Εθνικο Μετσοβιο Πολυτεχνειο

Σχολη Ηλεκτρολογων Μηχανικων και Μηχανικων Υπολογιστων

Τομεας Σηματων, Ελεγχου και Ρομποτικης

Παιγνιοθεωρητική Ανάλυση Διάχυσης Ανταγωνιστικών Προϊόντων σε Κοινωνικά Δίκτυα

Διπλωματικη Εργασια

του

ΒΑΣΙΛΕΙΟΥ Δ. ΤΖΟΥΜΑ

Επιβλέπων: Τρύφων Κουσιουρής
Καθηγητής Ε.Μ.Π.

Συνεπιβλέπων: Ευάγγελος Μαρκάκης
Λέκτορας Ο.Π.Α.

Αθήνα, Ιούνιος 2012

Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών
Τομέας Σημάτων, Ελέγχου και Ρομποτικής

Παιγνιοθεωρητική Ανάλυση Διάχυσης Ανταγωνιστικών Προϊόντων σε Κοινωνικά Δίκτυα

Διπλωματική Εργασία

του

ΒΑΣΙΛΕΙΟΥ Δ. ΤΖΟΥΜΑ

Επιβλέπων: Τρύφων Κουσιουρής
Καθηγητής Ε.Μ.Π.

Συνεπιβλέπων: Ευάγγελος Μαρκάκης
Λέκτορας Ο.Π.Α.

Εγκρίθηκε από την τριμελή εξεταστική επιτροπή την $1^{η}$ Ιουνίου 2012.

(Υπογραφή)	*(Υπογραφή)*	*(Υπογραφή)*
....	....	....
Τρύφων Κουσιουρής	Ευάγγελος Μαρκάκης	Δημήτριος Φωτάκης
Καθηγητής Ε.Μ.Π.	Λέκτορας Ο.Π.Α.	Λέκτορας Ε.Μ.Π.

Αθήνα, Ιούνιος 2012

(Υπογραφή)

....
Τζουμας Δ. Βασιλειος
Διπλωματούχος Ηλεκτρολόγος Μηχανικός και Μηχανικός Υπολογιστών Ε.Μ.Π.

Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών
Τομέας Σημάτων, Ελέγχου και Ρομποτικής

Περίληψη

Μελετάμε τη διάδοση ανταγωνιστικών προϊόντων σε κοινωνικά δίκτυα μέσω ενός μη συνεταιριστικού παιχνιδιού μεταξύ ανταγωνιζομένων επιχειρήσεων, που έχουν προϋπολογισμούς για να 'σπείρουν' την αρχική υιοθέτηση των προϊόντων τους σε ένα δίκτυο καταναλωτών. Οι ωφέλειες των επιχειρήσεων αυτών είναι ο τελικός αριθμός των υιοθετήσεων των προϊόντων τους με το τέλος της συμμετέχουσας διαδικασίας διάχυσης.

Επικεντρωνόμαστε σε παιχνίδια 2-παικτών, και μοντελοποιούμε την διαδικασία διάχυσης χρησιμοποιώντας το γνωστό linear threshold model, σε συνδυασμό με ντετερμινιστικά κριτήρια για την επίλυση τυχόν ισοπαλιών. Παραταύτα, πολλά από τα αποτελέσματά μας εξακολουθούν να ισχύουν και υπό ένα γενικότερο σύνολο υποθέσεων για τη διαδικασία αυτή.

Κατ΄ αρχάς, δείχνουμε ότι τα παιχνίδια αυτά δεν έχουν πάντοτε αμιγή σημεία ισορροπίας Nash (PSNE), και αποδεικνύουμε ότι το πρόβλημα απόφασης για την ύπαρξη PSNE σε ένα τέτοιο παιχνίδι είναι coNP-hard. Στη συνέχεια, δείχνουμε ότι δεν μπορούμε να ελπίζουμε ακόμη και για παιχνίδια πάνω σε δίκτυα με συγκεκριμένα in και out-degree distributions να είναι πιο ευσταθή από άλλα, σε σχέση, για παράδειγμα, με την μορφή των improvement paths, ή cycles που εμφανίζουν. Συνεχίζουμε με ικανές και αναγκαίες συνθήκες για την ύπαρξη ενός PSNE, και προτείνουμε ικανές συνθήκες για την ύπαρξη του ως προϋποθέσεις για την ύπαρξη μιας generalized ordinal potential. Στη συνέχεια, στοχεύοντας στην ποσοτικοποίηση της αστάθειας των εν λόγω παιχνιδιών, εξετάζουμε tight approximate generalized ordinal potentials. Περαιτέρω, υποκινούμενοι από μία απλή, αλλά ρεαλιστική κλάση κοινωνικών δικτύων, θεωρούμε μια ειδική περίπτωση παιχνιδιών 2-παικτών, την οποία και χαρακτηρίζουμε πλήρως όσον αφορά την ύπαρξη PSNE. Τέλος, επικεντρωνόμαστε σε παιχνίδια με αυθαίρετο αριθμό παικτών και τα μελετάμε ως προς το Price of Anarchy και Stability. Στη συνέχεια, προτείνουμε ένα νέο μέτρο που αποτυπώνει την αναποτελεσματικότητα ενός PSNE, όσον αφορά τις ωφέλειες των παικτών, καθώς ο αριθμός των παικτών αυξάνει: Το ονομάζουμε Price of Oligoply.

Λέξεις Κλειδιά

Κοινωνικό δίκτυο, Διάχυση Πληροφορίας, Θεωρία Παιγνίων

Abstract

We study the diffusion of competing products in social networks through a non-cooperative game between competing firms that have budgets to "seed" the initial adoption of their products within a network of consumers. The utilities of the firms are the eventual number of adoptions of their product by the end of this diffusion process.

We mainly focus on 2-player games, and we model the involved diffusion process using the known linear threshold model, enhanced with certain deterministic tie-breaking rules. Nonetheless, many of our results continue to hold under a more general framework for this process.

We first exhibit that these games do not always possess pure strategy Nash equilibria (PSNE), and we prove that deciding if PSNE exist is coNP-hard. Afterwards, we prove that we may not hope even for games over networks with special in and out-degree distributions to be more stable than others, concerning, for example, the form of the improvement paths, or cycles that they induce. We continue with the investigation of necessary and sufficient conditions for the existence of PSNE, and we propose sufficient conditions for the existence of PSNE as conditions for the existence of a generalized ordinal potential. Next, we target to quantify the instability of the games in question, and we examine approximate generalized ordinal potentials; tight bounds are offered. Further, motivated by simple but realistic classes of recommendation networks, we consider a special case of 2-player games, and we completely characterize them with regard to the existence of PSNE. Finally, we focus on games with an *arbitrary* number of players, and we first study their Price of Anarchy and Stability. Then, we propose a new measurement that captures the inefficiency of a PSNE, with regard to the utilities of the players, as the number of the involved players increases; we name it *Price of Oligopoly*.

Keywords

Social Networks, Information Diffusion, Game Theory, Economic Behavior

Acknowledgments

Our cognitive and psychological development is affected by a number of people some of which we may have never even met. Here, I would like to acknowledge some of those people that I have actually met and I am fortunate to know, for I would not stand at this extremely propitious point for me if they were absent from my life. Thus, assuming the soundness of this act under the heading “Acknowledgments”, I continue (in a rather chronological order).

My brother Alexandros was, is, and will be one of my main characters of reference; not only he first ensconced me the interest for scientific reasoning, using some of the funniest and exciting approaches someone could think, but also his continuous support and encouragement constituted for me a firm ground towards a better-balanced personality. For the same latter reasons I also want to wholeheartedly thank Athina; her unexpectedly (for our age and experiences) consummate character had a drastically positive affect to me. Hence, I hope that the years that I will spend in the US will not eventually obstruct me keeping knowing my brother, as I also hope that Athina and I we will continue heartily our common adventurous course in the US, as we did here.

But I would not stand at this point without the tremendously generous support of several of my professors. Evangelos Markakis, as my advisor, devoted repeatedly much of his time to kindly evaluate any of my research results, as well as to contribute with the most useful suggestions. And taking into account his exceptional character, I clearly consider him an advisor and person-paradigm. Similarly, to be able within a few words to describe the equally extremely beneficial effect to me of Tryfon Koussiouris was overly hard and time-consuming; his relentless support and confidence in me will be as well a truly inspiring memory for me throughout all of my educational life.

Additionally to Tryfon Koussiouri and Evangelo Markaki, I definitely owe my future steps to the trustfulness of two other professors of mine: Kiriaki Kiriakie and Petros Maragos; their kind reference letters were undoubtedly catalytic for my successful leap to the US. Lastly, from my high-school years I want to deeply thank Nikolao Hioteli, Yorgio Roumeli, and Nikoleta Oikonomou.

Moreover, being a person that his friendships never faint in his mind, I want to engrave the importance for me of Yorgo Fatouro and Yorgo Palaiologou; together, and during our pre-university years, we made our first major steps to formulate well-defined purposes and methods for our lives.

Finally, I want to cordially thank Sonia Martinez and George Pappas; their messages were a source of unexpected and overwhelming excitement for me, excitement that among others expedited the fulfillment of this research work.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The recent pervasion of the Internet in our every-day life induced the emergence of a plethora of social media websites, such as the Facebook, Twitter, and the YouTube, as well as of a myriad of interlinked blogs, that already constitute a sturdy nexus between countless individuals, and a major conduit of information flow of any kind. And especially for the consumers, today's social platforms provoked an extremely favorable shift in their power; they can massively share their preferences, and influence each other, over a plethora of products at the click of a mouse [BBCW]. On the other hand, this emergent power several companies already pave the way to harness, and strategically turn it to their advantage through modernized word-of-mouth tactics [BBCW, BBCG, KATHP]. Hence, the birth of the Internet, and the subsequent development of social media websites, have in overall transformed viral marketing to an indispensable strategic tool for a company's growth and long-term success [BBCG].

Therefore, concerning this fervent movement toward viral marketing, and following a series of related influential articles [AFPT10, DR01, IKMW07, KKT03, M00] and recent books [C04, EK10, G07, VR07, J08], we define a non-cooperative game that captures the diffusion of competing trends, or products; given a number of competitive firms that have budgets to "sow" the initial adoption of their products within a static social network of consumers, we define a non-cooperative game, and we analyze several of its characteristics. To this end, we consider the strategy of each firm as the initial set of individuals that the firm targets in order to "infect" with its product. Explicitly, we assume that each firm can initially infect $k \in \mathbb{Z}_{>0}$ nodes. Additionally, we define the payoffs of the firms as the eventual number of adoptions of their product that are "yielded" through a competitive diffusion process over the network. Further, we consider this diffusion process as a sequence of deterministic and irreversible local interactions between the consumers that have already adopted a particular product and their friends/neighbors that have adopted still none. Particularly, we consider a local interaction scheme based on the widely used linear threshold model (`LTM`) [G78]. Moreover, according to this scheme, we propose a deterministic tie-breaking criterion for the cases where an individual can adopt more than one products, given, of course, that yet has none. Explicitly, we assume that the individuals within the social network share over the involved products a common order of preferences $R^{\prec}$; i.e., the distinctive quality of each of the products — which is assumed to exist — is perceived the same by each individual, as, for example, Morris [M00], and Immorlica et al. [IKMW07] similarly assume. Therefore, we solve such dilemmas with regard to $R^{\prec}$, which is a clearly deterministic criterion — recall at this point the also deterministic nature of the classic `LTM`. Finally, we note that in several of our results the use of the threshold model `LTM`, as well as, of the criterion $R^{\prec}$, is done without loss of generality.

Under this general framework, our work unfolds into four main parts. In the first part, we illustrate that games over social networks with even simple structure may possess no pure strategy Nash equilibria (PSNE). Next, we propose necessary and sufficient conditions for the existence of a PSNE, along with some further necessary conditions. Moreover, we prove that it is co-NP-complete to decide whether a PSNE exists given a particular game. Afterwards, we establish that over a certain class of social networks, with appropriate underlying structure, all possible game matrices are realizable, regarding the improvement paths that they induce. Additionally, we prove that the result holds under "almost" any consistent in and out-degree distributions that may characterize the structure of the social network.

In the second part, we turn our attention to 2-player games, and we propose necessary conditions for the existence of a generalized ordinal potential, conditions that involve the structure of the social network. We continue with a set of sufficient conditions, and we define a certain class of games where these conditions are also necessary. Then, further classes of games with a generalized ordinal potential are described.

In the third part, and again for the 2-player game, we move on to approximate ϵ-generalized ordinal potentials, as a novel method of measuring the "instability" of the games in question — the larger the parameter ϵ is, the more "unstable" the game can be characterized. Particularly, we prove that each game admits an ϵ-generalized ordinal potential, where ϵ equals the maximum value of the diffusion collision factor of the involved social network. Further, for the realistic special case of games with diffusion depth one, we provide their complete characterization. We also prove that they always admit a 1-generalized ordinal potential. On the other hand, we provide tight examples to our approximation results for all the possible cases of diffusion depth.

Finally, in the forth part, regarding the Price of Anarchy (PoA) and the Price of Stability (PoS) of these games, and for any number of players, we show that the PoA may obtain its worst value, even if PoS obtains its possible best. We end by illustrating that when more than two firms are involved in the game, then a PSNE can exist where the firm associated with the product of the "best" quality does not receive the greatest payoff among the involved players.

Summarizing, in this work we unveil several features inextricably intertwined with various structural characteristics of the underlying social networks that to the best of our knowledge have met no previous investigation. Therefore, with this study we also target to motivate further empirical and experimental research on a series of relevant and essential questions of interest; empirical research that can form a feedback loop that will inevitably drive theoretical approaches on even more realistic models of social networks, local interaction schemes, and tie-breaking criteria.

The rest of our work is organized as follows: In Chapter 2 we briefly survey a necessary relevant background, and we discuss several related works. In Chapter 3, we again summarize the major contributions of our work, and we continue with the main exposition of our results. We end the chapter, and our study, with a discussion on motivated future work, and with some further concluding remarks.

Chapter 2

Related Background and Work

Related Background

In this section we summarize several fundamental properties of social networks, and some basic concepts of the diffusion of information through them. We begin with some structural characteristics of social networks, like the *diameter* and the *clustering coefficient*. Additionally, we describe structural phenomena, such as the *small-world* and the *scale-free* phenomena, that are present in a plethora of real-world social networks. Afterwards, we present several basic *graph models* for such networks.

We continue summarizing several models of information diffusion over social networks. In the first place, we present the concept of a *local interaction game*, and we continue with some of the prevailing models of information diffusion in the recent literature: the *threshold* and the *cascade* models.

Furthermore, we present recent results on the *inferring of social networks* from a collection of observed data of diffusion and influence within such networks. We end with a rather different perspective by summarizing a series of empirical results that concern many of the previous topics on information diffusion.

2.1 On the Structure of Social Networks: Basic Characteristics

We begin by summarizing some of the basic structural properties of social networks.[1]

A social network can be described as a collection of individuals with some pattern of interactions or "bonds" between them. Friendships among a group of individuals, business relationships between companies, and epidemiological contact networks are all examples of such networks. Network analysis has a long history in sociology with the literature stretching back at least half a century to the pioneering work of Rapaport, Harary, and others in the 1940s and 1950s. Typically, network studies in sociology have been data-oriented, involving empirical investigation of real-world networks, that was followed by

[1] This introductory discussion heavily relies on the work of Newman et al. [NWS02].

graph-theoretical analysis, and they often aimed at determining the centrality, or influence of the various involved individuals in the network.

More recently, following a surge in interest in the network structure among mathematicians and physicists, another body of research investigated statistical properties of networks, as well as methods for modeling networks either analytically, or numerically [S01, AB02]. One important and fundamental result, which has emerged from these studies, concerns the number of bonds that each individual has with his friends/neighbors, the so-called *degree*; it has been found that in many networks the distribution of individuals" degrees is highly skewed, with a small number of individuals having an unusually large number of bonds. Further simulations and analytic work suggested that this property can have a crucial impact on the way in which communities operate, including the way information diffuse/travels within the network. Thus, concerning these suggestions, we continue by describing some important structural properties of real-world social networks.

2.1.1 Diameter of the Network

Recent work on social networks, within mathematics and physics, has focused on three distinctive features of the network structure. The first concerns its usually small *diameter* [D10], as highlighted in an early work by Stanley Milgram [M67]. Specifically, Milgram described an experiment that he performed, involving letters that were passed from acquaintance to acquaintance, from which he deduced that many pairs of apparently distant people are actually connected by a very short chain of intermediate acquaintances. Particularly, he found this chain to be of typical length of only six. Since Milgram's work, it has been repeatedly shown that numerous other social networks are characterized by a similar small-world phenomenon [WS98, AJB99, N01].

2.1.2 Small-World Phenomenon

Although Milgram [M67] talked about the *small-world* phenomenon, meaning the question of how two people can have a short connecting path of acquaintances in a network that has other social structure, such as insular communities or geographical and cultural barriers, in a more recent work, Watts and Strogatz [WS98] have used this phrase over networks that exhibit a combination of such short paths with a particular social structure that is defined by clustering coefficient.

2.1.3 Clustering Coefficient

The *clustering coefficient* is the second property of social networks that has been emphasized in the recent literature. Watts and Strogatz [WS98] showed that in many real-world networks the probability of the existence of a bond between two individuals is much greater if the two individuals in question have another mutual acquaintance, or even several. In other words, the probability that two of one's friends know one another is much greater than the probability that two people chosen randomly from the population know one another. Based on this fact, Watts and Strogatz defined the clustering coefficient as the probability that two acquaintances of a randomly chosen person are themselves acquainted, and they showed that for a variety of networks this coefficient took values anywhere from a few percent to forty or fifty percent — a fact supported by several other subsequent studies [N01, NSW01].

But how this clustering phenomenon can affect a word-of-mouth diffusion processes within a social network, and explicitly its duration? For a simple, but indirect answer, we can consider a social network with a high clustering coefficient, and a particular individual i in it — the initiator of the word-of-mouth process. Then, the number of distinct individuals that can be reached from i after k steps — following the minimum paths of acquaintances between i and each of the individuals in the network — cannot grow exponentially with k. Therefore, the clustering phenomenon can have a major impact to the information diffusion over social networks.

2.1.4 Scale-Free Phenomenon

The third and last structural property of interest concerns the highly skewed degree distribution that many social networks exhibit, as Albert et al. [AJB99, BA99] first emphasized it. Explicitly, [AJB99, BA99] exploring several databases describing the topology of large networks that span fields as diverse as the WWW, they analyzed them as undirected graphs, and they showed that the probability $P^d(k)$ that a vertex in the network interacts with k other vertices decays as a power law, following a $P^d(k) \propto k^{-\gamma}$, where γ ranges between 2 and 3. This fact implies that each node has a statistically significant probability of having a very large number of connections compared to the average connectivity and, as a result, there is no representative connectivity in the network. Thus, the term *scale-free* network was also coined.

2.2 On the Structure of Social Networks: Basic Graph Models

In this part we point our attention to some of the basic graph models for the structure of the social networks: the *Erdős-Rényi* and the *Scale-Free* models.

Before the observation of the scale-free properties in many of the social networks [2.1.4] the *Erdős-Rényi model* (Erdős and Rényi [ER59]) constituted a prevailing model for a variety of real-world networks, and numerous papers had discussed its properties. On the one hand, an Erdős- Rényi graph G(n,p) is simple to define; one takes some number n of nodes and places connections between them independently at random, such that each pair of nodes i, j has a connecting edge with probability p. On the other hand, this random process generates a graph with a degree distribution that converges to a Poisson distribution as $n \to \infty$; a property in strong discordance to the scale-free character of many of the real-world networks [2.1.4].

The inability of the Erdős-Rényi model to capture the scale-free phenomenon led Albert and Barabási [BA99, BB03] to devise a new random algorithm — preferential attachment — that can generate networks consonant with a power-law degree distribution. Later works substantiated additional characteristics over an extended collection of random networks, either undirected or directed, and under a variety of possible degree distributions [NSW01, KN09].

2.3 On Models of Diffusion over Social Networks

We continue by summarizing some of the basic models of diffusion over social networks.[2] All the discussed models, unless otherwise specified, concern the diffusion over a social network of a *single* product (trend).

Although the first empirical studies of diffusion on social networks began to appear in the middle of the 20th century [CKM66, RG43], formal mathematical models of diffusion were introduced a few decades later [G78, S78]. In this overview, we will mainly concentrate on *local interaction games* [B93, E93, G96], *threshold models* [G78, S78, AM11], and *cascade models* [GLM01a, GLM01b, KKT03].

To this end, we will model the *structure* of a social network with a graph $G = (V, E)$, either directed, or undirected. Each vertex $v \in V$ will represent an individual within the network. For convenience, throughout the exposition we will refer to the nodes in the graph and the individuals they represent interchangeably. In a directed graph, an edge $(u, v) \in E$ will denote that u has a direct influence on the decisions made by v. Similarly, in an undirected graph, the edge $(u, v) \equiv (v, u)$ will signify a mutual influence between u and v. Finally, we will denote by $N(v)$ the neighbors of v. Explicitly, $N(v) = \{u | u \in V, (u, v) \in E\}$.

We proceed by firstly describing three different models of information diffusion over social networks: local interaction games, threshold models and cascade models. Each of these models relies on different assumptions with regard to how a network's node is influenced by its neighbors. In other words, they constitute alternative models for the "inter-node" interactions.

2.3.1 Local Interaction Games

Before introducing the *local interaction games*, let us recall the class of *coordination games*, as the class of games with multiple pure strategy Nash equilibria in which players choose the same, or corresponding strategies. We are particularly interested in coordination games since the diffusion of a given product (trend) unfolds whenever individuals without any product eventually adopt it, influenced by their friends/neighbors that have already adopted it too. From a game-theoretic perspective, the former individuals choose to "play" the same "strategy" as at least some of their neighbors. Therefore, coordination games formalize the idea of coordination problems, which are clearly widespread in the

[2] This introductory discussion heavily relies on the work of Wortman [W08].

social sciences, meaning situations in which all the involved parties can realize mutual gains, but only by making mutually consistent decisions.

As an example of a coordination game, we consider the 2-player, 2-strategy game, with the following payoff matrix:

	A	B
A	$u_1(A,A), u_2(A,A)$	$u_1(A,B), u_2(A,B)$
B	$u_1(B,A), u_2(B,A)$	$u_1(B,B), u_2(B,B)$

where $\{A,B\}$ is the set of available strategies for the two players, and $u_1(S_1,S_2)$ ($u_2(S_1,S_2)$) denotes the payoff of `Player 1` (`Player 2`) when the a set of strategies $(S_1,S_2) \in \{A,B\}^2$ is played. Evidently, this game is a coordination game, if $u_1(A,A) > u_1(B,A)$ and $u_1(B,B) > u_1(A,B)$, as well as, if $u_2(A,A) > u_2(A,B)$ and $u_2(B,B) > u_2(B,A)$; then the strategy profiles (A,A), and (B,B) are pure strategy Nash equilibria (PSNE). Evidently, this setup can be straightforwardly extended for more than two strategies.

On this ground we introduce local interaction games, which were developed as an extension of the two-player coordination games to the setting in which there is a large, or even infinite population of players interacting along the edges of a social network. Variations of local interaction games were presented and studied in the early nineties by Blume [B93] and Ellison [E93], and they can be viewed as a precursor to the more general, widely studied class of graphical games [KKLO03, KLS01]. Here, we focus primarily on the version of local interaction games formalized by Morris [M00].

For our purposes, a local interaction game is a pair (G,q) where $G = (V,E)$ is a connected, undirected graph representing the underlying structure of the social network. The vertex set V and edge set E may be finite or infinite, but it is assumed that no node has more than a finite number M of neighbors. Here, q is a parameter specifying the *relative* goodness of the two actions A and B. Particularly, if a player chooses action B, then he receives a payoff of q for *each* of his neighbors that also chose B. If a player chooses A, he receives a payoff of $1-q$ for *each* neighbor that also chose A. Thus, we can think of each player as interacting with each of his neighbors via the game matrix

	A	B
A	$1-q, 1-q$	$0, 0$
B	$0, 0$	q, q

Consequently, the overall payoff for each player is the sum of all his payoffs from each one of these separate games. A certain restriction, therefore, should be stressed: each player can choose only one action among the available A and B, which will also be common over all these separate games.

Now, if all the players in the network cannot deviate to a better strategy, given the strategies of their neighbors, we say that a Nash equilibrium is obtained. A trivial example of a Nash equilibrium is the situation where all the players choose action A, (or B), as their strategy — if any player chooses to deviate he then receives a payoff of value zero.

Examining a single equilibrium, however, does not give much insight about how trends — here represented by strategies A and B — spread. Instead, and in order to model the information diffusion process, it is more useful to consider the evolution of the players" actions over time, where the players are assumed to play at each time step a best response strategy to the strategies of their friends/neighbors. Specifically, we assume that at an initial time $t = 1$ the (possibly infinite) set of nodes V start out in a particular configuration where a *finite* set of players A_1 adopt the "new" product — for example A —, while the remaining set of players $V \setminus A_1$ remain with the "preexisting" product — here B. Then, at each subsequent time t, each player plays a best response strategy to the actions chosen by his neighbors at time $t-1$ (we note that from this latter assumption, a direct relation to the concept of Stackelberg games is evident). Furthermore, we assume that any ties — i.e., situations where both A and B constitute a best response strategy for an individual — are broken in favor of the new product A. Therefore, examining the evolution over time of the set of nodes that have eventually adopted A, we can yield a dynamic view of how a new product, or trend may spread across the network.

Later, in [2.5.1], we will describe an extension to this previous model where the players are allowed to play an extra strategy AB [IKMW07] also. Specifically, this new strategy AB corresponds to the case where the products A and B are compatible with each other, and therefore the players can even adopt them both, (by incurring, however, some extra cost).

2.3.2 Threshold Models

Local interaction games can be viewed as a subclass of the more general class of threshold models. Recall that in a local interaction game, a node v chooses its action at time t as a best response to the actions of its neighbors at time $t-1$. If v chooses action A, its payoff is $1-q$ times the number of its neighbors who choose A; if it chooses action B, its payoff is q times the number of its neighbors who choose B. Thus, A is a best response for v if and only if a fraction q, or more of its neighbors choose action A. Formally, node v chooses action A at time t if and only if

$$\frac{1}{|N(v)|}\sum_{u\in N(v)} X_{u,t-1} \geq q,$$

where $X_{u,t-1}$ is 1, if u chose action A at time $t-1$, and 0 otherwise. Here the parameter q can be viewed as a fixed cutoff, or threshold. And hence, this viewpoint gives rise to the so-called threshold models.

Threshold models date back at least as far as Granovetter [G78] and Schelling [S78], with a recent extension by Apt and Markakis [AM11]. Originally, they concern the spread of a *single* product over a social network. One simple example is the *linear threshold model.* In this model, each node $v \in V$ has a nonnegative weight w_{uv} for every $u \in N(v)$, where $\sum_{u\in N(v)} w_{uv} \leq 1$, and a personal threshold value $\theta_v \in (0,1]$. This threshold can be hard-wired to a particular value, as in the case of local interaction games, or chosen at random at the start of the process.

Given these thresholds and an initial set A_1 of active nodes — i.e., nodes that have adopted the "new" product A —, the process unfolds deterministically in a sequence of steps. At the t-th time step, every node that was active at time $t-1$ remains active. On the other hand, each node v that was inactive at time $t-1$ becomes active at time t if and only if

$$\sum_{u \in N(v)} X_{u,t-1} w_{uv} \geq \theta_v,$$

where $X_{u,t-1}$ is again 1 if u was active at time $t-1$ and 0 otherwise. Hence, the weight w_{uv} captures how much v is influenced by u, and the threshold θ_v the personal resistance level of v to adopt a new product after its neighbors.

