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Monoidal Categories and Applications

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Εξεταστική Επιτροπή

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Στη μνήμη του παππού μου Παναγιώτη
In memory of my grandfather Panagiotis

Ευχαριστίες

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Abstract

In this master thesis, we present an overview of monoidal categories. Following a presentation of basic category theory, we examine monoidal categories through the lenses of graphical calculus and 2-dimensional category theory. We discuss the coherence theorems that justify the use of string diagrams. Subsequently, we analyze braided, symmetric, and closed structures, focusing on rigid, pivotal, ribbon and compact closed categories. We also introduce biproducts and internal algebraic structures, specifically monoids, comonoids, bimonoids, and Frobenius objects. Throughout the text, abstract concepts are illustrated via concrete examples from linear algebra, topology, and order theory. Using this framework, we derive the No-Cloning and No-Deleting theorems of Categorical Quantum Mechanics. Finally, we introduce dagger categories and indicate how these structures illuminate aspects of quantum theory.

Contents

Introduction

The introduction of Category Theory by Eilenberg and Mac Lane [EM45] marked a paradigm shift in mathematics, shifting the focus from the internal structure of mathematical objects to the structure of the relationships and transformations between them. This structuralist perspective elevated category theory to a foundational framework for geometry and physics, particularly through the work of Grothendieck and Lawvere, suggesting that the fundamental laws governing mathematical universes are functorial in nature.

Parallel to these developments, the algebraic and topological richness of categories was revealed through the study of Monoidal Categories. The rigorous treatment of such structures, where associativity and unitality hold only up to coherent isomorphism, was established by Bénabou's work on bicategories [Bén67], as well as the coherence theorems of Mac Lane [Mac63] and Kelly [KML71]. Furthermore, the interplay between algebra and low-dimensional topology was illuminated by Joyal and Street, whose geometry of tensor calculus [JS91], [JS95] demonstrated that the algebraic axioms of certain monoidal categories are intrinsically linked to the topology of knots and manifolds.

In the early 21st century, these structures found a significant application in the foundations of quantum physics. Abramsky and Coecke, in their seminal work on Categorical Quantum Mechanics [AC04], identified that essential phenomena of quantum theory, such as entanglement, teleportation, and unitarity, are not inextricably linked to Hilbert spaces, but can be viewed as structural properties of Dagger Compact Categories. This realization initiated a research program, further systematized by Selinger [Sel07], Heunen and Vicary [HV19], aimed at reformulating quantum theory in terms of monoidal process theories.

The primary motivation of this thesis is to provide a systematic and rigorous algebraic exposition of the structures underpinning this framework. Our investigation begins in Chapter 1 by establishing the necessary categorical foundations. We introduce the core concepts of functors and natural transformations, and provide a detailed treatment of universal constructions, including limits, colimits, ends, and coends, illuminated by the Yoneda lemma and the theory of representable functors. The chapter concludes with an examination of adjunctions, closed categories, and semi-additive structures, setting the stage for the more specialized frameworks required for quantum mechanics.

Chapter 2 shifts the focus to Monoidal Categories, the primary setting for modeling composite processes. After discussing motivating examples from topology and physics, we define the appropriate morphisms, monoidal functors and natural transformations, and address the crucial issue of coherence. We prove that every monoidal category is monoidally equivalent to a strict one, a result that provides the formal justification for the use of String Diagrams as a rigorous calculational tool. Furthermore, we provide an alternative view of monoidal categories by introducing 2-categories and bicategories. We explore how bicategories serve as a categorification of ordinary categories and a "many-object" generalization of monoidal categories, establishing the precise correspondence between higher-order morphisms and the monoidal functors of the one-object case.

Chapter 3 serves as the synthesis of our investigation, examining deeper structures within monoidal categories. We identify two distinct hierarchies of structure: the first concerns the commutativity of the tensor product, encompassing braided, balanced, and symmetric monoidal categories; the second concerns exponentials and duality, arising from internal adjunctions, which lead to closed, rigid, and pivotal categories. A key insight discussed here is that in all such categories, an internal version of Cayley's theorem holds. By combining duality with braiding, we obtain richer topological structures such as ribbon and compact closed categories. Finally, we introduce the Dagger structure, an identity-on-objects contravariant involution that operates independently of the monoidal product. The culmination of this work is the unification of these families into Dagger Compact categories, the structures that elegantly combine topological duality with Hermitian adjoints to provide the natural habitat for quantum mechanics.

Chapter 1

Elements of Basic Category Theory

The aim of this chapter is a presentation of basic category theory. We firstly provide the definition of a category, examples of universal constructions and also discuss cartesian and cocartesian categories. We then present functors and natural transformations, while establishing the basic terminology. The comma category construction is also discussed to provide an alternative view of the category of elements of a (co)presheaf, a notion examined in the presentation of the Yoneda lemma. We continue by examining representable functors, the Yoneda lemma and how these shed light to universal constructions such as ends, coends, limits and colimits. Subsequently, we introduce the notion of adjoint functors and use it to provide an alternative view of universal constructions such as general limits and colimits, but also exponentials. We provide a brief presentation of closed categories, a structure which will reappear in chapter 3. Finally, we give an overview of some of the theory of semi-additive categories, which serves as an example of a category with a monoidal structure.

Key Sources: [AHS09], [Awo10], [Bor94a], [?], [HV19], [Law63], [Lap77], [Lei21], [Lei14], [Lor21], [ML98], [Par70], [Rie17], [?] and various nlab entries.

1.1 What is a category

As it has been put by Tom Leinster in [Lei14]:

"A category is a system of related objects. The objects do not live in isolation: there is some notion of map between objects, binding them together."

Since there is no way to explain such a statement without actually giving a concrete definition, we proceed by doing so.

Definition 1.1.1. *A **category** $\mathcal{C}$, consists of:*

- *a class $\mathcal{C}_0$ of **objects**,*
- *for every $A, B \in \mathcal{C}_0$, a class $\mathcal{C}(A, B)$ of **morphisms** or **maps**, called a **hom-class** and also denoted $\hom_{\mathcal{C}}(A, B)$,*
- *for every $A \in \mathcal{C}_0$, an **identity** morphism $\mathrm{id}_A \in \mathcal{C}(A, A)$*
- *for each $A, B, C \in \mathcal{C}_0$, a **composition** law:*

$$\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C),$$

$$(f, g) \mapsto g \circ f$$

satisfying the following axioms:

- *for every* $f \in \mathcal{C}(A,B),\ g \in \mathcal{C}(B,C),\ h \in \mathcal{C}(C,D)$*:*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

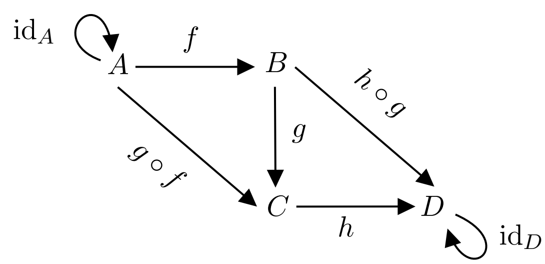
  also known as ***associativity*** *and*

- *for every* $A, B \in \mathcal{C}_0,\ f \in \mathcal{C}_0(A,B)$*:*

$$\mathrm{id}_B \circ f = f \text{ and } f = f \circ \mathrm{id}_A$$

  named ***identity laws****.*

Remark 1.1.2. 1. Category theorists use **commutative diagrams**. In this commutative diagram notation, objects are represented as points, or vertices, and morphisms are represented as arrows, or directed edges. Composition is represented path-wise and the word "commutative" indicates that different paths between the same objects are equal as morphisms. For example, associativity is depicted as the following commutative diagram:



2. In more than just a certain sense, category theory is a theory of structures. More precisely, an abstract algebraic theory of structures. This means that it suppresses the *thing-in-itself* aspect of objects and brings out the structural role an object plays inside the ambient category as a whole. This is achieved by adopting a view of the morphisms and their composition law as an abstraction from the corresponding laws governing functions. This fact will be highlighted in the following examples.

3. Following the previous remark, for every morphism f in a category $\mathcal{C}$ there are two definite objects $dom(f)$ and $cod(f)$, called the **domain** and the **codomain** of the morphism. So two morphisms $f, g \in \mathcal{C}$ are composable exactly when $cod(f) = dom(g)$.

Example 1.1.3. 1. The first example of a category is **Set**. In this category the objects are sets, the morphisms are functions between them, identity morphisms are identity functions and composition is given by the ordinary function composition. It is easy to check that all the axioms of a category are obeyed. The definition of this category makes sense exactly because the collection of all sets is a class. A remarkable thing to note here is that the collection of all functions between two sets is a set. Therefore, it is not just a class, but an object of this category!

2. Categories of algebraic structures are some of the most common examples of categories. Such a category is **Grp** the category of groups and homomorphisms between them. In such a case, the axioms of a category are satisfied. because they are already satisfied by functions and because the composition of homomorphisms is a homomorphism. Other examples of such categories are **Ring**, the category of rings and ring homomorphisms, **Mon** the category of monoids and monoid homomorphisms and so on.

   Another such category, also exhibiting a similar remarkable property to that of **Set**, is $\mathbf{Vect}_{\mathbb{F}}$, the category of vector spaces over the field $\mathbb{F}$ and of linear maps between them. As is already known from linear algebra courses the set of linear maps between two vector spaces is again a vector space.

3. Generalising the previous type of examples, there are categories whose objects are sets with additional, but not necessarily algebraic, structure. Examples of such categories are partially ordered sets with monotone functions between them, denoted **Pos**, preordered sets with monotone functions between them, denoted by **Ord**, topological spaces with continuous maps or just **Top**, smooth manifolds of order $p \in \mathbb{N}$ with p times differentiable maps noted $\mathbf{Man}^p$ and so on. Such categories, algebraic or not, are called **concrete** because their objects are sets with extra structure.

Every category exhibiting the exact same "remarkable" property of **Set**, namely that every hom-class is a set, is called a **locally small** category and for every $A, B \in \mathcal{C}_0$, $\mathcal{C}(A, B)$ is called a **hom-set**. Furthermore, a locally small category $\mathcal{C}$, such that $\mathcal{C}_0$ is a set and not a proper class, is called **small**. If $\mathcal{C}_0$ is a proper class, $\mathcal{C}$ is called **large**. Obviously a large category can either be locally small or not. From now on we will, either implicitly or explicitly, concern ourselves with locally small categories.

Definition 1.1.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. $\mathcal{C}'$ is a* ***subcategory*** *of $\mathcal{C}$ if*

- *$\mathcal{C}'_0$ is a subclass of $\mathcal{C}_0$ and*
- *for every $A, B \in \mathcal{C}'_0$,*
$$\mathcal{C}'(A, B) \subseteq \mathcal{C}(A, B).$$

$\mathcal{C}'$ will be called a ***full*** *subcategory of $\mathcal{C}$ if for every $A, B \in \mathcal{C}_0$,*
$$\mathcal{C}'(A, B) = \mathcal{C}(A, B).$$

The above examples were of categories whose objects were sets. Since studying categories means looking at how objects interact, rather than what they are in themselves, we give the first categorical definition of some already known concepts in mathematics.

Definition 1.1.5. *Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(A, B)$. We say that f is*

- *a* ***monomorphism*** *if it is left cancellable, meaning that for every $C \in \mathcal{C}_0$ and every $g, h \in \mathcal{C}(C, A)$,*
$$f \circ g = f \circ h \Rightarrow g = h,$$
- *an* ***epimorphism*** *if it is right cancellable, i.e. for every $D \in \mathcal{C}_0$ and every $g, h \in \mathcal{C}(B, D)$,*
$$g \circ f = h \circ f \Rightarrow g = h,$$
- *an* ***isomorphism*** *if there exists $g \in \mathcal{C}(B, A)$ such that:*
$$g \circ f = \mathrm{id}_A \text{ and } f \circ g = \mathrm{id}_B.$$

  Furthermore, we say that A and B are ***isomorphic*** *and write $A \cong B$, when there exists an isomorphism between them.*

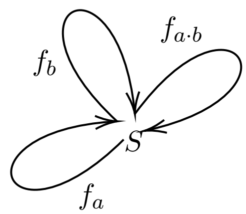
It is an easy exercise to see that the above definition coincides with the usual "one to one" and "onto" definitions in **Set**, but it also gives the right notion of isomorphism in every concrete category. For example in **Top** an isomorphism is not just a bijection, but a homeomorphism.

Having a notion of isomorphisms, we say that a category is **skeletal** whenever every two isomorphic objects are equal. This does not imply that every isomorphism is an identity map, but that there is no isomorphism between different objects. Thus, non-trivial automorphism groups are allowed. An example is the category $\mathbb{N}$, whose objects are the elements of the ordinal ω and the morphisms are the functions between them.

Another important sort of examples is that of non-concrete categories. That is, categories as structures rather than as classes of mathematical objects.

Example 1.1.6. 1. Let $\mathbf{B}M$ be a category with one object, namely $*$. From the axioms of a category it follows that $(\mathbf{B}M(*, *), \circ)$ is a monoid. Now let $(M, \cdot)$ be a monoid. We can consider a category $\mathbf{B}M$ with one object $*$ and create a morphism $f_a \in \mathbf{B}M(*, *)$ for every element $a \in M$. Then, define for every $a, b \in M$, $f_a \circ f_b = f_{a \cdot b}$.

Thus, for every monoid there is a corresponding one object category and to every one object category corresponds a monoid. In simpler terms we can identify one object categories with monoids and depict monoids as:



A useful thing to observe is that a category can be seen as a many-object monoid in the above sense.

2. In the spirit of the previous example, it is easy to see that a category with one object, $*$, where every arrow is an isomorphism is, essentially a group. Therefore to a group G we assign a category $\mathbf{B}G$, whenever $\mathbf{B}G(*, *)$ is isomorphic to G. By dropping the "one object" condition in the above "definition" of a group, we can define a new structure called a **groupoid**. So, a groupoid is a category where every arrow is an isomorphism. This is an instance of categorification, which, in broad terms, means using categories to generalize an algebraic structure.

3. A preordered set is a set P equipped with a binary relation " $\leq$ ", called preorder, such that:

   - for every $a \in P$, $a \leq a$ and,
   - for every $a, b, c \in P$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

   A preorder is actually an order without the antisymmetry condition. Moreover, we can say that every preorder, P, is (isomorphic to) a category $\tilde{P}$ with at most one morphism between any two objects. To be more specific we say that for every $a, b \in P$, $\tilde{P}(a, b)$ has only one element, namely $\leq$, when $a \leq b$, otherwise $\tilde{P}(a, b) = \emptyset$. The existence of an identity morphism for every object of $\tilde{P}$ guarantees the satisfaction of the first condition, while the composability of morphisms guarantees the satisfaction of the second one. Such categories are called **preorder categories**. Obviously, there is also a category whose objects are preorders and whose morphisms are monotone maps, but this is a concrete category.

4. Having defined a preorder in categorical terms we can say that an ordered set is actually a preordered set which is skeletal, i.e. every two isomorphic objects are equal, or in plain terms the antisymmetry condition is satisfied. Such categories are called **poset categories**.

An important thing to note about the last two examples is that every diagram commutes because of the "at most one morphism between objects" condition.

Opposite and product categories

In order theory, for every ordered (or preordered) set, there exists a dual or opposite one, which occurs by turning the inequalities around, or turning the corresponding Hasse diagram upside down. There is also the principle of duality, which states that every statement that holds for a given poset has a dual which occurs by reversing the inequalities and holds for the dual poset. The exact same notions can be transferred to categories.

Definition 1.1.7. *Let $\mathcal{C}$ be a category. Its **dual** category, $\mathcal{C}^{\text{op}}$, is defined as the category which obeys:*

1. $(\mathcal{C}^{\text{op}})_0 = \mathcal{C}_0$,
2. *for every* $A, B \in (\mathcal{C}^{\text{op}})_0,\ f \in \mathcal{C}(A, B) \Leftrightarrow f^{\text{op}} \in \mathcal{C}^{\text{op}}(B, A)$,
3. *for every* $A, B, C \in (\mathcal{C}^{\text{op}})_0$ *and* $f \in \mathcal{C}^{\text{op}}(B, A),\ g \in \mathcal{C}^{\text{op}}(C, B)$
$$f^{\text{op}} \circ g^{\text{op}} := (g \circ f)^{\text{op}}$$

The satisfaction of the associative and unit laws is implicit in the above definition, therefore $\mathcal{C}^{\text{op}}$ is a category. This category is actually the same as $\mathcal{C}$, but with the morphisms turned around. Reasonably, there exists a duality principle stating that for every proof, statement or construction holding for a category, there is a dual one holding for the opposite category.

The opposite category is the first "new category out of old one" construction. Another such construction is the product of two categories.

Definition 1.1.8. *Let $\mathcal{C}$, $\mathcal{D}$ be categories. The category $\mathcal{C} \times \mathcal{D}$, where:*

1. $(\mathcal{C} \times \mathcal{D})_0 := \mathcal{C}_0 \times \mathcal{D}_0$
2. *for every* $A, A' \in \mathcal{C}_0,\ B, B' \in \mathcal{D}_0$,
$$(\mathcal{C} \times \mathcal{D})((A, B), (A', B')) := \mathcal{C}(A, A') \times \mathcal{D}(B, B')$$
3. *for every* $(A, B) \in (\mathcal{C} \times \mathcal{D})_0$
$$\text{id}_{(A,B)} := (\text{id}_A, \text{id}_B)$$
4. *for every* $(A, B),\ (A', B'),\ (A'', B'') \in (\mathcal{C} \times \mathcal{D})_0$ *and*
$(f, g) : (A, B) \to (A', B'),\ (f', g') : (A', B') \to (A'', B'')$
$$(f', g') \circ (f, g) := (f' \circ f, g' \circ g)$$

*is called the **product** category of $\mathcal{C}$ and $\mathcal{D}$.*

We already used the term "category" in the above definition, since proving that this construction is a category is straightforward. In the next subsection we will see that the above definition is the "right definition" of a product in the categorical sense.

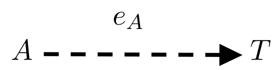
Universal properties

So far we have defined categories by giving a property of all their objects or all their morphisms, or even giving them explicitly in terms of an already given category. In this subsection we face the problem of defining something inside a given category, rather than defining a category itself.

One usual way to define something (at least in category theory), is to find some property that characterizes it. Such a property of an object in a category will often be articulated in terms of other objects and morphisms in that category, while imposing some commutativity conditions. Such kinds of properties are called **universal properties**. In a sense universal properties are properties that describe how an object relates/interacts with its neighbouring objects in an essentially unique way. We proceed with two examples of universal properties.

Definition 1.1.9. *Let $\mathcal{C}$ be a category. An object $T \in \mathcal{C}$ is called **terminal**, if for every $A \in \mathcal{C}$ there exists a unique morphism $e_A : A \to T$.*

The above definition tells us that T is terminal iff for every object $A \in \mathcal{C}$, $\mathcal{C}(A,T)$ is a one-element set. In diagrammatic notation we may draw the unique map to the terminal object as a dotted line like in the following diagram:



Lemma 1.1.10. *A terminal object, when it exists, is unique, up to unique isomorphism.*

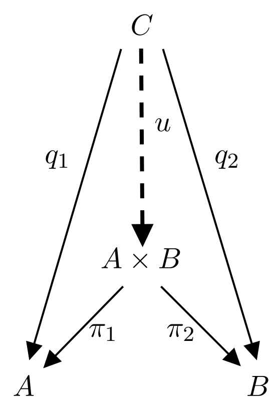
Proof. Let T, T' be terminal in $\mathcal{C}$. Then both $\mathcal{C}(T,T)$ and $\mathcal{C}(T',T')$ are one-element sets and since $\mathcal{C}$ is a category, $\mathcal{C}(T,T) = \{\mathrm{id}_T\}$ and $\mathcal{C}(T',T') = \{\mathrm{id}_{T'}\}$. By definition of T, there exists a unique $u : T \to T'$ and similarly there exists a unique $v : T' \to T$. Since $u \circ v \in \mathcal{C}(T',T')$ and $v \circ u \in \mathcal{C}(T,T)$, 1), these imply that $u \circ v = \mathrm{id}_{T'}$ and $v \circ u = \mathrm{id}_T$. Therefore u is an isomorphism and is also unique. □

This is the first example of a universal property, where we see that what is defined is a behaviour[1] rather than an object. Uniqueness up to unique isomorphism is something that holds for any universal property. We see that immediately in the next example.

Definition 1.1.11. *Let $A, B \in \mathcal{C}_0$. An object $X \in \mathcal{C}_0$ is called a* ***product*** *of A and B if it is equipped with two morphisms $\pi_1 : X \to A$, $\pi_2 : X \to B$ and satisfies the following universal property: for every object $C \in \mathcal{C}_0$ equipped with maps $q_1 : C \to A$, $q_2 : C \to B$, there exists a unique morphism $u : C \to X$ such that:*

$$q_1 = \pi_1 \circ u \text{ and } q_2 = \pi_2 \circ u.$$

Since a product always comes equipped with the above two morphisms, called **projections**, we sometimes say that (X, π_1, π_2) is a product. The following diagram is given in order to clarify the above definition.



The definition of a product captures the essence of a universal property. In this case, that is a statement of the form: "an object, possibly equipped with some morphisms, is universal if for every other object equipped with similar morphisms, there exists a unique morphism between them such that the occurring diagrams commute". Of course in this case the following lemma also holds.

Lemma 1.1.12. *A product of two objects, when it exists, is unique up to unique isomorphism.*

Proof. Let (X, π_1, π_2) and (Y, p_1, p_2) be products of $A, B \in \mathcal{C}_0$. Since X and Y are products there exist unique $v : X \to Y$ and unique $u : Y \to X$ such that:

$$p_1 \circ v = \pi_1, \ p_2 \circ v = \pi_2, \tag{1.1}$$

$$\pi_1 \circ u = p_1, \ \pi_2 \circ u = p_2. \tag{1.2}$$

[1]This specific behaviour can be interpreted in some contexts as deleting the object A, but such a statement will make sense after proposition 1.1.14.

There also exists a unique $f : X \to X$ such that $\pi_1 \circ f = \pi_1$ and $\pi_2 \circ f = \pi_2$ and since id_x functions exactly like f we have that $f = \mathrm{id}_X$. An exactly similar arguement holds for id_Y.

Proving that

$$p_1 \circ (v \circ u) = \pi_1,\ p_2 \circ (v \circ u) = \pi_2,$$
$$\pi_1 \circ (u \circ v) = \pi_1,\ \pi_2 \circ (u \circ v) = \pi_2,$$

which is obvious according to (1) and (2), shows that $u \circ v$ functions exactly like the f above, therefore $u \circ v = \mathrm{id}_X$. Similarly $v \circ u = \mathrm{id}_Y$, which concludes the proof. □

According to the above lemma, it is meaningful to talk about *the* product of two objects instead of a product. Therefore, we denote the product of A and B by $A \times B$. An immediate corollary to this lemma is that a product of two objects is commutative up to isomorphism, i.e. $A \times B \cong B \times A$.

Definition 1.1.13. *Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called* ***cartesian*** *if it is equipped with a terminal object $T \in \mathcal{C}_0$ and for every $A, B \in \mathcal{C}_0$, their product exists.*

An interesting property of cartesian categories is that we can duplicate objects in the following sense. For every object A in a cartesian category $\mathcal{C}$, there exists a unique morphism $d_A : A \to A \times A$ such that

$$\pi_1 \circ d_A = \mathrm{id}_A = \pi_2 \circ d_A,$$

where π_1, π_2 are the usual projections. This morphism is sometimes referred to as the **diagonal** and can be interpreted as duplicating A. Two final properties worth mentioning, before seeing what are the dual notions to product and terminal objects, are captured in the following propositions.

Proposition 1.1.14. *Let $\mathcal{C}$ be a cartesian category and let $T \in \mathcal{C}_0$ be terminal. Then for every object $A \in \mathcal{C}_0$*

$$A \times T \cong A \cong T \times A. \tag{1.3}$$

Proof. Let $A \in \mathcal{C}_0$. Using 1.1.12 it is enough to show that A is a product of T and A, because then it would be automatically isomorphic to both $A \times T$ and $T \times A$. To be more precise, we will prove that (A, id_A, e_A) is a product of A and T.

Let $C \in \mathcal{C}_0$ such that there exist $f : C \to A$ and $g : C \to T$. Then by terminality of T, we get $g = e_C$ and that for every $u : C \to A,\ e_A \circ u = e_C$. It is enough to show that $\mathrm{id}_A \circ f = f$ and that it is unique with this property. Both these statements are true, therefore the proposition holds. □

The above proposition tells us that a terminal object is essentially a unit for cartesian products.

