\int((\hat{\Psi}^\dagger(\mathbf{r})(-\frac{\hbar^2\nabla^2}{2m}[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),-\frac{\hbar^2\nabla^2}{2m}]\hat{\Psi}(\mathbf{r}))+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})](-\frac{\hbar^2}{2m}\nabla^2\hat{\Psi}(\mathbf{r}))$$

$$+\hat{\Psi}^\dagger(\mathbf{r})(V_{ext}(\mathbf{r})[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}(\mathbf{r},t)]+[\hat{\Psi}(\mathbf{r},t),V_{ext}(\mathbf{r})]\hat{\Psi}(\mathbf{r},t))+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})]V_{ext}(\mathbf{r})\hat{\Psi}(\mathbf{r},t))d\mathbf{r}$$

$$+\frac{1}{2}\int(\hat{\Psi}^\dagger(\mathbf{r}')(\hat{\Psi}^\dagger(\mathbf{r})[\hat{\Psi}(\mathbf{r},t),V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})]+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r})]V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}))$$

$$+[\hat{\Psi}(\mathbf{r},t),\hat{\Psi}^\dagger(\mathbf{r}')]\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}'))d\mathbf{r}d\mathbf{r}'$$

$$(4.26)$$

By using the bosonic field operators' commutators and the:

$$[\hat{\Psi},V_{ext}]=[\hat{\Psi},V(\mathbf{r}-\mathbf{r}')]=[\hat{\Psi},-\frac{\hbar^2}{2m}\nabla^2]=[\hat{\Psi},\hat{\Psi}]=[\hat{\Psi}^\dagger,\hat{\Psi}^\dagger]=0 \qquad (4.27)$$

$$\int\delta^3(\mathbf{r}-\mathbf{r}')V(\mathbf{r}-\mathbf{r}')d\mathbf{r}=V(\mathbf{r}'-\mathbf{r}) \qquad (4.28)$$

$$\int\delta^3(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r},t)d\mathbf{r}=\hat{\Psi}(\mathbf{r}',t) \qquad (4.29)$$

one gets that:

$$[\hat{\Psi},\hat{H}]=(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})+\int\hat{\Psi}^\dagger(\mathbf{r}',t)V(\mathbf{r}'-\mathbf{r})\hat{\Psi}(\mathbf{r}',t)d\mathbf{r}')\hat{\Psi}(\mathbf{r},t) \qquad (4.30)$$

We can now use the equations (4.30) and (4.25) for the mean field potential and the field operator in order to calculate the above integral:

$$\int(\Phi^*(\mathbf{r}')e^{i\mu t}+\Psi_k e{-i(\mathbf{k}\mathbf{r}'-(\mu+\omega)t}+\Psi_{-k}e^{i(\mathbf{k}\mathbf{r}'-(\mu-\omega)t)})$$

$$g\delta(\mathbf{r}'-\mathbf{r})(\Phi(\mathbf{r}')e^{-i\mu t}+\Psi_k e^{i(\mathbf{k}\mathbf{r}'-(\mu+\omega)t)}+\Psi_{-k}e^{-i(\mathbf{k}\mathbf{r}'-(\mu-\omega)t)})d\mathbf{r}' = \qquad (4.31)$$

$$g|\Psi(\mathbf{r},t)|^2$$

Finally we can set the pieces of the puzzle together:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = (-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2)\Psi(\mathbf{r},t) \qquad (4.32)$$

This is the Gross-Pitaevskii equation.If we study only the ground state the GP is:

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = (-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2)\Phi(\mathbf{r},t) \qquad (4.33)$$

4.4 The energy of the BEC in a cylindrical trap

The V_{ext} potential of the trap used in the experiment had a cylindrical symmetry:

$$V_{ext} = \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2 \qquad (4.34)$$

Therefor the Hamiltonian of the BEC is:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2 + g|\Phi(\mathbf{r},t)|^2 \qquad (4.35)$$