It is important to note that while the process is deterministic *given the set of thresholds*, the ability to randomize the thresholds' values allows an often natural injection of randomness into the process. For example, Kempe et al. [KKT03, KKT05], in order to model lack of prior knowledge about the true thresholds of each individual, assumed that the thresholds are chosen independently and uniformly from $[0, 1]$ for each individual.

The linear threshold model can be generalized further by replacing the term

$$\sum_{u \in N(v)} X_{u,t-1} w_{uv} \geq \theta_v$$

with an arbitrary function of the set of active neighbors of v [KKT03]. More specifically, let f_v be any *monotone increasing function* with $f_v(\emptyset) = 0$ that maps (active) subsets of $N(v)$ to the range $[0, 1]$. Similarly to the linear threshold model, in the general model at each time t each inactive node v becomes active if and only if $f_v(S) \geq \theta_v$, where $S = A_{t-1} \cap N(v)$ is the subset of $N(v)$ that is active at time $t-1$. As before, thresholds may be fixed, or chosen randomly. A notable work using this general model is that of Mossel and Roch [MR07].

Furthermore, note that for threshold models once a node has switched to a new action — here A —, it can never switch back to the previous action — B. This property is known as *progressiveness* [VNRT07]. Most of the dynamic processes that we examine here are progressive, either explicitly as in this model, or implicitly as a result of the actual dynamics of the model.

Finally, we note that these presented linear threshold models can be applied as they are whenever only *one* new product is introduced within the social network. On the other hand, Borodin et al. [BFO10], concerning the more competitive setting of *more than one* products, presented several appropriately enhanced versions.

A significant difference between this two types of models concerns the necessary existence of particular tie-breaking criteria for the latter case. Explicitly, these criteria must specify which product will be eventually adopted by each individual in the network, if he at any given time-step of the diffusion process can adopt more than one products at the same time. A detailed discussion on the subject can be found in [4].

2.3.3 Cascade Models

Inspired by research on interacting particle systems [L85], *cascade models* of diffusion were first studied in the context of marketing by Goldenberg et al. [GLM01a, GLM01b]. According to cascade models, each individual, when he becomes active, has a single, and probabilistic chance to activate each of his inactive neighbors. For the particular class of *independent cascade models*, the probability that an individual is activated by a newly active neighbor is independent of the set of neighbors who have attempted to activate her/him in the past. In overall, cascade models can capture directly the notion of contagion, combining the idea that an individual"s receptiveness to influence depends on the past history of interactions with his neighbors.

2.4 Influence Maximization

In this section we discuss influence maximization over social networks when a *single* trend, or product is involved. Influence maximization is the problem of choosing the "best" set of individuals within a network with regard to their potential to trigger a widespread adoption of a given product. This problem was first introduced by Domingos and Richardson [DR01], who noted that although data-mining techniques had been traditionally used in various direct marketing applications with great success, a wealth of network information was constantly remaining unused. Therefore, they suggested that marketing companies, instead of deciding if they should directly target a consumer over a particular product, and based solely on the expected profit that they could earn from his individual purchases of this product, they should take into account the effect that this consumer would have if she told her friends about it, and they, in turn, told their own friends, and so forth.

The influence maximization problem can be formalized in a number of different ways. Domingos and Richardson [DR01, RD02] modeled the problem as a Markov random field, and discussed heuristic techniques that aimed at finding a marketing strategy that approximately maximizes the global *expected lift in profit.* (Intuitively, the expected lift in profit — which was introduced by Chickering and Heckerman [CH00] in the context of direct marketing — is the difference between the expected profit obtained by employing a marketing strategy, and the expected profit obtained using no marketing at all.)

On the other hand, Kempe et al. [KKT03, KKT05], assuming a fixed marketing budget sufficient to target k individuals, studied the problem of directly identifying the "best" k individuals to target in the network. Particularly, they first showed that this problem is NP-hard — concerning the linear threshold and the independent cascade model as models of diffusion of the involved product —, and then they provided a simple greedy algorithm, analogous to the standard greedy set cover approximation algorithm [CLRS01], that is guaranteed to efficiently produce a $(1 - 1/e - \epsilon)$-optimal set, for any $\epsilon > 0$.

The problem of influence maximization regarding *two, or more* competing products was extensively examined by Borodin et al. [BFO10]. Specifically, they showed that the previously used greedy approach cannot be generally applied for this case. Nevertheless, they proposed alternative models of competitive diffusion that are amenable to this greedy approach. Additionally, they proved that under several such competitive influence models, it is NP-hard to achieve an approximation that is better than a square root of the optimal solution.

2.5 On Graph Properties Linked to Contagion

The previous section illustrated the problem of influence maximization. In this section, we turn to the related but fundamentally different study of the properties of infinite graphs that are linked to the spread of trends, or products.

Morris [M00] was one of the first to examine this question in depth. By studying best response dynamics on local interaction games, he aimed to uncover the diffusion properties of different classes of infinite graphs. Specifically, his work attempted to characterize the set of graphs, and the values of the local interaction game parameter q [2.3.1] for which there always exists a *finite* set of players A_1 such that: If only A_1 adopts action A at time $t = 0$, and all other individuals/players within the involved network choose the preexisting action B, then action A is eventually chosen everywhere in the network. In this sense, Morris' work was more focused on the analysis of a particular set of properties of graphs, as opposed to the algorithmic aspects of viral marketing with which Domingos and Richardson [DR01, RD02] and Kempe et al. [KKT03, KKT05] were mainly concerned.

Before presenting one of his main results, we first provide two useful definitions.

Definition (**Epidemic set, and contagion threshold**)**.** We say that A is **epidemic** on a graph G with parameter q, if there exists a finite set A_1 such that

$$\bigcup_{t \geq 1} A_t = V,$$

where for every $t \geq 1$, A_t is the set of players choosing action A at time t, assuming that each player plays a best response to their neighbors" actions at time $t - 1$, and under parameter q. Furthermore, we define the **contagion threshold** of a graph G as the maximum value of q for which A is epidemic on G with parameter q.

Note that this setting is not explicitly defined to be progressive. In other words, there is nothing preventing any nodes currently playing action A from switching back to action B, if B constitutes a best response. However, if it is assumed that the initial set of individuals A_1 sticks with action A, then it can be shown that no other individual ever switches from action A to action B.

On this ground, Morris [M00] proved that the contagion threshold is always less than, or equal to 1/2, given any graph G. Therefore, he showed that under his model it is never possible for a newly introduced product within a social network to overcome a stronger preexisting one.

2.5.1 Introducing Compatible Technologies

Immorlica et al. [IKMW07] introduced a model which we refer to as *compatible contagion games*. Compatible contagion games are a natural extension of local interaction games to the scenario in which the individuals within a social network may choose to adopt either product A, or product B as before, or to instead adopt both products A and B simultaneously at some additional cost c. This cost may be high, or low, depending on the inherent compatibility between products A and B.

Moreover, Immorlica et al. [IKMW07] assumed that the population of the individuals in the network is infinite, as Morris [M00] did. However, they also made the (possibly restrictive) assumption that every individual/node in the network has degree Δ, for some fixed Δ. Then, on the other hand, the cost c can be perceived as a per-neighbor cost of $r = c/\Delta$, and consequently each set of neighbors can be viewed as playing the modified coordination game presented in the following game matrix (Recall that the total payoff to a node is the sum of its payoffs from all the coordination games it plays with each of its neighbors).

	A	B	AB
A	$1-q, 1-q$	0, 0	$1-q, 1-q-r$
B	0, 0	q, q	$q, q-r$
AB	$1-q-r, 1-q$	$q-r, q$	$max\{q, 1-q\}-r, max\{q, 1-q\}-r$

The analog of the contagion threshold in the compatible contagion game setting is a two-dimensional epidemic region $\Omega(G)$ consisting of all (q, r) pairs for which A can become epidemic on G. Particularly, Immorlica et al. showed that the epidemic region can be surprisingly complex, even for very simple graphs.

Furthermore, they showed that compatible contagion games satisfy a property very similar to progressiveness. Specifically, Immorlica et al. [IKMW07] proved that if the initial set of active nodes A_1 is assumed to stick to action A, then no node ever switches from action A to action B, or AB. Furthermore, no node ever switches from action AB to action B. With these results in place, they subsequently proved that the existence of a particular "blocking structure" in a graph G is sufficient to fully determine whether, or not action A can become epidemic on G with parameters q and r.

An extension of this work was proposed recently by Oyama and Takahashi [OT10]. Explicitly, without adopting the restrictive assumption of a regular graph, they completely characterized when does a trend spread contagiously from a finite subset of players to the entire population in some network, and conversely, when a trend is never invaded by the other trend in any network. Generically, they showed that at least one convention spreads contagiously in some network, and for some range of payoff parameters, both conventions each spread contagiously in respective networks.

2.6 On Inferring Networks of Diffusion and Influence

Over the last decade a plethora of large scale data have emerged shedding some light to the patterns of influence in social networks [2.7]. On the other hand, the actual underlying networks over which the observed trends spread remained mostly unknown. A step towards to the solution of this challenge was made by Gomez-Rodriguez et al. it their recent work [RLK10, RLK10s]. There they developed a framework for tracing paths of diffusion and influence through networks, and inferring the networks over which contagions propagate. In particular, given the times when individuals adopt pieces of information, or become “infected”, they can identify the optimal network that best explains the observed infection times. Although they note that the the optimization problem is NP-hard to solve exactly, they develop an efficient approximation algorithm, called NETINF, that in practice gives near-optimal performance.

Therefore, their algorithm allows the study of several properties of real-world networks. Particularly, Gomez-Rodriguez et al. [RLK10, RLK10s] address several interesting questions, such as “What is the network over which the information propagates on the Web?”, “What is the global structure of such a network?”, “How do news media sites and blogs interact?”, or “What roles do different sites play in the diffusion process and how influential are they?”. As a forward step to obtain an answer, they evaluated NETINF on a large real data set of memes propagating across news websites and blogs. Then, they found that the inferred network exhibits a core-periphery structure with mass media influencing most of the blogosphere. Explicitly, they observed that clusters of sites related to similar topics existed (politics, gossip, technology, etc.), while a few sites with social capital were “acting” as interconnectors between these clusters, allowing, simultaneously, a potential diffusion of information among the sites of these different clusters.

In the next section we encounter a more detailed discussion with regard to the dynamics of information diffusion over social networks, based on some additional recent works over collected empirical data.

2.7 On Empirical Data of the Dynamics of Viral Marketing

In this part, we present a collection of mostly empirical studies over large-scale data, pertained to recommendation networks, the blogspace, and Internet chain-letters, that target to unveil several kinds of cascades that arise frequently in real life as well as how these kinds reflect the properties of their underlying network environment.

In the first place, Leskovec et al. [LSK06, LAH07] addressed these issues directly, regarding information cascades; information cascades are phenomena in which individuals adopt a new action, idea, or product due to influence by others. Particularly, Leskovec et al. considered information cascades in the context of recommendations, and more explicitly, they studied the patterns of cascading recommendations that arise in large social networks.

Thus, they investigated a large person-to-person recommendation network, by analyzing an online retailer's incentivised viral marketing program. The recommendation network consisted of four million people who made sixteen million recommendations to others over half a million products.

Then, they focused on the analysis of the topological aspects of the information cascades formed. To this end, they enumerated and counted cascade subgraphs on large directed graphs to disclose frequent patterns and they found that the distribution of cascade sizes follows approximately a power-law; cascades tend to be shallow, but occasional large bursts of propagation can occur. The cascade sub-patterns revealed to them mostly small tree-like subgraphs; however, they observed differences in connectivity, density, and the shape of cascades across product types. Indeed, the frequency of different cascade subgraphs, they calculated, was not a simple consequence of differences in size or density; rather, they found instances where denser subgraphs were more frequent than sparser ones. They summarized these results at [LSK06], p. 8, Table 2; for example, for the class of recommendation networks evolved over book products they identified 122,657 cascades, of which only 959 were topologically different. Moreover, they stressed that only 213 cascades between them occurred at least ten times. For the remaining classes of products (DVDs, music, videos) they tabulated similar results. Moreover, they found that the most common cascade represented the single recommendation; it accounted for 70% of all book cascades, 86.4% of all music cascades, 74% of all video cascades, but just 12.8% of DVD cascades, since DVD cascades, tended to be most densely linked.

Furthermore, Leskovec et al. [LMFGH07], working on the same data-set as previously in [LSK06], also noted that all formed cascade networks were very sparsely linked. For example, the typical size of the largest connected component that they calculated contained fewer than 5% of the nodes at the end-time of data over all kinds of networks products. Even more illustrative is Figure 3 ([LMFGH07], p. 13) where two typical product recommendation networks are depicted: (a) a medical study guide, and (b) a Japanese graphic novel. Explicitly, Figure 3(a) reveals the low connectivity of some of the examined networks as well as that the single recommendation costitutes the prevailing type. In Figure (b) we also notice bursts of recommendations; some nodes recommend to many friends, forming a star like pattern.

Finally, Leskovec et al. note that throughout the dataset they studied they observed similar patterns: "most product recommendation networks consist of a large number of small disconnected components where we do not observe cascades. Then, there is usually a small number of relatively small components with recommendations successfully propagating. This observation is reflected in the heavy tailed distribution of cascade sizes ([LAH07], p. 16 , Figure 6), having a power-law exponent close to 1 for DVDs in particular."

From a rather different perspective, concerning how do blogs cite and influence each other, Leskovec et al. in [LMFGH07], concluded again a series of similar results: the most popular shapes of the cascade networks examined were the "stars", that is, a single blog-post that is cited by several others, where, additionally, none of the citing posts are themselves cited. And more generally, they observed that most cascades followed tree-like shapes ([LMFGH07], p. 7, Figure 9, and p. 9, Figure 12).

Moreover, they made similar conclusions, as in [LSK06, LAH07] with regard to the topological aspects of the cascades formed when certain posts became popular and were linked by other posts. For example, they found that from a total of 2,092,418 cascades, 97% of them were trivial cascades (isolated posts), 1.8% were the smallest possible non-trivial cascades, and only the remaining 1.2% of the cascades were topologically more complex.

Furthermore, they showed that the diameter of the cascade networks analyzed increased logarithmically with the size of the cascade ([LMFGH07], p. 9, Figure 12), taking, thus, rather low values — less than 5 for the given dataset.

A last observation that Leskovec et al. made in [LMFGH07], strongly related to our work, concerns the enumeration of the collisions of cascades ([LMFGH07], p. 9, Figure 13), i.e., the cases where a particular post cites more than one different blog-post. They find out that 98% of all nodes belong to a single cascade, and that the rest of the distribution follows a power-law with exponent -2.2.

Finally, we discuss briefly the work of Liben-Nowell and Kleinberg [LK08] in order to illustrate that the kind of approach taken to analyze a particular network can yield rather different results concerning the structure of the emerged cascades, regardless of the actual structure of the underlying network. Particularly Liben-Nowell and Kleinberg traced information-spreading processes at a person-by-person level using methods to reconstruct the propagation of massively circulated Internet-chain letters. Surprisingly, they found that rather than fanning out widely, reaching many people in very few steps according to the "small-world" principles, the progress of these chain letters proceeded in a narrow but very deep tree-like pattern, continuing for several hundred steps. Explicitly, they derived a tree that had 18,119 nodes, of which 17,079 (94.26%) had exactly 1 child. Additionally, its median node depth was 288 and its the width 82.

This observation motivated Liben-Nowell and Kleinberg to suggest a new and more complex picture for the spread of this kind of information through a social network. Rather than assuming that the letter spreads in fixed unit time steps, they modeled each recipient as waiting a length of time τ before acting on the message, where τ is distributed according to a density function (related empirical observations have been presented extensively in the previously discussed work of Leskovec et al. [LAH07] over the case of recommendation networks). Then, they showed that this asynchronous pattern of response has a ""serializing"" effect in networks with large clustering coefficient [2.1.3]: "If the neighbors of a forwarding node are mutually connected, then they will forward the letter to each other as they act on it in order, producing a single long list with all of their names rather than many distinct shorter lists, each containing one of their names." In the observable tree, they further argued, this change would tend to produce deeper "runs"" of nodes in which each node has exactly one child — precisely the structure that they had actually observed.

2.8 Related Work

In this section we summarize some of the studies on competitive information diffusion over social networks.

Competitive information diffusion is encountered when there are at least two *competing* parties, external to the involved network, that aim to maximize their influence to the network's population; i.e., each party seeks to optimally promote its product (trend) over the network — by "seeding" the network with initial adoptions of its product — even at the expense of the rest of the parties. We note that in order to further differentiate these parties from the individuals within the involved social network, we will refer to them as **external players**.

Next, we begin with some of the recent studies that analyze this competing behavior of the external players using a *Stackelberg*-game [S34] framework. Afterwards, we continue with those studies that use a *simultaneous*-game approach. Particularly, and on the latter subject, we overview the only existing works on the subject — to the best of our knowledge — of Alon et al. [AFPT10], and in less detail of Goyal and Kearns [GK12].

2.8.1 External players: *Stackelberg Game* Approach

Morris [M00], as [IKMW07], and the recent [AM11, SA12], proposed several conditions for a newly introduced product (trend) A in a social network, adopted initially by a finite set S of individuals, to eventually spread to the whole network's population. For the following, let us assume that the selection S constitutes the strategy of an *external player* p^s the seeks to promote product A over the network. Moreover, let us consider that the preexisting product B has been catholicly spread due to a previous act of another also hypothetical external player — a first-mover p^f. Under this framework, we can define a sequential two-player game where p^s can play only a sufficient amount of time after p^f. Evidently, this class of games shares an essential characteristic with *Stackelberg games* [S34]: the sequentiality in playing.

In the same manner we can also interpret the works of Kempe et al. [KKT03], and Borodin et al. [BFO10] for example. In their studies they examined the Influence Maximization Problem [2.4], where over a social network an external player seeks to maximize its expected influence, given that initially he can "infects" a set of k individuals with a new product that she wants to promote. Therefore, and similarly to the previous discussion, this player corresponds to the second-mover p^s, while the hypothetical first-mover p^f is responsible for the preexisting product over the whole population.

On the other hand, actual instances of Stackelberg games were introduced with the recent studies of Carnes et al. [CNWZ07], Bharathi et al. [BKS07], and Kostka et al. [KOW08]. Carnes et al. studied the strategies of a company that wishes to invade an existing market and persuade people to buy their product. To this end, they defined a Stackelberg game where the first-mover player chooses a strategy in the first stage, which takes into account the likely reaction of the second-mover players (followers). In the second stage, the followers choose their own strategies having observed the first-mover's decision. Carnes et al. used models similar to the ones proposed in [KKT03] and showed that the second player faces an NP-hard problem if aiming at selecting an optimal strategy. Furthermore, the authors proved that a greedy hill-climbing algorithm leads to a $(1-1/e-\epsilon)$-approximation.

Bharathi et al. [BKS07] introduced roughly the same model, but in the context of competing rumors, and they also proved that there exists an efficient approximation algorithm for the second-mover player. Moreover, they presented an FPTAS for the single-player problem over trees.

Finally, under the model of Kostka et al. [KOW08] two players select a disjoint subset of nodes as initiators of the rumor propagation, seeking to maximize the number of persuaded nodes. They showed that computing the optimal strategy for both the first-mover and the second-mover is NP-complete, even in a most restricted model. Moreover, they proved that determining an approximate solution for the first player is NP-complete as well. Additionally, the authors analyzed several heuristics and showed that being the first to decide is not always an advantage; namely, networks exist where the second player can convince more nodes than the first, regardless of the first player"s decision.

2.8.2 External players: *Simultaneous Game* Approach

To the best of our knowledge, the work of Alon et al. [AFPT10] was the first to deal with the incentives of interested parties outside a social network under the framework of a simultaneous non-cooperative game. In particular, they supposed that several external players/firms there exist that strive to advertise their competing products over a social network using viral marketing. To this end, they assumed that all the firms target *simultaneously* at an initial step a small subset of individuals each, in the hope that subsequently the rumor about their products will spread throughout the network. Additionally, they assumed that an individual within the network that adopts a product cannot adopt another, and hence the campaign of one firm can negatively affect the success of the others. As a result, a competitive game-theoretic setting emerges.

Under this setting, and under the model of information diffusion over social networks they introduced, they studied the relation between the diameter of the involved network and the existence of pure Nash equilibria of the corresponding game. Particularly, they showed that if the diameter of a *modified* version of the original network (see errata [AFPT11], [THS12]) is at most two, then an equilibrium exists and can be found in polynomial time, whereas if the diameter is greater than two, then an equilibrium is not always guaranteed to exist.

We now present in detail the model of information diffusion over social networks that Alon et al. used in [AFPT10]. In the first place, the underlying social network is represented by an *undirected* graph $G^{ud} = (V, E)$, where the vertices model the individuals of the network, and the edges the (assumed mutual) bonds between them. Furthermore, the set of the external players/firms, is denoted as $\mathcal{M} = \{1, \ldots, m\}$, $m \in \mathbb{Z}_{>0}$. On the other hand, the diffusion process unfolds as follows: There are $n + 2$ colors: a color for each firm $i \in \mathcal{M}$, as well as the `white` and `gray`. Initially, at time $t = 1$, some of the vertices are colored in the colors of $\mathcal{M}$, while the others are colored `white`. Moreover, at time $t+1$ each `white` vertex that has neighbors colored in color i, but does not have neighbors colored in color j, for any $j \in \mathcal{M} \setminus \{i\}$, is colored in color i. Additionally, a `white` vertex that has two, or more neighbors colored by two, or more distinct colors $(i, j) \in \mathcal{M}$, $i \neq j$m — i.e. faces a dilemma between two different products —, is colored `gray`. In other words, Alon et al. assumed that if two firms compete for an individual at the same time, they "cancel out", and the user is *removed* from the game. Finally, the process continues until it reaches a fixed point, that is, all the remaining `white` vertices are unreachable due to the colored `gray` vertices.

Now, a game $\Gamma = (G^{ud}, \mathcal{M})$ is induced by a graph G^{ud}, and the set of agents $\mathcal{M}$. The strategy space of each agent is the set of vertices V in the graph, that is, each agent i selects a *single* node that is colored in color i at time $t = 1$. At this point Alon et al. [AFPT10] noted that if two, or more firms select the same vertex at time $t = 1$, then that vertex is colored automatically `gray`. Furthermore, Alon et al. defined as a strategy profile a vector $\mathbf{x} = (x_1, \ldots, x_m) \in V^m$, where $x_i \in V$, $i \in \{1, \ldots, |V|\}$, is the initial vertex selected by agent i. Also, they considered $\mathbf{x_{-i}} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$.

Moreover, given a strategy profile $\mathbf{x} \in V^m$, the utility of agent $i \in \mathcal{M}$, denoted $u_i(\mathbf{x})$, is the number of nodes that are colored in color i when the diffusion process terminates. Further, a strategy profile $\mathbf{x}$ is a pure strategy Nash equilibrium (PSNE) of the game Γ if an agent cannot benefit from unilaterally deviating to a different strategy, i.e., if for every $i \in \mathcal{M}$ and $x'_i \in V$ it holds that $u_i(x'_i, \mathbf{x}_{-i}) \leq u_i(\mathbf{x})$.

Finally, concerning the definition of the *diameter* of an undirected graph: Given an undirected graph G^{ud}, and a pair of nodes $(u, v) \in V^2$, let $d_{min}(u, v)$ be the length of the shortest path between u and v (in terms of the number of edges). Then, the **diameter** of the graph G^{ud}, denoted $DM(G^{ud})$, is the maximum shortest path between any pair of vertices, i.e., $DM(G^{ud}) = max_{(u,v) \in V^2} d_{min}(u,\, v)$.

Now we restate the main results of Alon et al. [AFPT10]. The first demonstrates that for underlying graphs with small diameter — and under certain modifications of the involved graph — the existence of a PSNE is guaranteed. On the other hand, the second illustrates that for greater diameters a PSNE may not exist. Explicitly:

Theorem ([AFPT10], errata [AFPT11, THS12]). *Let G^{ud} be an undirected graph, and let $\mathcal{M} = \{1, \ldots, m\}$ such that $DM(G^{ud'}) \leq 2$, for every $G^{ud'}$ that is obtained from G^{ud} by removing $m - 1$ vertices along with their neighbors. Then the game $\Gamma = (G^{ud}, \mathcal{M})$ admits a Nash equilibrium, which can be found in polynomial time.*

Theorem ([AFPT10]). *Let $\mathcal{M} = \{1, 2\}$. Then, there exists a graph G^{ud} with $DM(G^{ud}) = 3$ such that the game $\Gamma = (G^{ud}, \mathcal{M})$ does not admit a Nash equilibrium.*

This latter theorem can be easily extended to any larger number of agents, or to any (finite, or infinite) diameter greater than three, thus coinciding with the general result we have previously described.

Below, we turn our attention to the recent work of Goyal and Kearns [GK12]. Explicitly, they analyzed a simultaneous game where two firms simultaneously choose to allocate their resources on subsets of a consumers' social network, firstly, in order to "seed" the network with initial adoptions, next, to trigger a competitive diffusion process, and finally, to eventually each maximize — even at the expense of the other — the total number of consumers that will adopt its product. To this end, and similarly to Alon et al. [AFPT10], they considered the payoffs to the firms as the eventual number of adoptions of their product through a competitive stochastic diffusion process in the network. Then, they identified a general property of the adoption dynamics for which the Price of Anarchy is uniformly bounded above, across all equilibria and networks. They also showed that if this property is even slightly violated, the Price of Anarchy can be unbounded, thus yielding sharp threshold behavior for a broad class of dynamics.

They also introduced a new notion, the Price of Budgets, that measures the extent that imbalances in player budgets can be amplified at equilibrium. Then, they again identified a general property of the adoption dynamics for which the (pure) Price of Budgets is uniformly bounded above, across all equilibria and all networks. Finally, they showed that even a slight departure from this property can lead to unbounded Price of Budgets, again yielding sharp threshold behavior for a broad class of dynamics.

Chapter 3

Contributions

Given a number of competitive firms that have budgets to "sow" the initial adoption of their products within a static social network of consumers, we define a non-cooperative game, and we analyze several of its characteristics. To this end, we consider the strategy of each firm as the initial set of individuals that the firm targets in order to "infect" with its product. Explicitly, and in contrast to Alon et al. [AFPT10], we assume that each firm can initially infect $k \in \mathbb{Z}_{>0}$ nodes, instead of only one. Additionally, and similarly to Alon et al. [AFPT10], we define the payoffs of the firms as the eventual number of adoptions of their product that are "yielded" through a competitive diffusion process over the network. Further, we consider this diffusion process as a sequence of deterministic and irreversible local interactions between the consumers that have already adopted a particular product and their friends/neighbors that have adopted still none. Particularly, we consider an enhanced to Alon et al. [AFPT10] scheme, based on the widely used threshold model (`LTM`) [G78]. Moreover, according to this scheme, instead of "coloring" gray the nodes that are on a dilemma between two or more products, as in Alon et al. [AFPT10], we assume that the individuals within the social network share over the involved products a common order of preferences $R^{\prec}$; i.e., the distinctive quality of each of the products — which is assumed to exist — is perceived the same by each individual, as, for example, Morris [M00], and Immorlica et al. [IKMW07] similarly assume. Therefore, we solve such dilemmas with regard to $R^{\prec}$, which is clearly a deterministic criterion — recall at this point the also deterministic nature of the classic `LTM`. Finally, we note that in several of our results the use of the threshold model `LTM`, as well as, of the criterion $R^{\prec}$, is done without loss of generality.