Proposition 1.1.15. *Let $\mathcal{C}$ be a cartesian category. Then for every $A, B, C \in \mathcal{C}_0$:*

$$A \times (B \times C) \cong (A \times B) \times C. \tag{1.4}$$

The proof of the above proposition follows directly from the universal property of a (3-ary) product. It is a matter of induction to prove that a category having all binary products has also all n-ary products. Therefore, a cartesian category has all finite products, where in this case terminal objects are considered as nullary products.

The definion of the dual notions to terminal and product objects occurs, by turning the arrows in the other direction. This is the same as carrying out the same constructions in the opposite categories. In either case the resulting constructions are again instances of universal properties. By "dualizing" a terminal object we get an initial object, while by dualizing a product we get a co-product.

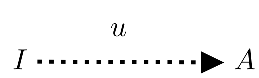
Definition 1.1.16. *Let $\mathcal{C}$ be a category.*

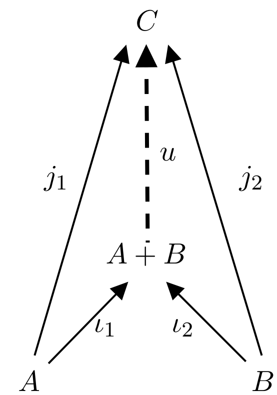
1. *An object $I \in \mathcal{C}_0$ is called* ***initial****, if for every object $A \in \mathcal{C}$ there exists a unique morphism $u_A : I \to A$.*

2. *Let $A, B \in \mathcal{C}_0$. An object $X \in \mathcal{C}_0$ is called a **coproduct** of A and B if it is equipped with two morphisms $\iota_1 : X \to A,\ \iota_2 : X \to B$ and for every other object $C \in \mathcal{C}_0$ equipped with maps $j_1 : A \to C,\ j_2 : B \to C$ there exists a unique morphism $u : X \to C$ such that:*

$$j_1 = u \circ \iota_1 \text{ and } j_2 = u \circ \iota_2.$$

The ι maps in the co-product definition are called **injections** and since duals to all the above lemmas hold, we may speak of *the* initial object of a category and *the* coproduct of two objects, denoted by $A + B$. The diagrammatic depiction of the above is given below.





Examples of terminal objects are the one element sets for **Set**, the zero dimensional vector space $\{0\}$, for $\mathbf{Vect}_{\mathbb{F}}$, the top element for a poset category and the one point space for **Top**. Examples of initial objects are the empty set $\emptyset$ for **Set**, the empty topological space in **Top**, the zero dimensional vector space $\{0\}$ for $\mathbf{Vect}_{\mathbb{F}}$ and the bottom element for a poset category. Examples of products are the cartesian product in **Set**, the product topological space in **Top**, the direct product for $\mathbf{Vect}_{\mathbb{F}}$ and the greatest lower bound (or meet) for a poset category. Finally a co-product in **Set** is the disjoint union of sets, the disjoint union topological space in **Top**, while a co-product in $\mathbf{Vect}_{\mathbb{F}}$ is the direct sum and, unsurprisingly, for a poset category the lower upper bound (or join) corresponds to co-product.

The dual notion to a cartesian category is a **cocartesian** category, which is a category containing an initial object and all the co-products of pairs of objects.

Very interesting notions occur when the above dual notions coincide.

Definition 1.1.17. *Let $\mathcal{C}$ be a category. An object $0 \in \mathcal{C}_0$ is called a **zero object**, if it is both terminal and initial in $\mathcal{C}$.*

Given a category with a zero object and two objects A, B, there exist unique morphisms $e_A : A \to 0$ and $u_A : 0 \to B$, since 0 is both initial and terminal. The composite $0_{A,B} = u_A \circ e_A : A \to B$ is called a **zero morphism** and it is the unique morphism that factors through the zero object. There are also categories, such as $\mathbf{Vect}_{\mathbb{F}}$, in which products and coproducts coincide, but this situation is more complicated than it seems at first, so for more on zero objects and biproducts, we refer to the appendix 1.6.

Finally, to see how to derive a product of morphisms rather than just objects, let $\mathcal{C}$ be a category, $A, A', B, B' \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, A'),\ g \in \mathcal{C}(B, B')$. Then by the universal property of $A' \times B'$ and since $f \circ \pi_1 : A \times B \to A',\ g \circ \pi_2 : A \times B \to B'$ there exists a unique map $u : A \times B \to A' \times B'$ such that

$$\pi_1' \circ u = f \circ \pi_1 \text{ and } \pi_2' \circ u = g \circ \pi_2.$$

We call u the product morphism of f and g and write $u = f \times g$. This way we have created a function $\mathcal{C}(A, A') \times \mathcal{C}(B, B') \to \mathcal{C}(A \times B, A' \times B')$. This is actually something more than a function. It is part of a functor called a **product functor**. Of course the above carried out in the opposite category yields the coproduct of morphisms, resulting in a **corpoduct functor**. Finally, when the above two coincide and interact in a nice way, to be defined in a later section, we may speak of a **biproduct functor**.

1.2 Functors

Since category theory is about morphisms between objects, one might wonder what is the corresponding notion for categories. The answer to that question is the concept of a functor.

Covariant and contravariant functors

Definition 1.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{C}$ and $\mathcal{D}$ consists of the following data:*

- *a function $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, such that for every $A \in \mathcal{C}_0,\ A \mapsto FA$,*
- *for every $A, B \in \mathcal{C}_0$ a function $F_1 : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$*

satisfying the following conditions:

- *$F(\mathrm{id}_A) = \mathrm{id}_{FA}$, for every $A \in \mathcal{C}_0$ and*
- *$F(g \circ f) = F(g) \circ F(f)$, for every $A, B, C \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$.*

Proposition 1.2.2. *Let $\mathcal{C}, \mathcal{D}$ be categories, $A, B \in \mathcal{C}_0$, $f : A \to B$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If f is an isomorphism then $Ff : FA \to FB$ is an isomorphism.*

Proof. Let f be an isomorphism. Then there exists a unique $f^{-1} : FB \to FA$ such that

$$f \circ f^{-1} = \mathrm{id}_B \text{ and } f^{-1} \circ f = \mathrm{id}_A.$$

This is equivalent to

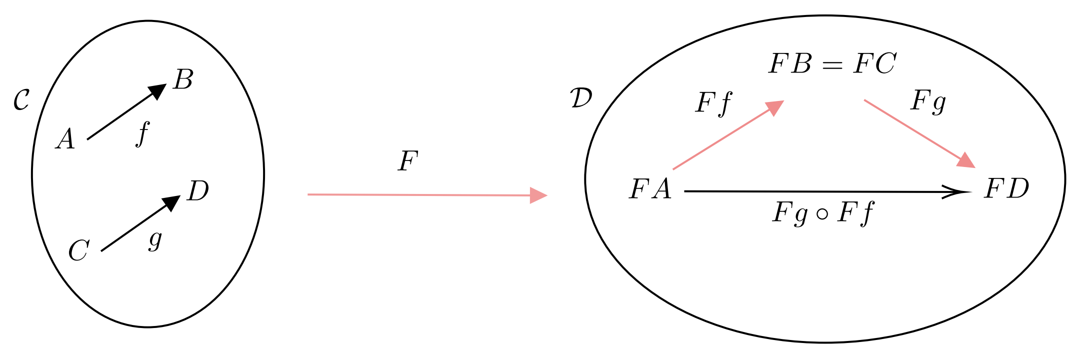
$$Ff \circ F(f^{-1}) = \mathrm{id}_F B \text{ and } F(f^{-1}) \circ Ff = \mathrm{id}_F A,$$

which by uniqueness of inverses implies that $F(f^{-1}) = (Ff)^{-1}$ and that Ff is an isomorphism, □

Remark 1.2.3. The above definition shows that, by preserving the identities and compositions of morphisms, a functor is the analogue of structure preserving maps between categories. There is also an identity functor for every category $\mathcal{C}$, denoted by $\mathbb{1}_\mathcal{C}$ and defined as $\mathbb{1}_\mathcal{C}(f : A \to B) = f : A \to B$.

In accordance to algebraic theories, the composition of two functors is defined in the obvious way and is again a functor. Moreover for every $F : \mathcal{C} \to \mathcal{D}$, $F \circ \mathbb{1}_\mathcal{C} = F = \mathbb{1}_\mathcal{D} \circ F$. This poses a question about the nature of the collection of all categories. The answer will be touched upon in the end of this subsection.

Contrary to algebraic theories though, the image of a category under a functor is not generally a subcategory of the target category. Such a situation is depicted in the following diagrams.



The image of F fails to be a category since $Fg \circ Ff \notin F\mathcal{C}$, but $Fg \circ Ff \in \mathcal{D}$. Sometimes in the literature though, $\mathrm{Im}(F)$ is considered to be the smallest subcategory of $\mathcal{D}$, through which F factors.

Example 1.2.4. 1. Categories often describe structured sets like groups, vector spaces, or topological spaces and structure preserving maps between them. The fact that every structured set is a set, allows one to form a functor from a concrete category to **Set**, by mapping a structured set to its underlying set and every structure preserving map to the underlying function. This functor is called **forgetful** and is denoted by U. Of course, generalising a bit, a forgetful functor need not "forget" all the structure of a set. For example one can form $U : \mathbf{Vect}_{\mathbb{F}} \to \mathbf{Ab}$, which sends every vector space $(V, +, \cdot)$ to the underlying abelian group $(V, +)$ and a linear map to the underlying group homomorphism.

2. Going the other direction, sets can be equipped with a structure. To achieve this, one creates formal elements needed so that the new set can bear the structure. This is more generally known as a **free** construction. It is called free because it does not impose any restrictions other than those needed to constitute the structure to be created. In other words it creates the smallest such structure. This is an example of a functor $F : \mathbf{Set} \to \mathcal{C}$, where $\mathcal{C}$ is any category of algebraic structures. Functoriality follows from the fact that functions are mapped to their uniquely defined extensions that preserve the structure. These kinds of functors are generally called **free** functors.

   An example of a free functor is given by the free vector space on a set construction. In linear algebra and functional analysis, one may create a vector space FS given a set S, by considering (finite) formal linear combinations of the elements of S. That is $F(S) = \{\Sigma\lambda_x x | x \in S,\ \lambda_x \in \mathbb{F} \text{ and } \lambda_x \neq 0 \text{ for finitely many } x\}$, where $\mathbb{F}$ is the ground field. Furthermore, it is a standard result that every function between sets uniquely determines a linear transformations between the free vector spaces on these sets and this condition guarantees functoriality of F.

3. If **Cat** is the category of all small categories, then there is a functor $(-)_0 : \mathbf{Cat} \to \mathbf{Set}$ sending every small category $\mathcal{C}$ to its set of objects $\mathcal{C}_0$ and every functor, F, between such categories to the function F_0 between the corresponding sets of objects. Analogously, there is a functor $\mathrm{Ar} : \mathbf{Cat} \to \mathbf{Set}$ sending a small category $\mathcal{C}$ to its set of morphisms $\mathrm{Ar}\mathcal{C}$, and a functor $F : \mathcal{C} \to \mathcal{D}$ to the function F_1 sending the (disjoint) union of $\mathcal{C}$'s hom-sets to the corresponding one of $\mathcal{D}$.

4. Let $\mathcal{C}, \mathcal{D}$ be categories and $A \in \mathcal{D}_0$. There exists a functor $\Delta_A : \mathcal{C} \to \mathcal{D}$ such that for every $X, Y \in \mathcal{C}_0$ and $f \in \mathcal{C}(X, Y)$, $\Delta_A(X) = A$ and $\Delta_A(f) = \mathrm{id}_A$. Such a functor is called a **constant functor**.

5. Another interesting example is a functor, F, from a category, $\mathbf{B}G$, corresponding to a group G, to **Set**. This functor is obliged to send the one object, $*$, of $\mathcal{C}$ to a set $S = F(*)$, but also to be a homomorphism from $G \cong \mathbf{B}G(*, *)$ to the automorphism group of S. Therefore we may identify such a functor with a (left) group action on a set. Similarly, considering a functor from $\mathbf{B}G$ to $\mathbf{Vect}_{\mathbb{F}}$ we get a representation of this group on the vector space $F(*)$.

6. A functor between one object categories $\mathcal{C}$ and $\mathcal{D}$ is a monoid homomorphism. Similarly, if $\mathcal{C}$ and $\mathcal{D}$ correspond to groups, then a functor between them is a group homomorphism.

7. If $\mathcal{C}$ and $\mathcal{D}$ are poset or preorder categories a functor F between them is a monotone map. That is because for every $A, B \in \mathcal{C}_0$:

$$\begin{aligned} A \leq B &\Leftrightarrow \mathcal{C}(A, B) \neq \emptyset \\ &\Rightarrow \mathcal{D}(FA, FB) \neq \emptyset \\ &\Leftrightarrow FA \leq FB. \end{aligned}$$

In order theory there is a dual notion to a monotone map, that of order reversing or antitone map. Let $\mathcal{C}$ and $\mathcal{D}$ be poset categories. An antitone map between them is a functor, $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Generalising this notion to arbitrary categories we get the following definition.

Definition 1.2.5. *Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$.*

According to the above definition, given categories $\mathcal{C}$ and $\mathcal{D}$ and a contravariant functor between them, we see that the composition of morphisms in $\mathcal{C}$ gets reversed, (as does the inequality in the poset categories case). To be precise, for every composable f, g in $\mathcal{C}$ we get:

$$F(g \circ f) = F(f) \circ F(g). \tag{1.5}$$

In this light, namely the reversion of arrows, the construction of the opposite category can be seen as a (trivial) contravariant functor from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$. Sometimes the word **covariant** is used to characterize ordinary functors and to emphasize the distinction between them and the contravariant ones.

Example 1.2.6.

1. In accordance with the "covariant" examples above a contravariant functor from a monoid or a group category, $\mathbf{B}G$, to **Set** is essentially a right G-action.
2. A contravariant functor between preorder or poset categories is an antitone map.
3. There is a contravariant functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ called the powerset functor, defined as follows. For every $X, Y \in \mathbf{Set}$ and $f : X \to Y$ we define $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ such that $\mathcal{P}(f) := f^{-1}$.

   This construction can be transferred unaltered to **Top** since the fact that f^{-1} sends open sets to open sets is the definition of continuity of f. Thus, we get the topology functor $\mathcal{T}$, attaching to every topological space $(X, \mathcal{T}(X))$ its topology $\mathcal{T}(X)$ and to any continuous function $f : X \to Y$ its preimage $\mathcal{T}(f) = f^{-1} : \mathcal{T}(Y) \to \mathcal{T}(X)$.

Very interesting and useful examples of both types of functors are given by carrying out the following construction.

Example 1.2.7. Let $\mathcal{C}$ be a (locally small) category and $A \in \mathcal{C}_0$.

1. There is a functor $H_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that

   - $H_A(X) := \mathcal{C}(X, A)$ for every $X \in \mathcal{C}_0$, and
   - for every $f : X \to Y$, $H_A(f) := - \circ f$, i.e. for every $u \in \mathcal{C}(Y, A)$, $H_A(f)(u) := u \circ f$, also known as *pre-composition.*

   H_A is a contravariant functor since for every $u \in \mathcal{C}(Z, A)$ and for every $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$:

$$\begin{aligned} H_A(g \circ f)(u) &= u \circ g \circ f \\ &= (- \circ f) \circ (- \circ g)(u) \\ &= H_A(f) \circ H_A(g)(u) \end{aligned}$$

   Figuratively, this functor sends the objects of $\mathcal{C}$ to the multitude of ways they can be transformed to A.

2. There is, also a similar functor $H^A : \mathcal{C} \to \mathbf{Set}$ such that

   - $H^A(X) := \mathcal{C}(A, X)$ for every $X \in \mathcal{C}_0$, and
   - for every $f : X \to Y$, $H^A(f) := f \circ -$, i.e. for every $u \in \mathcal{C}(X, A)$, $H^A(f)(u) := f \circ u$, also known as *post-composition.*

   H^A is a covariant functor since for every $u \in \mathcal{C}(A, X)$ and for every $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$:

$$\begin{aligned} H^A(g \circ f)(u) &= g \circ f \circ u \\ &= H^A(g)(f \circ u) \\ &= H^A(g) \circ H^A(f)(u) \end{aligned}$$

   Figuratively, H^A is the category $\mathcal{C}$ from the perspective of A.

3. Combining the above two examples, there is a functor $Hom_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \textbf{Set}$, called the **hom-functor**, defined as follows:

   - for every $(X, Y) \in (\mathcal{C}^{\text{op}} \times \mathcal{C})_0$, $Hom_{\mathcal{C}}(X, Y) = \mathcal{C}(X, Y)$ and
   - for every $(f^{\text{op}}, g) : (X, Y) \to (X', Y')$, $Hom_{\mathcal{C}}(f^{\text{op}}, g) := g \circ - \circ f$, i.e. for every $u \in \mathcal{C}(X, Y)$, $Hom_{\mathcal{C}}(f^{\text{op}}, g)(u) = g \circ u \circ f$.

   This functor is contravariant in the first argument and covariant in the second one.

In general contravariant functors to **Set**, like H_A, are called **presheaves**, while covariant ones, like H^A, are called **co-presheaves**. Functors, like the hom-functor, from a product category, $\mathcal{C} \times \mathcal{D}$, are sometimes called **bifunctors**, but this naming is not of much use since a bifunctor is just a functor from a product category. Either way, we can fix one of the arguments of a bifunctor by picking a certain object, A, and its identity morphism, id_A. This way we get an ordinary functor from one of the two categories, either $\mathcal{C}$ or $\mathcal{D}$. For example

$$H_A = \mathcal{C}(-, A) \text{ and } H^A := \mathcal{C}(A, -).$$

If $\mathcal{C}$ is a cartesian category, the product construction on objects and morphisms is easily seen to be a (bi)functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ taking every $A, B \in \mathcal{C}$ to $A \times B$ and every $(f, g) : (A, B) \to (A', B')$ to $f \times g$. This is a functor because of the uniqueness condition, in the universal property of the product, imposing that $(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g)$.

For every functor $F : \mathcal{C} \to \mathcal{D}$ there is an associated functor, $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$, called the **opposite functor**, which is defined as follows: $F^{\text{op}}A := FA$ and $F^{\text{op}}f^{\text{op}} := (Ff)^{\text{op}}$, for every $A, B \in \mathcal{C}^{\text{op}}$ and $f \in \mathcal{C}(A, B)$.

Category theory is a generalisation of set theory in many ways. One way in which categories generalise sets occurs by viewing a category as a set of objects together with a set of morphisms. Then a category equipped only with identity morphisms is essentially a set. Such categories are called **discrete** and in the case where their collection of objects is of cardinality $N \in \mathbb{N}$, they will be denoted by $\mathcal{D}_N$.

From this perspective, a functor between any two categories is a generalisation of a function. The notions of injection and surjection can effortlessly be transferred to categories by specifying injectivity and surjectivity on objects. On the other hand, there are also other similar notions which don't appear in set theory, due to the absence of morphisms.

Definition 1.2.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is*

1. ***faithfull**, if for every $f, g \in \mathcal{C}(A, B)$, $Ff = Fg$ implies $f = g$ (injective on homsets),*
2. ***full**, if for every $g \in \mathcal{D}(FA, FB)$ there exists $f \in \mathcal{C}(A, B)$ such that $Ff = g$ (surjective on homsets),*
3. ***essentially surjective on objects** if for every $D \in \mathcal{D}_0$ there exists an $A \in \mathcal{C}$ such that $FA \cong D$.*
4. *an **isomorphism** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that:*

$$G \circ F = \mathbb{1}_{\mathcal{C}} \text{ and } F \circ G = \mathbb{1}_{\mathcal{D}}.$$

To conclude this section, we mention that the collection of all small categories and functors between them is a category denoted by **Cat**. This is achieved by observing that the composition of functors is again a functor. For someone who doesn't care about foundational issues or embraces some other set theory than ZFC, there is also the category of all locally small categories and morphisms between them, denoted by **CAT**[2]. The category of categories equipped with the product of categories and with the one object preorder category **1** as a terminal object, is cartesian.

Finally, if two categories, $\mathcal{C}$ and $\mathcal{D}$, are picked, then the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ is denoted by $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$, rather than $\textbf{CAT}(\mathcal{C}, \mathcal{D})$. Actually, $[\mathcal{C}, \mathcal{D}]$ will turn out to be a category on its own.

[2]This does not create any paradox, since **CAT** is not locally small

Diagrams and generalised elements

Let A be an object of **Set** and $\{*\}$ a one element set. Then there is a bijection between elements of A and functions from $\{*\}$ to A. Using this we might write for every $a \in A$, $a : \{*\} \to A$ exactly when $a(*) = a$. Rephrasing, we have that for every set A

$$\mathbf{Set}(\{*\}, A) \cong A. \tag{1.6}$$

Abusing our notation a bit, we get that for every set B and every $f : A \to B$

$$f \circ a = f(a).$$

Thus we see that in set theory the "thing-in-itself" side of an object and its interaction-with-other-objects side are in balance.

Observing that $\{*\}$ is terminal in **Set**, we can generalise this to an arbitrary category $\mathcal{C}$ with a terminal object T. Given $A \in \mathcal{C}_0$ we say that a is a **global element** or a **point** of A and write $a \in A$, if $a \in \mathcal{C}(T, A)$. To generalise even further, we don't need to restrict ourselves to morphisms from terminal objects to get generalised elements, as is seen in the following definition.

Definition 1.2.9. *Let $\mathcal{C}$ be a category and $T, A \in \mathcal{C}_0$. We say that p is a* ***T-shaped element of*** *A or a* ***generalised element of shape T*** *and write $p \in_T A$, if*

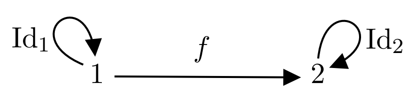
$$p : T \to A.$$

An equivalent statement to the above, in the case of locally small categories, would be that $p \in_T A$ if and only if $p \in H^T(A)$.

Remark 1.2.10. Duallity implies that there is a dual notion to generalised element. This could be called a *generalised coelement*. Formally a generalised coelement of A is an element $q \in H_T(A)$.

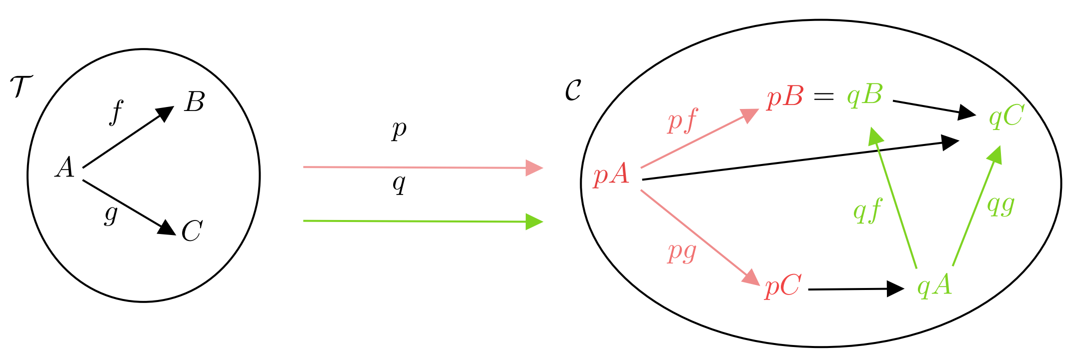
Perhaps, the best example of the above definition is the category of small categories **Cat** (and also **CAT**). Given a small category $\mathcal{C}$ and a one object category $\mathcal{T}$, we can identify $\mathcal{C}_0$ with $\mathbf{Cat}(\mathcal{T}, \mathcal{C})$, in the exact same way we did for **Set**. Note that a one object category is terminal in both **Cat** and **CAT**.

The crucial difference between **Set** and **Cat**, i.e. the fact that not all categories are discrete, makes the above definition meaningful. Given a small category, $\mathcal{C}$, and the poset category **2**, as defined below, we see that the set of all morphisms of $\mathcal{C}$ is actually $[\mathbf{2}, \mathcal{C}]$.



Poset category **2**

The following example is meant to illustrate the above point and to justify the word "shape" used in the definition of generalised element.



In the above picture we see two images of $\mathcal{T}$ inside $\mathcal{C}$ under the functors p and q. Their images are easily seen to have the same "shape" as $\mathcal{T}$. Of course there are degenerate cases where p or q might not be faithful or even injective on objects, but this is not a problem.

Using the notion of generalised elements we can reformulate the notion of monomorphism, as in the following proposition.

Proposition 1.2.11. *Let $\mathcal{C}$ be a category, and $f \in \mathcal{C}(A, B)$. Then f is a monomorphism if and only if*

$$f(a) = f(b) \Rightarrow a = b,$$

for every $T \in \mathcal{C}_0$ and for every $a, b \in_T A$.