We shall set that $\Lambda = (\frac{\omega_z}{\omega_r})^2$ where Λ is the anisotropy constant of the cylindrical oscillator potential.The eigenvalues' equation of the $\hat{H}$ is :

$$\hat{H}\Phi(\mathbf{r}) = E\Phi(\mathbf{r}) \qquad (4.36)$$

This is a non linear differential equation which can not be solved exactly. Thus we will have to make it simpler in order to find an approximate solution. We will assume that the non linear term $g|\Phi(\mathbf{r})|^2$ acts only on the z direction since the size of trap indicates that $z \gg r$ [80]. We will proceed by separating the wave function into:

$$\Phi(\mathbf{r}) = P(r)U(z) \qquad (4.37)$$

By substituting the last equation into the Schrödinger, we derive two differential equations, one for each variable:

$$-\frac{\hbar^2}{2m}\nabla_\perp^2 P(r) + \frac{1}{2}mr^2 P(r) = c_1 P(r) \qquad (4.38)$$

$$-\frac{\hbar^2}{2m}U''(z) + \frac{1}{2}m\Lambda z^2 U(z) + g|U(z)|^2 = c_2 U(z) \qquad (4.39)$$

where c_1, c_2 are constants.The first equation of these is the Schrödinger equation of the harmonic oscillator.In order to solve the second one we will assume that the potential of the z axis is changing slowly : $\frac{1}{2}m\Lambda z^2 \simeq constant = 0$ [80].So the differential equation we have to solve is:

$$-\frac{\hbar^2}{2m}U''(z) + g|U(z)|^2 = c_2 U(z) \qquad (4.40)$$

The corresponding time depended is:

$$i\hbar\frac{\partial U(z,t)}{\partial t} + \frac{\hbar^2}{2m} - g|U(z)|^2U(z) = 0 \tag{4.41}$$

We can suppose a solution which respects the Schrödinger and the bounded particle boundary conditions at $z \to \infty$. [80] The bellow solution serves our goals:

$$U(z,t) = Asech[\eta(z-vt)]e^{i(kz-\omega t)} \tag{4.42}$$

where ω is a time frequency , η, k have wavelength units and $|A|^2$ has the same units with the density function :

$$|U(z,t)|^2 = \frac{dN}{dz} \tag{4.43}$$

Therefor, the number of the atoms of the BEC is:

$$N = \int_{-\infty}^{+\infty} |U(z,t)|^2dz \tag{4.44}$$

4.4.1   Calculation of the atoms' number

We are going to need the integral:

$$\int \frac{dz}{cosh^n(az)} = \frac{sinh(az)}{a(n-1)cosh^{n-1}(az)} + \frac{n-2}{n-1}\int \frac{dz}{cosh^{n-2}(az)}, \forall n \neq 1 \tag{4.45}$$

So

$$N = |A|^2 \int sech^2[\eta(z-vt)]dz = |A|^2\frac{sinh[\eta(z-vt)]}{\eta cosh[\eta(z-vt)]}|_{-\infty}^{+\infty} = \frac{2|A|^2}{\eta} \tag{4.46}$$

4.4.2   Calculation of the energy

Now we are going to search for the energy.Firstly we have to mention some theory from the Classical Theory. If you are surprised by this , please remember that the whole concept of the Quantum Field Theory, from which the GP derived, is based on the Classical Fields. A system is called Hamiltonian if there exists a function $\tilde{H}$ (classical Hamiltonian)[82]:

$$\tilde{H} = \int_{-\infty}^{+\infty} h(U, U^*, t)dz \tag{4.47}$$

so as :

$$\frac{\delta\tilde{H}}{\delta U^*} = \sum_{n=0}^{+\infty}(-1)^n\frac{\partial^n}{\partial z^n}\left(\frac{\partial h}{\partial(\frac{\partial^n U^*}{\partial z^n})}\right) = \frac{\partial U}{\partial t} \tag{4.48}$$

The above equation is one of the Hamilton equations for the adjoint variables q=U and p=U* (generalized position and momentum).The function $h(U, U^*, t)$ is called *energy function*. The energy of the system is given as [80]:

$$E = -i\hbar\tilde{H} \tag{4.49}$$

But we already know that the $\frac{\partial U}{\partial t}$ is given by the time depended GP equation (4.41).Through these equations one can verify that the energy of all the atoms of the BEC is given by the:

$$E = \frac{1}{2} \int_{-\infty}^{+\infty} (\frac{\hbar^2}{m}|\frac{\partial U}{\partial z}|^2 + g|U|^4)dz \qquad (4.50)$$