Under this general framework, our work unfolds into four main parts. In the first part, we illustrate that games over social networks with even simple structure may possess no pure strategy Nash equilibria (PSNE). Next, we propose necessary and sufficient conditions for the existence of a PSNE, along with some further necessary conditions. Moreover, we prove that it is co-NP-complete to decide whether a PSNE exists given a particular game. Afterwards, we establish that over a certain class of social networks, with appropriate underlying structure, all possible game matrices are realizable, regarding the improvement paths that they induce. Additionally, we prove that the result holds under "almost" any consistent in and out-degree distributions that may characterize the structure of the social network.

In the second part, we turn our attention to 2-player games, and we propose necessary conditions for the existence of a generalized ordinal potential, conditions that involve the structure of the social network. We continue with a set of sufficient conditions, and we define a certain class of games where these conditions are also necessary. Then, further classes of games with a generalized ordinal potential are described.

In the third part, and again for the 2-player game, we move on to approximate ϵ-generalized ordinal potentials, as a novel method of measuring of the "instability" of the games in question — the larger the parameter ϵ is, the more "unstable" the game can be characterized. Particularly, we prove that each game admits an ϵ-generalized ordinal potential, where ϵ equals the maximum value of the diffusion collision factor of the involved social network. Further, for the realistic special case of games with diffusion depth one, we provide their complete characterization. We also prove that they always admit a 1-generalized ordinal potential. On the other hand, we provide tight examples to our approximation results for all the possible cases of diffusion depth.

Finally, in the forth part, regarding the Price of Anarchy (PoA) and the Price of Stability (PoS) of these games, and for any number of players, we show that the PoA may obtain its worst value, even if PoS obtains its possible best. We end by illustrating that when more than two firms are involved in the game, then a PSNE can exist where the firm associated with the product of the "best" quality does not receive the greatest payoff among the involved players.

Summarizing, in this work, under a general, realistic and deterministic framework, and using novel methods, we unveil several features inextricably intertwined with various structural characteristics of the underlying social networks that to the best of our knowledge have met no previous investigation. Therefore, with this study we also first target to motivate further empirical and experimental research on a series of relevant and essential questions of interest; empirical research that can form a feedback loop that will inevitably drive theoretical approaches on even more realistic models of social networks, local interaction schemes, and tie-breaking criteria. For a detailed discussion: [8].

Chapter 4

Preliminaries

We define a *non-cooperative* game between competitive firms that simultaneously allocate their resources on subsets of consumers within a social network; firstly in order to "seed" the network with initial adoptions, next, to trigger a competitive diffusion process, and finally, to eventually each maximize — even at the expense of the others — the total number of consumers that will adopt its product.

4.1 Social Networks

The underlying structure of the social network is assumed *static*, and is modeled by a fixed *directed graph* $G = (V, E)$ with no parallel edges and no self-loops. We denote the cardinality of V as $|V|$— and similarly for any other involved set. We also assume $|V|, |E| < \infty$. Furthermore, each node $v \in V$ represents an individual within the social network, while each directed edge $(u, v) \in E$ represents that v can be influenced by u over a set C of m available products (trends), where $m \in \mathbb{Z}_{>0}$, and $m < \infty$. To each such product we assign a unique **color**, such that $C = \{c_1, \dots, c_m\}$. Throughout this work, we shall use the terms *product* and *color* interchangeably. Moreover, we assume that m **firms** exist, each one associated with a *different* color from C. We denote the set of firms by $\mathcal{M} = \{1, \dots, m\}$. Furthermore, we shall use the terms *firm* and ***external player*** interchangeably. Also, we shall call the firm associated with the color c, simply, firm c. Further, the colors are assumed *incompatible* with each other, and *no* node that has already adopted one will adopt any other. Additionally, as with most of the literature, we assume that all adoption are final; no node that has adopted a particular color shall later alter its decision.

Moreover, the **neighbors** of $v \in V$ is the set $\{u \in V | (u, v) \in E\}$ — i.e., the set of nodes that can influence v. Also, we denote as d_v^{out} (d_v^{in}) its out-degree (in-degree). How the influence is exerted between any two neighbor-nodes is described by a *local interaction scheme*.

Definition 1 (**Local Interaction Scheme `LIS`**)**.** Given a directed graph $G = (V, E)$, and any node $v \in V$, a **local interaction scheme** (`LIS`) defines *how* v can be influenced by any subset of its neighbors over an available set of colors C.

A particular example of a `LIS` is the linear threshold model (`LTM`) [G78, S78], [2.3.2]: firstly, it describes that each individual within a social network can be influenced by his neighbors over the single color in the market, color that at least some of them have already adopted, and which he has still not. Additionally, it defines how this influence is quantified through the node's threshold θ_v, and the neighbors' edge weights w_{uv}, $\forall u \in V$ such that $(u, v) \in E$.

Additionally, whenever $m \geq 2$ — i.e., $|C| \geq 2$ — we shall call the diffusion process concerning the involved m competitive products as **competitive diffusion process**.

When a node adopts a color, we shall refer to it as **colored**. Also, we shall call a node `white` if it is still uncolored.

We assume that at an initial step $t = 0$ each firm $i \in \mathcal{M}$ targets a subset $S \subseteq V$ of at most $k \in \mathbb{Z}_{>0}$ nodes, where $k < |V|$ — and in practice k is much smaller than $|V|$. Specifically, S constitutes a **strategy** for firm i. We note that throughout this work, we shall use the phrases "strategy S" and "subset S" interchangeably. Furthermore, we denote as $\mathcal{S}(k) \equiv \{S : |S| = k\}$ the set of **available strategies**, which is assumed the same for each firm. We shall refer to parameter k as the **maximum strategies' cardinality**. Additionally, we call as pure **strategy profile** the vector $\mathbf{s} = (S_1, \ldots, S_m) \in \mathcal{S}^m$, where S_i corresponds to the strategy played by player $i \in \mathcal{M}$. Also, we consider $\mathbf{s_{-i}} := \{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_m\}$. Moreover, we define the set s^{dj} of strategy profiles **of disjoint strategies** as $s^{dj} = \{\mathbf{s} | \mathbf{s} \in \mathcal{S}^m, \forall (S, S') \in \mathbf{s}, S \cap S' \neq \emptyset\}$. We note at this point that under the appropriate context, we shall often consider the set of available strategies $\mathcal{S}$ as $\{S_i\}_{i \in \{1, \ldots, |\mathcal{S}|\}}$ — thus, under this notation, we shall use strategy S_i as the "i-th strategy within $\mathcal{S}$", and *not* as the "strategy of player $i \in \mathcal{M}$".

Moreover, we refer to step $t = 0$ as the **initiation** step, and to the eventually colored nodes as **initiated** nodes. On the other hand, we shall call those nodes that adopt a color at any later step $t \geq 1$ — i.e., during the competitive diffusion process — **infected**. Further, we shall say that a `white` node faces a **dilemma** at a particular step t, $t \geq 0$, if it *can* adopt at t more than one color.

Specifically, and to completely define the game, we use two tie-breaking rules over such dilemmas: One at the initiation step, and one during the competitive diffusion process. At the initiation step, two, or more external players may target to color overlapping subsets of nodes, i.e., given a pair of firms $\exists (i, j) \in \mathcal{M}^2$, $i \neq j$, and their corresponding strategies $(S_i, S_j) \in \mathcal{S}^2$, it may happen $S_i \cap S_j \neq \emptyset$. In other words, each node in $S_i \cap S_j$ faces a dilemma between the colors c_i, c_j. Thus, a **tie-breaking criterion `TBC1`**, that resolves such dilemmas, should assumed; for example, `TBC1` may capture the actual order of preferences of the involved nodes over the set C of available colors.

Similarly, during the competitive diffusion process, and at a particular step $t \geq 1$, a `white` node $v \in V$ may exists that faces a dilemma between two, or more colors, due to its neighbors earlier adoptions. Specifically, we assume that at step t a `white` node decides if it shall adopt a color by considering only the colors that its neighbors have in step $t - 1$.

At this point, we stress that each diffusion step t is assumed to have *zero duration*, i.e., to be **instantaneous**. Therefore, to solve such dilemmas, an additional **tie-breaking criterion** `TBC2` should be considered, similar, or same to `TBC1`. Such a criterion we have tacitly assumed that was incorporated in our previous Definition 1 of a `LIS`, whenever $m \geq 2$.

A particular instance of the `TBC1`, `TBC2` criteria, that we shall often invoke in our work, is the following: All the individuals within the social network share a *common* **order of preference** $R^{\prec} \equiv c_1 \succ \ldots \succ c_m$ over the available products C — i.e., all the individuals solve their dilemmas between any involved products by adopting the one that is most preferred with regard to $R^{\prec}$. Therefore, under $R^{\prec}$, we shall refer to the product c_1 as the product with the **highest**, (or **best**) **quality**.

Next, we formal define a social network.

Definition 2 (**Social Network**)**.** A **social network** $\mathcal{N}$ is defined through the tuple $(G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2})$.

4.2 Strategic Games

A **game** $\Gamma \equiv (\mathcal{N}, \mathcal{M}, k)$ is induced by a social network $\mathcal{N}$ and the set of firms $\mathcal{M}$. Given a strategy profile $\mathbf{s} \in \mathcal{S}^m$, the **utility**(**/payoff**) of firm $i \in \mathcal{M}$, denoted by $u_i(\mathbf{s})$, is the number of nodes that have been colored in c_i by the end of the competitive diffusion process. We note that whenever we consider the 2-player game — $\mathcal{M} = \{1, 2\}$ — we shall alternatively denote the players' utilities, given a strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}$, as u_i^{12}, instead of $u_i(S_1, S_2)$, $\forall i \in \mathcal{M} = \{1, 2\}$. Similarly, whenever we consider the set of available strategies $\mathcal{S}$ as $\{S_x\}_{x \in \{1,\ldots,|\mathcal{S}|\}}$, we shall often denote the players' utilities, given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, as u_i^{xy}, instead of $u_i(S_x, S_y)$, $\forall i \in \mathcal{M} = \{1, 2\}$.

Next, we denote as $\Pi(\Gamma)$ the associated **game matrix**. Also, for the 2-player game, we denote the **game matrix restriction** over the sets of strategies S_A, $S_B \subseteq \mathcal{S}$ as $\Pi(S_A, S_B)$.

Furthermore, given any firm $i \in \mathcal{M}$, any pair of its available strategies $(S_i, S_i') \in \mathcal{S}$, and any joint strategy $\mathbf{s_{-i}} \in \mathcal{S}^{m-1}$ of its opponents, we call S_i a **better response** to $\mathbf{s_{-i}}$ *with regard to* S_i' if $u_i(S_i, \mathbf{s_{-i}}) \geq u_i(S_i', \mathbf{s_{-i}})$. Similarly, we call S_i a **best response** if $\forall S_i' \in \mathcal{S}$ it is $u_i(S_i, \mathbf{s_{-i}}) \geq u_i(S_i', \mathbf{s_{-i}})$.

Additionally, we call a pure strategy profile $\mathbf{s}$ a **pure strategy Nash equilibrium (PSNE)** of game Γ if no agent can benefit from unilaterally deviating to a different strategy, i.e., if for every $i \in \mathcal{M}$ and $S_i' \in \mathcal{S}$ it holds that $u_i(S_i', \mathbf{s_{-i}}) \leq u_i(\mathbf{s})$. Similarly, a strategy profile $\mathbf{s}$ is an ϵ**-PSNE** of Γ if no agent can benefit more than ϵ from unilaterally deviating to a different strategy, i.e., if for every $i \in \mathcal{M}$ and $S_i' \in \mathcal{S}$ it holds that $u_i(S_i', \mathbf{s_{-i}}) \leq u_i(\mathbf{s}) + \epsilon$. Moreover, as $\mathcal{NE}(\Gamma)$ we denote the set of existing PSNEs of Γ, and as as ϵ-$\mathcal{NE}(\Gamma)$ its set of ϵ-PSNEs. We note that in this work we do *not* deal with mixed strategy Nash equilibria.

We continue with the formal definitions of the *generalized ordinal potential* [MS96], and ϵ*-generalized ordinal potential* [CS11]. A function $P : \mathcal{S}^m \mapsto \mathbb{R}$ is a **generalized**

ordinal potential for Γ if $\forall i \in \mathcal{M}$, $\forall \mathbf{s_{-i}} \in \mathcal{S}^{m-1}$, and $\forall x, z \in \mathcal{S}$,

$$u_i(x, \mathbf{s_{-i}}) > u_i(z, \mathbf{s_{-i}}) \Rightarrow P(x, \mathbf{s_{-i}}) > P(z, \mathbf{s_{-i}}).$$

Similarly, a function $P : \mathcal{S}^m \mapsto \mathbb{R}$ is an **ϵ-generalized ordinal potential** for Γ if $\forall i \in \mathcal{M}$, $\forall \mathbf{s_{-i}} \in \mathcal{S}^{m-1}$, and $\forall x, z \in \mathcal{S}$,

$$u_i(x, \mathbf{s_{-i}}) > u_i(z, \mathbf{s_{-i}}) + \epsilon \Rightarrow P(x, \mathbf{s_{-i}}) > P(z, \mathbf{s_{-i}}).$$

We shall refer to a game Γ that admits an ϵ-generalized ordinal potential as **ϵ-unstable**. Evidently, a 0-unstable game Γ admits a generalized ordinal potential. We note, however, that in order to capture the actual instability of the involved games, we should also consider the relative value of ϵ with regard to the typical value of the utilities of the players in these games. Below, an illustrative example follows:

Example. Consider the game matrices of two 2-player, 2-strategy games, as in Table 4.3. We call the game associated with Table 4.4a L, and the one with Table 4.4b R.

It can be verified that both games do not have any PSNE, but they both have two 1-PSNEs: the strategy profiles (A, B), and (B, A). Moreover, they both admit the same 1-generalized ordinal potential. Nevertheless, observe that the game R is far more stable than game L is in the following sense: In game R, in order for the 1-PSNEs to exist, `Player 1` must be compromised with a decrease to his payoff of only 0.1%. On the other hand, in game L, `Player 1` must be compromised with a decrease of 50%.

Therefore, we could redefine an **ϵ-unstable** 2-player game as **$\bar{\epsilon}$-unstable**, where

$$\bar{\epsilon} \equiv \frac{\epsilon}{max_{\mathbf{s} \in \epsilon\text{-}\mathcal{NE}} \frac{u_1(\mathbf{s}) + u_2(\mathbf{s})}{2}}.$$

Evidently, an $(\bar{\epsilon} = 0)$-unstable game Γ admits a generalized ordinal potential, as an $(\epsilon = 0)$-unstable does.

	A	B
A	2,0	1,1
B	1,1	2,0

(a) Game matrix of game L.

	A	B
A	1000,0	999,999
B	999,999	1000,0

(b) Game matrix of game R.

Table 4.3: The game matrices of two 2-player, 2-strategy games that both admit the same 1-generalized ordinal potential.

Thereby, under this more concise definition for our case, the game R is 1/999-unstable — i.e., it is essentially 0-unstable —, whereas the game L is 1-unstable. □

Moreover, we shall call **path**, in the strategy space $\mathcal{S}^m$, any sequence $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots)$ of strategy profiles $\mathbf{x}_k \in \mathcal{S}^m$ such that for all $k = 1, 2, \ldots$ the strategy combinations $\mathbf{x}_k$

and $\mathbf{x}_{k+1}$ differ in exactly one coordinate, say the $i(k)$-th. Then, a path is called an **improvement path** if $u_{i(k)}(\mathbf{x}_k) < u_{i(k)}(\mathbf{x}_{k+1})$, $\forall k = 1, 2, \ldots$, and it is called a **best response improvement path** if additionally $u_{i(k)}(\mathbf{x}_{k+1}) = max_{x \in \mathcal{S}} u_{i(k)}(x, (\mathbf{x}_{k+1})_{-i(k)})$, $\forall k = 1, 2, \ldots$. Moreover, a *finite* path $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, $n \in \mathbb{Z}_{>1}$, is called an **improvement cycle** if it is an improvement path, also $\mathbf{x}_1 = \mathbf{x}_n$, and $u_{i(k)}(\mathbf{x}_k) < u_{i(k)}(\mathbf{x}_{k+1})$ for some $k \in \{1, \ldots, n-1\}$.

Next, for any 2-player game, i.e., $\mathcal{M}\{1,2\}$, we define the *strictly component-wise dominance* between any two sets of strategies $X, Y \subseteq \mathcal{S}$.

Definition 3 (**Strictly component-wise dominance**)**.** For any 2-player game, given two sets of strategies $X, Y \subseteq \mathcal{S}$, we say that X **strictly compo- nent-wise dominates** Y, if at least one of the following holds:

i For all strategy profiles $(a, b) \in Y \times (Y \cup X)$, there exists $x \in X$ such that $u_1(x, b) > u_1(a, b)$, and for all strategy profiles $(a, b) \in X \times Y$, there exists $x \in X$ such that $u_2(a, x) > u_2(a, b)$.

ii For all strategy profiles $(a, b) \in (Y \cup X) \times Y$, there exists $y \in X$ such that $u_2(a, y) > u_1(a, b)$, and for all strategy profiles $(a, b) \in Y \times X$, there exists $x \in X$ such that $u_1(x, b) > u_1(a, b)$.

In the following, given a game $\Gamma(\mathcal{N}, \mathcal{M}, k)$, we denote the **set of strictly component-wise undominated strategies** as $\mathcal{S}_D$.

Moreover, given a strategy profile $\mathbf{s}$, we call the sum $SW(\mathbf{s}) = \sum_{j=1}^{m} u_j(\mathbf{s})$ the **social welfare** of $\mathbf{s}$. When the social welfare of $\mathbf{s}$ is maximal, we call $\mathbf{s}$ a **social optimum**. Furthermore, the *price of anarchy* [KP99], and the *price of stability* [ADKTWR04] are defined as follows.

Definition 4 (**Price of anarchy**)**.** Given a finite game that has a Nash equilibrium, its **price of anarchy** (**PoA**) is the ratio $SW(\mathbf{s})/SW(\mathbf{s}')$ where $\mathbf{s}$ is a social optimum, and $\mathbf{s}'$ is a Nash equilibrium with the lowest social welfare.

Definition 5 (**Price of stability**)**.** Given a finite game that has a Nash equilibrium, its **price of stability** (**PoS**) is the ratio $SW(\mathbf{s})/SW(\mathbf{s}')$ where $\mathbf{s}$ is a social optimum, and $\mathbf{s}'$ is a Nash equilibrium with the highest social welfare.

In the rest of this section, we introduce some further essential definitions. In the first place, given a firm in $\mathcal{M}$ that plays a strategy $S \in \mathcal{S}$, we consider in the following definition the hypothetical situation where *no* other firm participates in the game — i.e., we assume that *no* other strategy is played over the involved social network by any other player in $\mathcal{M} \setminus \{i\}$. Then, we denote as $d_S \in \mathbb{Z}_{\geq 0}$ the **last time step** of the diffusion process **initiated by strategy** S. Therefore, if $d_S = 1$, the associated diffusion process has only one step. We note that in general d_S is upper-bounded by the diffusion depth D [Definition 7]. Next, we define the *ideal spread of S at time step t* of the diffusion process, the *ideal cumulative spread of S at time step t* of the diffusion process, as well as, the *ideal cumulative spread of S*.

Definition 6. i. Assume that only one player from $\mathcal{M}$ participates in the game, and let S be one of its available strategies. We define as **ideal spread of S at time step t** of the diffusion process, denoted by I_S^t, the set of nodes in V that adopt the associated with this player color under strategy S, and *at* the particular time step t only. Further, we assume $I_S^0 := S$, as well as, $I_S^t := \emptyset$, whenever $t \in \{d_S + 1, \dots, D\}$.

ii. Assume that only one player from $\mathcal{M}$ participates in the game, and let S be one of its available strategies. We define as **ideal cumulative spread of S at time step t** of the diffusion process, denoted by H_S^t, the set of nodes in V that adopt the associated with this player color under strategy S, and *until* the particular time step t, i.e., $H_S^t \equiv \bigcup_{i=0}^t I_S^i$.

iii. Define as **ideal cumulative spread of S**, denoted by H_S, the set of nodes in V that will eventually adopt the associated with player i color under strategy S — i.e., $H_S \equiv H_S^{d_S}$.

For our purposes, given a strategy $S_i \in \mathcal{S}$, where $i \in \{1, \dots, |\mathcal{S}|\}$, we shall alternatively denote $I_i^t := I_{S_i}^t$, $H_i^t := H_{S_i}^t$, and $H_i := H_{S_i}$.

Furthermore, we set $|H_{max}| := max_{S \in \mathcal{S}} \{|H_S|\}$.

Now, we shall say that a set $S' \subseteq V$ is **reachable** from a strategy $S \in \mathcal{S}$ if and only if $S' \subseteq H_S$.

Additionally, given a social network $\mathcal{N}$, a strategy $S \in \mathcal{S}$, and a node $u \in V$, we define the **diffusion distance** *of u from the associated subset S* as

$$\widehat{d}(S, u) = argmin_{t \in \mathbb{Z}_{\geq 0}} \left\{ u \subseteq H_S^t \right\}.$$

Further, if u is not reachable from S, we assume $\widehat{d}(S, u) = \infty$. Evidently, the diffusion distance is a non-symmetric relation.

Finally, we define the *diffusion depth* of a game Γ as the maximum number of steps that the competitive diffusion may take place, given any strategy profile $\mathbf{s} \in \mathcal{S}^m$.

Definition 7 (Diffusion depth). The **diffusion depth** $D(\Gamma)$ of a game $\Gamma = (\mathcal{N}, \mathcal{M}, k)$ is defined as $D \equiv max_{S \in \mathcal{S}, u \in V} \widehat{d}(S, u)$.

Observe that the diffusion depth can take values either lower, equal, or greater than the diameter of underlying graph G.

Chapter 5

Existence: Examples and Complexity

For simplicity, we shall mainly consider the games ($\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k)$, unless otherwise specified.

We first present several examples of games without a pure strategy Nash equilibrium (PSNE). Then, we consider a game that although it has a PSNE, it does not admit a generalized ordinal potential. Afterwards, we repeat this discussion, but for games restricted over the set s^{dj} of strategy profiles of disjoint strategies. Finally, we prove that deciding whether the game ($\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k)$ has a PSNE is coNP-hard.

5.1 Examples of games with no PSNE

Example 1 (**Cycle**)**.** Consider the social network in Figure 5.1: As n_i, $\forall i \in \{1, \ldots, |V|\}$, we denote single nodes. Further, we assume that all of them have threshold 1. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, |V|\}$, are annotated with their corresponding weight with regard to the LTM.

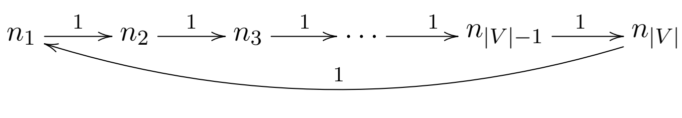


Figure 5.1: A social network ($G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}$) with underlying structure a cycle.

Next, let $k = 1$. Then, the involved game matrix is as in Table 5.1. Therefore, no PSNE exists. □

Example 2 (**Acyclic chain**)**.** Consider the social network in Figure 5.2: As n_i, $\forall i \in \{1, \ldots, |V|\}$, we denote single nodes. Further, we assume that all of them have thresh-

	n_1	n_2	$\cdots$	$n_{\lvert V\rvert-1}$	$n_{\lvert V\rvert}$
n_1	$\lvert V\rvert, 0$	$1, \lvert V\rvert - 1$	$\cdots$	$\lvert V\rvert - 2, 2$	$\lvert V\rvert - 1, 1$
n_2	$\lvert V\rvert - 1, 1$	$\lvert V\rvert, 0$	$\cdots$	$\lvert V\rvert - 3, 3$	$\lvert V\rvert - 2, 2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$n_{\lvert V\rvert-1}$	$2, \lvert V\rvert - 2$	$3, \lvert V\rvert - 3$	$\cdots$	$\lvert V\rvert, 0$	$1, \lvert V\rvert - 1$
$n_{\lvert V\rvert}$	$1, \lvert V\rvert - 1$	$2, \lvert V\rvert - 2$	$\cdots$	$\lvert V\rvert - 1, 1$	$\lvert V\rvert, 0$

Table 5.1: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 1)$, where the social network $\mathcal{N}$ is as in Figure 5.2.

old 1. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, |V|\}$, are annotated with their corresponding weight with regard to the `LTM`.

$$n_1 \xrightarrow{1} n_2 \xrightarrow{1} n_3 \xrightarrow{1} \cdots \xrightarrow{1} n_{|V|-1} \xrightarrow{1} n_{|V|}$$

Figure 5.2: A social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ with underlying structure an acyclic chain.

Next, let $k = 1$. Then, from the involved game matrix it can be verified that no PSNE exists. □

Example 3 (**Clique**)**.** Consider the social network in Figure 5.3: As n_i, $\forall i \in \{1, \ldots, 3\}$, we denote single nodes. Further, we assume that all of them have threshold $1/2$. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 3\}$, are annotated with their corresponding weight with regard to the `LTM`.

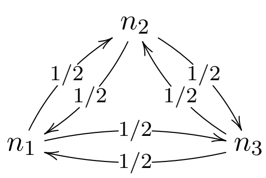


Figure 5.3: A social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ with underlying structure a clique.

Next, let $k = 1$. Then, from the involved game matrix it can be verified that no PSNE exists.

We note that the same result holds for any number of nodes $|V| \in \mathbb{Z}_{\geq 3}$ that form a clique, and have thresholds and edge-weights equal to $1/(|V| - 1)$. □

Example 4 (**Equitable graph**)**.** Consider the social network in Figure 5.4: As n_i, $\forall i \in \{1, \ldots, 3\}$, we denote single nodes. Further, we assume that all of them have threshold $1/2$. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 3\}$, are annotated with their corresponding weight with regard to the `LTM`.

Next, let $k = 2$: Therefore, there exist six available strategies, the $\{n_1n_2, n_1n_3, n_1n_4, n_2n_3, n_2n_4, n_3n_4\}$, where, for example, strategy n_1n_2 corresponds to the set (n_1, n_2). Then, the involved game matrix is as in Table 5.2. Evidently, no PSNE exists. □

	n_1n_2	n_1n_3	n_1n_4	n_2n_3	n_2n_4	n_3n_4
n_1n_2	4,0	3,1	3,1	3,1	3,1	2,2
n_1n_3	2,1	2,0	2,1	2,1	2,2	2,1
n_1n_4	2,1	2,1	2,0	2,2	2,1	2,1
n_2n_3	2,1	2,1	2,2	2,0	2,1	2,1
n_2n_4	2,1	2,2	2,1	2,1	2,0	2,1
n_3n_4	2,2	2,1	2,1	3,1	3,1	3,0

Table 5.2: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 2)$, where the social network $\mathcal{N}$ is as in Figure 5.4.

Next, we consider a game that although it has a PSNE, it does not admit a generalized ordinal potential.

Example 5. Consider the social network in Figure 5.5: As n_i, $\forall i \in \{1, \ldots, 9\}$, we denote single nodes. Further, we assume that all of them have threshold 1, except of node n_9 that has 1/2. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 9\}$, are annotated with their corresponding weight with regard to the `LTM`.

Next, let $k = 1$. Then, the involved game matrix, after eliminating the strictly dominated strategies $\{n_5, \ldots, n_9\}$, is as in Table 5.3. It can be verified that the strategy profile $(n1, n2)$ constitutes a PSNE. On the other hand, the following improvement cycle exists: $(n1, n1) \to (n1, n4) \to (n2, n4) \to (n2, n1) \to (n1, n1)$. Note that this path is also a *best response improvement path.* □

5.1.1 Examples of restricted games over s^{dj} with no PSNE

Next, we consider similar examples but for games $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k)$ restricted over the set s^{dj} of strategy profiles of disjoint strategies. To this end, we assume that $\forall \mathbf{s} \notin s^{dj}$ it is $u_i(\mathbf{s}) = 0$, $\forall i \in \mathcal{M}$.