Proof. Let $T \in \mathcal{C}_0$ and $a, b \in_T A$. Applying the definitions of monomorphism and of generalised elements we get:

$$f(a) = f(b) \Leftrightarrow f \circ a = f \circ b \Rightarrow a = b.$$

□

An example that illustrates the beauty of the generalised element idea is the following: Consider a cartesian category $\mathcal{C}$. For every two objects $A, B \in \mathcal{C}_0$, there exists an object $A \times B \in \mathcal{C}_0$, which is their product, equipped with two projections $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$. For $T \in \mathcal{C}_0$, we can consider T-shaped elements $a \in_T A, b \in_T B$. By the universal property of the product, there exists a unique T-shaped element $(a, b) \in_T A \times B$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Thus, we see that the T-shaped elements of a binary product object are pairs of T-shaped elements of its constituent objects.

Recalling the example 1.2.7, when referring to H_A and H^T, we might speak of a functor of generalized elements of A, since H_A is a functor and for every object $T \in \mathcal{C}_0$, $H_A(T)$ is the set of T-shaped elements of A. Dually, given any $T \in \mathcal{C}_0$, we can speak of a functor of T-shaped elements of objects of $\mathcal{C}$, since for every $A \in \mathcal{C}_0$, $H^T(A)$ is the set of T-shaped elements of A.

Using the above functors we can reformulate the universal property of the product as follows: *"There exists a natural isomorphism*

$$H_A(T) \times H_B(T) \cong H_{A \times B}(T),$$

for every T". But, what does "natural" mean?

1.3 Natural transformations and comma categories

In the last section we saw that functors in $[\mathcal{C}, \mathcal{D}]$ can either be seen as "constructions" on $\mathcal{D}$, like the representations of a group or a free structure, or as diagrams of shape $\mathcal{C}$ inside $\mathcal{D}$. Natural transformations offer a way to compare different constructions or to move a specified diagram inside $\mathcal{D}$ using only the tools (morphisms) provided by $\mathcal{D}$. In principle there should be no restriction to comparing diagrams/constructions, even if they originate from different categories. Comma categories are ubiquitous among category theory and offer a formulation of such an idea.

Natural Transformations

We now give the definition of natural transformations and some examples.

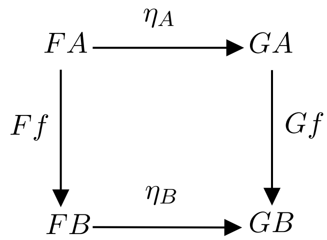
Definition 1.3.1. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between categories. A natural transformation $\alpha : F \Rightarrow G$ is a family of morphisms of $\mathcal{D}$ indexed by objects of $\mathcal{C}$, i.e $(\alpha_A)_{A \in \mathcal{C}_0}$, where*

$$\alpha_A : FA \to GA \text{ such that}$$

$$\alpha_B \circ Ff = Gf \circ \alpha_A$$

for all $A, B \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, B)$.

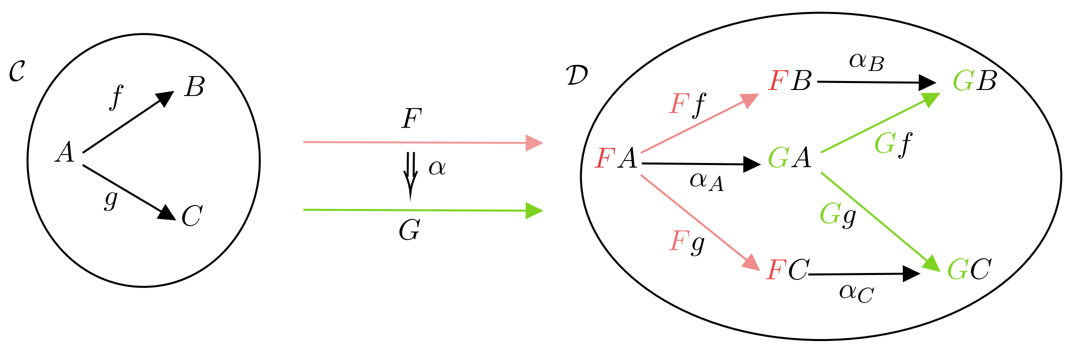
We write $\alpha = (\alpha_A)_{A \in \mathcal{C}_0}$, every α_A is called a component of α and the condition these components are required to obey is called **naturality condition** or just **naturality**. The naturality condition is better understood as the following commutative diagram:



where we see that the image of F, as a whole, is transformed to the image of G using, where applicable and in a coherent way, morphisms from $\mathcal{D}$. Such a diagram is sometimes called a **naturality square**.

When we have a natural transformation $\alpha : F \Rightarrow G$, between functors with two variables $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, we get two families of natural transformations, $\alpha_{(A,-)} : F(A,-) \Rightarrow G(A,-)$ and $\alpha_{(-,B)} : F(-,B) \Rightarrow G(-,B)$, indexed by the objects $A \in \mathcal{C}_0$ and $B \in \mathcal{D}$.

Before giving some examples, we proceed by giving a visualisation.



The above diagrams commute since every square is a naturality square.

Example 1.3.2.

1. Let M be a monoid and $\mathbf{B}M$ its corresponding one object category. We have already seen that a functor $F : \mathbf{B}M \to \mathbf{Set}$ is essentially an $M-$set, or a set equipped with an $M-$action. A natural transformation, η, between two M-sets consists of a morphism $\eta_* : F(*) \to G(*)$, where $*$ is the one object of $\mathbf{B}M$. Let $m : * \to *$, then the naturality square for η_* is equivalent to:

$$\eta_* \circ F(m) = G(m) \circ \eta_*,$$

meaning that η_* is an M-equivariant map.

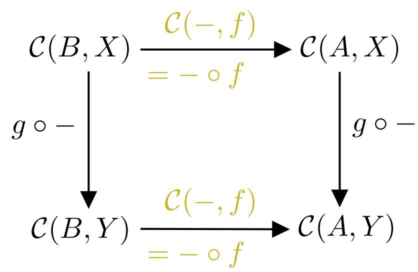
2. In the case that $M = G$ is a group, such a natural transformation is again a G-equivariant map. Furthermore if $F, G : \mathbf{B}G \to \mathbf{Vect}_{\mathbb{F}}$, and $\eta : F \Rightarrow G$, then η is an intertwining operator between the representations of G on $F(*)$ and $G(*)$.

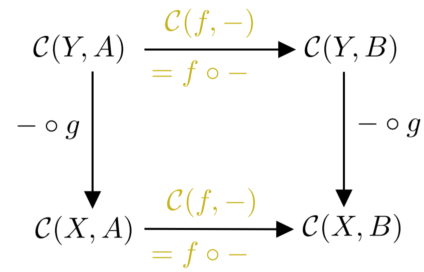
3. Let P, Q be preorder categories and $F, G : P \to Q$ be functors. Then a natural transformation $\eta : F \Rightarrow G$ consists of $\eta_A : F(A) \to G(A)$ for every $A \in P_0$ or equivalently $F(A) \leq G(B)$. Moreover the naturality condition is automatically satisfied, since every diagram in a preorder category commutes. Therefore there exists at most one such natural transformation and it exists if and only if $F(A) \leq G(A)$ for every $A \in P_0$. Note that if P and Q are small, then the set $[P, Q]$ becomes a preorder.

4. Recalling the generalised elements functor of example 1.2.7 we can define the following natural transformations. For $f : A \to B$ a morphism of a locally small category $\mathcal{C}$, there are two natural transformations, $\mathcal{C}(f,-) : H^B \Rightarrow H^A$ and $\mathcal{C}(-,f) : \mathcal{C}(-,A) \Rightarrow \mathcal{C}(-,B)$ associated to f. The components of these natural transformations at $X \in \mathcal{C}_0$ are defined as follows:

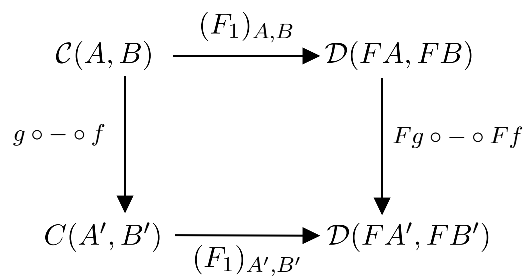
- $\mathcal{C}(f, X) := (- \circ f : \mathcal{C}(B, X) \to \mathcal{C}(A, X))$, such that for every $h \in \mathcal{C}(B, X)$, $h \mapsto h \circ f$.
- $\mathcal{C}(X, f) := (f \circ - : \mathcal{C}(X, A) \to \mathcal{C}(X, B))$, such that for every $h \in \mathcal{C}(X, A)$, $h \mapsto f \circ h$.

Naturality follows from the commutativity of the diagrams below for every $g \in \mathcal{C}(X, Y)$.





5. A conceptually interesting example, is the case of a functor. Functors can be seen as natural transformations in the following way. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Observe that for $\mathcal{C}$ and $\mathcal{D}$ locally small the functions F_1 associated to F provide functions between the homsets $\mathcal{C}(A, B)$ and $\mathcal{D}(FA, FB)$, for every $A, B \in \mathcal{C}_0$. So the functoriality conditions imply these functions, namely $(F_1)_{A,B}$, are natural in A, B, or equivalently that the following diagram is commutative for every $(f^{\mathrm{op}}, g) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}((A, B), (A', B'))$:



Conversely, demanding that the above diagram is commutative, one retrieves the functoriality conditions either by setting g or f or both equal to the identity morphism.

Remark 1.3.3. A natural transformation $\alpha : F \Rightarrow G$, betwen functors $F, G : \mathcal{C} \to \mathcal{D}$, consists of an assignement of morphisms of $\mathcal{D}$ to every object of $\mathcal{C}$, that satisfy the naturality condition. We may create a characterisation of natural transformations as assignements of morphisms of $\mathcal{D}$ to morphisms of $\mathcal{C}$. Concretely, let $(\alpha_f)_{f \in Mor\mathcal{C}}$ be a family of morphisms in $\mathcal{D}$ indexed by morphisms in $\mathcal{C}$ such that for every $f \in \mathcal{C}(A, B)$

$$\alpha_f = \alpha_{\mathrm{id}_B} \circ Ff = Gf \circ \alpha_{\mathrm{id}_A}.$$

This characterisation can be easily seen to be equivalent to the previous definition through the following equation:

$$\alpha_A = \alpha_{\mathrm{id}_A}.$$

We will revisit this characterisation in the appendix when talking about dinatural transformations.

Natural transformations are kinds of morphisms between functors. Therefore a reasonable question is whether they are part of some sort of category. To answer this question we define the following.

Definition 1.3.4. *Let $\mathcal{C}$, $\mathcal{D}$ be categories.*

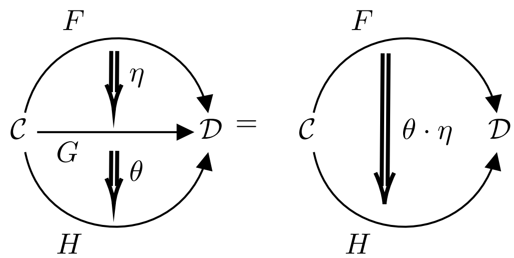
- *Let $F \in [\mathcal{C}, \mathcal{D}]$. The natural transformation* $\mathrm{Id}_F : F \Rightarrow F$ *defined by:*

$$(\mathrm{Id}_F)_A := \mathrm{id}_{FA},$$

for every $A \in \mathcal{C}_0$, is called the ***identity natural transformation on*** *F.*

- *Let* $F, G, H \in [\mathcal{C}, \mathcal{D}]$ *and* $\eta : F \Rightarrow G,\ \theta : G \Rightarrow H$. *We define a natural transformation* $\theta \cdot \eta$ *with components:*
$$(\theta \cdot \eta)_A := \theta_A \circ \eta_A,$$
for every $A \in \mathcal{C}_0$.

The composition $''\cdot''$ defined above is sometimes called **vertical composition**. The reason for this terminology has its roots in the following diagram:



We now show that based on the vertical composition we can create a hom-category instead of just a (hom-)set of functors.

Proposition 1.3.5. *Let* $\mathcal{C}, \mathcal{D}$ *be categories and* $F, G, H, L \in [\mathcal{C}, \mathcal{D}]$. *Then:*

1. Id_F *is a natural transformation,*
2. *for every* $\eta : F \Rightarrow G,\ \theta : G \Rightarrow H$, *their composition* $\theta \cdot \eta : F \Rightarrow H$ *is a natural transformation.*
3. *for every* $\eta : F \Rightarrow G,\ \eta \cdot \mathrm{Id}_F = \eta = \ \mathrm{Id}_G \cdot \eta$,
4. *for every* $\eta : F \Rightarrow G,\ \theta : G \Rightarrow H,\ \beta : H \Rightarrow L$, *we have*
$$(\beta \cdot \theta) \cdot \eta = \beta \cdot (\theta \cdot \eta).$$

Proof. We only need to prove the statements component-wise.

1. Let $f \in \mathcal{C}(A, B)$. Then
$$\begin{aligned}(\mathrm{Id}_F)_B \circ Ff &= \mathrm{id}_{FB} \circ Ff \\ &= Ff \\ &= Ff \circ \mathrm{id}_{FA} \\ &= Ff \circ (\mathrm{Id}_F)_A,\end{aligned}$$
which is the naturality condition for Id_F.
2. Let $f \in \mathcal{C}(A, B)$. Then
$$\begin{aligned}(\theta \cdot \eta)_B \circ Ff &= \theta_B \circ \eta_B \circ Ff \\ &= \theta_B \circ Gf \circ \eta_A \text{ (by naturality of } \eta) \\ &= Hf \circ \theta_A \circ \eta_A \text{ (by naturality of } \theta) \\ &= Hf \circ (\theta \cdot \eta)_A,\end{aligned}$$
which is the naturality condition for $\theta \cdot \eta$.

3. For every $A \in \mathcal{C}_0$ we have:

$$\begin{aligned}(\eta \cdot \mathrm{Id}_F)_A &= \eta_A \circ \mathrm{id}_{FA} \\ &= \eta_A \\ &= \mathrm{id}_{GA} \circ \eta_A \\ &= (\mathrm{Id}_G \cdot \eta)_A\end{aligned}$$

4. Let $A \in \mathcal{C}_0$. Then

$$\begin{aligned}[(\beta \cdot \theta) \cdot \eta]_A &= (\beta \cdot \theta)_A \circ \eta_A \\ &= (\beta_A \circ \theta_A) \circ \eta_A \\ &= \beta_A \circ (\theta_A \circ \eta_A) \\ &= \beta_A \circ (\theta \cdot \eta)_A \\ &= [\beta \cdot (\theta \cdot \eta)]_A\end{aligned}$$

which is equivalent to $(\beta \cdot \theta) \cdot \eta = \beta \cdot (\theta \cdot \eta)$.

□

Based on the above proposition, $[\mathcal{C}, \mathcal{D}]$ is a category with functors as objects and natural transformations as morphisms. The first two properties in the above definition guarantee the existence of identity morphisms and closure of $[\mathcal{C}, \mathcal{D}]$ under composition. The last two are the axioms of a category about identity and associativity laws. Such categories are called **functor categories** and are sometimes denoted by $\mathcal{D}^{\mathcal{C}}$.

Example 1.3.6.

1. Let $\mathbf{B}G$ be a group as a one object category. Then $[\mathbf{B}G, \mathbf{Vect}_{\mathbb{F}}]$ has as objects functors corresponding to representations of G and as morphisms natural transformations corresponding to intertwining operators between representations. Therefore $[\mathbf{B}G, \mathbf{Vect}_{\mathbb{F}}]$ is the category of representations of G.

2. Let $\mathcal{C}$ be an arbitrary category and $\mathbf{2}$ the order category with two objects, $1, 2$, and a morphism, $f : 1 \to 2$ between them. Then $[\mathbf{2}, \mathcal{C}]$ is the category whose objects are functors corresponding to morphisms of $\mathcal{C}$ and whose morphisms are natural transformations corresponding to pairs of morphisms, η_1, η_2 of $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} F1 & \xrightarrow{\eta_1} & G1 \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ F2 & \xrightarrow{\eta_2} & G2 \end{array}$$

Such functor categories are called **Arrow Categories** and, in a sense, their objects are morphisms and their morphisms are commutative diagrams.

3. Let $\mathcal{C}$ be a category and $\mathbf{2}_{\mathcal{D}}$ be the discrete two object category. Then $[\mathbf{2}_{\mathcal{D}}, \mathcal{C}]$ is essentially the product category $\mathcal{C} \times \mathcal{C}$. We will show this in two steps using the universal property of the product in **CAT**. Firstly, we equip $[\mathbf{2}_{\mathcal{D}}, \mathcal{C}]$ with the functors: $\mathrm{Ev}_i : [\mathbf{2}_{\mathcal{D}}, \mathcal{C}] \to \mathcal{C}$, for $i = 1, 2$, satisfying

$$\mathrm{Ev}_i(\alpha : F \Rightarrow G) = \alpha_i : F(1) \to G(1),$$

for every $F, G : 2_{\mathcal{D}} \to \mathcal{C}$ and $\alpha : F \Rightarrow G$. Secondly, we show that $([\mathbf{2}_{\mathcal{D}}, \mathcal{C}], \mathrm{Ev}_1, \mathrm{Ev}_2)$ satisfies the universal property of the product. So let X be a (locally small) category and $Q_i : X \to \mathcal{C}$ be functors for $i = 1, 2$. Consider the assignements $U_0 : X_0 \to [\mathbf{2}_{\mathcal{D}}, \mathcal{C}]_0$ and for every $f \in X(A, B)$, $U : X(A, B) \to [\mathbf{2}_{\mathcal{D}}, \mathcal{C}](U(A), U(B))$, defined by:

- for every $A \in X_0$, $U_0(A) : 2 \to \mathcal{C}$ is the functor $U(A)(i) = Q_i(A)$ and $U(A)(\mathrm{id}_i) = \mathrm{id}_{Q_i(A)}$, for $i = 1, 2$
- $U(f)(i) = Q_i(f)$, for $i = 1, 2$.

It is easy to check that the above assignments constitute a functor from X to $[\mathbf{2}_{\mathcal{D}}, \mathcal{C}]$, such that $\mathrm{Ev}_i \circ U = Q_i$, for $i = 1, 2$. We now show the uniqueness of U. Let $V : X \to [\mathbf{2}_{\mathcal{D}}, \mathcal{C}]$ be a functor such that $\mathrm{Ev}_i \circ V = Q_i$, for $i = 1, 2$. Then for every $A \in X_0$ and $i \in @_{\mathcal{D}}$:

$$
\begin{aligned}
V(A)(i) &= (\mathrm{Ev}_i \circ V)(A) = Q_i(A) = U(A)(i) \\
V(A)(\mathrm{id}_i) &= \mathrm{id}_{V(A)(i)} = \mathrm{id}_{Q_i(A)} = u(A)(\mathrm{id}_i) \text{ and} \\
V(f)(i) &= Q_i(f) = U(f)(i)
\end{aligned}
$$

which means that $U = V$. Therefore, U exists and is unique and thus, $([\mathbf{2}_{\mathcal{D}}, \mathcal{C}], \mathrm{Ev}_1, \mathrm{Ev}_2)$ satisfies the universal property of the product $\mathcal{C} \times \mathcal{C}$.

4. Let $\mathcal{C}$ be a cartesian category. Since the product operation is a functor it is easy to check, using the universal properties of products and terminals, that the following diagrams commute,

$$
\begin{array}{ccc}
X & \xrightarrow{d_X} & X \times X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times f} \\
Y & \xrightarrow[d_Y]{} & Y \times Y
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{e_X} & \Delta_T(X) = T \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Delta_T(f) = \mathrm{id}_T} \\
Y & \xrightarrow[e_Y]{} & \Delta_T(Y) = T
\end{array}
$$

where T is terminal in $\mathcal{C}$, d_X, d_Y are the duplication morphisms and Δ_T stands for the constant functor. Since the above diagrams are commutative squares, we can conclude that $d : \mathbb{1}_{\mathcal{C}} \Rightarrow - \times - \circ d_{\mathcal{C}}$ and $e : \mathbb{1}_{\mathcal{C}} \Rightarrow \Delta_T$, where $d := (d_X)_{X \in \mathcal{C}_0}$ and $e := (e_X)_{X \in \mathcal{C}_0}$, are natural transformations. One interesting thing to note is that $e_T = \mathrm{id}_T$ by terminality of T. These natural transformations are called **uniform copying** and **uniform deleting** respectively, and they play an important role characterising cartesian categories among "monoidal" ones.

5. Let $\mathcal{C}$ be a category, then $[\mathcal{C}, \mathbf{Set}]$ is its **copresheaf category** and $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is its **presheaf category**.

Since functor categories are categories, it is sensible to wonder to what the general categorical definitions amount to, in this context.

Definition 1.3.7. *Let $F, G \in [\mathcal{C}, \mathcal{D}]$ and $\eta : F \Rightarrow G$. We say that η is a **natural isomorphism** if for every $A \in \mathcal{C}_0$, η_A is an isomorphism. When there is an isomorphism between F and G we say that F and G are **isomorphic** and write $F \cong G$.*

Finally, given any $A \in \mathcal{C}_0$, we say that FA is naturally isomorphic to GA, or that $FA \cong GA$ naturally in A, when $F \cong G$.

Proposition 1.3.8. *Let $F, G \in [\mathcal{C}, \mathcal{D}]$ and $\eta : F \Rightarrow G$. Then η is a natural isomorphism if and only if it is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.*

Proof. Let η be a natural isomorphism. Then for every $A \in \mathcal{C}_0$ there exists $\epsilon_A : GA \to FA$ such that $\eta_A \circ \epsilon_A = \mathrm{Id}_{GA}$ and $\epsilon_A \circ \eta_A = \mathrm{Id}_{FA}$. Collecting all the ϵ_A's, we define $\epsilon := (\epsilon_A)_{A \in \mathcal{C}_0}$. We now need to prove that ϵ is natural, so let $f \in \mathcal{C}_0(A, B)$. Then naturality of η is equivalent to naturality of ϵ or:

$$
\begin{gathered}
Gf \circ \eta_A = \eta_B \circ Ff \Leftrightarrow \\
\epsilon_B \circ Gf \circ \eta_A \circ \epsilon_A = \epsilon_B \circ \eta_B \circ Ff \circ \epsilon_A \Leftrightarrow \\
\epsilon_B \circ Gf \circ \mathrm{id}_{GA} = \mathrm{Id}_{FB} \circ Ff \circ \epsilon_A \Leftrightarrow \\
\epsilon_B \circ Gf = Ff \circ \epsilon_A,
\end{gathered}
$$

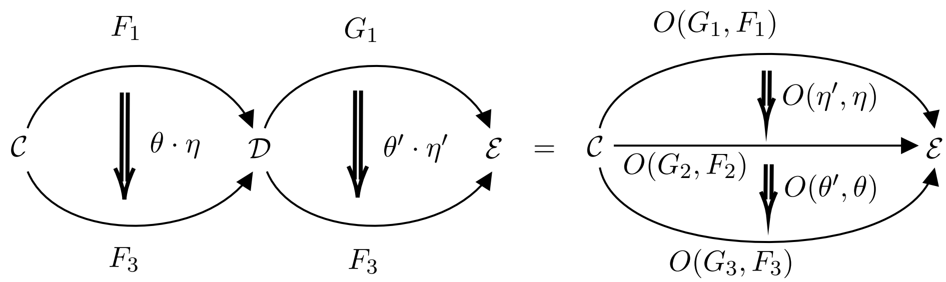
which proves that $\epsilon : G \Rightarrow F$ is natural.

For the other direction, since η is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ there exists an $\epsilon : G \Rightarrow F$ such that $\eta \cdot \epsilon = \mathrm{Id}_G$ and $\epsilon \cdot \eta = \mathrm{Id}_F$. This is equivalent to the statement that for every $A \in \mathcal{C}_0,\ \eta_A \circ \epsilon_A = \mathrm{id}_{GA}$ and $\epsilon_A \circ \eta_A = \mathrm{id}_{FA}$, which makes η a natural isomorphism. □

An interesting example of such a functor is the following. Given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, a functor $\mathrm{O} : [\mathcal{D}, \mathcal{E}] \times [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{E}]$ would have to satisfy:

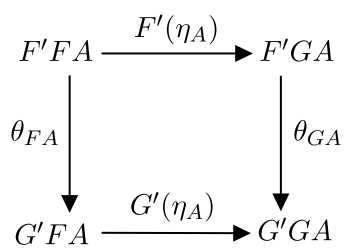
$$\mathrm{O}(\theta' \cdot \theta, \eta' \cdot \eta) = \mathrm{O}(\theta', \eta') \cdot \mathrm{O}(\theta, \eta),$$

for $F_1, F_2, F_3 \in [\mathcal{C}, \mathcal{D}]_0,\ G_1, G_2, G_3 \in [\mathcal{D}, \mathcal{E}]_0,\ \eta : F_1 \to F_2,\ \eta' : F_2 \to F_3$, $\theta : G_1 \to G_2,\ \theta' : G_2 \to G_3$. This condition actually looks like this:



This type of functors is interesting for two reasons. The first one is that it looks like, and in a later subsection we will see that it is, a composition of both functors and natural transformations, hence the name O. The second reason is that we can construct one such functor using the information provided by functor categories. This construction is given as follows:

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, $F, F' \in [\mathcal{C}, \mathcal{D}], G, G' \in [\mathcal{D}, \mathcal{E}]$ be functors and $\eta : F \Rightarrow F',\ \theta : G \Rightarrow G'$ be natural transformations. Then the following diagram commutes by naturality of θ.