For the calculation of this integral we will need the :

$$\int tanh^n(az)sech^2(az)dz = \frac{tanh^{n+1}(az)}{(n+1)a} \qquad (4.51)$$

$$\int sech^n(az)dz = \frac{sech^{n-2}(az)tanh(az)}{a(n-1)} + \frac{n-2}{n-1}\int sech^{n-2}(az)dz \qquad (4.52)$$

After some calculations the energy appears to be:

$$E = \frac{\hbar^2}{m}|A|^2(\frac{k^2}{\eta} + \frac{\eta}{3}) + \frac{2g}{3\eta}|A|^4 \qquad (4.53)$$

But we will not stop here because our target is to find the function $E \equiv E(N)$.From the equation (4.46) we get that:

$$\frac{|A|^2}{\eta} = \frac{N}{2} \qquad (4.54)$$

Finally the energy is:

$$E = \frac{\hbar^2}{2m}(k^2 + \frac{\eta^2}{3})N + \frac{g\eta}{6}N^2 \qquad (4.55)$$

We can clearly see now that the energy of the system is a non linear function of N due to the second term.

In order to understand the above result we will have to concider what is the basic idea of the second quantization? Suppose we have an one particle system where the energy levels are sepparated by the same enegy gap $\hbar\omega$ (i.e. harmonic oscillator). One particle excited in the Nth state of the oscillator will carry $N\hbar\omega$ amount of energy. This particle has the same energy with N particles of energy $\hbar\omega$ each. This idea gave birth to the second quantization language where the Nth energy level ,of **any** problem with many bosons, is constructed by a number of N bosons each of them carrying an $\hbar\omega$ amount of energy. One might wonder "Why bosons and not fermions?". This answer is obvious: because fermions can not share the same quantum state since the obey to the Pauli Principle. So if we have N bosons at almost zero temperature then we shall expect to find them all in the ground state of energy u. Thus the energy of the N bosons shall be:

$$E = uN. \qquad (4.56)$$

We notice that the above energy is a linear function $E \equiv E(N)$.

What happens if the particles are not bosons but are close to the bosonic behavior? Then we expect some particles to be excited in an upper energy level. This mean that the function $E \equiv E(N)$ will no longer be linear. The equation (4.55) ia a non linear function of N which indicates the divergence from the bosonic behavior of the atoms.

4.5   The order parameter of the Parafermion

We can describe the BECs by using the Bose-like Hamitonian. The Bose-like energy has been given in equation (2.3). All we have to do in order to find the λ, ρ, μ, ν is to set equal the equations (4.55) and (2.4) :

$$-\frac{\hbar\omega}{2}2\nu = 0 \Rightarrow \nu = 0 \tag{4.57}$$

$$\frac{\hbar\omega}{2}(-2\mu) = 0 \Rightarrow \mu = 0 \tag{4.58}$$

$$\frac{\hbar\omega}{2}(-2\lambda) = \frac{g\eta}{6} \Rightarrow \lambda = -\frac{g\eta}{6\hbar\omega} \tag{4.59}$$

$$\frac{\hbar\omega}{2}2\rho\lambda = \frac{\hbar^2}{2m}(k^2 + \frac{\eta^2}{3}) \Rightarrow \rho = -\frac{3\hbar^2}{gm\eta}(k^2 + \frac{\eta^2}{3}) \Rightarrow \rho = \frac{3\hbar^2}{|g|m\eta}(k^2 + \frac{\eta^2}{3}) \tag{4.60}$$

By definition the order parameter has to be $\rho \geq 1$. This is followed by a restriction for the coupling constant:

$$0 \leq |g| \leq \frac{3\hbar^2}{\eta m}(k^2 + \frac{\eta^2}{3}) \tag{4.61}$$

where $\frac{3\hbar^2}{\eta m}(k^2 + \frac{\eta^2}{3}) = g_{max}$. We must not worry about trespassing the limit of g_{max}, because in that case the gas is no longer dilute and the Gross-Pitaevskii is no longer valid. We examine only dilute gases in this study. Furthermore, if the order parameter ends to be less than one, $\rho < 1$, then we should try another Hamiltonian or another deformation function $\Phi(N)$.