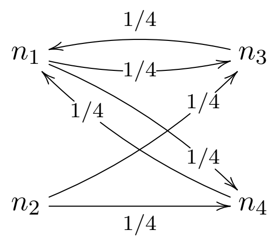


Figure 5.4: A social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ with underlying structure an equitable graph.

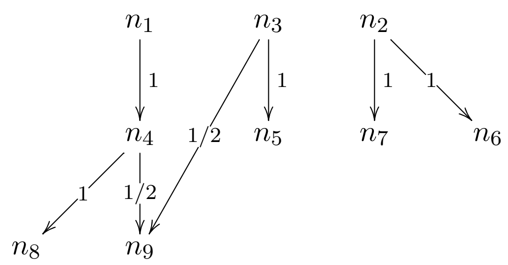


Figure 5.5: The social network (G, $\texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}$) of Example 5.

	n_1	n_2	n_3	n_4
n_1	4,0	4,3	3,3	1,3
n_2	3,4	3,0	3,3	3,3
n_3	3,3	3,3	3,0	3,2
n_4	3,1	3,3	3,2	3,0

Table 5.3: The game matrix, after eliminating the strictly dominated strategies $\{n_5, \ldots, n_9\}$, for the game $(\mathcal{N}, \mathcal{M}, k = 1)$, where the social network $\mathcal{N}$ is as in Figure 5.5.

Example 6. Consider the same social network in Figure 5.2. Next, let $k = 1$. Then, the involved game matrix is as in Table 5.4. Evidently, no PSNE exists. □

	n_1	n_2	$\cdots$	$n_{\lvert V\rvert-1}$	$n_{\lvert V\rvert}$
n_1	$0, 0$	$1, \lvert V\rvert - 1$	$\cdots$	$\lvert V\rvert - 2, 2$	$\lvert V\rvert - 1, 1$
n_2	$\lvert V\rvert - 1, 1$	$0, 0$	$\cdots$	$\lvert V\rvert - 3, 3$	$\lvert V\rvert - 2, 2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$n_{\lvert V\rvert-1}$	$2, \lvert V\rvert - 2$	$3, \lvert V\rvert - 3$	$\cdots$	$0, 0$	$1, \lvert V\rvert - 1$
$n_{\lvert V\rvert}$	$1, \lvert V\rvert - 1$	$2, \lvert V\rvert - 2$	$\cdots$	$\lvert V\rvert - 1, 1$	$0, 0$

Table 5.4: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 1)$, where the social network $\mathcal{N}$ is as in Figure 5.2.

Next, we consider a game over s^{dj} that although it has a PSNE, it does not admit a generalized ordinal potential.

Example 7. Consider the social network in Figure 5.6: As n_i, $\forall i \in \{1, \ldots, 5\}$, we denote single nodes. Further, we assume that all of them have threshold 1/2. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 5\}$, are annotated with their corresponding weight with regard to the `LTM`.

Next, let $k = 2$. Then, the involved game matrix is as in Table 5.5. It can be verified that the strategy profile $(n1n4, n2n3)$ constitutes a PSNE. On the other hand, the following improvement cycle exists: $(n3n4, n1n2) \to (n4n5, n1n2) \to (n4n5, n1n3) \to (n2n4, n1n3) \to (n2n4, n1n5) \to (n3n4, n1n5) \to (n3n4, n1n2)$. Note that this path is also a *best response improvement path.* □

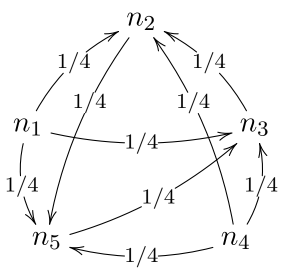


Figure 5.6: The social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ of Example 7.

	n_1n_2	n_1n_3	n_1n_4	n_1n_5	n_2n_3	n_2n_4	n_2n_5	n_3n_4	n_3n_5	n_4n_5
n_1n_2	0,0	0,0	0,0	0,0	0,0	0,0	0,0	3,2	2,2	2,3
n_1n_3	0,0	0,0	0,0	0,0	0,0	2,3	2,2	0,0	0,0	3,2
n_1n_4	0,0	0,0	0,0	0,0	3,2	0,0	3,2	0,0	3,2	0,0
n_1n_5	0,0	0,0	0,0	0,0	2,2	3,2	0,0	2,3	0,0	0,0
n_2n_3	0,0	0,0	2,3	2,2	0,0	0,0	0,0	0,0	0,0	2,2
n_2n_4	0,0	3,2	0,0	2,3	0,0	0,0	0,0	0,0	2,2	0,0
n_2n_5	0,0	2,2	2,3	0,0	0,0	0,0	0,0	2,2	0,0	0,0
n_3n_4	2,3	0,0	0,0	3,2	0,0	0,0	2,2	0,0	0,0	0,0
n_3n_5	2,2	0,0	2,3	0,0	0,0	2,2	0,0	0,0	0,0	0,0
n_4n_5	3,2	2,3	0,0	0,0	2,2	0,0	0,0	0,0	0,0	0,0

Table 5.5: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 2)$, where the social network $\mathcal{N}$ is as in Figure 5.6.

5.2 Complexity finding a PSNE

In this section, we prove that deciding whether the game $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k)$ has a PSNE is coNP-hard [Theorem 1].

We begin with an *illustration* of the involved proof over the 2-player game (the generalization to $m > 2$ players by introducing dummy areas is straightforward): We construct a social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ [Figure 5.7] that is composed by three main areas: The upper part, to node y_3, of the social network in Figure 5.8, the right part, to node y_3, of the social network in Figure 5.9, and the lower part, to node y_3, of the social network in Figure 5.8. The upper part, to node y_3, is constructed such that `Player 1` always initiates the set of nodes $S := \{n_1, \ldots, n_k\}$ [Figure 5.8], given that node y_3 is not initiated, or infected first by `Player 2`. The right part, to node y_3, is constructed with regard to any given 3CNF boolean formula Φ, and it is such that [Figure 5.9]:

i. `Player 2` always initiates an *appropriate* subset L of k nodes from the set of nodes $\{u_1, \neg u_1, \ldots, u_n, \neg u_n\}$, where $u_i, \neg u_i$, $i \in \{1, \ldots, n\}$, correspond to the literals that construct formula Φ. Specifically, if Φ is satisfiable, then L is composed by these nodes that correspond to those literals that constitute a satisfying assignment of Φ.

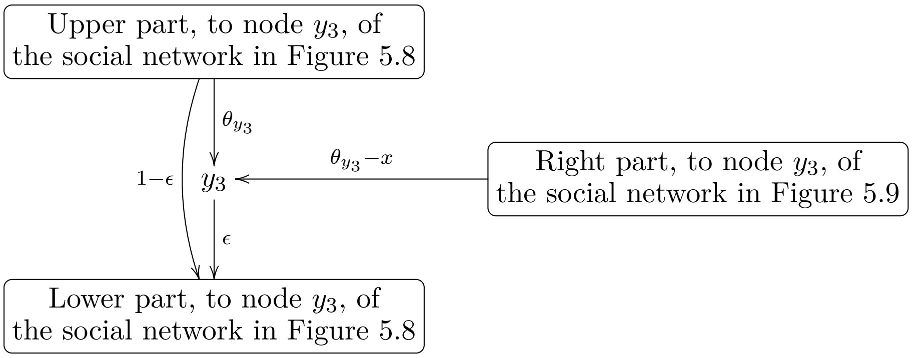


Figure 5.7: Overview of the constructed social network ($\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ for the proof of Theorem 1: Node y_3 is assumed to have a threshold θ_3. Moreover, the lower part, to node y_3, of the social network in Figure 5.8 is composed by a set of nodes that all have threshold one. The edges in this overview are annotated with the cumulative weight of the corresponding edges between the depicted areas, as well as, between the depicted areas and the node y_3. For example, note that the upper part of the social network in Figure 5.8 has out-going edges of cumulative weight $1 - \epsilon$ to each of the nodes in the lower part of it. Furthermore, note that ϵ is a constant in $(0, 1)$. On the other hand, the right part, to node y_3, of the social network in Figure 5.9 is constructed with regard to any given 3CNF boolean formula Φ in order for parameter x to take a value zero if Φ is satisfiable, and a positive value — lower than θ_3 — if not.

On the other hand, if Φ is not satisfiable, then L is composed by any set of k nodes from $\{u_1, \neg u_1, \ldots, u_n, \neg u_n\}$ where $\nexists i \in \{1, \ldots, n\}$ such that $u_i, \neg u_i \in L$.

ii. The parameter x equals to zero if Φ is satisfiable, and to some positive real value — lower, or equal to θ_3 — if not. Evidently, if $x = 0$, it is $\widehat{d}(L, y_3) = 2$ — y_3 is reachable from L at the $t = 2$ time-step of the involved diffusion process — otherwise $\widehat{d}(L, y_3) = \infty$ [Figure 5.9].

Next, the lower part, to node y_3, of the social network [Figure 5.8] guarantees the existence of a PSNE whenever Φ is not satisfiable [Figures 5.8, 5.9]: If Φ is satisfiable, then $\widehat{d}(L, y_3) = 2$, while $\widehat{d}(S, y_3) = 3$. In other words, `Player 2`, by playing L, infects first the node y_3, and as a result, `Player 1`, by playing S, cannot infect the nodes $f_1, \ldots, f_{\rho_2}$. Then, however, and due to the network's construction in Figure 5.7, `Player 1` prefer to deviate from S to L, and generally no PSNE exists. On the other hand, if Φ is not satisfiable, it is $\widehat{d}(L, y_3) = \infty$, while it still is $\widehat{d}(S, y_3) = 3$: Consequently, `Player 1`, by playing S, infects all the nodes in the social network of Figure 5.8, and due to the construction of the social network in Figure 5.9, `Player 2` continues to play L, since it infects all the nodes $b_1, \ldots, b_{\rho_3}$; i.e., under the network's construction in Figure 5.7, if Φ is not satisfiable, a PSNE exists where the `Player 1` plays S, and the `Player 2` plays L.

Theorem 1. *Deciding whether the game* $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k)$ *has a PSNE is coNP-hard.*

Proof. We prove the theorem using a reduction from the 3SAT problem. Specifically, consider any 3CNF boolean formula Φ: Denote as $C = \{c_1, \ldots, c_m\}$ the set of clauses that construct Φ, and as $U = \{u_1, \ldots, u_n\}$ the associated set of boolean variables.

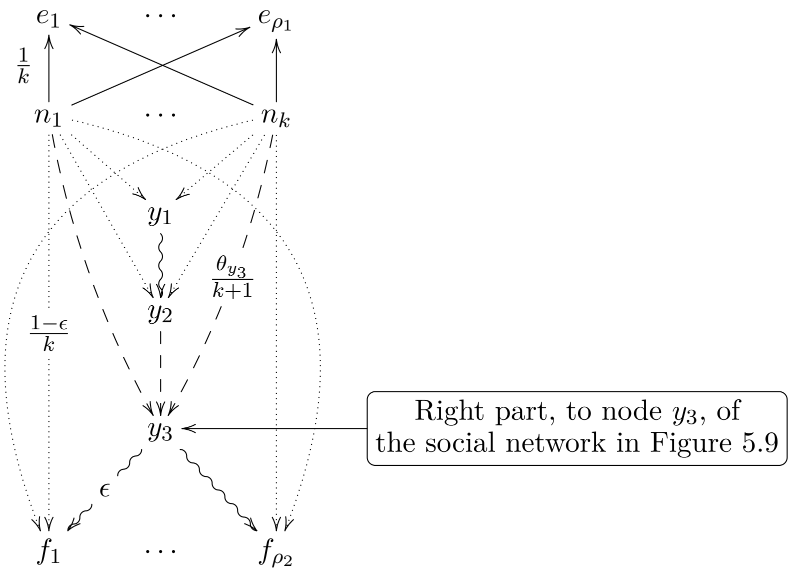


Figure 5.8: First part of the constructed social network ($\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ for the proof of Theorem 1: Edges of different styles are annotated with the different weight that corresponds to them: all the edges of the same style have also the same weight. Moreover, the "$\cdots$" between nodes e_1, and e_{ρ_1} denotes that $\rho_1 - 2$ same nodes exists between them, i.e., nodes that have the same in and out-going weighted edges as well as threshold as e_1, and e_{ρ_1} do (similarly for $n_1 \cdots n_k$, and $f_1 \cdots f_{\rho_2}$). Finally, note that this part shares the node y_3 with the right part in Figure 5.9 as in Figure 5.7.

Given any Φ we construct a certain social network that induces a game with at least one PSNE if and only if Φ is not satisfiable. We illustrate the construction through the following example: Let, without loss of generality, $\Phi = (u_1 \vee \neg u_2 \vee u_3) \wedge (u_1 \vee u_2 \vee u_3)$. Then, $C = \{c_1, c_2\}$, or $m = 2$, and $U = \{u_1, u_2, u_3\}$, or $n = 3$. Next, and as Figure 5.7 suggests, we construct the network as the combination of the networks in Figures 5.8, and 5.9 — note that these two networks share the y_3 node. Specifically, in Figures 5.8, and 5.9 only single nodes are depicted, with thresholds: $\forall i \in \{e_1, \ldots, e_{\rho_1}\} \cup \{n_1, \ldots, n_k\} \cup \{y_1, y_2\} \cup \{f_1, \ldots, f_{\rho_2}\} \cup \{u_1, \ldots, u_n\} \cup \{\neg u_1, \ldots, \neg u_n\}$ it is $\theta_i = 1$. Moreover, $\forall i \in \{a_1, \ldots, a_n\}$, it is $\theta_i = 1/2$, $\forall i \in \{b_1, \ldots, b_{\rho_3}\}$, it is $\theta_i = 2/3$, and $\forall i \in \{c_1, \ldots, c_m\}$, it is $\theta_i = 1/3$. Furthermore, for node y_3 it is $\theta_{y_3} = (k+1)(m+2n)/(5kn+2km+2m+5n)$. Additionally, it is $\epsilon = O(1) \in (0,1)$, $\rho_1 = 2k + 2m - 2$, $\rho_2 = m$, and $\rho_3 = k + m - 2$. On the other hand, all the edges are annotated with their corresponding weight with regard to the LTM. Specifically, for readability, each edge has a particular style associated with a unique weight, as it is designated in Figure is labeled with this weight.

Finally, let $k := n$ — note that $n \geq 3$, and $m \geq 1$; therefore $\rho_i > 0, \forall i \in \{1, 2, 3\}$.

For the first part of our proof, consider a *satisfiable* formula Φ, and denote as $\{t_1, \ldots, t_\lambda\}$ the set of its possible satisfying assignments, where $\lambda \in \mathbb{Z}_{\geq 1}$. Furthermore, denote as S_a the set of nodes $\{n_1, \ldots, n_k\}$. Also, denote as S_{t_i}, $i \in \{1, \ldots, \lambda\}$, the set of k nodes that correspond to those literals among the pairs $\{u_i, \neg u_i\}$, $i \in \{1, \ldots, n\}$, that are true with respect to the assignment t_i. For example, if $\Phi = (u_1 \vee \neg u_2 \vee u_3) \wedge (u_1 \vee u_2 \vee u_3)$, we may set as S_{t_1} the set $\{u_1, u_2, u_3\}$, as S_{t_2} the set $\{u_1, \neg u_2, u_3\}$, and so forth, given that we also

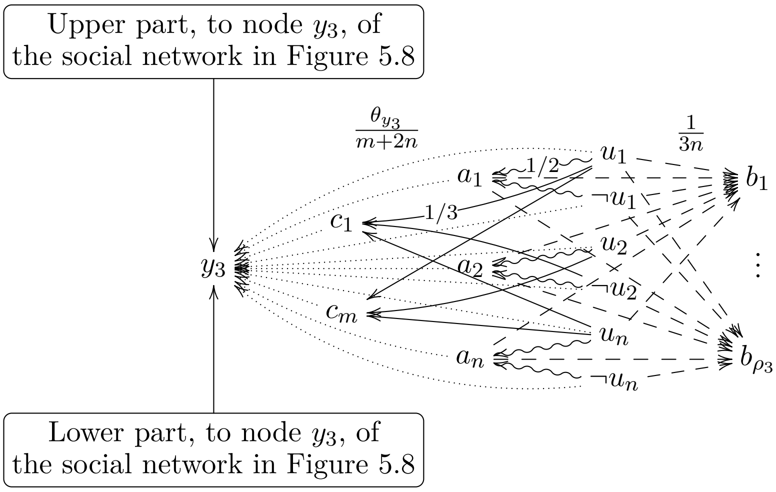


Figure 5.9: Second part of the constructed social network ($\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}$) for the proof of Theorem 1: Edges of different styles are annotated with the different weight that corresponds to them: all the edges of the same style have also the same weight. Moreover, the "$\cdots$" between nodes b_1, and b_{ρ_3} denotes that $\rho_3 - 2$ same nodes exists between them, i.e., nodes that have the same in and out-going weighted edges as well as threshold as b_1, and b_{ρ_3} do. Finally, note that this part shares the node y_3 with the upper part in Figure 5.8 as in Figure 5.7.

set $t_1 = \{u_1 = true, u_2 = true, u_3 = true\}$, and $t_2 = \{u_1 = true, u_2 = false, u_3 = true\}$ — both truthful evaluations of Φ.

Next, without loss of generality, consider the part of the induced game matrix as in Table 5.6, where $S_\beta \in \mathcal{S}_\beta := \mathcal{S} \setminus \{S_a, S_{t_1}, \ldots, S_{t_\lambda}\}$. Moreover, recall that given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x, S_y)$, $\forall i \in \mathcal{M} = \{1, 2\}$[4.2].

	S_a	S_{t_1}	$\cdots$	S_{t_λ}	S_β
S_a	$3K+3m+1, 0$	$3K+2m, 3K+3m-1$	$\cdots$	$3K+2m, 3K+3m-1$	$u_1^{a\beta}, u_2^{a\beta}$
S_{t_1}	$3K+3m-1, 3K+2m$	$3K+3m-1, 0$	$\cdots$	$3K+3m-1, u_2^{t_1 t_\lambda}$	$u_1^{t_1\beta}, u_2^{t_1\beta}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_{t_λ}	$3K+3m-1, 3K+2m$	$3K+3m-1, u_2^{t_\lambda t_1}$	$\cdots$	$3K+3m-1, 0$	$u_1^{t_\lambda\beta}, u_2^{t_\lambda\beta}$
S_β	$u_1^{\beta a}, u_2^{\beta a}$	$u_1^{\beta t_1}, u_2^{\beta t_1}$	$\cdots$	$u_1^{\beta t_\lambda}, u_2^{\beta t_\lambda}$	$u_1^{\beta\beta}, 0$

Table 5.6: Game matrix of the induced game over the social network constructed in the proof of Theorem 1.

We prove that no PSNE exists. Firstly, denote as $\mathcal{S}_\beta^A$ the set of the possible sets of k nodes $\{x_i\}_{i\in\{1,\ldots,n\}}$, (recall that k=n), such that $x_i \in \{u_i, \neg u_i\}$, and $\mathcal{S}_\beta^A \subseteq \mathcal{S}_\beta$. Moreover, let $\mathcal{S}_\beta^B \equiv \mathcal{S}_\beta \setminus \mathcal{S}_\beta^A$. Now, assume that $S_\beta \in \mathcal{S}_\beta^A$, and observe that $3k + m - 2 - k \leq u_1^{\beta(\cdot)} \leq 3k + 2m - 3$: The maximum value is obtained when `Player 1` achieves, among other, to color $m-1$ nodes from the set $\{c_1, \ldots, c_m\}$, while `Player 2` does not infect first nodes that would ideally be colored by `Player 1`. On the other hand, the minimum value is obtained when `Player 1` achieves to color zero node from the $\{c_1, \ldots, c_m\}$, while

`Player 2` achieves to first infect k nodes that `Player 1` was going to color at a later step. Recall that $S_\beta \in \mathcal{S}_\beta$, thereby, `Player 1` cannot possibly color all the m nodes $\{c_1, \ldots, c_m\}$. For the case that $S_\beta \in \mathcal{S}_\beta^B$, observe that $u_1^{\beta(\cdot)} \leq (3k + 2m - 3 - \rho_3) - 1 = 2k + m - 2$. Finally, note that the above discussion holds similarly for `Player 2`.

Now, observe that $u_1^{t_i t_i} = 3k + 3m - 1 > 3k + 2m - 3 \geq u_1^{\beta t_i}$, $i \in \{1, \ldots, \lambda\}$, given any $S_\beta \in \mathcal{S}_\beta$. Similarly, $u_1^{aa} > u_1^{\beta a}$, $S_\beta \in \mathcal{S}_\beta$. Next, let $S_x \in \mathcal{S}_\beta^B$, and $S_\beta \in \mathcal{S}_\beta$. Then, $u_1^{a\beta} \geq \rho_1 + 2 = 2k + 2m$, and $u_1^{x\beta} \leq 2k + m - 2$, thus $u_1^{a\beta} > u_1^{x\beta}$. On the other hand, if $S_x \in \mathcal{S}_\beta^A$, and $S_\beta \in \mathcal{S}_\beta^A$, it is $u_1^{a\beta} = 3k + 3m + 1 > 3k + 2m - 3 \geq u_1^{x\beta}$. Finally, if $S_x \in \mathcal{S}_\beta^A$, and $S_\beta \in \mathcal{S}_\beta^B$, it is $u_2^{x\beta} \leq 2k + m - 2 < 3k + 3m + 1 = u_2^{xa}$.

With regard to the payoff of `Player 2` we also have: $u_2^{a\beta} \leq 3k+2m-3 < 3k+3m-1 = u_2^{at_1}$, since $S_\beta \in \mathcal{S}_\beta$. Additionally, for $i \in \{1, \ldots, \lambda\}$, $u_2^{t_i\beta} \leq 3k + 2m - 3 < u_2^{t_i a} = 3k + 2m$.

In sum, never a pair of strategies $(S_x, S_y) \in \mathcal{S}^2$, where S_x, and/or S_y is in $\mathcal{S}_\beta$, can be a PSNE. Therefore, all the remaining candidate PSNEs lie within the pool of pairs $\{S_a, S_{t_1}, \ldots, S_{t_\lambda}\}^2$. However, $\forall (i,j) \in \{1, \ldots, \lambda\}^2$, $i \neq j$, it is $u_2^{t_i t_j} \leq k$, whereas $u_2^{at_j} = 3k + 2m$. Moreover, it can be verified from the above game matrix that the rest of the pairs of strategies cannot be a PSNE either. Hence, we have completed the first part of our proof: If Φ is satisfiable, there exists no PSNE.

For the reverse case, assume that Φ is *not* satisfiable; we prove that a PSNE always exists. First, let $\mathcal{S}_\omega = \times_{i=1}^{n} \{u_i, \neg u_i\}$, and set $\mu = |\mathcal{S}_\omega|$. Also, let $\mathcal{S}_\omega =: \left\{S_{b_1}, \ldots, S_{b_\mu}\right\}$, with $|H_{S_{b_1}}| \geq \ldots \geq |H_{S_{b_\mu}}|$.

We prove that the pair (S_a, S_{b_1}) constitutes a PSNE. On the one hand, it is $u_1^{ab_1} = 3k+3m+1 \geq u_1^{xb_1}$, $\forall S_x \in \mathcal{S} \setminus S_a$. On the other hand, $3k+m-1 \leq u_2^{ab_1} \leq 3k+2m-3$, and, by the definition of $\left\{S_{b_1}, \ldots, S_{b_\mu}\right\}$, $u_2^{ab_1} \geq u_2^{ab_i}$, $i \in \{1, \ldots, \mu\}$. Finally, let $S_x \in \mathcal{S} \setminus \mathcal{S}_\omega$. Then, $u_2^{ax} \leq 2k + m - 2 < 3k + m - 1 < u_2^{ab_1}$. Thus, if Φ is not satisfiable, a PSNE always exists, and our proof is completed. □

Chapter 6

Characterizations

In this section, we mainly consider the *2-player* game $(\mathcal{N}, \mathcal{M} = \{1, 2\}, k)$, unless otherwise specified.

6.1 Utilities quantification

We start with quantifying the utility functions $u_i : \mathcal{S}^2 \mapsto \mathbb{N}$, $i \in \mathcal{M}$, of the two players. We first need to introduce some important notions. Recall first the definition of H_S, in Section 4. This refers to the ideal spread of a product if the firm was playing on its own and used S as a seed. In the presence of a competitor, the firm will lose some of the nodes that belong to the ideal spread H_S. The losses happen due to three reasons. First, the competitor may have managed to infect a node at an earlier time step than the step that the firm would reach that node. Second, the firm may lose nodes if it happens that both firms are eligible to infect a node at the same time step due to the tie-breaking criteria. Finally, there may be nodes that belong to H_S, but the firm did not manage to infect enough neighbors so as to color them as well. These nodes either remain white, or are eventually infected by the other player. All these are captured below:

Orism'os 6.1. *Given a game $((G, \textit{LIS}, \textit{TBC1}, \textit{TBC2}), \mathcal{M} = \{1, 2\}, k)$, and a strategy profile $\mathbf{s} = (S_1, S_2)$,*

1. *for $i \in \{1, 2\}$, $\alpha_i(\mathbf{s})$ denotes the number of nodes that belong to H_{S_i}, and under profile $\mathbf{s}$, player i would be eligible to color them at some time step t but the other player has already infected them at some earlier time step $t' < t$ (e.g., this may occur under the threshold model when $\theta_v < 1/2$ for some node v).*

2. *for $i \in \{1, 2\}$, $\beta_i(\mathbf{s})$ denotes the number of nodes in H_{S_i}, such that under profile $\mathbf{s}$, both firms become eligible to infect them at the same time step, and due to tie-breaking rules, they get infected by the competitor of i.*

3. *for $i \in \{1, 2\}$, $\gamma_i(\mathbf{s})$ denotes the number of nodes that belong to H_{S_i}, but under $\mathbf{s}$, firm i never becomes eligible to infect them (because i did not manage to color the right neighbors under $\mathbf{s}$).*

Finally, we set $\alpha_{i,max}$ (respectively $\beta_{i,max}, \gamma_{i,max}$) to be the maximum value of $\alpha_i(\mathbf{s})$ over all valid strategy profiles and also $\alpha_{max} = \max\{\alpha_{1,max}, \alpha_{2,max}\}$ (similarly for β_{max}, and γ_{max}).

The following example serves as an illustration of these concepts.

Example. Consider the game $(\mathcal{N}, \mathcal{M} = \{1,2\}, k = 1)$ over the social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$ in Figure 6.1: As n_i, $\forall i \in \{1,\ldots,5\}$, we denote single nodes. Further, we assume that all of them have **threshold** one, i.e., $\theta_{n_i} = 1$, $\forall i \in \{1,\ldots,5\}$. Moreover, the edges (n_i, n_j), where $i,j \in \{1,\ldots,5\}$, are annotated with their corresponding **weight** with regard to the LTM. For example, $w_{n_1 n_5} = 1$, while $w_{n_3 n_4} = 1/2$.