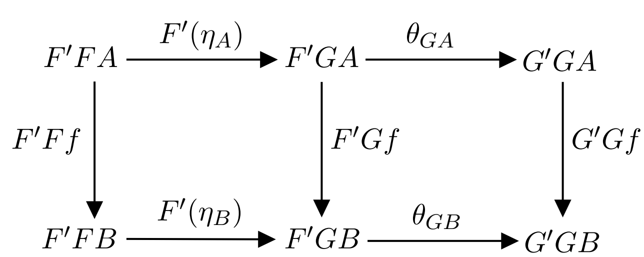
Thus for every $A \in \mathcal{C}_0$ we define $(\theta \circ \eta)_A := G'(\eta_A) \circ \theta_{FA} = \theta_{GA} \circ F'(\eta_A)$, which is the diagonal of the above naturality square. Now we need to prove that $\theta \circ \eta$ is a natural transformation from $F' \circ F$ to $G' \circ G$. Let $f \in \mathcal{C}(A, B)$. By naturality of η we get $\eta_B \circ Ff = Gf \circ \eta_A$, which implies that:

$$F'\eta_B \circ F'Ff = F'Gf \circ F'\eta_A.$$

Similarly by naturality of θ we get that:

$$G'Gf \circ \theta_{GA} = \theta_{GB} \circ F'Gf.$$

Therefore, both the following squares commute:

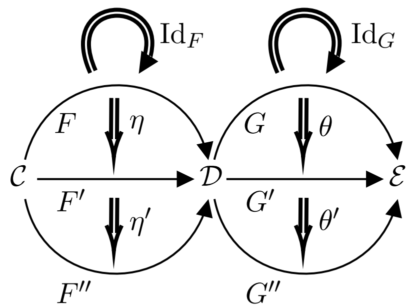


which translates to

$$\theta_{GB} \circ F'\eta_B \circ F'Ff = G'Gf \circ \theta_{GA} \circ F'\eta_A :\Leftrightarrow$$
$$(\theta \circ \eta)_B \circ F'Ff = G'Gf \circ (\theta \circ \eta)_A,$$

which is the naturality condition for $\theta \circ \eta$. The fact that $\theta \circ \eta$ is a natural transformation allows us to define $''\circ''$, the **horizontal composition** of natural transformations.

The final step is to prove that $''\circ''$ is a functor like the O ones defined above. To this end, let the following be categories, functors and natural transformations accordingly:

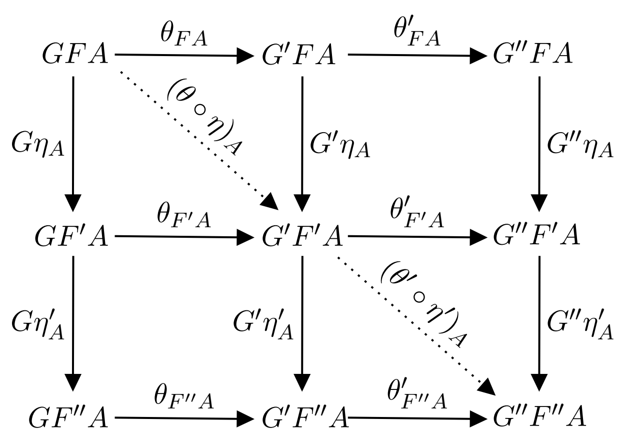


From the fact that

$$(\mathrm{Id}_G \circ \mathrm{Id}_F)_A := (\mathrm{Id}_G)_{FA} \circ G(\mathrm{id}_{FA}) = \mathrm{id}_{GFA} \circ \mathrm{id}_{GFA} = \mathrm{id}_{GFA}$$

we deduce that $(\mathrm{Id}_G \circ \mathrm{Id}_F) = \mathrm{Id}_{G\circ F}$, which is the first of the two functoriality conditions.

For the second one we need to prove that $(\theta' \circ \eta') \cdot (\theta \circ \eta) = (\theta' \cdot \theta) \circ (\eta' \circ \eta)$. Let $A \in \mathcal{C}_0$. From the commutativity of the following diagram:



we get

$$\begin{aligned}
(\theta' \circ \eta')_A \circ (\theta \circ \eta)_A =& (\theta'_{F''A} \circ \theta_{F''A}) \circ G(\eta'_A) \circ G(\eta_A) \\
=& (\theta' \cdot \theta)_{F''A} \circ G(\eta'_A \circ \eta_A) \\
=& (\theta' \cdot \theta)_{F''A} \circ G(\eta' \cdot \eta)_A :\Leftrightarrow \\
[(\theta' \circ \eta') \cdot (\theta \circ \eta)]_A =& [(\theta' \cdot \theta) \circ (\eta' \cdot \eta)]_A
\end{aligned}$$

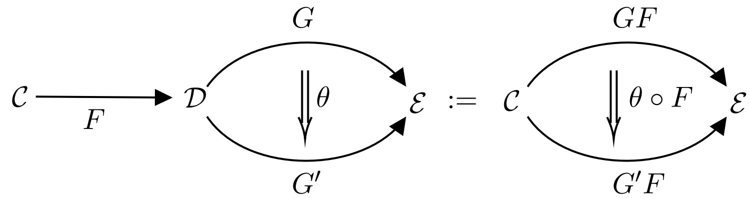
which is the second functoriality condition writen componetwise. Thus horizontal composition is a functor.

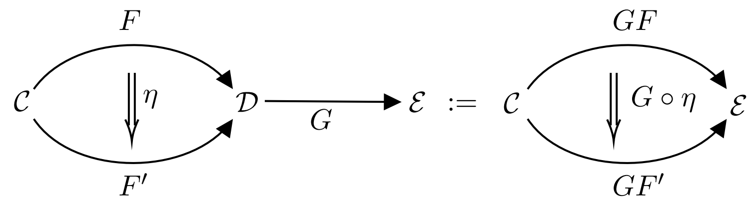
This horizontal composition functor has two interesting properties. The first one is a kind of "associativity". Specifically, for categories $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, functors $F, F' \in [\mathcal{B}, \mathcal{C}]$, $G, G' \in [\mathcal{C}, \mathcal{D}]$, $H, H' \in [\mathcal{D}, \mathcal{E}]$ and natural transformations $\alpha : F \Rightarrow F'$, $\beta : G \Rightarrow G'$, $\gamma : H \Rightarrow H'$ we have that $H \circ (G \circ F) = (H \circ G) \circ F$ and

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

This[3] is an easy statement to prove and therefore it is left as an exercise to the reader. Note that the discussion of an O functor was on an abstract level, while this "associative" property is very specific to horizontal composition. So, at least in principle, this property should not be expected to hold for any functor such as O.

The second one is called **whiskering**. Whiskering comes in two variants, the left and the right one. Below we see the left version on top and the right one at the bottom:





Writing the above equations, while introducing the appropriate terminology, we get that the left whiskering of θ by F is defined as: $\theta \circ F := \theta \circ \mathrm{Id}_F$, or componentwise for every $A \in \mathcal{C}_0$ as:

$$\begin{aligned}(\theta \circ F)_A &:= \theta_{FA} \circ \mathrm{id}_{GFA}\\ &= \theta_{FA}.\end{aligned}$$

Similarly the right whiskering of η by G is defined as $G \circ \eta := \mathrm{Id}_G \circ \eta$ and componentwise for every $A \in \mathcal{C}_0$ as:

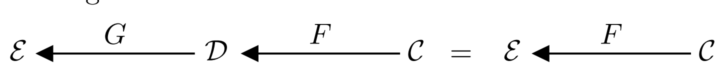
$$\begin{aligned}(G \circ \eta)_A &:= \mathrm{id}_{GF'A} \circ G(\eta_A)\\ &= G\eta_A.\end{aligned}$$

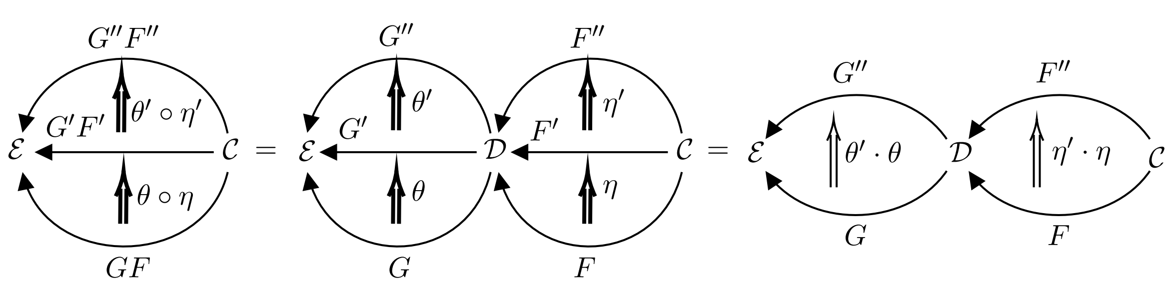
Finally, we can use whiskering to express horizontal composition in terms of vertical composition as follows:

$$\begin{aligned}\theta \circ \eta &:= G'\eta \cdot \theta F\\ &= \theta F' \cdot G\eta.\end{aligned}$$

String Diagrams for Functor Categories

Up until this point, to represent categorical notions involving categories and functors, we have used globular diagrams such as the following[4].





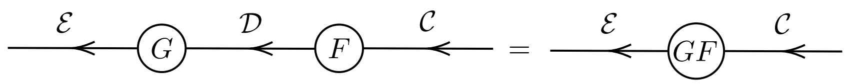
[3] We will revisit this property when discussing bicategories and 2-categories.

[4] Note that we present the diagrams with an unorthodox orientation for reasons having to do with the consistency of the presentation of the various string cacluli

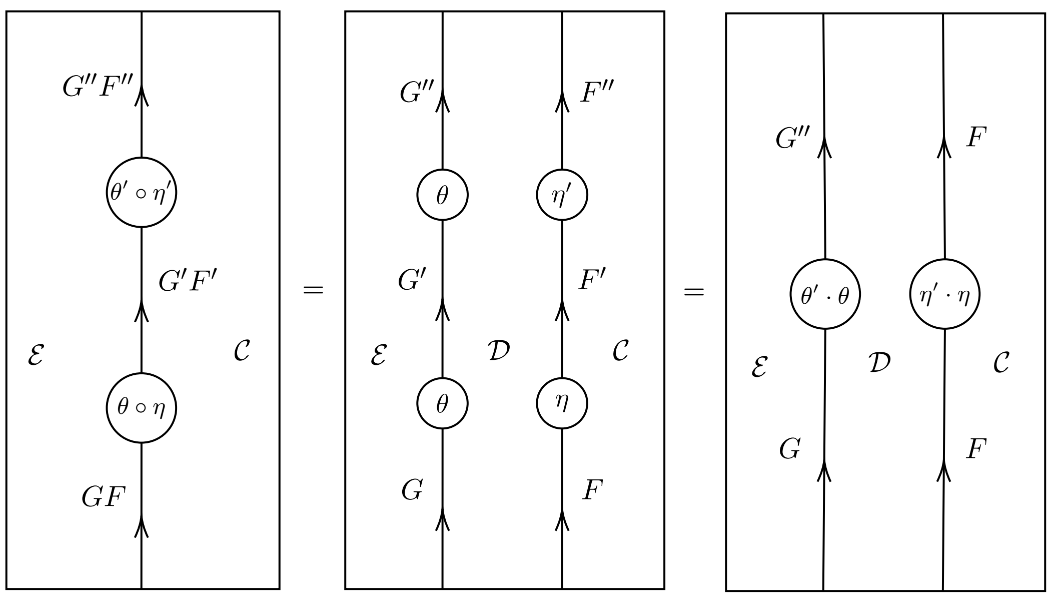
Poincare duals[5] of those preserve the same amount of information but are called **string diagrams**. In string diagrams not involving natural transformations, categories are represented by directed horizontal lines and functors by points, or rather circles. This representation reflects the one dimensionality of functor composition, as follows.

$$\mathcal{E} \xleftarrow{\quad G \quad} \mathcal{D} \xleftarrow{\quad F \quad} \mathcal{C} \; = \; \mathcal{E} \xleftarrow{\quad GF \quad} \mathcal{C}$$

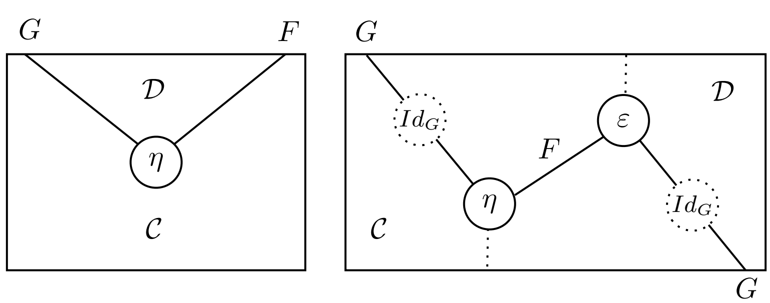
becomes



In string diagrams that do involve natural transformations, categories are represented by regions of the plane, functors by directed lines between such regions and natural tranformations as points (or rather circles), as shown below.



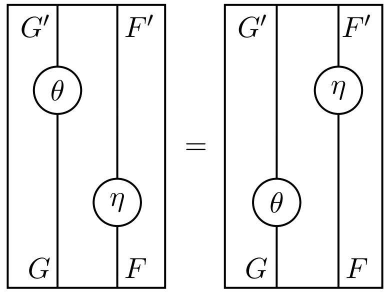
String diagrams have the advantage or representing horizontal and vertical composition of natural transformations more intuitively, mostly because horizontal composition is represented by putting strings next to each other. This fact forces us to represent the identity morphisms of categories without using any arrow. A general example illustrating this fact is the following.



Given that such diagrams should be read from bottom to top and from right to left, there is no ambiguity when drawing undirected lines for functors. Therefore, the above two diagrams represent $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow GF$ and $G\varepsilon \cdot \eta G : G \Rightarrow G$, where $\eta G : G \Rightarrow GFG$ and $G\varepsilon : GFG \Rightarrow G$, accordingly. Another thing worth noting is that whiskering is always implicit, in the sence that every point of every line can be interpreted as an

[5] A poincare dual of a two dimensional diagram is roughly another diagram with faces and vertices interchanged, while edges between vertices undergo a "90-degree rotation".

identity natural transformation. Therefore, the dotted circles of the second diagram above can be dropped. Another such example is the following:



where we see that $(\theta \cdot \mathrm{Id}_G) \circ (\mathrm{Id}_F \cdot \eta) = (\mathrm{Id}_G \cdot \theta) \circ (\eta \cdot \mathrm{Id}_F)$. Generally we see how intuitive whiskering looks when identity natural transformations are not depicted at all.

A last thing to note is that we can drop the ambient rectangle when using string diagrams to only describe endofunctors and natural transformations between them. This gives a view of functors as systems and natural transformations as processes between such systems. Adopting this viewpoint processes can be composed sequentially (horizontal composition), or in parallel (vertical composition). It is important to clarify though, that due to whiskering laws, as in the previous string diagram, there is no way to discern which of the two parallel processes happens first.

Comma and (co)slice categories

Comma categories are a construction that allows diagrams/ interpretations of two different categories to "be compared" within a third "interface" category. They provide a generalisation of arrow categories, as we introduced them in example 1.3.6, and are universal in a sense, at a higher categorical level.

Definition 1.3.9. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, and let $F : \mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ be functors. The category whose objects are triples (A, B, u), where $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$, $u : FA \to GB$ and whose morphisms are pairs $(f, g) \in \mathcal{A}(A, A') \times \mathcal{B}(B, B')$ such that the following diagram commutes:*

$$\begin{array}{ccc} FA & \xrightarrow{\;u\;} & GB \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gg} \\ FA' & \xrightarrow[\;u'\;]{} & GB' \end{array}$$

is called a ***comma category*** *and is denoted by $F \downarrow G$.*

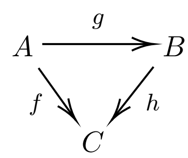
Obviously, the axioms of a category are satisfied by comma categories, since composition and identities are the same as in a product category.

Remark 1.3.10. Every comma category can be canonically equipped with two projection functors: $\pi_{\mathcal{A}} : F \downarrow G \to \mathcal{A}$ and $\pi_{\mathcal{B}} : F \downarrow G \to \mathcal{B}$, defined by collecting the first component of an object and a morphism, or the last ones, respectively. Therefore, by the universal property of the product of categories there is also a functor $(\pi_{\mathcal{A}}, \pi_{\mathcal{B}}) : F \downarrow G \to \mathcal{A} \times \mathcal{B}$.

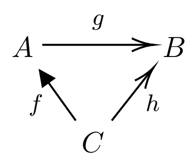
Example 1.3.11. In case $F = G = \mathbb{1}_{\mathcal{C}}$ we get that $\mathbb{1}_{\mathcal{C}} \downarrow \mathbb{1}_{\mathcal{C}}$ is isomorphic to $[\mathbf{2}, \mathcal{C}]$, the arrow category of $\mathcal{C}$.

Example 1.3.12. Consider the functor $\ulcorner C \urcorner : \mathbf{1} \to \mathcal{C}$, where $\ulcorner C \urcorner(*) = C \in \mathcal{C}_0$. We can now assume two categories. These capture how the object C witnesses the whole category $\mathcal{C}$ or how C is witnessed by the whole category $\mathcal{C}$. The first one corresponds to $\ulcorner C \urcorner \downarrow \mathbb{1}_{\mathcal{C}}$, which is called the **co-slice** or **under category** of $\mathcal{C}$ under C, and the second one to $\mathbb{1}_{\mathcal{C}} \downarrow \ulcorner C \urcorner$, which is called the **slice** or **over category** of $\mathcal{C}$ over C. These are sometimes denoted by $C/\mathcal{C}$ and $\mathcal{C}/C$, respectively.

Explicitly, the slice category of $\mathcal{C}$ over $C \in \mathcal{C}_0$ has pairs of the form $(A, f) \in \mathcal{C}_0 \times \mathcal{C}(A, C)$ as objects and the morphisms $(A, f) \to (B, h)$ are morphisms $g : A \to B$ such that the following diagram commutes:

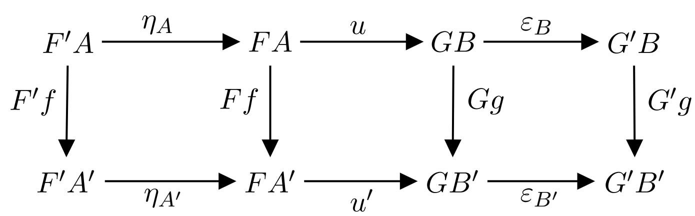


On the other hand, the co-slice category of $\mathcal{C}$ under $C \in \mathcal{C}_0$, has as objects $(f, A), (h, B)$, where $f \in \mathcal{C}(C, A)$, $g \in \mathcal{C}(C, B)$, and as morphisms commutative triangles of the form:



In addition, note that if $\mathcal{C}$ has a terminal object T, or an initial object I, then $\mathcal{C}/T$, or $I/\mathcal{C}$ respectively, are isomorphic to $\mathcal{C}$. Finally, note that in every slice category, $\mathcal{C}/C$, the object (C, id_C) is terminal and similarly, in every co-slice category, $C/\mathcal{C}$, the object (id_C, C) is initial.

Remark 1.3.13. Observe that given natural transformations $\eta : F' \Rightarrow F$, $\varepsilon : G \Rightarrow G'$, the following diagram commutes:



due to naturallity of η and ε. This implies that there exists a (bi-)functor $(- \downarrow \cdot)[\mathcal{A}, \mathcal{C}]^{\mathrm{op}} \times [\mathcal{B}, \mathcal{C}] \to \mathbf{Cat}$ taking $(F, G) \in [\mathcal{A}, \mathcal{C}]^{\mathrm{op}} \times [\mathcal{B}, \mathcal{C}]$ to $F \downarrow G$ and $(\eta^{\mathrm{op}}, \varepsilon)$ to $\eta \downarrow \varepsilon$. It is easy to check that functoriality of $\eta \downarrow \varepsilon$ follows from naturality of η, ε and of the identity natural transformations.

Remark 1.3.14. Given a category $\mathcal{C}$, every morphism $f \in \mathcal{C}(C, C')$ induces a natural transformation $\ulcorner f \urcorner : \ulcorner C \urcorner \Rightarrow \ulcorner C' \urcorner$. So, in the context of the previous remark, there is a functor $(\mathrm{Id}_{\mathbb{1}_\mathcal{C}} \downarrow \ulcorner f \urcorner) : \mathcal{C}/C \to \mathcal{C}/C'$ and a functor $(\ulcorner f \urcorner \downarrow \mathrm{Id}_{\mathbb{1}_\mathcal{C}}) : C'/\mathcal{C} \to C/\mathcal{C}$. These two functors transform the "apices" of the slice and the co-slice categories.

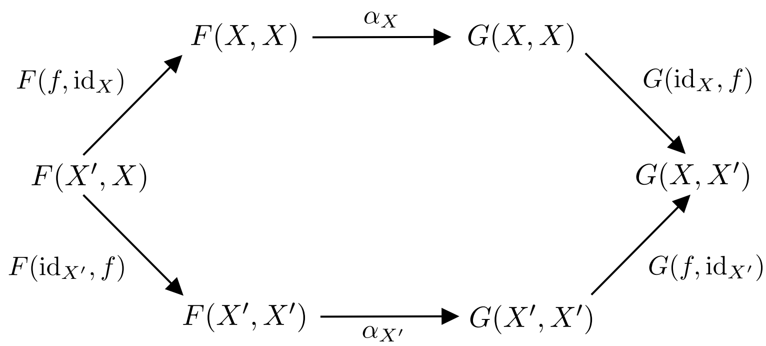
Dinatural Transformations and (Co)Wedges

There are cases where only the diagonal components of functors of two variables $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ are of interest. These diagonal components, considered as depending on one variable, are neither covariant nor contravariant. If we restrict ourselves to these diagonal components we see that the notion of natural transformation is not suitable to describe the morphisms between these restricted functors. A way to overcome this problem is given by the notion of dinatural transformation.

Definition 1.3.15. *Let $\mathcal{C}$ be a category and $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors with two variables. A family, $(\alpha_X)_{X \in \mathcal{C}}$, of morphisms of $\mathcal{D}$, where $\alpha_X : F(X, X) \to G(X, X)$, is called a* ***dinatural transformation*** *between F and G, denoted by*

$$\alpha : F \dot{\to} G$$

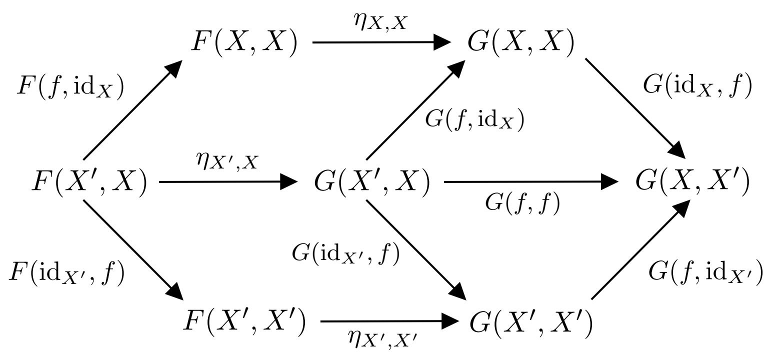
if the following diagram commutes:



for every $f \in \mathcal{C}(X, X')$.

Remark 1.3.16. We may denote the set of dinatural transformations between two functors with two variables F and G as $\mathrm{Dinat(F, G)}$.

Remark 1.3.17. Dinatural transformations are generalisations of natural transformations. This can be seen by the fact that for every $\eta : F \Rightarrow G$ and $f \in \mathcal{C}(X, X')$ the following inner diagrams commute, by naturality of η and functoriality of G.



So the outer diagram, which only involves the diagonal components of η, commutes. Thus every natural transformation is dinatural.