Now it is obvious that when the atoms do not interact at all , $g = 0$, the gas is an ideal bosonic gas with the order of the parafermion to be $\rho \to +\infty$. On the other hand, when the atoms interact the most, $g = g_{max}$, the order of the parafermion is $\rho = 1$. This is exactly what we expected! Up to now g was treated as a constant, but it is a variable which depends on the kind of gas we use and on the space confinement of the trap. It's value is proportional to the possibility for the atoms to interact. The more we push the atoms together, the more possible becames their interaction. Everything seems reasonable.

Moreover, this is a remarkable result for one more reason: I have already stated that *the divergence from the ideal bosonic behavior, at $T \approx 0K$, is signified by the non linear term of the $E \equiv E(N)$*. On the other hand the linear term indicates the ideal bosonic behavior of the gas. So let's assume that :

$$u = \frac{\hbar^2}{2m}(k^2 + \frac{\eta^2}{3}) \tag{4.62}$$

and

$$v = \frac{g\eta}{6} \tag{4.63}$$

so as the energy is:

$$E = uN + vN^2 \tag{4.64}$$

If the above statement is true, then the ratio $|\frac{u}{v}|$ measures the $\frac{bosonic-behavior}{fermionic-behavior}$. By calculating this ratio we get:

$$\frac{bosonic-behavior}{fermionic-behavior} = |\frac{u}{v}| = |\frac{3\hbar^2}{-g\eta m}(k^2 + \frac{\eta^2}{3})| = \rho \qquad (4.65)$$

This match is not a coincidence, because that's the very meaning of the order parameter. We confirm now that the above statement is true.

4.6 The Tonks-Girardeau gas

This kind of gas has been proposed forty years ago, but it has been created only after the cooling and the trapping techniques were excelled. Such a gas consists of bosonic particles that have been confined in one dimention. When the gas enters the Tonks-Giradeau (TG) region the particles start to repell each other. This is equivalent to the experiment we studied in the previous section.

One can find more details about the Tonks-Giradeau gas realization in the reference [83]. The value the experimentalists used to qualify whether the gas has entered the Tonks-Giradeau region or not was the ratio:

$$\gamma = \frac{I}{K}, \qquad (4.66)$$

where I is the energy of the interaction of the particles and K is their kinetic energy.

As far as concerns the meaning of this ration we say that:
a) if $\gamma << 1$ then the gas has not entered the TG region (bosonic gas)
b) if $\gamma >> 1$ then the gas has entered the TG region (fermionic gas)
c) if $\gamma \approx 1$ then the gas is neither bosonic nor fermionic

If the gas is homogenus then the ration is [37]:

$$\gamma = \frac{mg}{n\hbar^2}, \qquad (4.67)$$

where n is the concentration of particles.

At this point we shall compare the order parameter with the ratio γ. The η, which has shown up in the equation (4.41), has the same units with the concentration. So we can suppose that η is the concentration of the particles. Then :

$$\rho = \frac{1}{\gamma} + \frac{3\hbar^2 k^2}{|g|\eta m}. \qquad (4.68)$$

So if :
a)$\gamma << 1$ then $\rho >> 1$ (bosonic region)
b)$\gamma >> 1$ then $\rho \approx \frac{3\hbar^2 k^2}{|g|\eta m}$ (fermionic region)
c)$\gamma \approx 1$ then $\rho \approx 1 + \frac{3\hbar^2 k^2}{|g|\eta m}$ (nor bosonic nor fermionic region).

Chapter 5

Conclusion

The Generalised Parafermionic Oscillators' theory is a useful tool for the study of various many body problems such as nuclear, molecular and gas systems. The application of this theory on Bose-Einstein condensates was successful and led to the calculation of the order of the parafermion which is a measure of the bosonic behavior of the BEC.

The order of the parafermion coincides with the coefficient's ratio of the linear term, of the BEC's energy E(N), to the non linear term. Thus the more powerful is the non linearity of the energy the more fermionic is the gas.

Last but not least the order of the parafermion can be used as a measure on whether the gas has entered in the Tonks-Girardaeu region.

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