Now, verify that the **in-neighbors** of, e.g., n_4 form the set $N(n_4) = \{n_2, n_3\}$, while for n_1 it is $N(n_1) = \emptyset$. Moreover, it is $d_{n_3} = 0$, $d_{n_1} = 1$, and $d_{n_2} = 2$. Further, observe that the **diffusion depth** is $D = d_{n_2} = 2$. Additionally, the **ideal spread** of n_2 is $H_{n_2} = \{n_2, n_3, n_4\}$, i.e., n_3, and n_4 are both **reachable** by n_2. On the other hand, n_4 is not reachable from n_1, since $w_{n_3 n_4} = 1/2 < \theta_{n_4} = 1$. Also, the ideal spread of n_5 is $H_{n_5} = \emptyset$.

Next, observe that n_4 can be **infected** during the diffusion process by a player $i \in \mathcal{M}$, only if i has **colored** both nodes n_2 and n_3 — either at the **initiation** step, or later. Therefore, if the strategy profile $\mathbf{s} = (n_1, n_2)$ is played, node n_4 remains white after the termination of the competitive diffusion process: Player 1 infects first node n_3, according to $\texttt{TBC2} = R^{\prec}$. Thereby, given that Player 1 initiates n_1, and Player 2 n_2, it is $\gamma_2(\mathbf{s}) = 1$. On the other hand, it is $\gamma_1(\mathbf{s}) = 0$. Clearly, $\gamma_1(\mathbf{s}) \neq \gamma_2(\mathbf{s})$.

Similarly, $\alpha_1(\mathbf{s}) = 0$, $\alpha_2(\mathbf{s}) = 0$, $\beta_1(\mathbf{s}) = 0$, and $\beta_2(\mathbf{s}) = 1$. As a result, $u_1(\mathbf{s}) = |H_{n_1}| - \alpha_1(\mathbf{s}) - \gamma_1(\mathbf{s}) = 3 - 0 - 0 = 3$, while $u_2(\mathbf{s}) = |H_{n_2}| - \alpha_2(\mathbf{s}) - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) = 3 - 0 - 1 - 1 = 1$.

Finally, if both players choose to initiate the same node, e.g., n_1, then node n_1 will be colored only by Player 1, according to $\texttt{TBC1} = R^{\prec}$, and it will be $u_1(n_1, n_1) = |H_{n_1}| - \alpha_1(n_1, n_1) - \gamma_1(n_1, n_1) = 3 - 0 - 0 = 3$, while $u_2(n_1, n_1) = |H_{n_1}| - \alpha_2(n_1, n_1) - \beta_2(n_1, n_1) - \gamma_2(n_1, n_1) = 3 - 0 - 1 - 2 = 0$.

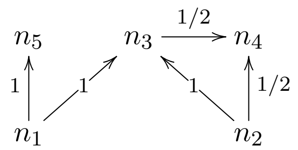


Figure 6.1: A social network $(G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$.

When we use $R^{\prec}$ for tie-breaking, clearly $\beta_1(\mathbf{s}) = 0$, hence the following is straightforward.

Lemma 1. *Consider the* 2-player *game* $((G, \textit{LIS}, \textit{TBC1} = R^{\prec}, \textit{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$. *The utility function of Player 1, given a strategy profile* $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$, *is*

$$u_1(\mathbf{s}) = |H_{S_1}| - \alpha_1(\mathbf{s}) - \gamma_1(\mathbf{s}), \tag{6.1}$$

Moreover, the utility function of Player 2 is

$$u_2(\mathbf{s}) = |H_{S_2}| - \alpha_2(\mathbf{s}) - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}). \tag{6.2}$$

In the following, whenever we consider more general games than the $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$, we shall resort to the fundamental definition of the payoff functions [4.2].

6.2 Conditions for the existence of a PSNE

Given a game $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$, we discuss several necessary and/or sufficient conditions for the existence of a PSNE.

Lemma 2 (**Necessary and sufficient conditions for the existence of a PSNE**). *Given a game* $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$*, a strategy profile* $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$ *is a PSNE, if and only if*

1. $\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s}) \leq \alpha_1(S_1', S_2) + \gamma_1(S_1', S_2) + |H_{S_1}| - |H_{S_1'}|, \forall S_1' \neq S_1, S_1' \in \mathcal{S}$.
2. $\alpha_2(\mathbf{s}) + \beta_2(\mathbf{s}) + \gamma_2(\mathbf{s}) \leq \alpha_2(S_1, S_2') + \beta_2(S_1, S_2') + \gamma_2(S_1, S_2') + |H_{S_2}| - |H_{S_2'}|, \forall S_2' \neq S_2, S_2' \in \mathcal{S}$.

Proof. For $\mathbf{s}$ it must:

$$u_1(\mathbf{s}) \geq u_1(S_1', S_2), \forall S_1' \neq S_1, S_1' \in \mathcal{S}, \tag{6.3}$$

$$u_2(\mathbf{s}) \geq u_2(S_1, S_2'), \forall S_2' \neq S_2, S_2' \in \mathcal{S}. \tag{6.4}$$

The proof completes by substituting Equations (6.1), (6.2) in Equations (6.3), (6.4), respectively. □

Theorem 2 (**Necessary condition for the existence of a PSNE**). *Given a game* $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$*, if the strategy profile* $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$ *is a PSNE, then it holds that* $|H_{S_1}| + |H_{S_2}| \geq |H_{max}| - \gamma_1(S_{max}, S_2)$*, where* $|H_{max}| \equiv max_{S \in \mathcal{S}} \{|H_S|\}$*, and* $S_{max} \equiv argmax_{S \in \mathcal{S}} \{|H_S|\}$.

Proof. For the following proof, given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, $(x, y) \in \{1, \dots, |\mathcal{S}|\}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x, S_y), \forall i \in \mathcal{M} = \{1, 2\}$.

In the following, we assume that $\mathcal{S} = \{S_i\}_{i \in \{1, \dots, |\mathcal{S}|\}}$, where $|H_{S_i}| \geq |H_{S_{i+1}}|, \forall i \in \{1, \dots, |\mathcal{S}| - 1\}$.

We first consider a strategy profile $\mathbf{s} = (S_i, S_j) \in \mathcal{S}^2$, where $i > j$. Then, if $|H_{S_i}| < |H_{S_j}|$, $\mathbf{s}$ cannot be a PSNE, since $u_1^{ij} \leq |H_{S_i}| < |H_{S_j}| = u_1^{jj}$. On the other hand, if $|H_{S_i}| = |H_{S_j}|$, for $\mathbf{s}$ to be a PSNE, it must $u_1^{ij} \geq u_1^{i'j}, \forall i' \in \{1, \dots, |\mathcal{S}|\}$. Thereby, by substituting in the previous inequality $u_1^{ij} = |H_{S_i}|$, and $u_1^{i'j} = |H_{S_i'}| - \alpha_1(S_i', S_j) - \gamma_1(S_i', S_j)$, we have $|H_{S_i}| \geq |H_{S_i'}| - \alpha_1(S_i', S_j) - \gamma_1(S_i', S_j)$, which for $i' = 1$ gives $|H_{S_i}| \geq |H_{max}| - \alpha_1(S_1, S_j) - \gamma_1(S_1, S_j)$. Moreover, it is $\alpha_1(S_1, S_j) \leq min\left\{|H_{S_j}|, |H_{max}|\right\} = |H_{S_j}|$, and as a result, from

$|H_{S_i}| \geq |H_{max}| - \alpha_1(S_1, S_j) - \gamma_1(S_1, S_j)$ we have $|H_{S_i}| + |H_{S_j}| \geq |H_{max}| - \gamma_1(S_1, S_j)$. In other words, if $\mathbf{s}$ is a PSNE, then it is $|H_{S_i}| + |H_{S_j}| \geq |H_{max}| - \gamma_1(S_{max}, S_j)$.

Now, we note that a strategy profile $\mathbf{s} = (S_i, S_i) \in \mathcal{S}^2$, cannot be a PSNE, since $u_2^{ii} = 0$, and even though, e.g., $|H_{S_1}| + |H_{S_1}| = 2|H_{max}| > |H_{max}|$.

Next, given a strategy profile $\mathbf{s} = (S_i, S_j)$, we assume $i < j$. Then, for $\mathbf{s}$ to be a PSNE, it must be $u_1^{ij} \geq u_1^{i'j}$, $\forall i' \neq i, (i, i') \in \{1, \ldots, |\mathcal{S}|\}^2$. Thus, the inequality must also hold $\forall i' \leq i$. Now, let $i' =: i - x$, $x \in \{0, \ldots, i-1\}$, and substitute $u_1^{yj} = |H_{S_y}| - \alpha_1(S_y, S_j) - \gamma_1(S_y, S_j)$, $y \in \{i, i-x\}$ to the previous inequality: We take $\alpha_1(S_i, S_j) + \gamma_1(S_i, S_j) \leq \alpha_1(S_{i-x}, S_j) + \gamma_1(S_{i-x}, S_j) + |H_{S_i}| - |H_{S_{i-x}}|$, $\forall x \in \{0, \ldots, i-1\}$. Thereby, we have $\alpha_1(S_{i-x}, S_j) + \gamma_1(S_{i-x}, S_j) + |H_{S_i}| - |H_{S_{i-x}}| \geq 0$. However, $\alpha_1(S_{i-x}, S_j) \leq min\left\{|H_{S_j}|, |H_{S_{i-x}}|\right\} = |H_{S_j}|$, due to our initial assumption and the fact that $i - x \leq i < j$. Thereby, it must hold $|H_{S_j}| + |H_{S_i}| \geq |H_{S_{i-x}}| - \gamma_1(S_{i-x}, S_j)$, $\forall x \in \{0, \ldots, i-1\}$, i.e., for $x = i-1$ must hold $|H_{S_j}| + |H_{S_i}| \geq |H_{S_1}| - \gamma_1(S_1, S_j) = |H_{max}| - \gamma_1(S_{max}, S_j)$, and the proof is completed. □

Next, we prove that a strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$ cannot be a PSNE if S_1 is reachable from S_2, .

Theorem 3. *A game* $((G, \mathtt{LIS}, \mathtt{TBC1} = R^{\prec}, \mathtt{TBC2}), \mathcal{M} = \{1, 2\}, k)$ *in which the strategies in* $\mathcal{S}$*, are all reachable one from the other has no PSNE.*

Proof. Assume that S_1 is reachable from S_2, i.e., $S_1 \subseteq H_{S_2}$. Then, $H_{S_2} \supseteq H_{S_1}$, and as a result, $u_1(S_2, S_2) = |H_{S_2}| \geq |H_{S_1}| > |H_{S_1}| - |S_2 \setminus S_1| \geq u_1(S_1, S_2)$, since $S_1 \neq S_2$, (i.e., since $S_1 \neq S_2$, Player 1, by playing S_1, cannot color at least one of the nodes in S_2). Thereby, the strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$ is not a PSNE.

At this point, note the trivial case where a single strategy S exists in the game, i.e., $k = |V|$ (contrary to our basic assumption in this paper that $k < |V|$): Then, a unique trivial PSNE (S, S) exists, where Player 1 has utility $|H_S|$, and Player 2 zero. On the other hand, S is reachable by itself, since $S \subseteq H_S$ by definition. □

6.3 Realizability of improvement paths

We first establish that over a certain class of social networks, with appropriate underlying structure, all possible game matrices are realizable with regard to the improvement paths that they induce [Theorem 4]. Afterwards, we prove that this result holds under "almost" any consistent in and out-degree distributions that may characterize the underlying structure of the social network [Corollary 1].

Recall that given a game $\Gamma(\mathcal{N}, \mathcal{M}, k)$, the set $\mathcal{S}_D$ constitutes the set of strictly componentwise undominated strategies of the involved players [4.2]. Here, we set $\mathcal{S}_D =: \{S_1, \ldots, S_r\} \subseteq \mathcal{S}$, where $r \in \mathbb{Z}_{\geq 2}$, and we further assume $|H_{S_1}| \geq \ldots \geq |H_{S_r}|$.

Furthermore, we denote as $G_d = (V_d, E_d)$ any graph from the class of graphs that is defined in the following proof of Theorem 4 by assuming a fixed set V_d, and a variable E_d.

Theorem 4 (**All improvement paths are realizable**). *Consider the games* $(\mathcal{N}(G_d,$ $\texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E_d), \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$, *where* $D \geq 2$, *for all the possible instances of* G_d. *Then, all improvement paths are realizable in* $\Pi(\mathcal{S}_D, \mathcal{S}_D)$.

Proof. For the following proof, given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, $(x,y) \in \{1,\ldots,|\mathcal{S}|\}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x, S_y)$, $\forall i \in \mathcal{M} = \{1,2\}$.

We construct a certain class of graphs, denoted by $\mathcal{G}_d$, that induce a class of social networks with the involved property. To this end, we consider without loss of generality that $k = 1$, and $D = 2$. Moreover, we assume that under our construction it holds $\alpha_1(S_1, S_2) = \alpha_2(S_2, S_1)$, and $\beta_2(S_1, S_2) = \beta_2(S_2, S_1)$, $\forall (S_1, S_2) \in \mathcal{S}$ (Definition 6.1). Thereby, we set for simplicity $\alpha_{S_2S_1} \equiv \alpha_1(S_1, S_2)$, i.e., $\alpha_{S_1S_2} = \alpha_2(S_1, S_2)$.

Additionally, we assume that $\texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E)$. Therefore, for $i \in \{1,2\}$, it is $\gamma_i(S_1, S_2) = 0$, $\forall (S_1, S_2) \in \mathcal{S}^2$, where $S_1 \neq S_2$. Also, recall that $\gamma_1(S,S) = 0$, $\forall S \in \mathcal{S}$, since $\texttt{TBC1} = R^{\prec}$.

Furthermore, we let $|H_{S_i}| = H$, $\forall i \in \{1,\ldots,r\}$, where H will be appropriately chosen to guarantee that the set $\{S_i\}_{i \in \{1,\ldots,r\}}$ is a sink in $\mathcal{S}$.

Next, we assume for the constructed class of social networks that:

i. $I^0_{S_i} \cap I^1_{S_j} = \emptyset$, $\forall (i,j) \in \{1,\ldots,r\}^2$, where $i \neq j$,

ii. $(H^1_{S_j} \cap I^2_{S_i}) \cap (H^1_{S'_j} \cap I^2_{S'_i}) = \emptyset$, $\forall (i,j,i',j') \in \{1,\ldots,r\}^4$, where $(i,j) \neq (i',j')$,

iii. $(H^1_{S_j} \cap H^1_{S_i}) \cap (H^1_{S'_j} \cap H^1_{S'_i}) = \emptyset$, $\forall (i,j,i',j') \in \{1,\ldots,r\}^4$, where $(i,j) \neq (i',j')$,

iv. $(H^1_{S_j} \cap H^1_{S_i}) \cap (I^2_{S'_j} \cap I^2_{S'_i}) = \emptyset$, $\forall (i,j,i',j') \in \{1,\ldots,r\}^4$, where $(i,j) \neq (i',j')$,

v. $(I^2_{S_j} \cap I^2_{S_i}) \cap (I^2_{S'_j} \cap I^2_{S'_i}) = \emptyset$, $\forall (i,j,i',j') \in \{1,\ldots,r\}^4$, where $(i,j) \neq (i',j')$, as well as,

vi. $H = (r-1)(2\alpha_{max} + \beta_{max}) + 1 + \mathtt{c}$, where $\mathtt{c}$ is appropriately chosen for the set $\{S_i\}_{i \in \{1,\ldots,r\}}$ to be a sink in $\mathcal{S}$.

Over these assumptions, we can construct a class of social networks, where $H^1_{S_j} \cap I^2_{S_i}$, $H^1_{S_j} \cap H^1_{S_i}$, and $I^2_{S_j} \cap I^2_{S_i}$ can be independently decided from one another. Particularly, without loss of generality, consider the pair of strategies (S_i, S_j), where $i \neq j$, and $(i,j) \in \{1,\ldots,r\}$. Then, recall that $\texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E)$, and consider the following three *disjoint* areas of H_{S_i} that can share nodes *only* with three corresponding areas of H_{S_j} (Figure 6.2):

A. Name this area A_i (Figure 6.3); through its structure we shall later show that the value of $\alpha_{S_iS_j}$ can be independently decided from the values of $\alpha_{S'_iS'_j}$, $\beta_2(S'_i, S'_j)$, $\forall (i',j') \in \{1,\ldots,r\}^2$. Assume, $A_i \subset H_{S_i}$, $|A_i| = \alpha_{max}$, and that $\forall k \in A_i, (S_i, k) \in E$, while $\nexists v \in V \setminus \{S_i, S_j\} : (v,k) \in E$. Additionally, assume $A_i = (H^1_{S_i} \cap I^2_{S_j}) \cup X_i$, where $X_i \equiv A_i \setminus (H^1_{S_i} \cap I^2_{S_j})$, and that $\forall k \in X_i, \nexists v \in V : (k,v) \in E$. Similarly, $\forall k \in$

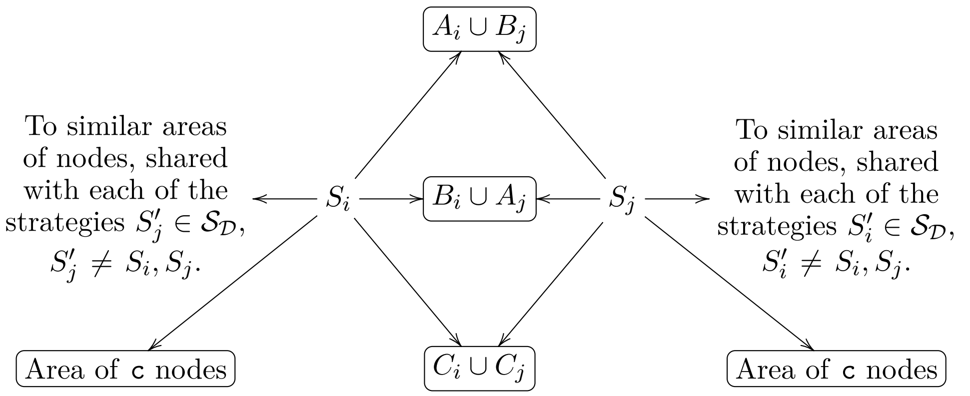


Figure 6.2: The partially shared areas $A_i \cup B_j$, $B_i \cup A_j$, and $C_i \cup C_j$ between the two strategies $(S_i, S_j) \in \mathcal{S_D}^2$, $S_i \neq S_j$. Moreover, the similar areas that exists between S_i, S_j and any other strategy in $\mathcal{S_D}^2$ are depicted, as well as, the sets of c nodes that each of them targets.

$H^1_{S_i} \cap I^2_{S_j}, \nexists v \in V : (k, v) \in E$. Next, note that it is $|H^1_{S_i} \cap I^2_{S_j}| = \alpha_{S_i S_j}$ by definition, therefore $|X_i| = \alpha_{max} - \alpha_{S_i S_j}$.

The corresponding region in H_{S_j} subset of which is the set $H^1_{S_i} \cap I^2_{S_j}$ is the area B_j (Figure 6.3). Below we define B_i.

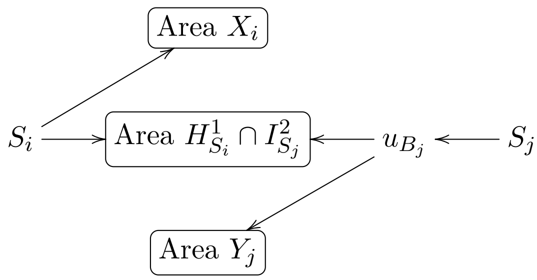


Figure 6.3: The partially shared area $A_i \cup B_j$ between the two strategies $(S_i, S_j) \in \mathcal{S_D}^2$, $S_i \neq S_j$: $A_i \equiv (H^1_{S_i} \cap I^2_{S_j}) \cup X_i$, and $B_j \equiv (H^1_{S_i} \cap I^2_{S_j}) \cup Y_j \cup \{u_{B_j}\}$. Moreover, note that $|H^1_{S_i} \cap I^2_{S_j}| = \alpha_{S_i S_j}$, and $|X_i| = \alpha_{max} - \alpha_{S_i S_j}$. Therefore, $|A_i| = \alpha_{max}$. Also, it is $|Y_j| = \alpha_{max} - \alpha_{S_i S_j}$, and as a result $|B_j| = \alpha_{max} + 1$.

B. Name this part B_i (Figure 6.4); through its structure we shall later show that the value of $\alpha_{S_j S_i}$ can be independently decided from the values of $\alpha_{S'_i S'_j}$, $\beta_2(S'_i, S'_j)$, $\forall (i', j') \in \{1, \ldots, r\}^2$. Assume, $B_i \subset H_{S_i}$, $|B_i| = 1 + \alpha_{max}$. Specifically, let $B_i = (H^1_{S_j} \cap I^2_{S_i}) \cup Y_i \cup \{u_{B_i}\}$, where $Y_i \equiv B_i \setminus (H^1_{S_j} \cap I^2_{S_i}) \cup \{u_{B_i}\}$, and u_{B_i} denotes a *single* node such that $(S_i, u_{B_i}) \in E$. Moreover, $\forall k \in (H^1_{S_j} \cap I^2_{S_i}) \cup Y_i$ we consider $(u_{B_i}, k) \in E$. Additionally, $\nexists v \in V \setminus \{u_{B_i}, S_j\} : (v, k) \in E$. Lastly, $\forall k \in B_i, \nexists v \in V : (k, v) \in E$, except u_{B_i} that has edges only to $(H^1_{S_j} \cap I^2_{S_i}) \cup Y_i$ as defined previously.

Evidently, subset of $(H^1_{S_j} \cap I^2_{S_i}) \cup Y_i$ is the set $H^1_{S_j} \cap I^2_{S_i}$; therefore, area B_i is constructed to give area A_j the end that similarly area A_i had (Figures 6.3, 6.4).

C. Name this part C_i (Figure 6.5); through its structure we shall later show that the value of $\beta_2(S_i, S_j)$ can be independently decided from the values of $\alpha_{S'_i S'_j}$, $\beta_2(S'_i, S'_j)$,

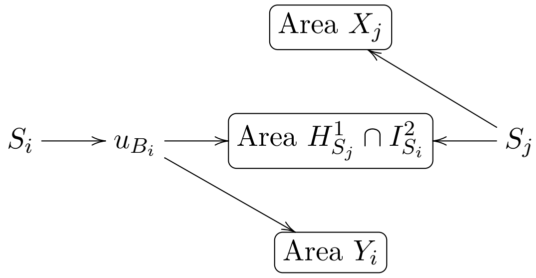


Figure 6.4: The partially shared area $B_i \cup A_j$ between the two strategies $(S_i, S_j) \in \mathcal{S_D}^2$, $S_i \neq S_j$: $B_i \equiv (H^1_{S_j} \cap I^2_{S_i}) \cup Y_i \cup \{u_{B_i}\}$, and $A_j \equiv (H^1_{S_j} \cap I^2_{S_i}) \cup X_j$. Moreover, note that $|H^1_{S_j} \cap I^2_{S_i}| = \alpha_{S_j S_i}$, and $|Y_i| = \alpha_{max} - \alpha_{S_j S_i}$. Therefore, $|B_i| = \alpha_{max} + 1$. Also, it is $|X_j| = \alpha_{max} - \alpha_{S_j S_i}$, and as a result, $|A_j| = \alpha_{max}$.

$\forall (i', j') \in \{1, \ldots, r\}^2$. Assume, $C_i \subset H_{S_i}$, $|C_i| = \beta_{max}$, and that $\forall k \in C_i, (S_i, k) \in E$, while $\nexists v \in V \setminus \{S_i, S_j\} : (v, k) \in E$. Additionally, assume $C_i = (H^1_{S_i} \cap H^1_{S_j}) \cup Z_i$, where $Z_i \equiv C_i \setminus (H^1_{S_i} \cap H^1_{S_j})$, and that $\forall k \in Z_i, \nexists v \in V : (k, v) \in E$. Similarly, $\forall k \in H^1_{S_i} \cap H^1_{S_j}, \nexists v \in V : (k, v) \in E$. For simplicity we have assumed $I^2_{S_i} \cap I^2_{S_j} = \emptyset$, but the generalization is straightforward. Therefore, note that it is $|H^1_{S_i} \cap H^1_{S_j}| = \beta_2(S_i, S_j)$, i.e., $|Z_i| = \beta_{max} - \beta_2(S_i, S_j)$.

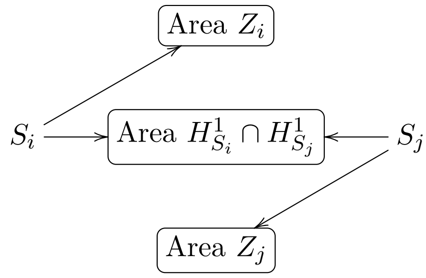


Figure 6.5: The partially shared area $C_i \cup C_j$ between the two strategies $(S_i, S_j) \in \mathcal{S_D}^2$, $S_i \neq S_j$: $C_i = (H^1_{S_i} \cap H^1_{S_j}) \cup Z_i$, and $C_j = (H^1_{S_i} \cap H^1_{S_j}) \cup Z_j$. Moreover, note that here $|H^1_{S_i} \cap H^1_{S_j}| = \beta_2(S_i, S_j)$, and $|Z_i| = |Z_j| = \beta_{max} - \beta_2(S_i, S_j)$. Therefore, $|C_i| = |C_j| = \beta_{max}$.

Now, recall that we assumed $\beta_2(S_i, S_j) = \beta_2(S_j, S_i)$. Therefore, an area similar to C_i exist also in H_{S_j}, i.e., an area C_j, that shares the same $H^1_{S_i} \cap H^1_{S_j}$ with C_i (Figure 6.5).

Evidently, $H^1_{S_j} \cap I^2_{S_i}$, $H^1_{S_j} \cap H^1_{S_i}$, and $I^2_{S_j} \cap I^2_{S_i}$ can be independently decided from one another.

We can now justify our selection for H. Specifically, rewrite H as $H = (r-1)(\alpha_{max} + (1+\alpha_{max}) + \beta_{max}) + 1 + \mathsf{c}$: The factor "$(r-1)$" is due to the fact that region H_{S_i} is connected with — at most — the rest $(r-1)$ of the $H_{S_{(\cdot)}}$ regions (Figure 6.2). Furthermore, the first factor in the sum, "α_{max}", is the cardinality of area A_i, while the second the cardinality of B_i, and the third the cardinality of C_i. Lastly, the unity corresponds to the node S_i, while c corresponds to the cardinality of a set X_i such that $\forall k \in X_i$, $(S_i, k) \in E$, while $\nexists v \in V \setminus \{S_i\} : (v, k) \in E$ and $\nexists v \in V : (k, v) \in E$. And as noted before, if c is sufficiently large, e.g. $\mathsf{c} \geq \alpha_{max}$, the set $\{S_i\}_{i \in \{1,\ldots,r\}}$ is a sink in $\mathcal{S}$. We note that an obvious modification to the structure of B_i can decrease c to zero.

As a final step to our proof, we prove that if

i $\alpha_{max} = r - 1$, and

ii $\beta_{max} = \alpha_{max} + r - 1$,

then all possible value orderings of $\left\{\alpha_{S_iS_j}\right\}_{(i,j)\in\{1,\ldots,r\}^2}$, $\{\beta_2(S_i, S_j)\}_{(i,j)\in\{1,\ldots,r\}^2}$ are achievable.