Remark 1.3.18. There is also another way in which a dinatural transformation is a generalisation of a natural transformation. We see this by restricting F and G. Observe that when F, G are *mute in* (i.e. do not depend on) the first variable (or the second one), then the dinaturality hexagon reduces to a naturality square. This is caused by the fact that $F(X', X) = F(X, X) \equiv F(X)$ and $F(f, \mathrm{id}_X) = \mathrm{id}_{F(X)}$ and similarly for G. For the second variable this goes accordingly.

Remark 1.3.19. Finally, there is a third way, maybe a little "twisted", in which a dinatural transformation generalises the notion of a natural transformation. This is in exact accordance with remark 1.3.3. A dinatural transformation may equivallently be defined as a family of morphisms $(\alpha_f) : F(X', X) \to G(X, X')$ indexed by morphisms in $\mathcal{C}$ such that

$$\alpha_f = G(f, \mathrm{id}_{X'}) \circ \alpha_{\mathrm{id}_{X'}} \circ F(\mathrm{id}_{X'}, f) = G(\mathrm{id}_X, f) \circ \alpha_X \circ F(f, \mathrm{id}_X).$$

A hint towards the proof of this equivalence is the equality

$$a_X = a_{\mathrm{id}_X}.$$

An interesting fact about dinatural transformations is that there is no easy way to compose them, but when precomposed or postcomposed (vertically) with a natural transformation dinaturality is preserved. We state this precisely in the following lemma.

Lemma 1.3.20. *Let $F, G, P, Q : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors with two variables, $\alpha : P \dot{\to} Q$ be a dinatural transformation and let $\eta : F \Rightarrow P$, $\theta : Q \Rightarrow G$ be natural transformations. Then the collections*

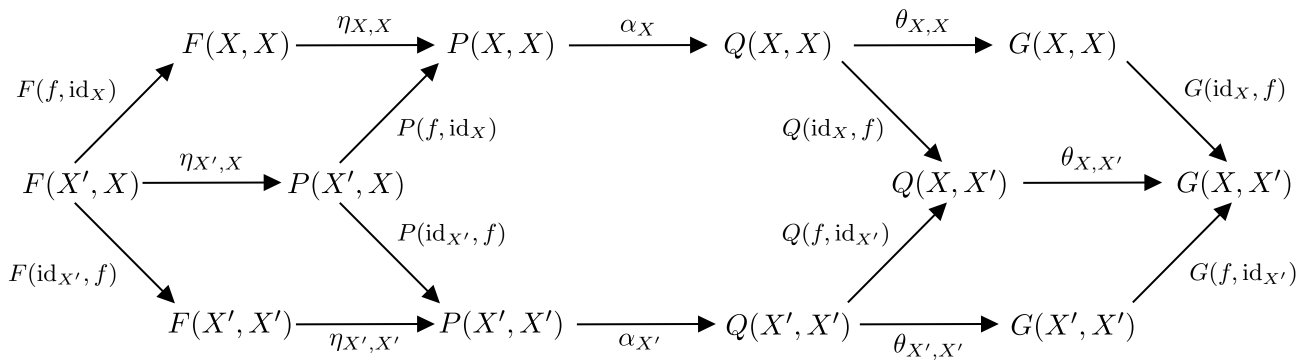
$$\alpha \cdot \eta := (\alpha_X \circ \eta_{X,X})_{X \in \mathcal{C}}$$

and

$$\theta \cdot \alpha := (\theta_{X,X} \circ \alpha_X)_{X \in \mathcal{C}}$$

are dinatural transformations of type $\alpha \cdot \eta : F \dot{\to} Q$ and $\theta \cdot \alpha : P \dot{\to} G$.

Proof. We only need to prove that the dinaturality condition holds for $\alpha \cdot \eta$ and $\theta \cdot \alpha$, so let $f \in \mathcal{C}(X, X')$. Then in the following diagram, the inner diagrams commute either by naturality of η and θ, or by dinaturality of α.



Thus, we get the dinaturality for $\alpha \cdot \eta$ and $\theta \cdot \alpha$, but also for $\theta \cdot \alpha \cdot \eta$, which is an extra! ☐

Something that should be noted is the form a dinaturality condition takes, when one of the two functors with two variables involved is a constant functor. Such dinatural transformations are of two separate types called *wedges* and *cowedges* respectively.

Definition 1.3.21. *Let $\mathcal{C}, \mathcal{D}$ be categories, $B \in \mathcal{D}_0$ be an object and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor with two variables. We call (B, α)*

- *a **wedge** for F, if α is a dinatural transformation $\alpha : \Delta_B \dot{\to} F$,*
- *a **cowedge** for F, if α is a dinatural transformation $\alpha : F \dot{\to} \Delta_B$.*

Remark 1.3.22. Wedges and cowedges are related by duality. In clear terms, a wedge for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is exactly a cowedge for the contravariant version of F, $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$, which sends (f, g) to $F(f, g)^{\mathrm{op}}$ in $\mathcal{D}^{\mathrm{op}}$. Somewhat informally, this means that a wedge in $\mathcal{D}$ is a cowedge in $\mathcal{D}^{\mathrm{op}}$.

Remark 1.3.23. If we restrict to functors with two variables that are mute in the first variable, then we get ordinary functors of one variable. In this case dinatural transformations are ordinary natural transforations. So wedges in such cases are called **cones** of F or *over* F and cowedges are called **cocones** of F or *under* F.

Remark 1.3.24. Observe that if (B, α) is a wedge for F, then given any morphism $f \in \mathcal{D}(A, B)$ we can create a new wedge $(A, \alpha \cdot f)$, since $\Delta_f : \Delta_A \Rightarrow \Delta_B$ is a natural transformation (see lemma 1.3.20). Similarly if (B, α) is a cowedge for F, for every $g \in \mathcal{D}(B, C)$, $g \cdot \alpha$ is again a cowedge for F.

"Turning this around" we can create a category $\mathcal{W}d(F)$ of wedges for a functor F, whose objects are wedges, say (B, α), (C, β), and whose morphisms from (B, α) to (C, β), are the natural transformations of the form Δ_f, for some $f \in \mathcal{D}(B, C)$, such that $\beta \cdot f = \alpha$. Similarly we can create a category $\mathcal{CW}d(F)$ of cowedges for a given functor F by considering postcomposition with morphisms of $\mathcal{D}$. In the restricted case of ordinary functors, in this context functors of two variables mute in the first variable, the wedge and cowedge categories for a given functor, F, give the categories $\mathcal{C}on(F)$ of cones over F and $\mathcal{C}o\mathcal{C}on(F)$ of cocones under F.

1.4 Representable functors and the Yoneda lemma

In category theory we are not apriori allowed to "see inside objects". Instead we have to deduce their inner structure (if any) through the ways they interact with their surrounding objects. Generalised elements play a very important role towards this objective. For a category $\mathcal{C}$, this role is captured by the functors of generalized elements $H_A : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ and $H^T : \mathcal{C} \to \mathbf{Set}$, for every $A, T \in \mathcal{C}_0$. These functors provide a set of generalised elements for every object of $\mathcal{C}$ and for every morphism a function between such elements. But not all functors to **Set** are of this type. This section expands on what is the relation of such arbitrary functors to generalised elements functors. Furthermore, the notions of ends, limits, coends and colimits are introduced via the category of elements construction through the representability lense.

Representable functors

Definition 1.4.1. *Let $\mathcal{C}$ be a localy small category and let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. F is called a* ***representable functor*** *or a* ***representable copresheaf*** *or just* ***representable****, if there exists a $T \in \mathcal{C}_0$ such that*

$$F \cong \mathcal{C}(T, -).$$

A choice (T, η) of a $T \in \mathcal{C}_0$ and an isomorphism

$$\eta : F \Rightarrow H^T$$

is called a ***representation*** *of F, while F is said to be* ***represented by*** *T. Dually, a functor $G : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is called a* ***representable functor*** *or a* ***representable presheaf*** *or just* ***representable****, if there exists an $A \in \mathcal{C}_0$ such that*

$$G \cong \mathcal{C}(-, A).$$

A choice of an $A \in \mathcal{C}_0$ together with an isomorphism

$$\theta : G \Rightarrow H_A$$

is called a ***representation*** *of G, while G is said to be* ***represented by*** *A.*

According to the above definition, a copresheaf of a category $\mathcal{C}$ is representable, if it essentially assigns to every object of $\mathcal{C}$ its set of T-shaped elements, for some $T \in \mathcal{C}_0$, and to every morphism a function between those sets. Dually, if a presheaf, G, essentially assigns to every object $T \in \mathcal{C}_0$ the set of T-shaped elements of a given $A \in \mathcal{C}_0$, and provides for every morphism $f : T \to T'$ a recast of T'-shaped elements of A in terms of T-shaped ones, then G is representable.

Example 1.4.2.

1. We have already seen in 1.6 that for every set A we get:

   $$\mathbf{Set}(T, A) \cong A,$$

   whenever T is a one element set. It is almost trivial to check that this equation is natural in A. Therefore, we have a natural isomorphism:

   $$\mathbf{Set}(T, -) \cong \mathbb{1}_{\mathbf{Set}},$$

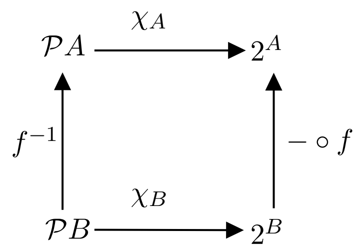
   thus the identity functor on **Set** is represented by a one element set.

2. Similarly, if $\mathbf{1}, \mathbf{2}$ are the one and two object order categories accordingly, we get that for every small category $\mathcal{C} \in \mathbf{Cat}$,

   $$\begin{aligned} &\mathbf{Cat}(\mathbf{1}, \mathcal{C}) \cong \mathcal{C}_0 \text{ and} \\ &\mathbf{Cat}(\mathbf{2}, \mathcal{C}) \cong \text{Ar}\mathcal{C}, \end{aligned} \tag{1.7}$$

   where $(-)_0$ and Ar are the object and morphism functors of example 1.2.4. It is an easy task to check that the above statement is natural in $\mathcal{C}$. Therefore, $(-)_0$ is represented by $\mathbf{1}$ and Ar by $\mathbf{2}$.

3. Let $A \in \mathbf{Set}$. Denote by $\chi_A : \mathcal{P}(A) \to 2^A$ the function that maps a subset of A to its characteristic function. This is a well known bijection between subsets, V, of A and characteristic functions, $\chi_A(V)$, of those subsets. Thus,
$$\mathbf{Set}(A, 2) \cong \mathcal{P}(A).$$
Letting $a \in A,\ f : A \to B$ and $V \subset B$ we get the following equivalence:
$$a \in f^{-1}[V] \Leftrightarrow f(a) \in V$$
which proves that for every $a \in A$ and $V \in \mathcal{P}(B)$,
$$\chi_A(f^{-1}V)(a) = \chi_B \circ f(a),$$
thus making the following diagram commute:



Therefore, χ is a natural isomorphism, which proves that the powerset functor is represented by the set $2 = \{0, 1\}$.

4. Equipping the set $2 = \{0, 1\}$ with the topology $\mathcal{S}(2) := \{\emptyset, \{1\}, 2\}$ we get a topological space called the Sierpinski space and denoted by S. This topological space has the remarkable property of representing the topology functor $\mathcal{T}$ we presented in example 1.2.6. That is:
$$\mathbf{Top}(-, S) \cong \mathcal{T}.$$
The proof is similar to the one in the powerset functor case above.

In example 1.3.2 we defined two natural transformations, $\mathcal{C}(f, -)$ and $\mathcal{C}(-, f)$, for every morphism f of a category $\mathcal{C}$. Some authors, like Borceaux and Stubbe [BS00], call these *representable natural transformations*. We will now establish the following two facts.

Proposition 1.4.3. *Let $\mathcal{C}$ be a localy small category, $X, Y, Z \in \mathcal{C}$, $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ and let $\mathcal{C}(-, f) : H_X \Rightarrow H_Y$ be the corresponding natural transformation. Then*

1. $\mathcal{C}(-, g) \cdot \mathcal{C}(-, f) = \mathcal{C}(-, g \circ f)$ and

2. $\mathcal{C}(-, \mathrm{id}_X) = \mathrm{Id}_{H_X}$.

Proof. This proof is computational and goes as follows:

1. For every $A \in \mathcal{C}$,
$$\begin{aligned}(\mathcal{C}(-, g) \cdot \mathcal{C}(-, f))_A &= \mathcal{C}(A, g) \circ \mathcal{C}(A, f) \\ &= (g \circ -) \circ (f \circ -) \\ &= (g \circ f) \circ - \\ &= \mathcal{C}(A, g \circ f).\end{aligned}$$

2. For $Y = Z$, $g = \mathrm{id}_Y$ and according to 1. we get:
$$\begin{aligned}\mathcal{C}(-, \mathrm{id}_Y) \cdot \mathcal{C}(-, f) &= \mathcal{C}(-, \mathrm{id}_Y \circ f) \\ &= \mathcal{C}(-, f)\end{aligned}$$
Similarly, $\mathcal{C}(-, f) \cdot \mathcal{C}(-, \mathrm{id}_X) = \mathcal{C}(-, f)$. Therefore, by uniqueness of the identity morphism in the monoid $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(H_X, H_X)$, we get $\mathcal{C}(-, \mathrm{id}_X) = \mathrm{Id}_{H_X}$ for every $X \in \mathcal{C}_0$.

$\square$

According to proposition 1.4.3 we can form a functor $よ_{\mathcal{C}} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ for every locally small category $\mathcal{C}$. This functor is called the **Yoneda embedding** of $\mathcal{C}$ in its presheaf category and is defined as follows:

$$よ_{\mathcal{C}} A := H_A \text{ and } よ_{\mathcal{C}} f := \mathcal{C}(-, f),$$

for every $A, B \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, B)$.

Dually, for a locally small category $\mathcal{C}$ we get that

$$\mathcal{C}(g, -) \cdot \mathcal{C}(f, -) = \mathcal{C}(f \circ g, -),$$

where $X, Y, Z \in \mathcal{C}_0$ and $g \in \mathcal{C}(Y, Z), f \in \mathcal{C}(X, Y)$. As a result, there is a contravariant version of the Yoneda embedding $よ^{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ defined as:

$$よ^{\mathcal{C}} A := H^A \text{ and } よ^{\mathcal{C}} f := \mathcal{C}(f, -),$$

for every $A, B \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, B)$. Therefore we can also embedd a locally small category in its copresheaf category.

The Yoneda lemma

We will now see how a representable functor relates to an arbitrary one. The answer to this question is given by the Yoneda lemma. This lemma has two versions, each dual to each other. Although the original statement is about presheaves, and the dual version is informally called the co-Yoneda Lemma, we proceed by presenting the one for covariant functors.

Theorem 1.4.4. *Let $\mathcal{C}$ be a locally small category and let $F : \mathcal{C} \to \mathbf{Set}$ be a copresheaf. Then there is an isomorphism:*

$$[\mathcal{C}, \mathbf{Set}](H^A, F) \cong FA,$$

which is natural in both F and A.

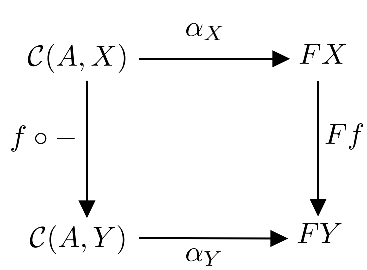
Before continuing with the proof, we note that for $A \in \mathcal{C}_0$ and $F \in [\mathcal{C}, \mathbf{Set}]$, the requested isomorphism, $FA \to [\mathcal{C}, \mathbf{Set}](H^A, F)$,, will be denoted by $\phi_{F,A}$ or just ϕ, and its inverse by $\xi_{F,A}$ or just ξ.

Proof. We will prove the statement in four steps. First, given an $\alpha \in FA$ we will define a candidate natural transformation $\phi_{F,A}(\alpha) = \tilde{\alpha}$ for its image. Then we will show that $\phi_{F,A}$ is an isomorphism. Afterwards, we will prove that it is natural in F and finally that it is also natural in A.

1. Let $\alpha \in FA$. We define $\phi_{F,A}(\alpha) = (\alpha_X)_{X \in \mathcal{C}_0} : H^A \Rightarrow F$ such that:

$$\alpha_A(\mathrm{id}_A) := \alpha \text{ and}$$
$$\alpha_X(f) := Ff(\alpha), \text{ for every } f \in H^A(X).$$

   To prove that $(\alpha_X)_{X \in \mathcal{C}_0}$ is a natural transformation we need to show that the following diagram commutes,



for every $f \in \mathcal{C}(X, Y)$, which it does since by definition for every $h \in H^A(X)$,

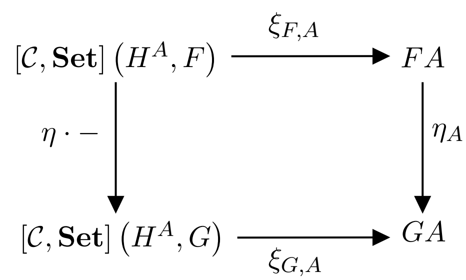
$$\begin{aligned} Ff \circ \alpha_X(h) &= F(f) \circ F(h)(\alpha) \\ &= F(f \circ h)(\alpha) \\ &= \alpha_Y(f \circ h) \end{aligned}$$

2. To prove injectivity of $\phi_{F,A}$ let us assume that $\phi_{F,A}(\alpha) = \phi_{F,A}(\beta)$, for $\alpha, \beta \in FA$. Then $\alpha = \alpha_A(\mathrm{id}_A) = \beta_A(\mathrm{id}_A) = \beta$, which proves that $\phi_{F,A}$ is injective. Now let $\eta : H^A \Rightarrow F$. Then there exists an $\alpha \in FA$ such that $\eta_A(\mathrm{id}_A) = \alpha$. Naturality of η forces its components to obey the following for every $f \in \mathcal{C}(A, X)$:

$$\begin{aligned} \eta_X(f) &= Ff \circ \eta_A(\mathrm{id}_A) \\ &= Ff(a) \\ &= \alpha_X(f) \end{aligned}$$

Therefore $\eta = (\eta_X)_{X \in \mathcal{C}_0} = (\alpha_X)_{X \in \mathcal{C}_0} = \phi_{F,A}(\alpha)$, or in other words $\phi_{F,A}$ is surjective. Thus $\phi_{F,A}$ is a bijection.

3. To show that $\phi_{F,A}$ is natural in F it is enough to prove that $\xi_{F,A}$ is so. Let $\eta : F \Rightarrow G$. Since F was arbitrary in 1. and 2. we also know that $\phi_{G,A}$ is an isomorphism. Thus, we need to prove that the following diagram commutes:

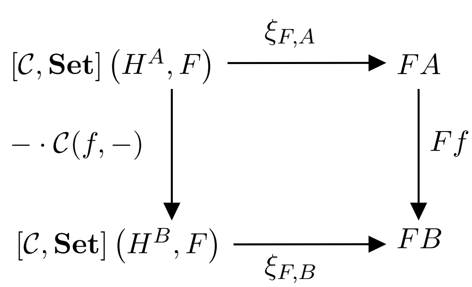


or equivalently that the following holds for every $\alpha : H^A \Rightarrow F$

$$\eta_A \circ \xi_{F,A}(\alpha) = \xi_{G,A}(\eta \cdot \alpha).$$

This is indeed the case since both $\xi_{G,A}(\eta \cdot \alpha) = (\eta \cdot \alpha)_A(\mathrm{id}_A) = \eta_A(\alpha_A(\mathrm{id}_A))$ and $\eta_A \circ \xi_{F,A}(\alpha) = \eta_A(\alpha_A(\mathrm{id}_A))$. Thus $\xi_{F,A}$ and also $\phi_{F,A}$ are natural in F.

4. Finally to prove naturality in A it is enough to show that the ξ's obey the following naturality condition for every $f : A \to B$.



Algebraically, the above is equivalent to

$$\xi_{F,B}(\alpha \cdot \mathcal{C}(f, -)) = Ff(\xi_{F,A}(\alpha)),$$

for every $\alpha : H^A \Rightarrow F$. This is indeed the case again, since:

$$\begin{aligned}\xi_{F,B}(\alpha \cdot \mathcal{C}(f,-)) &= (\alpha \cdot \mathcal{C}(f,-))_B(\mathrm{id}_B)\\ &= \alpha_B \circ \mathcal{C}(f,B)(\mathrm{id}_B)\\ &= \alpha_B(\mathcal{C}(f,B)(\mathrm{id}_B))\\ &= \alpha_B(\mathrm{id}_B \circ f)\\ &= \alpha_B(f)\\ &= Ff(\alpha_A(\mathrm{id}_A))\\ &= Ff(\xi_{F,A}(\alpha))\\ &= Ff \circ \xi_{F,A}(\alpha)\end{aligned}$$

thus proving the requested naturality.

Therefore, we have proved that every natural transformation from the representable copresheaf H^A to an arbitrary copresheaf F corresponds bijectively to a unique element of FA naturally in both A and F. □

Remark 1.4.5. Note that the isomorphism, $\phi_{F,A}$, and its inverse $\xi_{F,A}$, in the proof of the Yoneda Lemma, are given explicitly by the following formulas for $x \in FA$ and $\alpha : H^A \Rightarrow F$:

$$\phi_{F,A}(x) = F(-)(x) \text{ and } \xi_{F,A}(\alpha) = \alpha_A(\mathrm{id}_A).$$

By applying the same arguement to $\mathcal{C}^{\mathrm{op}}$ we get the Yoneda lemma for contravariant functors and presheaves.

Theorem 1.4.6. *Let $\mathcal{C}$ be a locally small category and let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. Then there is an isomorphism:*

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F) \cong FA,$$

which is natural in both F and A.

As weird or difficult the Yoneda lemma might seem at first glance, so immediate and beautiful are its corollaries.

Corollary 1.4.7. *Let $\mathcal{C}$ be a locally small category. Then:*

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, H_B) \cong \mathcal{C}(A, B). \tag{1.8}$$

The meaning of the above is that if two objects are isomorphic, so are their points of view, i.e. the copresheaves they represent. This is because the above natural bijection, (1.8), carries isomorphisms to natural isomorphisms and vice versa. This also allows us to check for natural isomorphisms between functors using the Yoneda lemma as follows.

Let $\mathcal{D}, \mathcal{C}$ be locally small categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors such that $\mathcal{D}(-, FY) \cong \mathcal{D}(-, GY)$ naturally in Y. Then by the above corrolary we may conclude that there exists a unique $\eta_Y : FY \to GY$, which is an isomorphism and $\eta_Y \circ -$ gives the natural isomorphism component $\mathcal{D}(-, FY) \cong \mathcal{D}(-, GY)$ above. Furthermore, since $(\eta_Y \circ -)_{Y \in \mathcal{C}_0}$ is a natural isomorphism we get that for every $f \in \mathcal{C}(Y, X)$, $\eta_{X'} \circ Ff \circ - = Gf \circ \eta_Y \circ -$, which, again by fully faithfulness of the Yoneda embedding, implies

$$\eta_{X'} \circ Ff = Gf \circ \eta_Y.$$

So $F \cong G$ if and only if $\mathcal{D}(-, F-) \cong \mathcal{D}(-, G-)$.

Moreover, again due to (1.8), the Yoneda embedding, $よ_{\mathcal{C}} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, is fully faithful i.e. embedds a category into its presheaf category. Thus, for every $f \in \mathcal{C}(A, B)$ the assignment $f \mapsto よ_{\mathcal{C}}(f) = \mathcal{C}(-, f) = f \circ -$ is one to one and onto. Reformulating this, we get that two morphisms $f, g \in \mathcal{C}(A, B)$ are the same, if and only if $f \circ - = g \circ -$.

There are some special kinds of categories worth mentioning at this point, that don't require checking the values of $f \circ -$ and $g \circ -$ at every generalised element to judge if they are equal. We give the definition of such a category, which will be of interest in the subsequent chapters.

Definition 1.4.8. *Let $\mathcal{C}$ be a category with a terminal object T. We say that $\mathcal{C}$ is* ***well pointed*** *if for every $f, g \in \mathcal{C}(A, B)$*

$$f = g \Leftrightarrow f(a) = g(a),$$

for every $a \in_T A$. (see Definition 1.2.9)

Given any presheaf $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ and any object $A \in \mathcal{C}_0$, one can "see inside" the sets FA. By the Yoneda lemma, every element $x \in FA$ corresponds to a natural transformation $\chi : H_A \to F$, via $\phi_{F,A}(x) = \chi$. In the case that F is a representable functor, represented by $(A,\ \chi)$, then x is called a **universal element** of F. We might wonder in what sense is such an element universal. We will answer to this question by introducing the notion of the category of elements of a presheaf.

Definition 1.4.9. *Let $\mathcal{C}$ be a category and F a presheaf. The category whose objects are pairs (A, x), where $A \in \mathcal{C}_0$ and $x \in FA$, and whose morphisms $f : (A, x) \to (B, y)$ are morphisms $f \in \mathcal{C}(A, B)$ that satisty:*

$$Ff(y) = x, \tag{1.9}$$

is called the ***category of elements*** *of F and is denoted by $\int F$.*

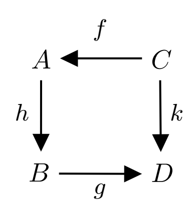
Remark 1.4.10. For every presheaf $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$, there exists a "projection" functor $\pi^F : \int F \to \mathcal{C}$, sending (A, x) to A and $f : (A, x) \to (B, y)$ to f. Although projection functors are reminiscent of products of categories, morphisms in the category of elements are more like a slice category. This is actually more than just an analogy.

By the Yoneda lemma, the objects $(A, a) \in \int F$ are in bijective correspondece with triples (A, F, α), where $\alpha = \phi_{F,A}(a) : H_A \Rightarrow F$. Also, naturality of $\phi_{F,A}$ in A, implies that condition (1.9) is equivalent to the following:

$$\begin{aligned} Ff(b) = a &\Leftrightarrow \\ \phi_{F,A}(Ff(b)) = \phi_{F,A}(a) &\Leftrightarrow \\ \phi_{F,B}(b) \circ \mathcal{C}(-, f) = \alpha &\Leftrightarrow \\ \beta \circ \mathcal{C}(-, f) = \alpha&, \end{aligned}$$

where $(B, b) \in \int F$ and $\phi_{F,B}(b) = \beta$. Therefore, the morphisms $f \in \int F((A, a), (B, b))$ are in bijection with morphisms $(f, \mathrm{Id}_F) \in \text{よ}_{\mathcal{C}} \downarrow \ulcorner F \urcorner$, where $\text{よ}_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the Yoneda embedding and $\ulcorner F \urcorner : \mathbf{1} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the functor picking F.

Example 1.4.11. The hom functor of a locally small category $\mathcal{C}$ is of type $\hom_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbf{Set}$, so it may be considered as a co-presheaf on $\mathcal{C}^{\text{op}} \times \mathcal{C}$. Its category of elements $\int \hom_{\mathcal{C}}$ has as objects pairs of the form $((A, B), h)$ and $((C, D), k)$ and as morphisms $(f, g) \in \mathcal{C}(C, A) \times \mathcal{C}(B, D)$, where $A, B, C, D \in \mathcal{C}_0$, $h \in \mathcal{C}(A, B)$ and $k \in \mathcal{C}(C, D)$, such that the following diagram commutes:



We define $\mathbf{tw}(\mathcal{C})$ as the category whose objects are morphisms in $\mathcal{C}$ and whose morphisms are pairs of oppositely oriented morphisms of $\mathcal{C}$ that make the above diagram commute. This category is called the **twisted arrow category** of $\mathcal{C}$. According to the above, the twisted arrow category of $\mathcal{C}$ is the category of elements of the hom functor of $\mathcal{C}$.

Example 1.4.12. As it was hinted at remark 1.3.24, we may view wedges of a functor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ as dinatural transformations of shape $\alpha : \Delta_A \dot{\to} F$, for a given $A \in \mathcal{C}_0$. Similarly, cones can be thought of as natural transformations of shape $\pi : \Delta_A \Rightarrow D$, for a given diagram $D : \mathcal{C} \to \mathcal{D}$. We may thus consider the presheaves $\mathrm{Dinat}(\Delta_{-}, \mathrm{F}) : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathcal{D}^{\mathcal{C}}(\Delta_{-}, D) : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$, that take an object $X \in \mathcal{D}_0$ to $\mathrm{Dinat}(\Delta_{\mathrm{X}}, \mathrm{F}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $\mathcal{D}^{\mathcal{C}}(\Delta_X, D) : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, accordingly, and a morphism $f : X \to Y$, to the precomposition with the natural transformation $\Delta_f : \Delta_X \to \Delta_Y$ (see Lemma 1.3.20). Note that the categories of elements, $\int \mathrm{Dinat}(\Delta_{-}, \mathrm{F})$ and $\int \mathcal{D}^{\mathcal{C}}(\Delta_{-}, D)$, of these two presheaves are isomorphic to the categories of wedges of F, $Wd(F)$, and of cones $\mathcal{C}on(D)$.

Dually, the category of elements of the copresheaf $\mathrm{Dinat}(\mathrm{F}, \Delta_{-})$, is the category of **cowedges** of F and the category of elements of the copresheaf $\mathcal{D}^{\mathcal{C}}(D, \Delta_{-})$ is category of **cocones** under F.

There is a remarkable property that characterizes representable functors via their categories of elements.

Proposition 1.4.13. *Let $\mathcal{C}$ be a locally small category and $F \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ a presheaf. The following are equivalent.*

i) F is representable.

ii) $\int F$ has a terminal object.

Proof. For i)⇒ii) let (A, η) be such that $\eta : H_A \Rightarrow F$ is a natural isomorphism. By the Yoneda lemma there exists a unique $x = \phi_{F,A}(\eta) \in FA$ such that $\eta_A(\mathrm{id}_A) = x$. We will show that (A, x) is terminal in $\int F$.

Let (B, y) be another object of $\int F$. Since η is an isomorphism, by definition, $\eta_B : \mathcal{C}(B, A) \to FB$ is also an isomorphism. Therefore, there is a unique $f \in \mathcal{C}(B, A)$ satisfying $\eta_B(f) = y$. Naturality of η implies that:

$$\begin{aligned} Ff(x) &= Ff(\eta_A(\mathrm{id}_A)) \\ &= \eta_B(\mathrm{id}_A \circ f) \\ &= \eta_B(f) \\ &= y \end{aligned}$$

Thus there exists a unique $f \in \mathcal{C}(B, A)$, such that $Ff(x) = y$.

For ii)⇒i) let $(A, x) \in \int F$ be terminal. By the Yoneda lemma, there exists a unique $\phi_{F,A}(x) = \eta : H_A \Rightarrow F$ such that $\eta_A(\mathrm{id}_A) = x$. To prove that F is representable it is enough to show that η_B is an isomorphism for every $B \in \mathcal{C}_0$. Equivalently, we need to show that for every $B \in \mathcal{C}_0$ and for every $y \in FB$, there exists unique $f \in \mathcal{C}(B, A)$, such that

$$\eta_B(f) = y.$$

But this is equivalent to the statement that for every $(B, y) \in \int F$ there exists a unique $f : (B, y) \to (A, x)$, which holds by terminality of (A, x). Therefore, η_B is an isomorphism for every $B \in \mathcal{C}_0$, which proves that η is a natural isomorphism and that (A, η) represents F. □

According to the above proposition, any universal element of a presheaf, F, is terminal in the category of elements $\int F$. Therefore, the reason behind the use of the word “universal” for an element is that it is characterised by a universal property.

By duality, a category of elements can also be defined for a copresheaf, $F \in \mathbf{Set}^{\mathcal{C}}$. The only difference is that a morphism $f : (A, x) \to (B, y)$ of $\int F$ is a morphism $f \in \mathcal{C}(A, B)$, such that $Ff(x) = y$. Furthermore, when F is representable, a universal element is again characterised by a universal property. In this dual case the universal element is initial in $\int F$.

Example 1.4.14. Let $\mathcal{C}$ be a (locally small) category and let $A, B \in \mathcal{C}_0$. Define the covariant functor $\mathcal{C}(A, -) \times \mathcal{C}(B, -) : \mathcal{C} \to \mathbf{Set}$ by: $X \mapsto \mathcal{C}(A, X) \times \mathcal{C}(B, X)$, for all $X \in \mathcal{C}_0$ and $u \mapsto (u, u) \circ -$, for every $X, Y \in \mathcal{C}_0$ and every $u \in \mathcal{C}(X, Y)$. Also define the contravariant functor $\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C} \to \mathbf{Set}$ by: $X \mapsto \mathcal{C}(X, A) \times \mathcal{C}(X, B)$, for every $X \in \mathcal{C}_0$ and $u \mapsto - \circ (u, u)$, for every $X, Y \in \mathcal{C}_0$ and $u \in \mathcal{C}(Y, X)$.

If the contravariant one is representable, then, by the previous proposition, there exists a terminal object $(X, (\pi_1, \pi_2))$ in the category $\int \mathcal{C}(-, A) \times \mathcal{C}(-, B)$. Equivalently, for every object $(Y, (q_1, q_2))$, there exists a unique $u : Y \to X$ such that

$$(\pi_1 \circ u, \pi_2 \circ u) = (q_1, q_2).$$

Thus, $(X,(\pi_1,\pi_2))$ satisfies the universal property of the product $A \times B$. Therefore, the existence of a product is equivalent to the representability condition of $\mathcal{C}(-,A)\times\mathcal{C}(-,B)$. Similarly, a universal element for the covariant functor $\mathcal{C}(A,-)\times\mathcal{C}(B,-)$ is $(A+B,(i_1,i_2))$, the coproduct of A, B. Of course it only exists, if and only if the functor $\mathcal{C}(A,-)\times\mathcal{C}(B,-)$ is representable.

Example 1.4.15. The formal machinery of representability and categories of elements allows us to characterize certain constructions, that are ubiquitous in category theory. These are a generalisation of the above example.

- Consider a diagram $D : \mathcal{C} \to \mathcal{D}$. We have seen that the category of cones over D, $\mathcal{C}on(D)$, is the category of elements of the presheaf $\mathcal{D}^{\mathcal{C}}(\Delta_{-}, D)$. If this presheaf is representable, its **universal element**, denoted by $(\lim D, p)$ is a terminal object in $\int \mathcal{D}^{\mathcal{C}}(\Delta_{-}, D)$. We call this representing object the **limit** of D and the components of the natural transformation associated with p through the Yoneda lemma are called the **projections** of the limit. Rephrasing, we see that a limit is a **terminal cone**. Dually, a **colimit** of a diagram $D : \mathcal{C} \to \mathcal{D}$, is an initial object, denoted by $(\mathrm{colim(D)}, \mathrm{i})$, in the category of elements $\int \mathcal{D}^{\mathcal{C}}(D, \Delta_{-})$. That is, a colimit is an **initial cocone** and the components of i are called the **coprojections** of the colimit. Thus, the existence of a limit or a colimit is equivalent to the following representability conditions accordingly:

$$\mathcal{D}^{\mathcal{C}}(\Delta_{-}, D) \cong \mathcal{D}(-, \lim D) \text{ and } \mathcal{D}^{\mathcal{C}}(D, \Delta_{-}) \cong \mathcal{D}(\mathrm{colimD}, -).$$

- Similarly, for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, the wedges of F form the category of elements of the presheaf $\mathrm{Dinat}(\Delta_{-}, \mathrm{F})$. If this presheaf is representable, its representing object is the universal wedge, and it is called the **end** of F, denoted by $(\int_{C\in\mathcal{C}} F(C,C), p)$ or just $\int_{\mathcal{C}} F, p)$, where p again gives the **projections**. Dually, the universal element of the copresheaf $\mathrm{Dinat}(\mathrm{F}, \Delta_{-})$, is called a **coend** of F, denoted by $(\int^{C\in\mathcal{C}} F(C,C), i)$ or just $(\int^{\mathcal{C}} F, i)$.

Before proceeding with some more examples, we give a proposition conserning the the projections and coprojections of ends and coends, and therefore limits and colimits.

Proposition 1.4.16. *Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor with two variables having $(\int_{\mathcal{C}} F, \pi)$ as an end and let $g, f : X \to \int_{\mathcal{C}} F$. Then*

$$\pi \cdot f = \pi \cdot g$$

implies $f = g$. Dually, if $(\int^{\mathcal{C}} F, \iota)$ is a coend of F and $g, f : \int^{\mathcal{C}} F \to Y$ are such that

$$f \cdot \iota = g \cdot \iota,$$

then $f = g$.

Proof. We will only prove the first statement, since the second one follows by duality. To this end observe that $(X, \pi \cdot f)$ and $(X, \pi \cdot g)$ are the same wedge for F. Thus, by the isomorphism

$$\mathrm{Dinat}(\Delta_{\mathrm{X}}, \mathrm{F}) \cong \mathcal{D}(\mathrm{X}, \int_{\mathcal{C}} \mathrm{F})$$

and the terminality of $\int_{\mathcal{C}} F$ $\pi \cdot f \in \mathrm{Dinat}(\Delta_{\mathrm{X}}, \mathrm{F})$ and $\pi \cdot g \in \mathrm{Dinat}(\Delta_{\mathrm{X}}, \mathrm{F})$ correspond to a unique morphism $u \in \mathcal{D}(X, \int_{\mathcal{C}} F)$, so

$$f = u = g.$$

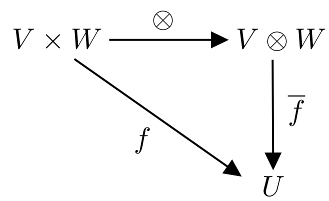
□

Remark 1.4.17. The above proposition also holds for limits and colimits, respectively. In any case, we see that, although projections of ends or limits might not be monomorphisms, their collection acts like one. In this case we say that the projections are **jointly monic**. Similarly, the coprojections of coends/colimits are **jointly epic**.

Example 1.4.18. Let $V, W \in \mathbf{Vect}_{\mathbb{F}}$. Define $\mathbf{Bilin}(V; W, -) : \mathbf{Vect}_{\mathbb{F}} \to \mathbf{Set}$ to be the functor assigning to a vector space U the set of bilinear maps from $V \times W$ to U and to every linear map $f : U \to U'$ the post-composition $f \circ -$, since the post-composition of a bilinear by a linear map is bilinear. Representability for this functor is, according to the previous proposition, equivalent to the existence of a universal element, $(V \otimes W, u)$ in the category of elements $\int \mathbf{Bilin}$. Since $\mathbf{Bilin}$ is a copreashef, $(V \otimes W, u)$ is initial, thus for every $f \in \mathbf{Bilin}(V; W, U)$ there exists a unique $\bar{f} \in \mathbf{Vect}_{\mathbb{F}}(V \otimes W, U)$ such that

$$\bar{f} \circ u = f.$$

Renaming $u \equiv \otimes$, we get the following commutative diagram:



which is the universal property of the tensor product of vector spaces.

Remark 1.4.19. We already saw, in remark 1.4.10, that the category of elements of a presheaf, $F \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ can equivalently be defined as the comma category $\text{よ}_{\mathcal{C}} \downarrow \ulcorner F \urcorner$. Another interesting fact is that the category of elements of a representable presheaf, H_A is a slice category. Note that the objects of $\text{よ}_{\mathcal{C}} \downarrow \ulcorner H_A \urcorner$ are of the form $(X, H_A, \alpha : H_X \Rightarrow H_A)$ for all $X \in \mathcal{C}_0$. But, the Yoneda embedding being fully faithful implies that these objects are in bijection with triples $(X, A, a : X \to A)$. Similarly, the morphisms $(\mathcal{C}(-, f)), \mathcal{C}(-, \mathrm{id}_A))$ are in bijection with pairs (f, id_A). This can easily be seen to be functorial, therefore there is an isomorphism $\int H_A \cong \mathbb{1}_{\mathcal{C}} \downarrow A$. Dually, there is an isomorphism $\int H^A \cong A \downarrow \mathbb{1}_{\mathcal{C}}$.

To conclude this subsection we examine how the category of elements construction is a faithful functor of type $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{CAT}$, for $\mathcal{C}$ locally small, and characterize its image.

Proposition 1.4.20. *Let $\mathcal{C}$ be a locally small category. The assignment $\int : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Cat}$ defined by $F \mapsto \int F$ and $\eta \mapsto \int \eta := \int \eta$, for every pair of presheaves $F, G \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and every natural transformation $\eta : F \Rightarrow G$, is functorial.*

Proof. Consider the category of elements construction as the assignement $\text{よ}_{\mathcal{C}} \downarrow \ulcorner - \urcorner)$. Then, by remarks 1.4.10 and 1.3.14, $\ulcorner - \urcorner$ and $(\mathrm{Id}_{\text{よ}_{\mathcal{C}}} \downarrow -)$ are functors. So, their composition, $(\mathrm{Id}_{\text{よ}_{\mathcal{C}}} \downarrow \ulcorner - \urcorner)$, is also a functor. □

Remark 1.4.21. By considering the category of elements as in definition 1.4.9, given a natural transformation $\eta : F \Rightarrow G$, $\int \eta$ is defined as follows:

- for every $(A, x) \in \int F$, $\int \eta(A, x) := (A, \eta_A(x))$ and
- for every $(A, x), (B, y) \in \int F$ and $f : (A, x) \to (B, y)$, $\int \eta\ (f) := f$.

Note that $\int \eta\ (f)$ is of type $(A, \eta_A(x)) \to (B, \eta_B(y))$ and that the condition (1.9) follows from naturality of η. That is

$$Gf(\eta_B(y)) = \eta_A(Ff(y)) = \eta_A(x).$$

Combining the above result and the fact that functors preserve isomorphisms (see proposition 1.2.2), we have the following corrolary.

Corollary 1.4.22. *Let $F, G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be presheaves such that $F \cong G$. Then $\int F \cong \int G$.*

Proposition 1.4.23. *The functor $\int : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{CAT}$ is faithful.*

Proof. Let F, G be presheaves on $\mathcal{C}$ and $\theta, \eta : F \to G$ be natural transformations such that $\int \eta = \int \theta$. Then, for every $A \in \mathcal{C}_0$ and every $x \in F(A)$, we have that

$$\int \eta\ (A, X) = \int \theta\ (A, X),$$

which is equivalent to

$$(A, \eta_A(x)) = (A, \theta_A(x)),$$

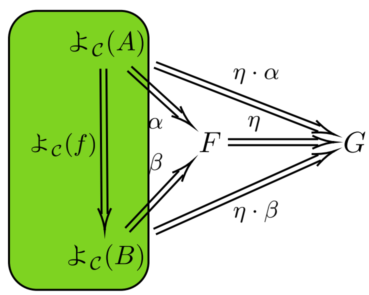
thus $\eta_A(x) = \theta_A(x)$. □

Remark 1.4.24. The same results hold for co-presheaves by duality.

Finally, we can consider the category of elements as a structure canonically equipped with a projection functor. This way we can characterise the functors between categories of elements of presheaves over the same base category.

Proposition 1.4.25. *Let $\mathcal{C}$ be a category, $F, G \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ be presheaves and $(\int F, \pi^F), (\int G, \pi^G)$ be their respective catgegories of elements. Then the homset $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(F, G)$ is in bijection with the subset of $\mathbf{Cat}(\int \mathrm{F}, \int \mathrm{G})$ of functors commuting with the projections.*

Proof. Let $\eta : F \Rightarrow G$. Then the functor $\int \eta$ is such that the following diagram commutes:



for every $f \in \mathcal{C}(A, B)$. The part shown as green above, not including α, β, is the preimage/fiber of both the projections at $f : A \to B$, therefore $\pi^G \circ \int \eta = \pi^F$.

On the other hand let $K : \int F \to \int G$ be a functor such that $\pi^G \circ K = \pi^F$. Firstly, denote $K(A, x)$ by $(KA, K_A x)$, for every object $(A, x) \in \int F$. Then, for every $f : (A, Ff(y)) \to (B, y)$ we get:

$$\pi^G \circ K((A, Ff(y)) \xrightarrow{f} (B, y)) = \pi^F((A, Ff(y)) \xrightarrow{f} (B, y)) \Leftrightarrow$$
$$\pi^G((KA, K_A Ff(y)) \xrightarrow{Kf} (KB, K_B y)) = (A \xrightarrow{f} B) \Leftrightarrow$$
$$(KA \xrightarrow{Kf} KB) = (A \xrightarrow{f} B)$$

thus $KA = A$ and $Kf = f$. Furthermore, by definition we have that $K_A Ff(y) = Gf K_B(y)$, which is a naturality condition, for K_A, if we consider $(K_A : FA \to GA)_{A \in \mathcal{C}_0}$ as a collection of functions. It is easy to check that the above processes are mutually inverse, therefore we have a bijection. □

1.5 Equivalences and adjunctions

One of the things touched upon previously on this chapter was a way to express a strict notion of similarity[6] between categories. This went by the name of isomorphism between categories. Functors played a crucial role in constructing such a notion. In this section we will see weakenings of the notion of isomorphism of categories, resulting in different and more fruitful ways of expressing similarity. These are equivalences and adjunctions.

[6]By similarity, we mean an equivalence relation in **CAT** that expresses structural similarity.

Equivalence of Categories

An isomorphism of categories consists of a pair of functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, such that:

$$F \circ G = \mathbb{1}_{\mathcal{D}} \text{ and } G \circ F = \mathbb{1}_{\mathcal{C}}.$$

Replacing the above equalities with natural isomorphisms results in what is called an equivalence of categories.

Definition 1.5.1. *Two categories $\mathcal{C}, \mathcal{D}$ are called **equivalent**, denoted by $\mathcal{C} \simeq \mathcal{D}$, if there exist functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow GF$, $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{D}}$. The functors F, G are called **equivalences** and the quadruple $(F, G, \eta, \varepsilon)$ is called an **equivalence**.*

A rephrasing of the above definition is that two categories are equivalent, when they are *isomorphic up to natural isomorphism.* Sometimes G is called a **weak inverse** of F and vice and both functors are sometimes said to be **equivalences**.

Remark 1.5.2. Equivalence of categories is an equivalence relation between categories. This statement follows from the properties of whiskering and the fact that every component of Id_F is an isomorphism for every functor F.

Equivalence of categories lets us define a general notion of duality between categories.

Definition 1.5.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. $\mathcal{C}$ is **dual** to $\mathcal{D}$ when $\mathcal{C}^{\mathrm{op}}$ is equivalent to $\mathcal{D}$. If $\mathcal{C}$ is equivalent to $\mathcal{C}^{\mathrm{op}}$, then $\mathcal{C}$ is called **self-dual**.*

To characterize a functor $F : \mathcal{C} \to \mathcal{D}$ as an equivalence one does not need to provide an η, an ϵ and a weak inverse. Instead, one can prove that the conditions of the following proposition hold.

Proposition 1.5.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is an equivalence if and only if F is fully faithful and essentially surjective on objects.*

Proof. Let $(F, G, \eta, \varepsilon)$ be an equivalence. From the fact that $\eta_B \circ - \circ \eta_A^{-1} : \mathcal{C}(A, B) \to \mathcal{C}(GFA, GFB)$ is an isomorphism for every $A, B \in \mathcal{C}_0$, we get that for $f, g \in \mathcal{C}(A, B)$

$$f \neq g \Rightarrow GFf \neq GFg.$$

This implies that $Ff \neq Fg$, which proves that F *is faithful.* Similarly, we get that G is faithful.

To prove fullness, let $h \in \mathcal{D}(FA, FB)$. Using the above isomorphism $\eta_B \circ - \circ \eta_A^{-1}$, there exists a unique $f \in \mathcal{C}(A, B)$ such that $\eta_B \circ f \circ \eta_A^{-1} = Gh$, or equivalently $\eta_B \circ f = Gh \circ \eta_A$. Naturality of η implies that $GFf \circ \eta_A = Gh \circ \eta_A$ and the fact that it is an isomorphism gives

$$GFf = Gh.$$

Having proven that G is faithful we get that $Ff = h$, which means that F *is full.*

Now let $B \in \mathcal{D}_0$. Then $GB \in \mathcal{C}_0$ and $FGB \in \mathcal{D}_0$. Since $\varepsilon_B : B \to FGB$ is an isomorphism $FGB \cong B$. Thus for every $B \in \mathcal{D}_0$ there exists an $A = GB \in \mathcal{C}_0$ such that $FA \cong B$, i.e. F *is essentially surjective on objects.*

To prove the other direction we will construct a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{D}}$.

Since F is essentially surjective on objects we know that for every $B \in \mathcal{D}_0$ there exists a, not necessarily unique, $A \in \mathcal{C}_0$ such that $FA \cong B$. We choose such an isomorphism and denote it by $\varepsilon_B : FA \to B$. Then by picking such an A we define $GB := A$. Now let $g \in \mathcal{D}(B_1, B_2)$. Then F being fully faithfull and $\varepsilon_{B_2}^{-1} \circ g \circ \varepsilon_{B_1} \in \mathcal{D}(FGB_1, FGB_2)$, imply that there exists a unique $f : FB_1 \to FB_2$ such that $Ff = \varepsilon_{B_2}^{-1} \circ g \circ \varepsilon_{B_1}$. Therefore we define $Gg := f$.