First, we justify our selection for α_{max}. Recall that in our social network for $u_1^{ij} \geq u_1^{i'j}$, $i' \in \{1,\ldots,r\} \setminus \{i\}$ to hold, it must $\alpha_{S_jS_i} \leq \alpha_{S_jS'_i}$. Moreover, $\alpha_{S_jS_i} \in \{0, 1, \ldots, r-1\}$, i.e. it can take a value from a pool of r different ones. On the other hand, $\alpha_{S_jS_i}$ is compared with exactly $(r-1)$ other $\alpha_{S_jS'_i}$, since $i' \in \{1,\ldots,r\} \setminus \{i\}$. Thereby, there exist a set of surjective functions defined from the set $\left\{\alpha_{S_iS_j}\right\}_{(i,j)\in\{1,\ldots,r\}^2}$ to the set $\{0, 1, \ldots, r-1\}$ that achieves all the possible value orderings of $\left\{\alpha_{S_iS_j}\right\}_{(i,j)\in\{1,\ldots,r\}^2}$ and, in accordance, all the possible value orderings of $u_1^{(\cdot)j}$, $\forall j \in \{1,\ldots,r\}$ (cf. Remark 2).

Next, given that $\left\{\alpha_{S_iS_j}\right\}_{(i,j)\in\{1,\ldots,r\}^2}$ has been decided, we can validate our selection for β_{max}. First, recall that in our social network for $u_2^{ij} \geq u_2^{ij'}$, $j' \in \{1,\ldots,r\} \setminus \{j\}$ such that $j \neq i$, and $j' \neq i$, to hold, it must $\beta_2(S_i, S_j) \leq \alpha_{S_iS'_j} - \alpha_{S_iS_j} + \beta_2(S_i, S'_j)$, $\forall j' \in \{1,\ldots,r\} \setminus \{j\}$ such that $j \neq i$, and $j' \neq i$, (i.e., we do not consider the strategy profiles (S, S), since $u_2(S, S) = 0$, while $u_2(S_i, S_j) \geq 1$, $\forall S_i, S_j \in \mathcal{S}$, where $S_i \neq S_j$ — also, recall from the begining of the proof that $\gamma_2(S_i, S_j) = 0$, $\forall (S_i, S_j) \in \mathcal{S}^2$, where $S_i \neq S_j$). Therefore, $\beta_2(S_i, S_j)$ is compared with exactly $(r-1)$ other $\beta_2(S_i, S'_j)$, since $j' \in \{1,\ldots,r\} \setminus \{j\}$. Now, we are able to give an algorithm that achieves all the possible value orderings of $\{\beta_2(S_i, S_j)\}_{(i,j)\in\{1,\ldots,r\}^2}$ and, in accordance, all the possible value orderings of $u_2^{i(\cdot)}$, $\forall i \in \{1,\ldots,r\}$, for any given $\left\{\alpha_{S_iS_j}\right\}_{(i,j)\in\{1,\ldots,r\}^2}$. Specifically,

1. For fixed $i \in \{1,\ldots,r\}$, for all $(j, j') \in 1, \ldots, r^2, j \neq j'$ compute $\alpha_{S_iS'_j} - \alpha_{S_iS_j}$.
2. Let $\beta = max_{(j,j')\in 1,\ldots,r^2} \left\{\alpha_{S_iS'_j} - \alpha_{S_iS_j}\right\}$.
3. Define the set of surjective functions from the set $\{\beta_2(S_i, S_j)\}_{(i,j)}$, where $(i, j) \in \{1,\ldots,r\}^2$, to the set $\{\beta, \ldots, \beta + (r-1)\}$ that achieves all the possible value orderings of $\{\beta_2(S_i, S_j)\}_{(i,j)\in\{1,\ldots,r\}^2}$.
4. **return** this set of functions.

Since, $\beta \leq \alpha_{max}$, and $\beta_{max} = \alpha_{max} + (r-1)$, the above algorithm works for all possible values of β. The fact completes our proof. □

Given any game $\Gamma = (\mathcal{N}, \mathcal{M}, k)$, Theorem 4 suggest that in the involved game matrix, or its restriction $\Pi(\mathcal{S}_D, \mathcal{S}_D)$, any improvement path may exists. Therefore, it indicates that it is generally hard to identify a unique set of conditions that capture simultaneously all games Γ that have at least one PSNE, or even admit a generalized ordinal potential.

Remark 1. We stress that the above constructed graph is a DAG.

Remark 2. We note with regard to the condition $\alpha_{S_j S_i} \leq \alpha_{S_j S'_i}$, $\forall i' \neq i, i' \in \{1, \ldots, r\}$, that $\forall x \in \{1, \ldots, r\}$ it is $u_1(S_x, S_x) = H$, i.e., $a_{S_x S_x} = 0$ and $u_1(S_x, S_x)$ is always greater, or equal to any of the $u_1(S'_x, S_x)$, $x' \in \{1, \ldots, r\} \setminus \{x\}$; a restriction that should be considered whenever we state that "all improvement paths are realizable". On the other hand, this limitation can be nullified through simple modifications to the constructed network: *(Sketch of proof)* It is sufficient to letting $|H_{S_i}|$, $\forall i \in \{1, \ldots, r\}$, differ slightly from H, and increasing appropriately the $\alpha_{max}, \beta_{max}$.

Conjecture 1 (**Generalization of Theorem** 4)**.** Classes of networks similar to the one in Theorem 4 can be constructed over a greater variety of games ($\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M}, k = 1)$.

On the other hand, an indirect generalization step for $k > 1$ is given in the following Lemma 3, while for $D = 1$, we characterize the 2-player game in Theorem 9.

Lemma 3 (**Generalization step from** $k > 1$ **to** $k = 1$)**.** *Consider the game* $(\mathcal{N} = (G, \texttt{LIS} = \textit{LTM}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k > 1)$*, and its game matrix* Π*. A graph* $G_s \equiv (V_s, E_s)$*,* $G_s \supset (V, E)$*, exists, such that the game* $(\mathcal{N}' = (G_s, \texttt{LIS} = \textit{LTM}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k = 1)$*, with diffusion depth* $D + 1$*, induces a game matrix* Π_s *with a restriction* $\Pi_s(X, X)$ *that: Except of its entries on the diagonal that correspond to the utility of* `Player 2`*, and which have zero value, as the ones in* Π*, all its remaining entries have value equal to the* increased by one *value of the corresponding entries in* Π*. Specifically, it is* $G_s = (V_s := V \cup X, E_c := E \cup E_X)$*, and*

i. $V \cap X = \emptyset$*, where* X *denotes a set of* $\binom{|V|}{k}$ *new nodes added to* G*.*

ii. $E \cap E_X = \emptyset$*, where* $E_X = \{(v, u) \mid v \in X, u \in V\}$*, and* $\forall (u, v) \in E_X$*,* $w_{uv} = \theta_v$*.*

Next, we provide some necessary definitions, and then we prove that Theorem 4 holds under "almost" any consistent in and out-degree distributions that may characterize the underlying structure of the social network.

Definition 8. Given a graph $G = (V, E)$, let $k \in \mathbb{Z}_{\geq 0}$, and $\delta \equiv \delta(E) \in \{0, \ldots, |V|\}$. We define as $\mathcal{P}^G_{in} \equiv \mathcal{P}^G_{in}(r, \alpha_{max}, \beta_{max})$ a set of *in-degree* distributions over a graph G, such that $\forall P^G_{in} \in \mathcal{P}^G_{in}$,

$$P^G_{in}(k) \geq \begin{cases} r, & \text{if } k = 0 \\ r + r(r-1), & \text{if } k = 1 \\ 0, & \text{if } k \geq 2 \end{cases}$$

Similarly, we define as $\mathcal{P}^G_{out} \equiv \mathcal{P}^G_{out}(r, \alpha_{max}, \beta_{max}, \delta)$ a set of *out-degree* distributions such that $\forall P^G_{out} \in \mathcal{P}^G_{out}$,

$$P^G_{out}(k) \geq \begin{cases} \delta, & \text{if } k = 1 \\ r(r-1), & \text{if } k = \alpha_{max} \\ r, & \text{if } k = (r-1)(1 + \alpha_{max} + \beta_{max}) \\ 0, & \text{otherwise} \end{cases}$$

Definition 9 (Restriction of an in-degree (out-degree) distribution to $\mathcal{P}_{in}^{G}$ ($\mathcal{P}_{out}^{G}$)). Given any in and out-degree distributions P_{in}, P_{out} *consistent with each other*, we denote as P_{in}^{G}, P_{out}^{G} their *consistent with each other* restrictions in $\mathcal{P}_{in}^{G}$ and $\mathcal{P}_{out}^{G}$ respectively — i.e. $P_{in}^{G} \in \mathcal{P}_{in}^{G}$, and $P_{out}^{G} \in \mathcal{P}_{out}^{G}$, and consistent with each other.

Furthermore, we denote as G'_d an augmented version of G_d [Theorem 4]; specifically, the class of all possible graphs G'_d is defined in the following proof of Corollary 1.

Corollary 1 (All improvement paths are realizable under "almost" any consistent in and out-degree distributions). *Consider any in and out-degree distributions P_{in}, P_{out} consistent with each other. Moreover, consider the games* ($\mathcal{N} = (G'_d, \texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E'_d), \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1$) *of diffusion depth $D \geq 2$, where G'_d has as in and out-degree distributions the P_{in}^{G} and P_{out}^{G}, respectively. Then, all improvement paths are realizable in* $\Pi(\mathcal{S}_D, \mathcal{S}_D)$.

Proof. For the following proof, recall that given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, $(x,y) \in \{1,\ldots,|\mathcal{S}|\}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x, S_y)$, $\forall i \in \mathcal{M} = \{1,2\}$[4.2]. Furthermore, recall that we consider $\alpha_{xy} := \alpha_{S_x S_y}$, as well as, $\beta_{xy} := \beta_{S_x S_y}$[4.2].

We construct a certain class of social networks with the involved property, based on the social networks of Theorem 4.

Firstly, we construct r new *disjoint* graphs Y_i, $i \in \{1,\ldots,r\}$, which we will later append to the graph G_d of Theorem 4:

1. For each $i \in \{1,\ldots,r\}$, construct a directed rooted tree $Y_i = (V_{Y_i}, E_{Y_i})$, such that $|V_{Y_i}| = n_Y$, $|E_{Y_i}| = n_Y - 1$, and let r_{Y_i} denote the root.

   Thereby, all nodes in V_{Y_i} have in-degree one, except of the root, and out- degree one, except of the leaves. The number of leaves defines the value of $\delta(E'_d)$ [Definition 8].

2. Given Y_i, add edges between the nodes of V_{Y_i}, according to the P_{in} and P_{out} distributions, bearing in mind the preexisting structure of the graph of Theorem 4.

3. Next, add the edge (S_i, r_{Y_i}).

4. Finally, as in Theorem 4, set $w_{ij} = \theta_j, \forall (i,j) \in E'_d$.

Thus, we obtain a modified graph G'_d that has in-degree distribution P_{in}^{G}, and out-degree distribution P_{out}^{G}, consistent with each other.

To complete our proof, we prove that the set of strategies $\{S_i\}_{i \in \{1,\ldots,r\}}$ is strictly component-wise undominated over the set $\{S_i\}_{i \in \{r+1,\ldots,|V|\}}$. We first consider the strictly component-wise dominance of $\{S_i\}_i$, $i \in \{1,\ldots,r\}$, over $\{S_i\}_i$, $i \in \{r+1,\ldots,|V|\}$, from `Player's 2` perspective. To this end, let $(\lambda, \lambda') \in \{r+1,\ldots, \ |V|\}^2$, and $(i,j,j') \in \{1,\ldots,r\}^3$, where $i \neq j, i \neq j', j \neq j'$, as well as, $\widehat{d}(S_i, S_\lambda) = \infty$, $\widehat{d}(S_j, S_{\lambda'}) = \infty$. Next, consider the following part of the game matrix:

	S_j	$S_{j'}$	S_λ
S_i	u_2^{ij}	$u_2^{ij'}$	$u_2^{i\lambda}$
S_j	0	$u_2^{ij'}$	$u_2^{j\lambda}$
$S_{\lambda'}$	$u_2^{\lambda' j}$	$u_2^{\lambda' j'}$	$u_2^{\lambda' \lambda}$

We want,

i. $u_2^{ij} > u_2^{i\lambda}$: Equivalently, we write $\alpha_{ij} + \beta_{ij} < \alpha_{i\lambda} + \beta_{i\lambda} + |H_{S_j}| - |H_{S_\lambda}|$. However, $\alpha_{i\lambda} + \beta_{i\lambda} = 0$, since $\widehat{d}(S_i, S_\lambda) = \infty$. Thus, we have $\alpha_{ij} + \beta_{ij} < |H_{S_j}| - |H_{S_\lambda}|$. But, $|H_{S_j}| - |H_{S_\lambda}| \geq N = (r-1)(2\alpha_{max} + \beta_{max}) + 1 + \mathtt{c} > \alpha_{max} + \beta_{max} = max_{(i,j)\in\{1,\ldots,r\}^2}\{\alpha_{ij} + \beta_{ij}\}$. Therefore, $u_2^{ij} > u_2^{i\lambda}$.

ii. $u_2^{jj'} > u_2^{j\lambda}$: Replace in the previous analysis i with j, and j with j'.

iii. $u_2^{\lambda' j} > u_2^{\lambda' \lambda}$: Equivalently we write $\alpha_{\lambda' j} + \beta_{j\lambda'} < \alpha_{\lambda'\lambda} + \beta_{\lambda\lambda'} + |H_{S_j}| - |H_{S_\lambda}|$. However, $\alpha_{\lambda' j} + \beta_{j\lambda'} = 0$, since $\widehat{d}(S_j, S_{\lambda'}) = \infty$. Thus, we have $\alpha_{\lambda'\lambda} + \beta_{\lambda\lambda'} + |H_{S_j}| - |H_{S_\lambda}| > 0$, which always holds, and the proof is completed.

Thereby, the set $\{S_j, S_{j'}\}$ strictly component-wise dominates S_λ.

Now, let $\mu \in \{r+1, \ldots, |V|\}$, while $(i,j) \in \{1, \ldots, r\}^2$, $i \neq j$. We can limit our analysis, from the `Player's 1` perspective, to the following part of the game matrix:

	S_1	$\cdots$	S_j	$\cdots$	S_r
S_i	u_1^{i1}	$\cdots$	u_1^{ij}	$\cdots$	u_1^{ir}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
S_μ	$u_1^{\mu 1}$	$\cdots$	$u_1^{\mu j}$	$\cdots$	u_1^{ir}

It is $u_1^{\mu j} \leq |H_{S_\mu}|$. On the other hand, it is $u_1^{jj} = |H_{S_j}| \geq N + |H_{S_\mu}| > |H_{S_\mu}|$, since $(S_j, r_{Y_j}) \in E$. Thereby, $u_1^{jj} > u_1^{\mu j}$, and as a result $\{S_i\}_{i\in\{1,\ldots,r\}}$ strictly component-wise dominates $\{S_i\}_{i\in\{r+1,\ldots,|V|\}}$.

The fact completes our proof. □

Remark 3. Assuming that $n_Y >> N$, and since $N = O(r^2)$, P_{in}^G (P_{out}^G) and P_{in} (P_{out}) can be considered essentially the same.

6.4 Generalized ordinal potentials

In this section, we first propose necessary and/or sufficient conditions for the existence of a generalized ordinal potential over a game $(\mathcal{N}, \mathcal{M} = \{1,2\}, k)$ — conditions that also involve the underlying structure of the social network $\mathcal{N}$. Next, we continue with certain classes of networks that induce a 2-player game that admits a generalized ordinal potential.

At this point, recall that if a game admits a generalized ordinal potential, then it has at least one PSNE. Therefore, we target to classify games that has a PSNE by identifying if they admit a generalized ordinal potential.

We note that although the following unfold over the whole game matrix Π, similarly we could consider the existence of a generalized ordinal potential only within the restriction $\Pi(\mathcal{S}_D, \mathcal{S}_D)$, where $\mathcal{S}_D$ is the set of strictly component-wise undominated strategies. The main reason for such a direction is to avoid taking into consideration relative small areas in the social network that, on the one hand, generate improvement cycles, and, on the other hand, they are actually never played by any of the players. A representative example follows [Example 8].

Remark 4. We may interpret the results of this section *not* with regard to the standard generalized ordinal potential for *better response* paths, but for *best response* paths [V00].

Example 8. Consider the game $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k = 1)$ over the social network in Figure 6.6: As n_i, $\forall i \in \{1, \ldots, 5\}$, and as a_i, $\forall i \in \{1, \ldots, 200\}$ we denote single nodes. Further, we assume that all of them have threshold 1. Moreover,all the edges are annotated with their corresponding weight with regard to the LTM.

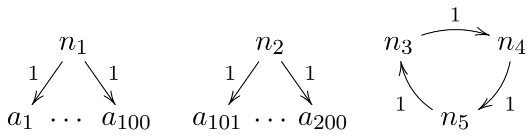


Figure 6.6: Social network for Example 8.

Evidently, over the game matrix Π [Table 6.1] there is **no** generalized ordinal potential, due to the clique of nodes n_3, n_4, and n_5. On the other hand, any player would never play n_3, n_4, and n_5, in comparison to n_1, n_2. Therefore, we prefer to restrict our analysis in $\Pi(\{n_1, n_2\}, \{n_1, n_2\})$ for the existence of a generalized ordinal potential, as we would restrict our analysis in $\Pi(\{n_1, n_2\}, \{n_1, n_2\})$ if we were interested on a generalized ordinal potential not for *better response* paths, but for *best response* paths. □

	n_1	n_2	n_3	n_4	n_5
n_1	101,0	101,101	101,3	101,3	101,3
n_2	101,101	101,0	101,3	101,3	101,3
n_3	3,101	3,101	3,0	1,2	2,1
n_4	3,101	3,101	2,1	3,0	1,2
n_5	3,101	3,101	1,1	2,1	3,0

Table 6.1: The game matrix for the game $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k = 1)$ over the social network in Figure 6.6.

Lemma 4 (Necessary conditions for the existence of a generalized ordinal potential). *Let* $\{S_i\}_{i \in \{1, \ldots, |\mathcal{S}|\}}$ *be the set of strategies of a game* $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M} = \{1, 2\}, k)$. *Then,* Γ *cannot admit a generalized ordinal potential if* *i.* $\exists (i,j) \in \{1,\ldots,|\mathcal{S}|\}^2$, $i \neq j$, *such that* S_j *is reachable from* S_i, *and* S_i *is reachable from* S_j.

ii. $\exists (i,j) \in \{1,\ldots,|\mathcal{S}|\}^2$, $i \neq j$, *such that* $|H_{S_i}| = |H_{S_j}|$.

iii. $\exists (i,j) \in \{1,\ldots,|\mathcal{S}|\}^2$, $i \neq j$, *such that* $|H_{S_i}| > |H_{S_j}|$, *and* $u_1(S_i,S_j) < |H_{S_j}|$.

Proof. For the following proof, recall that given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, $(x,y) \in \{1,\ldots,|\mathcal{S}|\}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x,S_y)$, $\forall i \in \mathcal{M} = \{1,2\}$[4.2].

We treat only the first and the third case. The proofs for the second case is similar.

For the first case, let $(i,j) \in \{1,\ldots,|\mathcal{S}|\}^2$ such that S_j is reachable from S_i, and S_i is reachable from S_j. Then, we deduce $H_{S_i} = H_{S_j}$. Consequently, it is also $|H_{S_i}| = |H_{S_j}| =:$ N. Now, consider the following part of the game matrix:

	S_i	S_j
S_i	$N, 0$	u_1^{ij}, u_2^{ij}
S_j	u_1^{ji}, u_2^{ij}	$N, 0$

From `Player's 1` perspective, it is $u_1^{ij} < |H_{S_i}| = N$, and similarly for u_1^{ji}. From `Player's 2` perspective, it is $u_2^{ij} > 0$ and, similarly for u_2^{ij}. Therefore, when S_j is reachable from S_i, and S_i is reachable from S_j there is an improvement cycle. Thus, a generalized ordinal potential cannot exist.

For the third case, consider the following part of the game matrix:

	S_i	S_j
S_i	$\lvert H_{S_i}\rvert, 0$	u_1^{ij}, u_2^{ij}
S_j	u_1^{ji}, u_2^{ij}	$\lvert H_{S_j}\rvert, 0$

It is $u_1^{ji} \leq |H_{S_j}| < |H_{S_i}|$. Thus, if $u_1^{ij} < |H_{S_j}|$ there is an improvement cycle. The fact completes our proof. ☐

The following corollary, provides one more sufficient condition for the special case where $k = 1$, and follows from Lemma 4(i). When $k = 1$, a player has to pick a single node. Then the only reasonable strategies are the nodes u for which there exists at least one edge (u,v) such that $w_{uv} \geq \theta_v$.

Corollary 2. *If the game* $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$ *admits a generalized ordinal potential, then*

1. *the graph contains a DAG that includes the set* $\{u | \exists v \in V, v \neq u, \text{such that } w_{uv} \geq \theta_v\}$.

2. *if* $w_{uv} \geq \theta_v$ *for every edge* $(u,v) \in E$, *then* G *has to be a DAG.*

Next, we prove a set of sufficient conditions, useful for the proof of Theorem 5.

Lemma 5 (**Sufficient conditions for** $u_1(S_1, S_2) \geq u_1(S_1', S_2)$**,** $S_1 \neq S_1'$**, and** $u_2(S_1, S_2) \geq u_2(S_1, S_2')$**,** $S_2 \neq S_2'$**).** *Consider a game* $\Gamma = ((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M} = \{1,2\}, k)$*, and that* $(S_1, S_1', S_2, S_2') \in \mathcal{S}^4$*, for* $S_1 \neq S_1'$ *and* $S_2 \neq S_2'$*. If* $|H_{S_1}| \geq |H_{S_1'}| + |H_{S_2}| + \gamma_1(S_1, S_2)$*, then* $u_1(S_1, S_2) \geq u_1(S_1', S_2)$*, and if* $|H_{S_2}| \geq |H_{S_2'}| + |H_{S_1}| + \gamma_2(S_1, S_2)$*, then* $u_2(S_1, S_2) \geq u_2(S_1, S_2')$*.*

Proof. For the following proof, given a strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}$, we denote the players' utilities as u_i^{12}, instead of $u_i(S_1, S_2)$, $\forall i \in \mathcal{M} = \{1, 2\}$.

Let $S_1 \neq S_1'$, and consider the following part of the game matrix, from Player's 1 perspective:

	S_2
S_1	u_1^{12}
S_1'	$u_1^{1'2}$

It is $u_1^{12} \in \{max\{0, |H_{S_1}| - |H_{S_2}| - \gamma_1(S_1, S_2)\}, \ldots, |H_{S_1}|\}$. The lower bound is obtainable if Player 2, by playing S_2, colors the whole region H_{S_2}, even if Player 1 plays S_1. On the other hand, the upper bound is achievable when Player 1, by playing S_2, colors the whole region H_{S_1}, regardless the fact that Player 2 plays S_2. Similarly, it is $u_1^{1'2} \in \left\{max\left\{0, |H_{S_1'}| - |H_{S_2}| - \gamma_1(S_1', S_2)\right\}, \ldots, |H_{S_1'}|\right\}$. Now, assume $|H_{S_1}| - |H_{S_2}| - \gamma_1(S_1, S_2) \geq |H_{S_1'}|$, i.e., $|H_{S_1}| \geq |H_{S_1'}| + |H_{S_2}| + \gamma_1(S_1, S_2)$. Then, $u_1^{12} \geq u_1^{1'2}$.

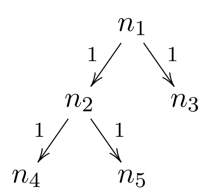
Now, let $S_2 \neq S_2'$, and consider the following part of the game matrix, from Player's 2— perspective:

	S_2	S_2'
S_1	u_2^{12}	$u_2^{12'}$

It is $u_2^{12} \in \{max\{0, |H_{S_2}| - |H_{S_1}| - \gamma_2(S_1, S_2)\}, \ldots, |H_{S_2}|\}$. Similarly for $u_2^{12'}$. Now, assume $|H_{S_2}| - |H_{S_1}| - \gamma_2(S_1, S_2) \geq |H_{S_2'}|$, i.e., $|H_{S_2}| \geq |H_{S_2'}| + |H_{S_1}| + \gamma_2(S_1, S_2)$. Then, $u_2^{12} \geq u_2^{12'}$, and the proof is completed. □

Through the next example we present the basic intuition behind Theorem 5.

Example 9 (**Introductory example for Theorem 5**)**.** Consider a game $\Gamma = (\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M} = \{1,2\}, k = 1)$ over the following social network: As n_i, $\forall i \in \{1, \ldots, 5\}$, we denote single nodes. Further, we assume that all of them have threshold 1. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 5\}$, are annotated with their corresponding weight with regard to the LTM.



Next, verify that $|H_{S_{n_1}}| = 5$, whereas $|H_{S_{n_2}}| = 3$. Now, observe that $|H_{S_{n_2}}| > |H_{S_{n_1}}|/2$, and as a result $|H_{S_{n_2}}| = 3 > |H_{S_{n_1}}| - |H_{S_{n_2}}| = 2$. Therefore, if `Player 1` initiates node n_1, and `Player 2` node n_2, then the latter player will eventually achieve a higher payoff than the former. On the other hand, if `Player 1` initiates n_2, then `Player 2` would prefer to initiate n_1. And then, `Player 1` would prefer again to play n_1, and so forth.

In other words, this game does not admit a generalized ordinal potential since the strategy with the second highest ideal spread can first infect more than half the number of nodes that the strategy with the highest ideal spread also can. □

Next, we have the following related theorem.

Theorem 5 (**Sufficient condition for the existence of a generalized ordinal potential).** *Consider a game* $((G, \mathit{LIS}, \mathit{TBC1} = R^{\prec}, \mathit{TBC2}), \mathcal{M} = \{1, 2\}, k)$*. If there is an ordering of the available strategies* $\left\{S_1, \ldots, S_{|\mathcal{S}|}\right\}$*, such that for all i*

$$|H_{S_{i+1}}| \leq \frac{\lfloor |H_{S_i}| + max\left\{\gamma_1(S_i, S_{i+1}), \gamma_2(S_i, S_{i+1})\right\}\rfloor}{2},$$

the game admits a generalized ordinal potential. Moreover, all its PSNE have the form (S_{max}, S_2)*, where* $S_{max} \equiv argmax_{S\in\mathcal{S}}\left\{|H_S|\right\}$*.*

Proof. For the following proof, given a strategy profile $\mathbf{s} = (S_x, S_y) \in \mathcal{S}^2$, $(x, y) \in \{1, \ldots, |\mathcal{S}|\}^2$, we denote the players' utilities as u_i^{xy}, instead of $u_i(S_x, S_y)$, $\forall i \in \mathcal{M} = \{1, 2\}$.

Moreover, we assume for simplicity that $|H_S|+max\left\{\gamma_1(S, Y), \gamma_2(Y, S)\right\}$ is even, $\forall (S, Y) \in \mathcal{S}^2$.

Assume $(i, i', j) \in \{1, \ldots, |\mathcal{S}|\}^3$, and set $j > i' > i$. We shall compare u_1^{ij} and $u_1^{i'j}$: Based on the hypothesis, and the assumption that $j > i' > i$, it is $|H_{S_i}| \geq 2(|H_{S_{i+1}}| + \gamma_1(S_i, S_{i+1})) > |H_{S_j}|+\gamma_1(S_i, S_j)+|H_{S_{i'}}|+\gamma_1(S_i, S_i') > |H_{S_j}|+|H_{S_{i'}}|+\gamma_1(S_i, S_j)$. Thereby, $u_1^{ij} > u_1^{i'j}$, according to Lemma 5.

Now, set $j = i' > i$. We shall compare u_1^{ij} and u_1^{jj}: It is $|H_{S_i}| \geq 2(|H_{S_{i+1}}| + \gamma_1(S_i, S_{i+1})) \geq |H_{S_j}| + |H_{S_j}| + \gamma_1(S_i, S_j)$. Thus, $u_1^{ij} \geq u_1^{jj}$. However, it also is $u_2^{jj} = 0$, therefore the strategy profile (S_j, S_j) cannot be a PSNE.