Functoriality of G is evident from the fact that given $g_1 : B_1 \to B_2$ and $g_2 : B_2 \to B_3$, $G(g_2 \circ g_1)$ is $\varepsilon_{B_3}^{-1} \circ g_2 \circ g_1 \circ \varepsilon_{B_1} = \varepsilon_{B_3}^{-1} \circ g_2 \circ \varepsilon_{B_2} \circ \varepsilon_{B_2}^{-1} \circ g_1 \circ \varepsilon_{B_1}$ and $G(g_i)$ is $\varepsilon_{B_{i+1}}^{-1} \circ g_i \circ \varepsilon_{B_i}$, for $i = 1, 2$. Similarly, $G(\mathrm{id}_{B_1})$ is $\varepsilon_{B_1}^{-1} \circ \varepsilon_{B_1} = \mathrm{id}_{FGB_1} = F(\mathrm{id}_{GB_1})$ and, since F is a faithful functor, $G(\mathrm{id}_{B_1}) = \mathrm{id}_{GB_1}$.

The collection $(\varepsilon_B)_{B \in \mathcal{D}_0}$ is readily seen to constitute a natural isomorphism $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{D}}$ by the very definition of G.

Finally, since ε is a natural isomorphism we get that for every $A \in \mathcal{C}_0$ the morphism $\varepsilon_{FA}^{-1} : FA \to FGFA$ is an isomorphism. Therefore, by fullness of F and proposition 1.2.2, there exists an isomorphism $\eta_A : A \to GFA$ such that $F\eta_A = \varepsilon_{FA}^{-1}$. Similarly, for every $f \in \mathcal{C}(A, B)$ we get that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\varepsilon_{FA}^{-1}} & FGFA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle FGFf} \\ FB & \xrightarrow[\varepsilon_{FB}^{-1}]{} & FGFB \end{array}$$

By fullness and faithfulness of F we get that the unique "preimage" of this diagram under F commutes i.e. $\eta_B \circ f = FGf \circ \eta_A$, thus proving that $\eta : \mathbb{1}_{\mathcal{C}} \to GF$ is natural isomorphism. □

Remark 1.5.5. An equivalence between skeletal categories is forced, by essential surjectivity on objects, to be an isomorphism. Similarly, every category is equivalent to a skeletal one.

Proposition 1.5.6. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a fully faithful functor. Then $\mathcal{C}$ is equivalent to a full subcategory $\mathcal{C}'$ of $\mathcal{D}$.*

Proof. Define category $\mathcal{C}'$ as the full subcategory of $\mathcal{D}$ whose class of objects is $\{X \in \mathcal{D}_0 | \exists A \in \mathcal{C}_0 (FA \cong X)\}$. Define a functor $G : \mathcal{C} \to \mathcal{C}'$ such that $G(A) = F(A)$ and $G(f) = F(f)$, for every $A, B \in \mathcal{C}_0$ and $f \in \mathcal{C}(A, B)$. Then, by definition of $\mathcal{C}'$, G is essentialy surjective on objects. Furthermore, G is fully faithful, since F is, therefore $G : \mathcal{C} \to \mathcal{C}'$ is an equivalence. □

Example 1.5.7. Categories such as **Set** and **Vect** have finite or finite dimensional versions, respectively, called **FinSet** and **FdVect**. **FinSet** has finite sets as objects and all functions between them as morphisms, while **FdVect**'s objects and morphisms are finite dimensional vector spaces and all linear maps between them. According to the previous propositions there are equivalences Fin : **FinSet** $\to \mathcal{C}$ and Fd : **FdVect** $\to \mathcal{D}$, where $\mathcal{C} \hookrightarrow$ **Set** and $\mathcal{D} \hookrightarrow$ **Vect** are full subcategories.

The same holds for the skeleton, ω, of **FinSet**. Its set of objects is $\mathbb{N}$, i.e. the natural numbers viewed either as sets or as well-ordered sets, and its morphisms are the corresponding functions between them, not only the order preserving ones.

Example 1.5.8. There is a kind of categories called **cliques**. These are categories equivalent to the terminal category **1**. Thus, they are many object versions of the terminal category and their hom-sets have exactly one morphism, due to equivalences being fully faithful. Such unique morphisms between different objects are necessarily isomorphisms due to essential surjectivity of equivalences. Surprisingly every object of such a category is terminal, but also all terminal objects of a category, including **Cat**, form a clique.

Example 1.5.9. A good guess is that every first year student learning linear algebra is wondering either why do we need vector space theory when matrix theory is sufficient enough, or the converse. This question hints at a deep connection between linear transformations and matrices, at the level of their corresponding categories.

To make this connection clear and rigorous, define $\mathbf{Mat}_{\mathbb{C}}$ to be the category whose objects are natural numbers, whose morphisms $f : n \to m$ are $m \times n$ matrices with complex entries and their composition is the usual matrix multiplication. The identity morphisms in this category are the unit matrices. In this setting we recover[7] column vectors as morphisms of type $1 \to n$ and row vectors as morphisms of type $n \to 1$.

From linear algebra, we know that for every finite dimensional complex vector space, V, there exists unique $n \in \mathbb{N}$ such that $V \cong \mathbb{C}^n$. We also know that every matrix $A : n \to m$ corresponds functorialy to a unique linear map, $f_A : \mathbb{C}^n \to \mathbb{C}^m$. So a functor $F : \mathbf{Mat}_{\mathbb{C}} \to \mathbf{Vect}$ such that $Fn = \mathbb{C}^n$ and $FA = f_A$ is fully faithful. Finally, since every finite dimensional vector space is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, we see that F is also essentially surjective on objects, thus an equivalence. Therefore $\mathbf{Mat}_{\mathbb{C}} \simeq \mathbf{Vect}$.

Two aspects of adjunctions

Adjunctions are another (weaker) way of expressing "similarity" between categories. As we saw in the case of equivalences, there arises some sort of asymmetry between equivalent categories. Adjunctions give rise to a greater asymmetry as we will see below. In general there are three ways of thinking of adjunctions, two of which will be presented in what follows.

Adjunction as a natural transformation

Definition 1.5.10. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is **left adjoint** to G(equivalently G is **right adjoint** to F) and write $F \dashv G$ when there exists a natural isomorphism*

$$\overline{(\ )} : \mathcal{D}(F-, -) \Rightarrow \mathcal{C}(-, G-).$$

*We say that F and G are a pair of **adjoint functors** and that they form an **adjunction**.*

The naturality condition in the definition above means that for all morphisms $(f^{op}, g) : (A, B) \to (A', B')$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$, the following diagram in **Set** commutes:

$$\begin{array}{ccc} \mathcal{D}(FA, B) & \xrightarrow{\overline{(\ )}} & \mathcal{C}(A, GB) \\ {\scriptstyle g \circ - \circ Ff} \downarrow & & \downarrow {\scriptstyle Gg \circ - \circ f} \\ \mathcal{D}(FA', B') & \xrightarrow[\overline{(\ )}]{} & \mathcal{C}(A', GB') \end{array}$$

which means that for every $h \in \mathcal{D}(FA, B)$,

$$\overline{g \circ h \circ Ff} = Gg \circ \overline{h} \circ f. \tag{1.10}$$

Denoting the inverse of $\overline{(\ )}$ by $\overline{(\ )}$ again, we get that a double bar is like no bar at all. Then by the above diagram we also get that for every $u \in \mathcal{C}(A, GB)$:

$$\overline{Gg \circ u \circ f} = g \circ \overline{u} \circ Ff. \tag{1.11}$$

Substituting once f and once g with identities on A and B respectively, we get the following pairs of equations for $h \in \mathcal{D}(FA, B)$ and $u \in \mathcal{C}(A, GB)$:

$$\overline{h \circ Ff} = \overline{h} \circ f \text{ and } \overline{g \circ h} = Gg \circ \overline{h}, \tag{1.12}$$

$$\overline{u \circ f} = \overline{u} \circ Ff \text{ and } \overline{Gg \circ u} = g \circ \overline{u} \tag{1.13}$$

The above equations express naturality in each variable seperately.

[7] This is analogous to the natural isomorphism between a finite dimensional vector space and its double dual.

Example 1.5.11. A fact about vector spaces is that for every function f from a basis S of a vector space FS to another vector space V, there exists a unique linear map $\overline{f} : FS \to V$ that extends f. To be more precise the linear map $\overline{f}$ extends the function $f : S \to UV$, where UV is the underlying set of V. Of course this is an adjunction between F, the free, and U the forgetful functors between $\mathbf{Vect}_{\mathbb{F}}$ and $\mathbf{Set}$. Thus we have $F \dashv U$.

Remark 1.5.12. The above example is an instance of a more general concept about free and forgetful functors. Such an adjunction arises not just between $\mathbf{Vect}_{\mathbb{F}}$ and $\mathbf{Set}$ but between $\mathbf{Set}$ and a concrete category whose objects have a certain algebraic structure and its morphisms are the appropriate homomorphisms. This is also the case between a category with less structured objects and a category with more structured ones. Examples of this fact are adjoint pairs of free and forgetful functors between **Set** and **Mon** or **Ab** and **Ring** and so on.

Example 1.5.13. Let $D : \mathbf{Set} \to \mathbf{Top}$ be the functor which sends a set X to the discrete topological space $(X, \mathcal{T}_D)$ and a function to itself. Since every function out of a discrete space is always continuous D is a functor. Considering $U : \mathbf{Top} \to \mathbf{Set}$ as a forgetful functor that "forgets" the topology on a set, we can easily check that:

$$\mathbf{Top}(DX, Y) \cong \mathbf{Set}(X, UY).$$

Thus, there exists an adjunction $D \dashv U$.

Similarly, let $I : \mathbf{Set} \to \mathbf{Top}$ be the functor which sends a set Y to the indiscrete topological space $(Y, \mathcal{T}_I)$ and a function between sets to itself. Since every function into an indiscrete space is continuous, we can see why I is a functor and that:

$$\mathbf{Top}(X, IY) \cong \mathbf{Set}(UX, Y).$$

Thus there exists an adjunction $U \dashv I$. Combining the above we get a so called **adjoint triple** $D \dashv U \dashv I$.

Example 1.5.14. In the case that both $\mathcal{C}$ and $\mathcal{D}$ are posetal categories, or preorders, an adjunction between them is called a **Galois connection**. To be precise let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be monotone functions. Then it is said that they form a galois connection if for every $x \in \mathcal{C},\ y \in \mathcal{D}$,

$$f(x) \leq y \Leftrightarrow x \leq g(y).$$

Remark 1.5.15. An interesting thing to notice is that for every category $\mathcal{C}$ we have an adjunction $\mathbb{1}_{\mathcal{C}} \dashv \mathbb{1}_{\mathcal{C}}$. Furthermore, given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and adjoint pairs $F_1 \dashv G_1$, $F_2 \dashv G_2$, such that $F_1 : \mathcal{C} \to \mathcal{D}$, $F_2 : \mathcal{D} \to \mathcal{E}$, $G_2 : \mathcal{E} \to \mathcal{D}$ and $G_1 : \mathcal{D} \to \mathcal{E}$, there exists an adjunction $F_2 \circ F_1 \dashv G_1 \circ G_2$. This is clear from the fact that:

$$\mathcal{E}(F_2 \circ F_1 A, B) \cong \mathcal{D}(F_1 A, G_2 B) \cong \mathcal{C}(A, G_1 \circ G_2 B),$$

for every $A \in \mathcal{C}_0, B \in \mathcal{E}_0$. Therefore, there is a large category with locally small categories as objects and adjoint pairs as morphisms, composed in the above way.

A valid question at this point is the following. Given a functor, $F : \mathcal{C} \to \mathcal{D}$, who has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, what functors, apart from G, could form an adjunction with F.

Proposition 1.5.16. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{C}$ form adjunctions $F \dashv G$ and $F \dashv G'$ (respectively $G \dashv F$ and $G' \dashv F$). Then $G \cong G'$.*

Proof. Since $F \dashv G$ and $F \dashv G'$, we have for every $A \in \mathcal{D}_0, B \in \mathcal{C}_0$ the following isomorphisms:

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B) \cong \mathcal{C}(A, G'B)$$

which are natural in both A and B. Therfore, $H_{GB} \cong H_{G'B}$ and since the Yoneda embedding is fully faithful we arrive at $GB \cong G'B$, naturaly in B. Thus $G \cong G'$. □

Universal constructions via adjunctions

Adjointness implies representability of a family of presheaves and a family of copresheaves.

Proposition 1.5.17. *Let $F \dashv G$ be an adjunction, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then the functors $\mathcal{C}(A, G-)$ and $\mathcal{D}(F-, B)$ are representable for every $A \in \mathcal{C}_0$ and $B \in \mathcal{D}_0$.*

Proof. For every $A \in \mathcal{C}_0$, $B \in \mathcal{D}_0$, $\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B)$ naturally in A and B. Therefore $\mathcal{C}(A, G-)$ is represented by FA. Similarly, $\mathcal{D}(F-, B)$ is represented by GB. □

Adjoint functors, provide a way of expressing universal constructions. We will show some aspects of this line of thought in the following cases.

Proposition 1.5.18. *Let $\mathcal{C}$ be a category and let $\mathbf{1}$ be the one object terminal category. The following are equivalent:*

1. *The unique functor $e_{\mathcal{C}} : \mathcal{C} \to \mathbf{1}$ has a left (respectively right) adjoint.*
2. *$\mathcal{C}$ has an initial (respectively terminal) object.*

Proof. Let $\ulcorner I \urcorner : \mathbf{1} \to \mathcal{C}$ be a functor such that $\ulcorner I \urcorner \dashv e_{\mathcal{C}}$. Then, for every $A \in \mathcal{C}$,

$$\mathcal{C}(\ulcorner I \urcorner(1), A) \cong \mathbf{1}(1, e_{\mathcal{C}}(A)) = \mathbf{1}(1, 1),$$

where 1 is the unique object of $\mathbf{1}$. Since $\mathbf{1}(1, 1)$ is a one element set, we get that for every $A \in \mathcal{C}$ there exists a unique $e_A = \overline{\mathrm{id}_1} : I(1) \to A$. So $\ulcorner I \urcorner(1)$ is initial in $\mathcal{C}$.

If $\mathcal{C}$ has an initial object I, then the functor $\ulcorner I \urcorner : \mathbf{1} \to \mathcal{C}$, such that $\ulcorner I \urcorner(1) = I$ is a left adjoint of $e_{\mathcal{C}}$. Indeed, in this case

$$\mathcal{C}(I, A) = \mathcal{C}(\ulcorner I \urcorner(1), A) \cong \mathbf{1}(1, 1) = \mathbf{1}(1, e_{\mathcal{C}}(A)),$$

holds, since initiality of I implies that $\mathcal{C}(I, A)$ is a one-element set. Dual reasoning proves that a right adjoint, to $e_{\mathcal{C}}$, exists if and only if $\mathcal{C}$ has a terminal object. □

In the above proposition, we see that two levels of abstraction interrelate. To state this clearly, we see that a construction, such as the terminal category, can be used to define universal constructions inside particular categories. A similar fact is also highlighted in the following.

Proposition 1.5.19. *Let $\mathcal{C}$ be a category. The following are equivalent.*

1. *$\mathcal{C}$ has all finite products (respectively coproducts).*
2. *The diagonal functor $d_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ has a right (respectively left) adjoint.*

Proof. If $\mathcal{C}$ has all finite products, then there exists a functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that for every $X, A, B \in \mathcal{C}_0$,

$$\mathcal{C}(X, A \times B) \cong \mathcal{C}(X, A) \times \mathcal{C}(X, B) = \mathcal{C} \times \mathcal{C}((X, X), (A, B)) = \mathcal{C} \times \mathcal{C}(d_{\mathcal{C}}(X), (A, B)).$$

Functoriality of the product and the hom functors implies that the above isomorphism is natural. Therefore, $d_{\mathcal{C}} \dashv \times$.

If $d_{\mathcal{C}}$ has a right adjoint, G, then

$$\begin{gathered}\mathcal{C} \times \mathcal{C}(d_{\mathcal{C}}(X), (A, B)) \cong \mathcal{C}(X, G(A, B)) \Leftrightarrow \\ \mathcal{C}(X, A) \times \mathcal{C}(X, B) \cong \mathcal{C}(X, G(A, B)),\end{gathered}$$

which is the universal property of the product. Since adjoints are unique up to natural isomorphism, we conclude that $G \cong \times$, that is $\mathcal{C}$ has all finite products. Dual reasoning implies that a left adjoint to the diagonal functor is actually a coproduct functor. □

In the above cases we see that adjoint functors can be used for the formalisation of universal properties. Note, though, that expressing universal properties through adjoints to special kinds of morphisms in **CAT**, such as $e_{\mathcal{C}}$ or $d_{\mathcal{C}}$ above, is restrictive in the following sense. In the case of the diagonal functor $d_{\mathcal{C}}$, a left or right adjoint exists, if and only if $\mathcal{C}$ has all finite (co)products. Since the existence of a (co)product of two objects does not guarantee the existence of all finite (co)products, an adjunction such as the above cannot capture the universal construction "locally". So in this sense adjunctions express universal constructions "globaly".

To formalise this, we firstly introduce the following definition, based on example 1.4.15.

Definition 1.5.20. *Let $\mathcal{I}$ be a small category and $\mathcal{C}$ a locally small category. We call $\mathcal{C}$ $\mathcal{I}$-**complete** if every diagram $D : \mathcal{I} \to \mathcal{C}$ has a limit. Dually, we call $\mathcal{C}$, $\mathcal{I}$-**cocomplete**, if every diagram $D : \mathcal{I} \to \mathcal{C}$ has a colimit. In each case, we call $\mathcal{I}$, an **index** category.*

Proposition 1.5.21. *Let $\mathcal{I}$ be an index category and $\mathcal{C}$ be locally small. Then $\mathcal{C}$ is $\mathcal{I}$-complete if and only if the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ has a right adjoint.*

Proof. Assume that $\mathcal{C}$ is $\mathcal{I}$-complete. Then for every $D :\in \mathcal{C}^{\mathcal{I}}$, the functor $\mathcal{C}^{\mathcal{I}}(\Delta_{-}, D)$ is representable. This means that there exists an assignement of objects $\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, which can easily be seen to constitute a functor, such that

$$\mathcal{C}^{\mathcal{I}}(\Delta_X, D) \cong \mathcal{C}(X, \lim D),$$

naturally in $X \in \mathcal{C}_0$. Thus $\Delta_{-} \dashv \lim$.

Now assume that Δ_{-} has a right adjoint, say $\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$. Then by proposition 1.5.17

$$\mathcal{C}^{\mathcal{I}}(\Delta_X, D) \cong \mathcal{C}(X, \lim D),$$

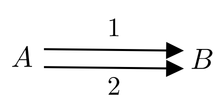
naturally in $X \in \mathcal{C}_0$ and $D \in \mathcal{C}^{\mathcal{I}}$. Thus, $\mathcal{C}^{\mathcal{I}}(\Delta_{-}, D)$ is representable for every $D \in \mathcal{C}^{\mathcal{I}}$, which implies that every diagram of shape $\mathcal{I}$ has a limit. □

Remark 1.5.22. Dually, a category is $\mathcal{I}$-cocomplete if and only if the diagonal $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ has a left adjoint.

Remark 1.5.23. Note that uniqueness of adjoints up to isomorphism implies uniqueness of limits and colimits up to isomorphism.

We now introduce two special cases of limits, aside from products and terminal objects, by picking suitable index categories.

Example 1.5.24 ((Co)Equalizers). Let $\mathcal{I}$ be the category with two objects and two non-identity morphisms, given in diagrammatic form as follows:



So a functor $D : \mathcal{I} \to \mathcal{C}$ to an arbitrary category, $\mathcal{C}$, is an $\mathcal{I}$-shaped diagram inside $\mathcal{C}$. Thus a cone over D, is a pair (X, η) such that

$$\eta_B = D_1 \circ \eta_A \text{ and } \eta_B = D_2 \circ \eta_A$$

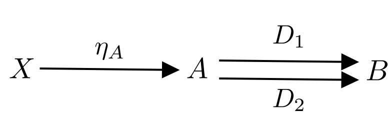
and a cocone under D is a pair (Y, θ) such that

$$\theta_A = \theta_B \circ D_1 \text{ and } \theta_A = \theta_B \circ D_2.$$

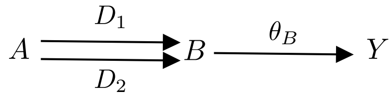
Therefore we get

$$D_1 \circ \eta_A = D_2 \circ \eta_A \text{ and } \theta_B \circ D_1 = \theta_B \circ D_2$$

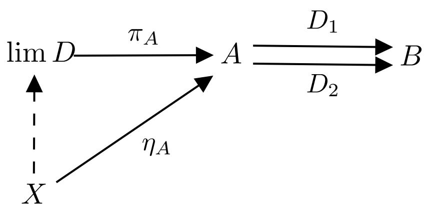
and a cone over such a diagram looks as follows:

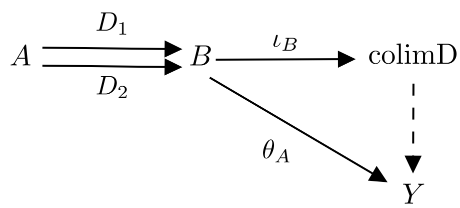


while a cocone under F looks like:



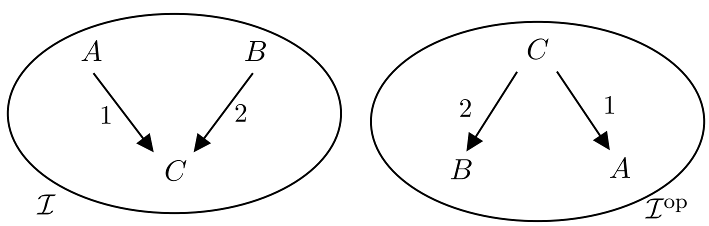
but in any of the above cases one of the two (co)projections of the (co)cone is redundant. A limit of D is a terminal cone $(\lim D, \pi)$ and a colimit is an initial cocone (colimD, ι) satisfying the following universal properties:



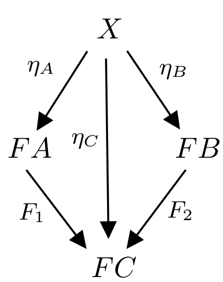


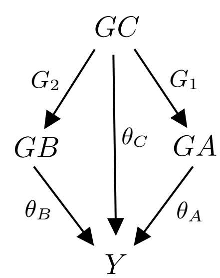
These two universal constructions are called the *equalizer* and the *coequalizer* of D_1 and D_2, repsectively. Note that in this case, proposition 1.4.16, gives that the projection of an equaliser, π_A, is a monomorphism, while the coprojection of a coequaliser is an epimorphism.

Example 1.5.25 (Pullbacks-Pushouts). Let $\mathcal{I}$ be the category with three objects and two non-identity morphisms and $\mathcal{I}^{\mathrm{op}}$ its opposite category given as follows:

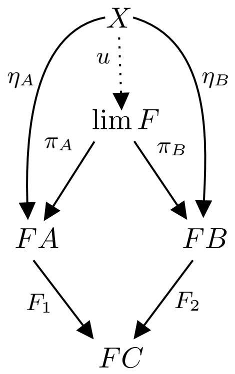


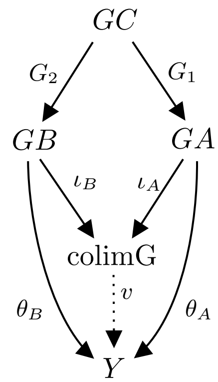
Then a functor $F : \mathcal{I} \to \mathcal{D}$ is a diagram of shape $\mathcal{I}$ inside $\mathcal{D}$ and a functor $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{D}$ is a $\mathcal{I}^{\mathrm{op}}$-shaped diagram in $\mathcal{D}$. A cone over F is a pair (X, η) and a cocone under G is a pair (Y, θ) such that the following diagrams commute inside $\mathcal{D}$.





From the naturality of η and θ we see that their C-component is redundant leaving us with two commutative squares. The limit of F and the colimit of G are pairs $(\lim F, \pi)$ and (colimG, ι) such that for every (X, η) and (Y, θ) as above, there exist unique morphisms $u : X \to \lim F$ and $v : \mathrm{colimG} \to \mathrm{Y}$ such that the following diagrams commute:





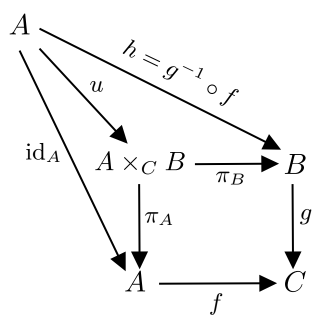
These two universal constructions, $\lim F$ and colimG, are named a **pullback** and a **push-out**, respectively. A common notation for the apex of a *pullback of* A *and* B *with respect to* C is $A \times_C B$ and for their push-out it is $A +_C B$. The apex of pullback in a diagram is sometimes denoted with a "⌟" and for the pushout it is "⌜".