In sum, Player 1, from any strategy profile $(S_{i'}, S_j)$ such that $i' < j$, can deviate to any strategy profile (S_i, S_j), $i < i'$, and increase his utility.

Similarly, we can prove that Player 2, from any strategy profile $(S_i, S_{j'})$ such that $j' \leq i$, can deviate to any strategy profile (S_i, S_j), $j < j'$, and increase his utility.

Now, due to the previous discussion no improvement cycles exist — i.e., a generalized ordinal potential exists [MS96, VN97] — and the candidate sets of strategy profiles, where a PSNE can exist, are the

1. (S_1, S_j), $\forall j \in \{2, \ldots, |\mathcal{S}|\}$, (for $j = 1$ it is $u_2^{11} = 0$), and

2. (S_i, S_1), $\forall i \in \{1, \ldots, |\mathcal{S}|\}$. However, $u_1^{11} = |H_{S_1}| > u_1^{i1}$, $\forall i \in \{2, \ldots, |\mathcal{S}|\}$, since $u_1^{i1} \leq |H_{S_i}|$, and $|H_{S_i}| < |H_{S_1}|$ from the hypothesis.

Thereby, all PSNE have the form (S_{max}, S_j), and our proof is completed. □

Example 10. *Let a game* $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E), \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$*, where* G *is a DAG such that* $\forall k \in V$ *it is* $d_k^{out} = d$, $d \in \mathbb{Z}_{\geq 2}$*. Then, this game admits a generalized ordinal potential function, and all PSNEs have the form* (S_{max}, S_j)*, where* $S_{max} := argmax_{S \in \mathcal{S}} \{|H_S|\}$*, and* $S_j \in \mathcal{S} \setminus \{S_{max}\}$*.*

Sketch of Proof. It follows from Theorem 5. □

In the following example, we present a game $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E), \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$, where G is a DAG such that $\forall k \in V$ it is $d_k^{out} = 1$, that has no PSNEs, and as a result, it does not admit a generalized ordinal potential.

Example 5. Consider the game $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$ over the following social network: As n_i, $\forall i \in \{1,2,3\}$, we denote single nodes. Further, we assume that all of them have threshold 1. Moreover, the edges (n_i, n_j), where $i, j \in \{1, \ldots, 5\}$, are annotated with their corresponding weight with regard to the LTM.

$$n_1 \xrightarrow{1} n_2 \xrightarrow{1} n_3$$

Then, the induced game matrix is:

	S_{n_1}	S_{n_2}	S_{n_3}
S_{n_1}	3,0	1,2	2,1
S_{n_2}	2,1	2,0	1,1
S_{n_3}	1,2	1,1	1,0

Evidently, there is no PSNE, and as a result no generalized ordinal potential. □

We end the section with some necessary and sufficient conditions for the existence of a generalized ordinal potential over games $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M} = \{1,2\}, k)$ that have symmetric game matrices.

Theorem 6 (Necessary and sufficient conditions for the existence of a generalized ordinal potential over symmetric games). *A game* $(\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M} = \{1,2\}, k)$ *that has a symmetric game matrix admits a generalized ordinal potential if and only if it is* $u_1(S_i, S_j) \geq u_1(S'_i, S_j)$, $\forall (i, i', j) \in \{1, \ldots, |\mathcal{S}|\}^3$*, such that* $|H_{S_i}| \geq |H_{S_{i'}}|$*. Moreover, all PSNEs have the form* (S_{max}, S_j)*, where* $S_{max} := argmax_{S \in \mathcal{S}} \{|H_S|\}$*, and* $S_j \in \mathcal{S} \setminus \{S_{max}\}$*.*

Proof. For the following proof, recall that given a strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}$, we denote the players' utilities as u_i^{12}, instead of $u_i(S_1, S_2)$, $\forall i \in \mathcal{M} = \{1,2\}$[4.2].

Without loss of generality, consider the following part of the game matrix, where $|H_{S_1}| \geq |H_{S_2}| \geq |H_{S_3}|$:

	S_1	S_2	S_3
S_1	$\|H_{S_1}\|, 0$	u_1^{12}, u_1^{21}	u_1^{13}, u_1^{31}
S_2	u_1^{21}, u_1^{12}	$\|H_{S_2}\|, 0$	u_1^{23}, u_1^{32}
S_3	u_1^{31}, u_1^{31}	u_1^{32}, u_1^{23}	$\|H_{S_3}\|, 0$

In the first place, in order for a generalized ordinal potential to exist, the condition (iii.) of Lemma 4 must hold: Therefore, $u_1^{12} \geq |H_{S_2}|$, $u_1^{13} \geq |H_{S_3}|$ and $u_1^{23} \geq |H_{S_3}|$ (on the other hand $u_1^{21} < |H_{S_2}| \leq |H_{S_1}|$, and similarly $u_1^{31} < |H_{S_1}|$, $u_1^{32} < |H_{S_2}|$). Moreover, $u_1^{12} \geq |H_{S_2}| \geq |H_{S_3}| > u_1^{32}$, and similarly for the other cases. Thus, $u_1^{12} > u_1^{32}$. Consequently, if it is $u_1^{13} < u_1^{23}$, the condition (iv.) of Lemma 4 cannot hold, and a generalized ordinal potential cannot exist. □

6.5 ϵ-generalized ordinal potentials

We move on to approximate ϵ-generalized ordinal potentials, as a method of measuring the "instability" of the games in question — the larger the parameter ϵ is, the more "unstable" the game can be characterized. Therefore, recall that we shall characterize a game Γ that admits an ϵ-generalized ordinal potential as ϵ-unstable [4.2].

We first obtain such a potential function for games that have diffusion depth $D = 1$, based on the ideal spread of a player's strategy.

Theorem 7. *Any game* $\Gamma = ((G, \mathit{LIS}, \mathit{TBC1} = R^{\prec}, \mathit{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k)$*, where* $D(\Gamma) = 1$*, admits the function* $P(\mathbf{s}) = (1 + \beta_{max} + \gamma_{max})|H_{S_1}| + |H_{S_2}| - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s})$*,* $\forall \mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$*, as a k-generalized ordinal potential. Moreover, a k-PSNE can be computed in polynomial time.*

Proof. Firstly, assume $(S_1, S_1', S_2, S_2') \in \mathcal{S}^4$, where $S_1 \neq S_1'$, and $S_2 \neq S_2'$, and set $\mathbf{s} = (S_1, S_2)$. Now, let Player 1 diverge from S_1 to a better strategy S_1'. Specifically, set $\mathbf{s}' = (S_1', S_2)$, and let $K_1 \geq k + 1$ be the increasing step, such that

$$\begin{aligned} u_1(\mathbf{s}') &= u_1(\mathbf{s}) + K_1 \Leftrightarrow \\ |H_{S_1'}| - |S_2 \cap (H_{S_1'} \setminus S_1')| &= |H_{S_1}| - |S_2 \cap (H_{S_1} \setminus S_1)| + K_1 \Rightarrow \\ |H_{S_1'}| &\geq |H_{S_1}| + 1, \end{aligned}$$

since $|H_{S_1'}| - |S_2 \cap (H_{S_1'} \setminus S_1')| \leq |H_{S_1'}|$, and $K_1 - |S_2 \cap (H_{S_1} \setminus S_1)| \geq 1$. Moreover, $\beta_2(\mathbf{s}') + \gamma_2(\mathbf{s}') \leq \beta_{max} + \gamma_{max} + \beta_2(\mathbf{s}) + \gamma_2(\mathbf{s})$. Therefore, after substitution we get:

$$\begin{aligned} P(\mathbf{s}') &= (1 + \beta_{max} + \gamma_{max})|H_{S_1'}| + |H_{S_2}| - \beta_2(\mathbf{s}') - \gamma_2(\mathbf{s}') \\ &\geq (1 + \beta_{max} + \gamma_{max})|H_{S_1}| + 1 + \beta_{max} + \gamma_{max} + |H_{S_2}| - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) - \beta_{max} - \gamma_{max} \\ &> P(\mathbf{s}). \end{aligned}$$

Now, let Player 2 diverge from S_2 to a better strategy S_2'. Specifically, set $\mathbf{s}' = (S_1, S_2')$, and let $K_2 \geq k+1$ be the increasing step, so that

$$\begin{aligned} u_2(\mathbf{s}') &= u_2(\mathbf{s}) + K_2 \Leftrightarrow \\ |H_{S'_2}| - \alpha_2(\mathbf{s}') - \beta_2(\mathbf{s}') - \gamma_2(\mathbf{s}') &= |H_{S_2}| - \alpha_2(\mathbf{s}) - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) + K_2 \Rightarrow \\ |H_{S'_2}| - \beta_2(\mathbf{s}') - \gamma_2(\mathbf{s}') &\geq |H_{S_2}| - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) + 1, \end{aligned}$$

since $K_2 + \alpha_2(\mathbf{s}') - \alpha_2(\mathbf{s}) \geq 1$. Therefore,

$$\begin{aligned} P(\mathbf{s}') &= (1 + \beta_{max} + \gamma_{max})|H_{S_1}| + |H_{S'_2}| - \beta_2(\mathbf{s}') - \gamma_2(\mathbf{s}') \\ &\geq (1 + \beta_{max} + \gamma_{max})|H_{S_1}| + |H_{S_2}| - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) + 1 \\ &> P(\mathbf{s}). \quad \square \end{aligned}$$

Example 11 (Approximation $\epsilon = \mathbf{k}$ in Theorem 7 is tight). Consider the game $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$, where $D = 1$, over the social network in Figure 6.7, where we have assumed without loss of generality that $|V| = 2$, and $k = 1$ (generally, we could consider a clique of $|V| = 2k$ nodes, where $k \geq 1$, and such that $w_{uv} = \theta_v$, $\forall (u,v) \in E$, with regard to the LTM). Then, the induced game matrix is as in Table 6.2; evidently, there is no PSNE; nevertheless, the game has a 1-generalized ordinal potential.

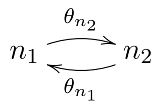


Figure 6.7: Tight example for Theorem 7: The edges are annotated with their corresponding weight with regard to the LTM.

	n_1	n_2
n_1	2,0	1,1
n_2	1,1	2,0

Table 6.2: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 1)$, where the social network $\mathcal{N}$ is as in Figure 6.7.

$\square$

Next, we turn our attention to the games $\Gamma = (\mathcal{N} = (G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k)$, where $D \geq 2$.

For games with higher diffusion depth, we define below an important parameter that captures the quality of approximation we can achieve via ϵ-generalized ordinal potentials.

Definition 10.

1. Given a 2-player game, and two strategy profiles $\mathbf{s} = (S_1, S_2), \mathbf{s}' = (S'_1, S_2)$, the **diffusion collision factor** of player 1 for strategy S'_1 compared to S_1, given S_2, is defined as $DC_1(S'_1, S_1|S_2) \equiv (\alpha_1(\mathbf{s}') + \gamma_1(\mathbf{s}')) - (\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s}))$.

2. Similarly, for two strategy profiles $\mathbf{s} = (S_1, S_2), \mathbf{s}' = (S_1, S'_2)$, the diffusion collision factor of Player 2 for S'_2 compared to S_2 compared to S_1 is defined as $DC_2(S'_2, S_2|S_1) \equiv (\alpha_2(\mathbf{s}') + \gamma_2(\mathbf{s}')) - (\alpha_2(\mathbf{s}) + \gamma_2(\mathbf{s}))$.

Evidently, the diffusion collision factor of any player is a non-symmetric function. Moreover, for $i \in \{1,2\}$, we denote as $DC_{i,max}$ the maximum diffusion collision factor that can be achieved by player i over all his strategy pairs. Also, we set $DC_{max} = \max\{DC_{1,max}, DC_{2,max}\}$.

Theorem 8. *Any game* $\Gamma = ((G, \mathit{LIS}, \mathit{TBC1} = R^{\prec}, \mathit{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k)$, *where* $D(\Gamma) \geq 2$, *admits the function* $P(\mathbf{s}) = x_1|H_{S_1}| + |H_{S_2}| - \beta_2(\mathbf{s})$, $\forall \mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$, *as a* DC_{max}*-generalized ordinal potential, where* x_1 *is any number satisfying* $x_1 > \beta_{max}$.

Proof. Firstly, assume $(S_1, S_1', S_2, S_2') \in \mathcal{S}^4$, where $S_1 \neq S_1'$, and $S_2 \neq S_2'$, and set $\mathbf{s} = (S_1, S_2)$. Now, let Player 1 diverge from S_1 to a better strategy S_1'. Specifically, set $\mathbf{s}' = (S_1', S_2)$, and let $K_1 \geq DC_{max} + 1$ such that

$$\begin{aligned} u_1(\mathbf{s}') &= u_1(\mathbf{s}) + K_1 \Leftrightarrow \\ |H_{S_1'}| &= |H_{S_1}| + (\alpha_1(\mathbf{s}') + \gamma_1(\mathbf{s}')) - (\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s})) + K_1 \Rightarrow \\ |H_{S_1'}| &\geq |H_{S_1}| + 1, \end{aligned}$$

since $K_1 \geq DC_{max} + 1$. Moreover, $\beta_2(\mathbf{s}') \leq \beta_{max} + \beta_2(\mathbf{s})$. Therefore,

$$\begin{aligned} P(\mathbf{s}') &= x_1|H_{S_1'}| + |H_{S_2}| - \beta_2(\mathbf{s}') \\ &\geq x_1|H_{S_1}| + x_1 + |H_{S_2}| - \beta_2(\mathbf{s}) - \beta_{max} \\ &> P(\mathbf{s}). \end{aligned}$$

Now, let Player 2 diverge from S_2 to a better strategy S_2'. Specifically, set $\mathbf{s}' = (S_1, S_2')$, and let $K_2 \geq DC_{max} + 1$ such that

$$\begin{aligned} u_2(\mathbf{s}') &= u_2(\mathbf{s}') + K_2 \Leftrightarrow \\ |H_{S_2'}| - \alpha_2(\mathbf{s}') - \beta_2(\mathbf{s}') - \gamma_2(\mathbf{s}') &= |H_{S_2}| - \alpha_2(\mathbf{s}) - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}) + K_2 \Rightarrow \\ |H_{S_2'}| - \beta_2(\mathbf{s}') &\geq |H_{S_2}| - \beta_2(\mathbf{s}) + 1, \end{aligned}$$

since $K_2 \geq DC_{max} + 1$. Therefore,

$$\begin{aligned} P(\mathbf{s}') &= x_1|H_{S_1}| + |H_{S_2'}| - \beta_2(\mathbf{s}') \\ &\geq x_1|H_{S_1}| + |H_{S_2}| - \beta_2(\mathbf{s}) + 1 \\ &> P(\mathbf{s}). \quad \square \end{aligned}$$

Example 12 (Approximation $\mathbf{DC_{max}}$ in Theorem 8 is tight). Consider the game $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k)$, such that $D = 2$, over the social network in Figure 6.8, where we have assumed without loss of generality that $|V| = 3$, and $k = 1$ (for the description of the general case, see below). Then, the induced game matrix is as in Table 6.3; evidently, there is no PSNE; nevertheless, the game has an ϵ-generalized ordinal potential **only** if $\epsilon = DC_{max} = 2$ (for $\mathbf{s} = (n_2, n_2)$, and $\mathbf{s}' = (n_1, n_2)$, it is $DC_{max} = DC_1(n_1, n_2|n_2) = (\alpha_1(\mathbf{s}') + \gamma_1(\mathbf{s}')) - (\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s})) = 2$, since it is $\alpha_1(\mathbf{s}') = 2$, and $\alpha_1(\mathbf{s}) = \gamma_1(\mathbf{s}') = \gamma_1(\mathbf{s}) = 0$).

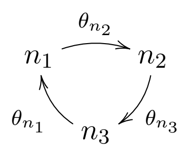


Figure 6.8: Tight example for Theorem 8: The edges are annotated with their corresponding weight with regard to the LTM.

	n_1	n_2	n_3
n_1	3,0	1,2	2,1
n_2	2,1	3,0	1,2
n_3	1,2	2,1	3,0

Table 6.3: The game matrix for the game $(\mathcal{N}, \mathcal{M}, k = 1)$, where the social network $\mathcal{N}$ is as in Figure 6.8.

Generally, we could consider a social network similar to the one in Figure 6.8, where the nodes n_1, n_2, and n_3 are replaced by the sets of k nodes S_1, S_2, and S_3, respectively, and the edges (n_1, n_2), (n_2, n_3), and (n_3, n_1), with the following: For $i \in \{1, 2, 3\}$, and $j = (i \bmod 3) + 1$, we consider $\forall v_i \in S_i$, $\forall v_j \in S_j$, that $w_{v_i v_j} = \theta_{v_j}/k$ — i.e., only if all the nodes in S_i are colored by a unique color, they can infect any node in S_j. Moreover, no other edges exist in the network. □

6.6 Special Case: Diffusion Depth, and Maximum Strategies' Cardinality One

In this section, motivated by recent empirical studies over the underlying structure of real-world recommendation networks [LSK06, LAH07, LMFGH07] [2.7], we consider the games $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k = 1)$, where $D = 1$ — i.e., games where both D, and k equal one.

Although these games are apparently simple, several examples exist with no PSNE: For example, the clique in Example 3, Section 5.1. On the other hand, given such a game, it is easy to decide whether a PSNE exists — $O(n^2)$.

Finally, observe that according to Lemma 3, the games $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k > 1)$, where $D = 1$, are essentially equivalent to games where $D \geq 2$, and $k = 1$.

6.6.1 Underlying network structure

We restrict our analysis over a weakly connected component of G. Evidently, because the diffusion diameter, as well as, the parameter k are equal to one, we assume $\texttt{LTM}(w_{vu} \geq \theta_u, \forall (v, u) \in E)$, without loss of generality.

Next, we consider a weakly connected component Δ of graph G, and some node $v \in V$ such that $d_v^{out} \geq 1$. Then, we distinguish between the following:

1. *No other node $u \in \Delta$ has $d_u^{out} \geq 1$* [Figure 6.9]: As a result, the component is like a star with center node the node v: $\forall u \in \Delta$, where $u \neq v$, it is $(v, u) \in E$.

2. *There exists at least one additional node $u \in \Delta$, $u \neq v$, that has $d_u^{out} \geq 1$*: We separate further our analysis: Specifically, for each such pair of nodes u, v we consider the

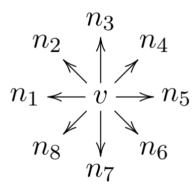


Figure 6.9: A weakly connected component Δ with only one node v such that $d_v^{out} \geq 1$.

connections between u, v and their out-neighbors, i.e., the nodes that they influence:

i. *$(v, u) \notin E$, and $(u, v) \notin E$* [Figure 6.10]. Therefore, the component has at least two stars — one for each node with out-degree more than one. Moreover, due to the connectivity of Δ, these stars must share with each other at least one leaf: For example, in Figure 6.10 the node n_5 is a shared leaf.

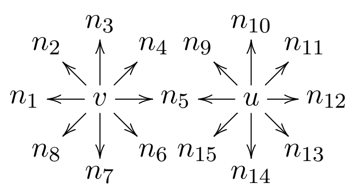


Figure 6.10: A weakly connected component Δ with two nodes v, u such that $(v, u), (u, v) \notin E$, and $d_v^{out} \geq 1$, and $d_u^{out} \geq 1$: v, u must share at least one of their leaves $n_1, \ldots, n_{15}$ — here only the n_5.

ii. *$(v, u) \in E$, whereas $(u, v) \notin E$* [Figure 6.11]. Now, since $D = 1$, node v must have out-going edges to all the nodes that u also has.

Note that $|H_u \setminus \{u\}| \geq 1$, since u has at least one out-going edge; therefore, $|H_v \setminus \{v\}| \geq 2$, since v points not only to u, but also to each node that u does. Generally, (recall that $v \in H_v$ by definition),

$$H_u \setminus \{u\} \subseteq H_v \Rightarrow$$
$$|H_u| \geq |H_v| + 1$$

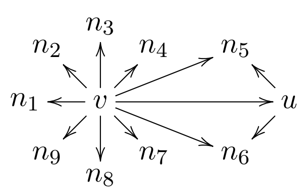


Figure 6.11: A weakly connected component Δ with two nodes v, u such that $(v, u) \in E$, $(u, v) \notin E$, and $d_v^{out} \geq 1$, and $d_u^{out} \geq 1$: v must have out-going edges to all the nodes that u also has.

iii. *$(v,u) \notin E$, whereas $(u,v) \in E$.* This case is symmetric to the previous one.

iv. *$(v,u) \in E$, and $(u,v) \in E$* [Figure 6.12]. By combining the two previous cases, it is

$$N := |H_u| = |H_v|.$$

We note that the induced game matrix between the two corresponding strategies S_v, and S_u is

	S_v	S_u
S_v	$N, 0$	$N-1, 1$
S_u	$N-1, 1$	$N, 0$

Evidently, an improvement cycle exists.

6.6.2 Game characterization

For our purposes, we assume that the set of available strategies $\mathcal{S}$ for the two players in $\mathcal{M}$ constitutes from all the nodes $v \in V$ such that $d_v^{out} \geq 1$ — i.e., we associate with each such v a strategy S_v. In other words, we ignore any weakly dominated strategies that correspond to leaves-nodes, i.e., nodes with out-degree zero. Finally, we note that we shall refer to strategy S_v, and node v interchangeably.

In the following lemma, we consider a class of games where $\{S_v\} \cap I^1_{S_u} = \emptyset$, $\forall (v,u) \in V^2$, while $u \neq v$: Particularly, this assumption states that in the involved network do not exist any nodes $(u,v) \in V^2$ such that $d_v^{out}, d_u^{out} \geq 1$, and $(u,v) \in E$, or $(v,u) \in E$.

Lemma 6. *Consider a game $\Gamma = (\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1,2\}, k = 1)$, where $D = 1$, and $\{S_v\} \cap I^1_{S_u} = \emptyset$, $\forall (v,u) \in V^2$, where $u \neq v$. Then, Γ admits a generalized ordinal potential $P(\mathbf{s}) = (1 + |V|)|H_{S_1}| + |H_{S_2}| - |H_{S_1} \cap H_{S_2}|$, $\forall \mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$. Moreover, all PSNEs have the form (S_{max}, S_j), where $S_{max} := argmax_{S \in \mathcal{S}} \{|H_S|\}$, and $S_j \in \mathcal{S} \setminus \{S_{max}\}$.*

Proof. In the first place, observe that under the hypothesis that $\{S_v\} \cap I^1_{S_u} = \emptyset$, $\forall (v,u) \in V^2$, where $u \neq v$, the player's utilities [Lemma 1] can be written as follows, given a strategy

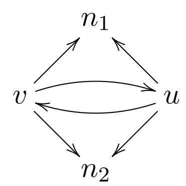


Figure 6.12: A weakly connected component Δ with two nodes v, u such that $(v,u), (u,v) \in E$, and $d_v^{out} \geq 1$, and $d_u^{out} \geq 1$: v must have out-going edges to all the nodes that u also has, and similarly, u must have out-going edges to all the nodes that v also has.

profile $\mathbf{s} \in \mathcal{S}^2$:

$$u_1(\mathbf{s}) = |H_{S_1}| \quad (6.5)$$
$$u_2(\mathbf{s}) = |H_{S_2}| - |H_{S_1} \cap H_{S'_2}| \quad (6.6)$$

Now, assume $(S_1, S'_1, S_2, S'_2) \in \mathcal{S}^4$, where $S_1 \neq S'_1$, and $S_2 \neq S'_2$. Now, let `Player 1` diverge from S_1 to a better strategy S'_1. Specifically, set $\mathbf{s}' = (S'_1, S_2)$, recall that $|\{S_2\} \cap I^1_{S_1}| = |\{S_2\} \cap I^1_{S'_1}| = 0$ from the hypothesis, and let $K_1 \geq 1$ be the increasing step such that

$$\begin{aligned} u_1(\mathbf{s}') &= u_1(\mathbf{s}) + K_1 \Rightarrow \\ |H_{S'_1}| &= |H_{S_1}| + K_1 \Rightarrow \\ |H_{S'_1}| &\geq |H_{S_1}| + 1. \end{aligned}$$

In other words, whenever `Player 1` digresses to a better strategy S'_1, it increases its ideal cumulative spread $|H_{S'_1}|$. Moreover, $|H_{S'_1} \cap H_{S_2}| \leq |H_{S_1} \cap H_{S_2}| + |V|$. Therefore,

$$\begin{aligned} P(\mathbf{s}') &= |V||H_{S'_1}| + |H_{S'_1}| + |H_{S_2}| - |H_{S'_1} \cap H_{S_2}| \\ &\geq |V||H_{S_1}| + |V| + |H_{S_1}| + 1 + |H_{S_2}| - |H_{S_1} \cap H_{S_2}| - |V| \\ &> P(\mathbf{s}). \end{aligned}$$

Now, let `Player 2` diverge from S_2 to a better strategy S'_2. Specifically, set $\mathbf{s}' = (S_1, S'_2)$, recall that $|\{S_2\} \cap I^1_{S_1}| = |\{S'_2\} \cap I^1_{S_1}| = 0$ from the hypothesis, and let $K_2 \geq 1$ be the increasing step such that

$$\begin{aligned} u_2(\mathbf{s}') &= u_2(\mathbf{s}) + K_2 \Rightarrow \\ |H_{S'_2}| - |H_{S_1} \cap H_{S'_2}| &= |H_{S_2}| - |H_{S_1} \cap H_{S_2}| + K_2 \Rightarrow \\ |H_{S'_2}| - |H_{S_1} \cap H_{S'_2}| &\geq |H_{S_2}| - |H_{S_1} \cap H_{S_2}| + 1. \end{aligned}$$

Moreover, $|H_{S'_1} \cap H_{S_2}| \leq |H_{S_1} \cap H_{S_2}| + |V|$. Therefore,

$$\begin{aligned} P(\mathbf{s}') &= |V||H_{S_1}| + |H_{S_1}| + |H_{S'_2}| - |H_{S_1} \cap H_{S'_2}| \\ &\geq |V||H_{S_1}| + |H_{S_1}| + |H_{S_2}| - |H_{S_1} \cap H_{S_2}| + 1 \\ &> P(\mathbf{s}). \end{aligned}$$

□

At this point, recall that $|H_{max}| := max_{S \in \mathcal{S}} \{|H_S|\}$.

Theorem 9. *For a game* $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k = 1)$*, where* $D = 1$*, either*

i. all its PSNEs have the form (S_{max}, S_j)*, where* $S_{max} := argmax_{S \in \mathcal{S}} \{|H_S|\}$*, and* $S_j \in \mathcal{S} \setminus \{S_{max}\}$*, or*

ii. it has no PSNE.

Additionally, it always admits a 1-generalized ordinal potential [Theorem 7].

Proof. Recall that the players' set of available strategies constitutes from all the nodes $v \in V$ such that $d_v^{out} \geq 1$. Therefore, instead of $(v, u) \in E$, we shall write (S_v, u) to denote the out-going edge from a node v, that has $d_v^{out} \geq 1$, to a node $u \in V$, $u \neq v$.

In the following, we assume that the strategy $S_v \in V$ — associated with node $v \in V$ — achieves the maximum ideal spread $|H_{max}|$.

i. *Assume that* $\{S_i\} \cap I^1_{S_j}, \forall (i,j) \in V^2$: Then, according to Lemma 6, the game Γ admits a generalized ordinal potential.

ii. *Assume that exists at least one pair* $(S_v, S_u) \in V^2$ *such that* $(S_v, S_u) \in E$, *and* $(S_u, S_v) \notin E$: Then $\nexists k \in V$ such that $(S_k, S_v) \in E$ if also $(S_v, S_k) \notin E$, since that would imply $|H_{S_k}| > |H_{S_v}| = |H_{max}|$ (recall that the assumption $D = 1$ implies that k connects to all the out-neighbors of v if $(S_k, S_v) \in E$), which contradicts our assumption that $|H_{S_v}| = |H_{max}|$. We have the following further cases:

   (a) *Assume that* $\exists k \in V, (S_v, S_k) \in E$, *and* $(S_k, S_v) \notin E$: Then $|H_{S_v}| > |H_{S_k}|$. Assume that `Player 2` plays S_k, and let `Player 1` play S_v. Then, `Player 1` shall have utility $u_1(S_v, S_k) = |H_{max}| - 1$, and `Player 2` a utility $u_2(S_v, S_k) = 1$. Now, if `Player 2` does not have a better strategy to diverge to, then the strategy profile (S_v, S_k) is a PSNE.

   On the other hand, if `Player 2` has an alternative S to play, then he will always obtain a utility greater, or equal to one, while `Player 1` will obtain, by playing S_v, a utility $u_1(S_v, S) = |H_{max}|$, if $S \notin H_{S_v}$, or $u_1(S_v, S) = |H_{max}| - 1$, if $S \in H_{S_v}$. Particularly, `Player 2` can obtain a higher utility only if there is a strategy S such that $(S_v, S) \notin E$, which means that the utility of `Player 1` will then be $|H_{max}|$, and that the strategy profile (S_v, S) is a PSNE.

   Next, assume that `Player 2` prefers to play S_k, given that `Player 1` plays S_v, and that `Player 1` has an alternative strategy $S'_v \neq S_v$, that gives him utility $u_1(S'_v, S_k) = |H_{max}|$. However, that means that $k \notin H_{S'_v}$, or else it would be $|H_{S'_v}| = |H_{max}| + 1$: contradiction. Therefore, there exists a node $b \in H_{S'_v}$, that it is not in H_{S_v}, such that $u_1(S'_v, S_k)$ can be equal to $|H_{max}|$. Next observe that $v \notin H_{S'_v}$, or else it must be $k \in H_{S_v}$, due to the fact that $D = 1$. As a result, given that `Player 1` plays strategy S_v (and not the alternative S'_v), if `Player 2` picks S'_v, then `Player 1` obtains a utility $|H_{max}|$, and `Player 2` a utility of at least 2, since he infects the nodes v', and b: a contradiction. Hence, the strategy profile (S_v, S_k) is a PSNE.

   (b) *Assume* $(S_v, S_k) \in E$, *and* $(S_k, S_v) \in E$. Then, it is also $|H_{S_k}| = |H_{max}|$. The proof is similar to the previous case. □

Chapter 7

Quantifying Inefficiency

In the section we first study the price of anarchy and stability of the games $(\mathcal{N}, \mathcal{M}, k)$, and then, after of Theorem 12(ii), we propose a new measurement that we call "*price of oligopoly*".

7.1 Price of Anarchy and Stability

We focus our attention on the price of anarchy and stability of the games $(\mathcal{N}, \mathcal{M}, k)$. In the first place, concerning their price of anarchy, it is

$$1 \leq \text{price of anarchy} \leq \frac{|V|}{mk}$$

Therefore, we set $PoA_{max} := |V|/(mK)$. Similarly, concerning their price of stability, it is

$$1 \leq \text{price of stability} \leq \text{price of anarchy}.$$

We have the following theorems.

Theorem 10. *The price of anarchy for the games* $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$ *can be* PoA_{max}*, even if the price of stability is one.*

Proof. We construct a social network such that the induced game over it has price of anarchy $PoA_{max} = |V|/(mK)$, and price of stability 1. Specifically, we consider the social network in Figure 7.1: As V we assume the union of following sets of nodes: $X_0 \cup X_1 \cup \ldots \cup X_{m-1} \cup X_r$, where $|X_0| = |X_1| = \ldots = |X_{m-1}| = k$. Moreover, as E we assume the union of following sets of edges:

i. $\forall u \in X_0, \forall v \in X_1 \cup \ldots \cup X_{m-1}$ there is an edge (u, v) with $w'_{uv} = 1/k$.

ii. $\forall u \in X_1 \cup \ldots \cup X_{m-1}, \forall v \in X_r$ there is an edge (u, v) with $w'_{uv} = 1/\left[(m-1)\,k\right]$.

Furthermore, in Figure 7.1, the edges (X_0, X_i), where $i \in \{1, \ldots, m-1\}$, are annotated with the *accumulated* corresponding weight of the underlying edges between each of the

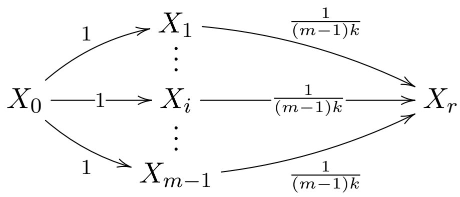


Figure 7.1: The social network for the proof of Theorem 10 ($i \in \{2, \ldots, m-2\}$).

nodes in X_0 and each of the nodes in X_i, and with regard to the LTM. In other words, X_0 accumulatively points with weight 1 to each node in X_i. Similarly, for the edges (X_i, X_r).

Additionally, all nodes in V are assumed to have threshold 1.

Thereby, only if the entire set X_0 is uniquely colored by a single firm it can color all the nodes in each of the $X_i, i \in \{1, \ldots, m-1\}$. On the other hand, X_0 cannot color any other node in V. Additionally, only if the entire $X_1 \cup \ldots \cup X_{m-1}$ area is uniquely colored it can color X_r. Nevertheless, it cannot color any other subset of the graph.

Furthermore, the maximal social welfare of the induced game is exactly $|V|$. Specifically, given $i \in \mathcal{M}$, for any strategy profile $(S_i, S_{-i}) \in \mathcal{S}^m$, such that $S_i = X_0$, and $S_j \subset X_r$, $\forall j \in \mathcal{M} \setminus \{i\}$, player i colors initially X_0, then $X_1 \cup \ldots \cup X_{m-1}$ and afterwards $X_r \setminus \bigcup_{j \in \mathcal{M} \setminus \{i\}} S_j$. Thereby, the whole network is eventually colored, and a maximal social welfare is obtained.

Also, the lowest social welfare is obtained for the PSNE where the firm associated with the best quality product, according to $R^{\prec}$, picks the area X_0 as its strategy, and the remaining $m-1$ players pick the areas $X_1, \ldots, X_{m-1}$. Moreover, this lowest value is mK, and as a result the price of anarchy for the involved game is $|V|/(mK)$.

On the other hand, the highest social welfare is also obtained for the equilibrium where the firm associated with the best quality product, according to $R^{\prec}$, picks X_0, and the remaining $m-1$ players pick any of the mutual exclusive subsets of X_r. This best value is $|V|$, and as a result the price of stability for the involved game is 1.

The previous facts complete our proof. □

Theorem 11. *The price of anarchy and the price of stability for the games* $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$ *can be equal to* $\frac{k}{k+1} PoA_{max}$.

Proof. The proof is similar to the proof of Theorem 10, but over the social network in Figure 7.2.

□

7.2 Price of Oligopoly

We first prove that if at least three firms are involved in a game $(\mathcal{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$, then a PSNE may exist where the firm associated

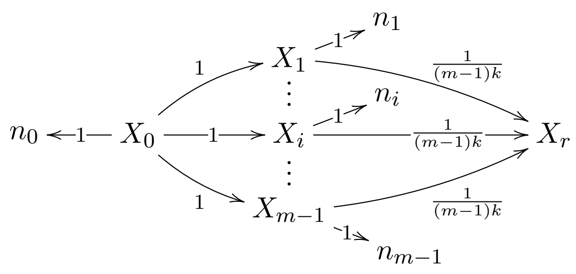


Figure 7.2: The social network for the proof of Theorem 10 ($i \in \{2, \dots, m-2\}$).

with the *best quality* product does not necessarily receive the highest payoff among all the involved players. Then, we propose a relevant measurement under the name "price of oligopoly".

Recall that $\mathcal{M} = \{1, \dots, m\}$, $R^{\prec} = 1 \succ \dots \succ m$, and that we denote as $\mathcal{NE}(\Gamma)$ the set of existing PSNEs of game Γ.

Theorem 12. *Consider the class of games* $(\mathcal{N} = (G, \texttt{LIS}, R^{\prec}, \texttt{TBC2}), \mathcal{M}, k)$.

i. If $m = 2$*, then for all strategy profiles* $\mathbf{s} \in \mathcal{NE}$ *it is* $u_1(\mathbf{s}) \geq u_2(\mathbf{s})$.

ii. If $m \geq 3$*, and* $\texttt{LIS} = \texttt{LTM}$*,* $\texttt{TBC2} = R^{\prec}$*, then a game* Γ *exists that has a strategy profile* $\mathbf{s} \in \mathcal{NE}$ *such that* $u_i(\mathbf{s}) < u_j(\mathbf{s})$*, although* $i \succ j$ *with regard to* $R^{\prec}$.

Proof. i. Assume that a PSNE $\mathbf{s} := (S_1, S_2)$ exists such that $u_1(\mathbf{s}) < u_2(\mathbf{s})$. Then, `Player 1` can deviate to $S_1' := S_2$, and obtain utility $u_1(S_2, S_2) \geq u_2(\mathbf{s}) > u_1(\mathbf{s})$. Thus, $\mathbf{s}$ cannot be a PSNE.

ii. Assume $R^{\prec} = 1 \succ 2 \succ 3$, and consider the social network in Figure 7.3: As n_i, $\forall i \in \{1, \dots, 8\}$, and as a_i, $\forall i \in \{1, \dots, \nu\}$, where $\nu \in \mathbb{Z}_{>k}$, we denote single nodes. Further, we assume that all of them have threshold 1, except of node n_3 that has $\theta_{n_3} = 1/2$, and node n_8 that has $\theta_{n_8} = 1/2$. Moreover, as S_i, $\forall i \in \{1, 2, 3\}$, we denote sets of k nodes in V, with no edges among them, and such that $S_i \cap S_j = \emptyset$, $\forall (i, j) \in \{1, 2, 3\}^2$, whenever $i \neq j$. Finally, the edges (n_i, n_j), where $i, j \in \{1, \dots, 5\}$, and the edges (n_8, a_i), where $i \in \{1, \dots, \nu\}$, are annotated with their corresponding weight with regard to the `LTM`. On the other hand, the edges (S_i, n_j), where $i \in \{1, 2, 3\}$, and $j \in \{1, \dots, 8\}$, and the edges (S_2, a_i), where $i \in \{1, \dots, \nu\}$, are annotated with the *accumulated* corresponding weight of the underlying edges between each of the nodes in S_i and node n_j, and each of the nodes in S_2 and node a_i with regard to the `LTM`.

We shall prove that the strategy profile $\mathbf{s} := (S_1, S_2, S_3)$ constitutes a PSNE, even though it is $u_2(\mathbf{s}) = k+\nu+4$, $u_1(\mathbf{s}) = k+3$, $u_3(\mathbf{s}) = k+1$ — i.e., $u_2(\mathbf{s}) > u_1(\mathbf{s}) > u_3(\mathbf{s})$, while $1 \succ 2 \succ 3$: Firstly, `Player 1`, by deviating to S_3, receives utility $u_1(S_3, S_2, S_3) = k+3 = u_1(\mathbf{s})$. Moreover, by deviating to S_2, he receives utility $u_1(S_2, S_2, S_3) = k+3 =$

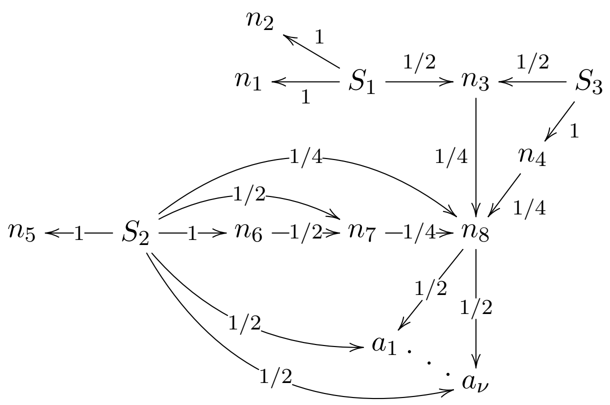


Figure 7.3: The social network for the proof of Theorem 12(ii): All single nodes have threshold 1, except of node n_3 that has $\theta_{n_3} = 1/2$, and node n_8 that has $\theta_{n_8} = 1/2$. Moreover, the "$\cdots$" between nodes a_1, and a_ν, where $\nu \in \mathbb{Z}_{>k}$, denotes that $\nu - 2$ same nodes exists between them, i.e., nodes that have the same in and out-going weighted edges, as well as, threshold as a_1, and a_ν do.

$u_1(\mathbf{s})$, and as a result, he fails again to increase his utility. Finally, `Player 1` cannot achieve utility higher than k by initiating any other combination of k nodes from V.

Similarly, it can be verified that neither `Player 2`, nor `Player 3` can deviate to a better strategy, given that the strategy profile $\mathbf{s} := (S_1, S_2, S_3)$ is initially played. Hence, $\mathbf{s}$ constitutes a PSNE, and the fact completes our proof. □

Remark 6. With regard to the first part of Theorem 12(i): It follows that for $m = 2$, $\nexists \mathbf{s} \in \mathcal{NE}$ such that $u_2(\mathbf{s}) > |V|/2$. With regard to the second part of Theorem 12(ii): Firstly, the assumption $\texttt{TBC2} = R^{\prec}$ can be abandoned. Additionally, we conjecture that the assumption $\texttt{LIS} = \texttt{LTM}$ can be also relaxed.

Now, with regard to the game over the social network in Figure 7.3 [Theorem 12(ii)]: Given the PSNE $\mathbf{s} = (S_1, S_2, S_3)$, observe that $u_2(\mathbf{s}) = k + \nu + 4 > u_1(\mathbf{s}) + u_3(\mathbf{s}) = 2k + 4$, since $\nu \in \mathbb{Z}_{>k}$. Thereby, if firm 1 is affiliated with firm 3, while their products are marketed as competing and incompatible, firm 1 is incentivized to withdraw firm 3 from the game: Firm 1, by participating in the 2-player game between itself and firm 2, achieves at the unique PSNE (S_1, S_2) the maximum possible utility — i.e., $u_1(S_1, S_2) = k + \nu + 4$. Moreover, notice that in this 2-player game, firm 1 initiates only k nodes in order to achieve this maximum utility. On the other hand, in the original 3-player game, firms 1 and 3 initiate k nodes *each*, and still they achieve a lower sum of utilities at the PSNE $\mathbf{s}$. Thus, firm 1, by nullifying the affiliated firm 3, not only induces a game with a unique PSNE, where its utility equals the maximum possible utility with regard to both the induced 2-player game and the original 3-player game, but also it achieves this utility by initiating only k nodes in V.

Our previous discussion over Theorem 12(ii) indicates the necessity of a new measurement that captures the motivation of a player in $\mathcal{M}$ to either merge with other players in

$\mathcal{M}$, or to divide itself to several new players that, although they are affiliated, they are still non-cooperative within the induced game. Hence, we suggest for this measurement the name **"price of oligopoly"**.

Chapter 8

Conclusions and Future Work

In this work we analyzed a non-cooperative game between firms that compete to promote their products over a social network. Particularly, we unveiled several of its properties, and we demonstrated that they are inextricably intertwined with various structural characteristics of the underlying network — characteristics that to the best of our knowledge have met no previous investigation. Therefore, with this study we first target to motivate further empirical and experimental research on relevant essential questions of interest as: What is the general structure of common social networks, for example, of consumers? How much influence each individual exerts to his neighbors, and which parameter values capture these influences, with regard to a threshold model for example? And even more importantly, are there any representative values concerning decisive structural features such as the diffusion depth, and the maximum diffusion collision factor? In other words, how unstable are the induced games over such networks? Or, do they generally possess a PSNE, or even admit a generalized ordinal potential?

Moreover, which values of k can be considered optimal with regard to the existence and the efficiency of the PSNE? Or, what is the optimal number of firms, (possibly affiliated), that should participate in the game? Equivalently, when the price of oligopoly is minimized/maximized? On the other hand, which are the typical values for the price of anarchy, and stability? Additional compelling questions may concern the robustness of the discussed characteristics to network changes. For example, how the introduction of new individuals in the network can affect the answers to all the previous questions?

Finally, we stress the exciting prospect of relaxing the assumption — common in the related bibliography [M00, IKMW07] — that a unique reputation ordering holds over the existing products, and throughout the population. Nevertheless, such a richer framework may render randomized tie-breaking criteria unavoidable, due to the intractability of the deterministic approach. Therefore, we believe that risk analysis should constitute an essential part of such future works.

Bibliography

[AB02] Albert, R., and Barabási, A.-L. (2002) Statistical mechanics of complex networks. *Reviews of modern physics* 74, 47-97.

[ADKTWR04] Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, E., Wexler, T. , and Roughgarden, T. (2004) The price of stability for network design with fair cost allocation. *In Proceedings of 45th FOCS*, 295-304.

[AJB99] Albert, R., Jeong, H.M and Barabási, A.-L. (1999) Diameter of the world-wide web. *Nature* 401, 130-131.

[AFPT10] Alon, N., Feldman, M., Procaccia, A. D., and Tennenholtz, M. (2010) A note on competitive diffusion through social networks. *Information Processing Letters*, 110(6):221-225.

[AFPT11] Alon, N., Feldman, M., Procaccia, A. D., and Tennenholtz, M. (2011) Erratum: A Note on Competitive Diffusion Through Social Networks. *http://www.cs.cmu.edu/ arielpro/papers/diffusion.err.pdf.*

[AM11] Apt, K. R., and Markakis, E. (2011) Diffusion in social networks with competing products. In *Proc. 4th International Symposium on Algorithmic Game Theory (SAGT11).* Lecture Notes in Computer Science Series, vol. 6982. Springer, 212-223.

[BB03] Barabási, A.-L., Bonabeau, E. (2003) Scale-Free Networks. *Scientific American* 288, 50-59.

[B93] Blume, L. E. (1993) The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387-424.

[BA99] Barabási, A.-L., and Albert, R. (1999) Emergence of scaling in random networks. *Science* 286, 509-512.

[BBCG] Graham, F. (2010) Viral ticketing: Catching a bigger audience on Facebook. *http://www.bbc.co.uk/news/business-11643405.*

[BBCW] Weber, T. (2010) Why companies watch your every Facebook, YouTube, Twitter move. *http://www.bbc.co.uk/news/business-11450923.*

[BFO10] Borodin, A., Filmus, Y., and Oren, J. (2010) Threshold models for competitive influence in social networks. *In Proc. 6th International Workshop on Internet and Network Economics.*

[BKS07] Bharathi, S., Kempe, D., and Salek, M. (2007) Competitive Influence Maximization in Social Networks. *In International Workshop on Internet and Network Economics (WINE)*, 306-311.

[C04] Chamley, C. P. (2004) *Rational herds: Economic models of social learning.* Cambridge University Press.

[CH00] Chickering, D. M. and Heckerman, D. (2000) A decision-theoretic approach to targeted advertising. *In Proceedings of Sixteenth Conference on Uncertainty in Artificial Intelligence.*

[CKM66] Coleman, J. S., Katz, E., and Menzel, H. (1966) *Medical innovation: A Diffusion Study.* Bobbs-Merrill, Indianapolis, MN.

[CLRS01] Cormen, T. H., Leiserson, C. E., Rivest,R. L., and Stein, C. (2001) *Introduction to Algorithms, Second Edition.* MIT Press and McGraw-Hill.

[CNWZ07] Carnes, T., Nagarajan, C., Wild, S. M., and Zuylen, V. (2007) Maximizing Influence in a Competitive Social Network: A Follower"s Perspective. *International Conference on Electronic Commerce*, 351-360.

[CS11] Chien, S., and Sinclair, A. (2011) Convergence to approximate Nash equilibria in congestion games. *Games and Economic Behavior*, 71(2):315-327.

[D10] Diestel, R. (2010) *Graph Theory.* Springer-Verlag, Heidelberg Graduate Texts in Mathematics, Volume 173.

[DR01] Domingos, P. and Richardson, M. (2001) Mining the network value of customers. *In Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD).*

[E93] Ellison, G. (1993) Learning, local interaction, and coordination. *Econometrica*, 61(5):1047-1071.

[EK10] Easley, D., Kleinberg, J. (2010) *Networks, Crowds, and Markets.* Cambridge University Press.

[ER59] Erdős, P., and Rényi, A. (1959) On random graphs. *Publicationes Mathematicae* 6, 290-297.

[G07] Goyal, S. (2007) *Connections: An introduction to the economics of networks.* Princeton University Press.

[G96] Goyal, S. (1996) Interaction structure and social change. *Journal of Institutional and Theoretical Economics*, 152:472-495.

[G78] Granovetter, M. S. (1978) Threshold models of collective behavior. *The American Journal of Sociology*, 83(6):1420-1443.

[GK12] Goyal, S., and Kearns, M. (2012) Competitive Contagion in Networks. To appear, *STOC 2012*.

[GLM01a] Goldenberg, J., Libai, B., and Muller, E. (2001) Using complex systems analysis to advance marketing Theory development: Modeling heterogeneity effects on new product growth through stochastic cellular automata. *Academy of Marketing Science Review*, [Online] 1(9).

[GLM01b] Goldenberg, J., Libai, B. and Muller, E. (2001) Talk of the network: A complex systems look at the underlying process of word-of-mouth. *Marketing Letters*, 12(3):209-21.

[IKMW07] Immorlica, N., Kleinberg, J., Mahdian, M., and Wexler, T. (2007) The role of compatibility in the diffusion of technologies through social networks. *In Proceedings of the Eighth ACM Conference on Electronic Commerce (EC).*

[J08] Jackson, M. (2008) *Social and Economic Networks.* Princeton University Press.

[KATHP] Papavassileios, N. (2012) Marketing using social networks. *http://news.kathimerini.gr/4dcgi/_w_articles_economy_1_28/01/2012_470442* (In Greek).

[KKLO03] Kakade, S., Kearns, M., Langford, J., and Ortiz, L. (2003) Correlated equilibria in graphical games. *In Proceedings of the Fourth ACM Conference on Electronic Commerce (EC).*

[KLS01] Kearns, M., Littman, M., and Singh, S. (2001) Graphical models for game theory. *In Proceedings of the 17th Conference in Uncertainty in Artificial Intelligence (UAI).*

[KKT03] Kempe, D., Kleinberg, J., and Tardos, É. (2003) Maximizing the spread of influence in a social network. *In Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD).*

[KKT05] Kempe, D., Kleinberg, J., and Tardos, É. (2005) Influential nodes in a diffusion model for social networks. *In Proceedings of the 32nd International Colloquium on Automata, Languages and Programming (ICALP).*

[KN09] Karrer, B., and Newman, M. E. J. (2009) Random graph models for directed acyclic networks. *Physical Review* E 80, 046110.

[KOW08] Kostka, J., Oswald„ Y. A., and Wattenhofer, R. (2008) Word of Mouth: Rumor Dissemination in Social Networks. *In SIROCCO*, 185-196.

[KP99] Koutsoupias, E., Papadimitriou, C. (1999) Worst-case equilibria. *In Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science*, 404-413

[L85] Liggett, T. M. (1985) *Interacting Particle Systems.* Springer.

[LAH07] Leskovec, J., Adamic, L., and Huberman, B. (2007) The Dynamics of Viral Marketing. *ACM Transactions on the Web (TWEB)*, 1(1).

[LK08] Liben-Nowell, D., and Kleinberg, J. (2008) Tracing Information Flow on a Global Scale Using Internet Chain-Letter Data. *Proceedings of the National Academy of Sciences*, 105(12):4633-4638.

[LMFGH07] Leskovec, J., McGlohon, M., Faloutsos, C., Glance, N., and Hurst, M. (2007) Cascading Behavior in Large Blog Graphs. *SIAM International Conference on Data Mining (SDM).*

[LSB11] Liu, Y.-Y., Slotine, J.-J., and Barabási, A.-L. (2011) Controllability of complex networks. *Nature* 43, 123-248.

[LSK06] Leskovec, J., Singh, A., and Kleinberg, J. (2006) Patterns of Influence in a Recommendation Network. *Pacific-Asia Conference on Knowledge Discovery and Data Mining (PAKDD).*

[M67] Milgram, S. (1967) The small world problem. *Psychology Today* 2, 60-67.

[M00] Morris, S. (2000) Contagion. *The Review of Economic Studies*, 67(1):57-78.

[MR07] Mossel, E., and Roch, S. (2007) On the submodularity of influence in social networks. In *STOC*, 128-134.

[MS96] Monderer, D., and Shapley, L. S. (1996) Potential Games. *Games and Economic Behavior* 14, 124-143.

[N01] Newman, M. E. J. (2001) The structure of scientific collaboration networks. *Proceedings of the National Academy of Sciences of the United States of America* 98, 409-415.

[NSW01] Newman, M. E. J., Strogatz, S. H., and Watts, D. J. (2001) Random graphs with arbitrary degree distributions and their applications. *Physical Review* E 64, 026118.

[NWS02] Newman, M. E. J., Watts, D. J., and Strogatz, S. H. (2002)Random graph models of social networks. *Proceedings of the National Academy of Sciences of the United States of America* 99, 2566-2572.

[OT10] Oyama, D., and Takahashi, S. (2010) Contagion and Uninvadability in Social Networks with Bilingual Option. *Working paper.*

[RD02] Richardson, M. and Domingos, P. (2002) Mining knowledge-sharing sites for viral marketing. *In Proceedings of the Eighth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD).*

[RG43] Ryan, B. and Gross, N. C. (1943) The diffusion of hybrid seed corn in two Iowa communities. *Rural Sociology*, 8:15-24.

[RLK10] Gomez-Rodriguez, M., Leskovec, J., and Krause, A. (2010) Inferring networks of diffusion and influence. *Proceedings of the 16th ACM SIGKDD international conference on Knowledge discovery and data mining.*

[RLK10s] Supporting website for [RLK10]. http://snap.stanford.edu/ netinf

[S34] Stackelberg, H. V. (1934) *Marktform und Gleichgewicht.* Julius Springer, Heidelberg.

[S78] Schelling, T. C. (1978) *Micromotives and Macrobehavior.* Norton, New York, NY.

[S01] Strogatz, S. H., (2001) Exploring complex networks. *Nature* 410, 268-276.

[SA12] Simon, S., and Apt, K. (2012) Choosing Products in Social Networks. *To appear.*

[THS12] Takehara, R., Hachimori, M., and Shigeno, M. (2012) A comment on pure-strategy Nash equilibria in competitive diffusion games. *Information Processing Letters*, 112(3):59-60.

[V00] Voorneveld, M. (2000) Best response potential games. *Economics Letters*, 66, 289-295.

[VN97] Voorneveld, M. and Norde, H. (1997) A characterization of ordinal potential games. *Games and Economic Behavior*, 19, 235-242.

[VNRT07] Vazirani, V., Nisan, N., Roughgarden, T., Tardos, E. (2007) *Algorithmic Game Theory.* Cambridge University Press.

[VR07] Vega-Redondo, F. (2007) *Complex Social Networks*. Cambridge University Press.

[W08] Wortman, J. (2008) *Viral Marketing and the Diffusion of Trends on Social Networks*. University of Pennsylvania Technical Report MS-CIS-08-19.

[WS98] Watts, D. J., and Strogatz, S. H. (1998) Collective dynamics of “small-world” networks. *Nature* 393, 440-442.