It is easy using the universal property of the pullback and the pushout to prove that they are commutative up to isomorphism, as is the case for ordinary products and co-products.

Before proceeding with other constructions formalised by the notion of adjunction, we give a usefull property of pullbacks.

Proposition 1.5.26. *Let* $f : A \to C,\ g : B \to C$ *inside a category* $\mathcal{C}$ *and let* $(A \times_C B, \pi_A, \pi_B)$ *be a pullback. If* g *is an isomorphism, then* π_A *is an isomorphism.*

Proof. Since g is an isomorphism, there is a map $h : A \to B$, defined as $h := g^{-1} \circ f$, such that (A, id_A, h) is a cone over the diagram defined by A, B, C, f and g. So, by the universal property of the pullback, there exists a unique $u : A \to A \times_C B$ such that the following diagram commutes:



So, $\pi_A \circ u = \mathrm{id}_A$ and $\pi_B \circ u = h$. Now notice that $\pi_A \circ u \circ \pi_A = \pi_A$ and

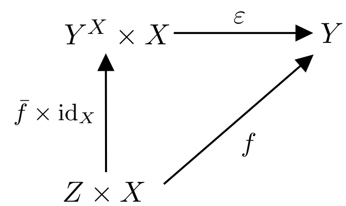
$$\begin{aligned} \pi_B \circ u \circ \pi_A &= h \circ \pi_A \\ &= g^{-1} \circ f \circ \pi_A \\ &= g^{-1} \circ g \circ \pi_B \\ &= \pi_B, \end{aligned}$$

which means that $u \circ \pi_A$ is the unique endomorphism of $A \times_C B$, such that $(A \times_C B, \pi_A, \pi_B)$ is a pullback. Thus, $u \circ \pi_A = \mathrm{id}_{A \times_C B}$, which proves that π_A is an isomorphism. □

Cartesian Closed Categories

There is another universal construction, which is easily expressed by the adjunction concept.

Definition 1.5.27. *Let* $\mathcal{C}$ *be a category with all finite products and let* $X, Y \in \mathcal{C}_0$*. A pair* (Y^X, ε)*, where* $Y^X \in \mathcal{C}_0$ *and* $\varepsilon : Y^X \times X \to Y$*, is called an* ***exponential object*** *if for every pair* (Z, f)*, such that* $Z \in \mathcal{C}_0$ *and* $f : (Z \times X) \to Y$*, there exists a unique morphism* $\bar{f} : Z \to Y^X$ *which makes the following diagram commute:*



An object $Y \in \mathcal{C}_0$ is called ***exponentiating*** *if for every object $X \in \mathcal{C}_0$, Y^X exists. An object $X \in \mathcal{C}_0$ is called* ***exponentiable*** *if for every $Y \in \mathcal{C}_0$, Y^X exists. A cartesian category, every object of which is exponentiable (or equivalently exponentiating), is called* ***cartesian closed****.*

Remark 1.5.28. At this point it is useful to observe that the above definition can be seen as naming a bijection between hom sets, i.e.

$$\overline{(-)} : \mathcal{C}(Z, Y^X) \to \mathcal{C}(Z \times X, Y).$$

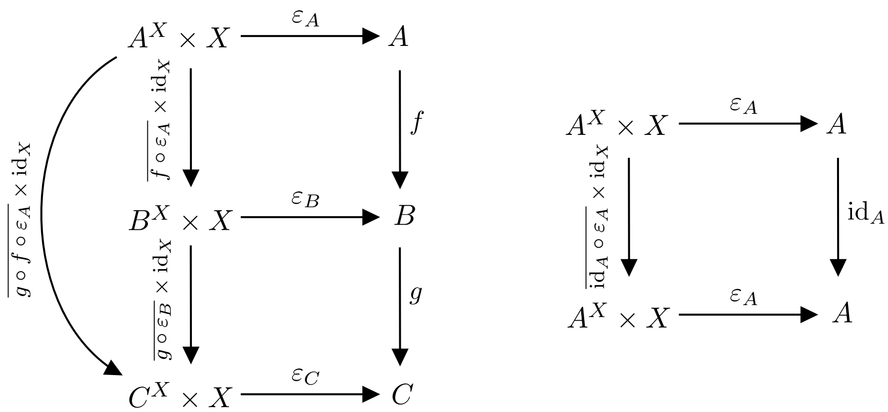
This isomorphism is defined as

$$\overline{(-)} := \varepsilon \circ (- \times \mathrm{id}_X)$$

and its inverse is also noted as a $\overline{(-)}$. In the following we will show that this bijection is natural in X, Y and Z, therefore an adjunction.

Proposition 1.5.29. *Let $\mathcal{C}$ be a cartesian category and $X, Y \in \mathcal{C}_0$. If X is exponentiable then $(-)^X : \mathcal{C} \to \mathcal{C}$ is a covariant functor. If Y is exponentiating, then Y^- is a contravariant functor.*

Proof. Let $A, B, C \in \mathcal{C}_0$, $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$. If X is exponentiable then by the universal property of exponentials we get that the following diagrams commute.



Defining $f^X := \overline{f \circ \varepsilon_A} = \overline{\varepsilon_B \circ (\overline{f \circ \varepsilon_A} \times \mathrm{id}_X)}$, we get that $g^X := \overline{g \circ \varepsilon_B}$, $(g \circ f)^X = \overline{g \circ f \circ \varepsilon_A}$ and $(\mathrm{id}_A)^X = \overline{\varepsilon_A}$. Now the uniqueness condition of the universal property implies the following equalities:

$$(g \circ f)^X = g^X \circ f^X \text{ and } (\mathrm{id}_A)^X = \mathrm{id}_{A^X},$$

which shows that $(-)^X$ is a covariant functor.

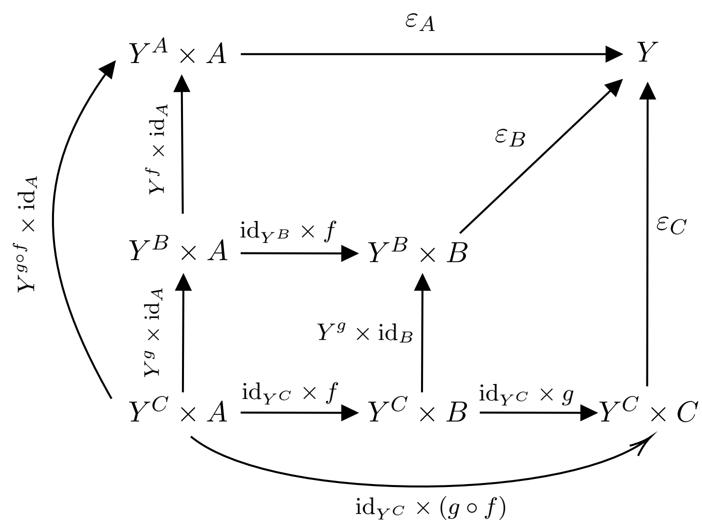
Similarly, given $f : A \to B$ we define $Y^f : Y^B \to Y^A$, as

$$Y^f := \overline{\varepsilon_B \circ (\mathrm{id}_{Y^B} \times f)}.$$

Well definiteness of Y^f is provided by the universal property of exponentials. By this definition and the universal property of the product it is immediate that

$$Y^{\mathrm{id}_A} = \overline{\varepsilon_A \circ (\mathrm{id}_{Y^A} \times \mathrm{id}_A)} = \overline{\varepsilon_A} = \mathrm{id}_{Y^A}.$$

Again by this definition and the universal property of exponentials the following diagrams commute,



which shows that $Y^{g\circ f} = Y^f \circ Y^g$, thus Y^- is a contravarinat functor. □

Proposition 1.5.30. *Let $\mathcal{C}$ be a cartesian category. The following are equivalent:*

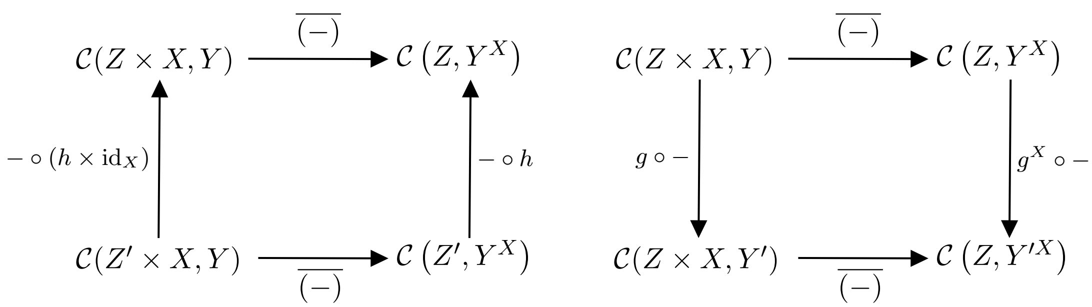
1. *$\mathcal{C}$ is cartesian closed.*

2. *For every object $X \in \mathcal{C}_0$, the functor $- \times X$ has a right adjoint G_X.*

Proof. Let $\mathcal{C}$ be cartesian closed. Then, since every object X is exponentiable we get the following isomorphsim

$$\overline{(-)} : \mathcal{C}(Z \times X, Y) \to \mathcal{C}(Z, Y^X),$$

which is natural in Z and Y. Indeed, functoriality of the products, the exponentiables and of the hom functors, implies the commutativity of the following naturality squares:



for $h \in \mathcal{C}(Z, Z')$, $g \in \mathcal{C}(Y, Y')$. Therefore, we see that the functor $- \times X$ has $(-)^X$ as a right adjoint, for every $X \in \mathcal{C}_0$.

Let $- \times X$ have a right adjoint G_X, for every $X \in \mathcal{C}_0$. Then, for every $Z, Y \in \mathcal{C}_0$, we have a natural isomorphism

$$\mathcal{C}(Z \times X, Y) \cong \mathcal{C}(Z, Y^X),$$

which by the uniqueness of adjoints is identified with the exponential functor. Therefore, we have that every object of $\mathcal{C}$ is exponentiable, so $\mathcal{C}$ is cartesian closed. □

Remark 1.5.31. In a cartesian closed category, $\mathcal{C}$, we can interpret an exponential object, Y^X, as an object of morphisms, sometimes called an **internal hom** and denoted by $[X, Y]$. Based on the isomorphism $T \times X \cong X$ we can see that

$$\mathcal{C}(X, Y) \cong \mathcal{C}(T \times X, Y) \cong \mathcal{C}(T, Y^X).$$

Thus, the global elements of the object Y^X are in natural bijection with the set of morphisms $\mathcal{C}(X, Y)$.

If we shift our focus to generalised elements of Y^X and of $Z \times X$, say $\bar{g} \in_Z Y^X$ and $(a, x) \in_A Z \times X$, then the universal property of exponentials yields

$$\varepsilon_Y(\bar{g}(a), x) = g(a, x).$$

This is the reason why every member of the family $(\varepsilon_A)_{A \in \mathcal{C}}$ is called an evaluation morphism, which is actually a component of the natural transformation $\varepsilon : (-)^X \times X \Rightarrow \mathbb{1}_{\mathcal{C}}$.

Having proven that both Y^- and $(-)^X$ are functors, for every X, Y in a cartesian closed category, we can reconstruct a bifunctor $(-)^- : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, called the **internal hom functor** of the cartesian closed category. This functor assigns to a pair of morphisms $(f, g) \in \mathcal{C}(X', X) \times \mathcal{C}(Y, Y')$ the morphism $g^f : Y^X \to Y'^{X'}$, which in **Set** corresponds to the function $g \circ - \circ f : Y^X \to Y'^{X'}$. Since the exponential comes with the evaluation morphism, while the internal hom functor is a bifunctor we will denote its evaluation morphism by ε_Y^X, for $(X, Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

Example 1.5.32. **Set** equipped with the usual cartesian product and a one element set as terminal, is a cartesian category. The universal property of the exponential in **Set** holds. Indeed, given $X, Y, Z \in \mathbf{Set}_0$ and a function $f : Z \times X \to Y$ there exists a unique function $\bar{f} : Z \to Y^X$ such that $\bar{f}(z)(x) := f(z, x)$. Therefore, any object of the form Y^X, for sets X, Y can be identified as the set of all functions $f : X \to Y$ and the map ε is just the evaluation map $\varepsilon(f, x) := f(x)$. The function, $\bar{f}$ is called the currying of f and its existence guarantees that **Set** is cartesian closed.

Example 1.5.33. The categories **CAT** and **Cat**, equipped with the category **1** as terminal and the cartesian product, are cartesian. In these cases an exponential object, $\mathcal{D}^{\mathcal{C}}$ corresponds to the functor category $[\mathcal{C}, \mathcal{D}]$. The evaluation functor $\varepsilon : \mathcal{D}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{D}$ takes a pair $(F, A) \in (\mathcal{D}^{\mathcal{C}} \times \mathcal{C})_0$ to FA and takes a natural transformation $\alpha : [\mathcal{C}, \mathcal{D}](F, G)$ and a morphism $f \in \mathcal{C}(A, B)$ to the diagonal of the following naturality square:

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\alpha_B} & GB \end{array}$$

Example 1.5.34. A similar example are the categories of preorders and posets. They are cartesian with the usual cartesian product, but also every object is exponetiable. To see this we define the exponential of two preorders (orders), $(X, \leq), (Y, \leq')$ as the set Y^X equipped with the relation $\preceq$ such that

$$f \preceq g :\Leftrightarrow f(x) \leq g(x) \ \forall x \in X.$$

It is fairly obvious that the currying of a monotone map, from a product of posets, is a monotone map $\tilde{(\ )} : \mathbf{Ord}(\mathrm{X} \times \mathrm{Y}, \mathrm{Z}) \to \mathbf{Ord}(\mathrm{Y}, \mathrm{Z}^{\mathrm{X}})$ and a natural isomorphism.

The categories **Vect** and **Hilb** cannot be examples of such a notion because their tensor product is not cartesian. They do have a similar property, though, but we would have to generalise the notion of closure to categories with a non-carteisan product functor.

To conclude this section we give some properties of cartesian closed categories. Firstly, since the cartesian product of any two objects, in a cartesian closed category $\mathcal{C}$, is symmetric up to isomorphism, then, for every $X, Y \in \mathcal{C}_0$, we get the isomorphism

$$\mathcal{C}(X \times Y^X, Y) \cong \mathcal{C}(Y^X \times X, Y),$$

and by the adjunction property of exponentials,

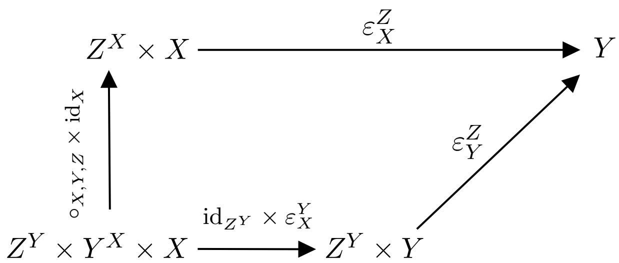
$$\mathcal{C}(X, Y^{Y^X}) \cong \mathcal{C}(Y^X \times X, Y).$$

Now this property in **Set** is quite trivial, since any $x \in X$ gives a function $ev_x : Y^X \to Y$ which acts by evaluation at x. Any such function for a given x has a unique representative $\varepsilon : Y^X \times X \to Y$, for which $\varepsilon(-, x) = ev_x$.

Secondly, internal homs come with internal composition. This means assigning a unique **composition morphism** to a product of two compatible hom objects. This goes as follows. Given a cartesian closed category, $\mathcal{C}$, according to the universal property of exponentials there exists a unique morphism $\circ_{X,Y,Z} : Z^Y \times Y^X \to Z^X$ such that

$$\circ_{X,Y,Z} := \overline{\varepsilon_Y^Z \circ (\mathrm{id}_{Y^Z} \times \varepsilon_X^Y)}.$$

This stems from the following commutative diagram:



and uniqueness is implied by the universal property of exponentials.

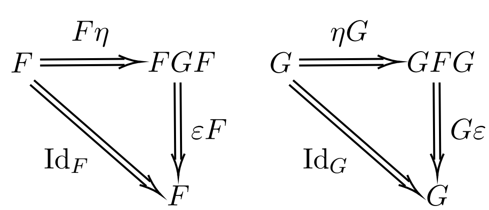
Adjunction via unit and counit

In the previous section we saw that adjunctions are provided by natural isomorphisms between specific hom-sets. To get a further insight into such a natural isomorphism one might use the same method used for the proof of the Yoneda lemma, i.e. look at what happens to identities. Therefore we give the following definition.

Definition 1.5.35. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors with $F \dashv G$. We define the **unit** of the adjunction as the family of morphisms $(\eta_A : A \to GFA)_{A \in \mathcal{C}_0}$, such that $\eta_A = \overline{\mathrm{id}_{FA}}$. The **counit** of the adjunction is defined as the family of morphisms $(\varepsilon_B : FGB \to B)_{B \in \mathcal{D}_0}$, such that $\varepsilon_B = \overline{\mathrm{id}_{GB}}$.*

Proposition 1.5.36. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors with $F \dashv G$ and let $(\eta_A)_{A \in \mathcal{C}_0}$, $(\varepsilon_B)_{B \in \mathcal{D}_0}$ be the unit and counit respectively. Then*

1. *the unit, η, and the counit, ε, are natural transformations and*
2. *satisfy the following identities:*



The above two identities are called **triangle identities**.

Proof. For 1. we need to show that given $f \in \mathcal{C}(A, A')$ and $g \in \mathcal{D}(B, B')$,

$$GFf \circ \eta_A = \eta_{A'} \circ f \text{ and } g \circ \varepsilon_B = \varepsilon_{B'} \circ FGg.$$

Using the $\overline{(\ )}$ isomorphism we get equivalently that

$$\overline{GFf \circ \eta_A} = \overline{\eta_{A'} \circ f} \text{ and } \overline{g \circ \varepsilon_B} = \overline{\varepsilon_{B'} \circ FGg},$$

or that

$$Ff \circ \overline{\eta_A} = \overline{\eta_{A'}} \circ Ff \text{ and } Gg \circ \overline{\varepsilon_{B'}} = \overline{\varepsilon_B} \circ Gg,$$

which holds by the definition of units and counits.

For the triangle identities it is enough to show that

$$\varepsilon_{FA} \circ F\eta_A = \mathrm{id}_{FA} \text{ and } G\varepsilon_B \circ \eta_{GB} = \mathrm{id}_{GB}$$

or, equivalently, that the transposed versions of the above equations hold for every $A \in \mathcal{C}_0,\ B \in \mathcal{D}_0$. This is indeed the case, since

$$\begin{aligned}\overline{\varepsilon_{FA} \circ F\eta_A} &= \overline{\varepsilon_{FA}} \circ \eta_A \\ &= \mathrm{id}_{GFA} \circ \eta_A \\ &= \overline{\mathrm{id}_{FA}}\end{aligned}$$

and also

$$\begin{aligned}\overline{G\varepsilon_B \circ \eta_{GB}} &= \varepsilon_B \circ \overline{\eta_{GB}} \\ &= \varepsilon_B \circ id_{FGB} \\ &= \overline{\mathrm{id}_{GB}},\end{aligned}$$

therefore the triangle identities hold, for every adjunction. □

The significance of units, counits and triangle identities is that they are enough data to form an adjunction, i.e. to create the $\overline{(\ )}$ natural isomorphism of hom-sets. The following proposition makes this precise.

Proposition 1.5.37. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D},\ G : \mathcal{D} \to \mathcal{C}$ be functors. Assume there exist natural transformations $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{D}}$ satisfying the triangle identities. Then the unit and the counit determine a unique adjunction $F \dashv G$.*

Proof. We will prove the statement in four steps. The first step will be the definition of the candidate $\overline{(\ )}$ function. The second step will be to show naturality and the third will be to show bijectivity. The fourth step will be to show that this natural isomorphism is unique, which will also mean that it is well defined.

1. Let $A \in \mathcal{C}_0, B \in \mathcal{D}_0$. Define

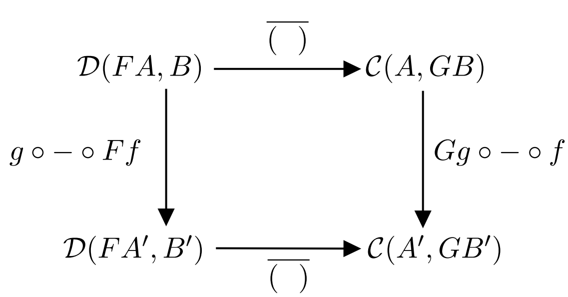
   $$\overline{h} := Gh \circ \eta_A \text{ and } \overline{u} := \varepsilon_B \circ Fu$$

   for every $h : FA \to B$ and $u : A \to GB$.

2. Let $f : A' \to A$ and $g : B \to B'$. Then for every $h : FA \to B$ and $u : A \to GB$, we have

   $$\begin{aligned}\overline{g \circ h \circ Ff} &= G(g \circ h \circ Ff) \circ \eta_{A'} && \\ &= Gg \circ Gh \circ GFf \circ \eta_{A'} && \text{by functoriality of } G \\ &= Gg \circ Gh \circ \eta_A \circ f && \text{by naturality of } \eta \\ &= Gg \circ \overline{h} \circ f && \text{by definition of } \overline{(\ )}\end{aligned}$$

   which makes the following naturality square commute:



Similarly, by functoriality of F and naturality of ε, we get that the mapping $u \mapsto \overline{u} = \varepsilon_B \circ Fu$, for every $u \in \mathcal{C}(A, GB)$ is natural in both A and B.

3. To prove bijectivity we only need to show that $\overline{\overline{h}} = h$ and $\overline{\overline{u}} = u$ for every $h : FA \to B$ and $u : A \to GB$. In the first case we have that:

$$\begin{aligned}
\overline{\overline{h}} &= \overline{Gh \circ \eta_A} \\
&= \varepsilon_B \circ F(Gh \circ \eta_A) \\
&= \varepsilon_B \circ FGh \circ F\eta_A && \text{by functoriality of } F \\
&= h \circ \varepsilon_{FA} \circ F\eta_A && \text{by naturality of } \varepsilon \\
&= h \circ \mathrm{id}_{FA} && \text{by the triangle identity for } F
\end{aligned}$$

Similarly, in the second case, the triangle identity for G combined with the functoriality of G and the naturality of η proves that $\overline{\overline{u}} = u$. Thus, $\overline{(\ )}$ is a natural isomorphism.

4. Now let $\tilde{(\ )} : \mathcal{D}(F-, -) \Rightarrow \mathcal{C}(-, G-)$ also form an adjunction between F and G. Then by the definition in 1. we get that

$$\tilde{h} = Gh \circ \eta_A \text{ and } \tilde{u} = \varepsilon_B \circ Fu$$

for every $h \in \mathcal{D}(FA, B), u \in \mathcal{C}(A, GB)$. Thus, by the definition of the $\overline{(\ )}$ isomorphism above we get that $\overline{h} = \tilde{h}$ and $\overline{u} = \tilde{u}$. Therefore $\tilde{(\ )} = \overline{(\ )}$.

□

Remark 1.5.38. Note that in the above proof we used a formula for $\overline{(\ )}$ and its inverse given using the unit and the counit. These are:

$$\overline{h} := Gh \circ \eta_A \text{ and } \overline{u} := \varepsilon_B \circ Fu$$

for every $h : FA \to B$ and $u : A \to GB$. We can use this idea to establish the following fact.

Proposition 1.5.39. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors and let $\mathcal{I}$ be a category. Then $F \dashv G$ if and only if the functor $F \circ - : \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ is left adjoint to $G \circ - : \mathcal{D}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$.*

Proof. Firstly, assume that $F \circ - \dashv G \circ -$ and consider $X \in \mathcal{C}_0$ and $Y \in \mathcal{D}_0$. Note that every natural transformation between constant functors $\Delta_X \Rightarrow \Delta_{X'}$ is given by a morphism $f : X \to X'$, that is $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is fully faithfull. By the adjunction $F \circ - \dashv G \circ -$ we get the following natural isomorphism

$$\mathcal{D}^{\mathcal{I}}(F \circ \Delta_X, \Delta_Y) \cong \mathcal{C}^{\mathcal{I}}(\Delta_X, G \circ \Delta_Y),$$

which is equivalent to

$$\mathcal{D}^{\mathcal{I}}(\Delta_{FX}, \Delta_Y) \cong \mathcal{C}^{\mathcal{I}}(\Delta_X, \Delta_{GY})$$

and by Δ providing fully faithful functors, we get

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY).$$

Now assume that $F \dashv G$. We will show that

$$\mathcal{D}^{\mathcal{I}}(F \circ X, Y) \cong \mathcal{C}^{\mathcal{I}}(X, G \circ Y),$$

naturally in $X \in \mathcal{C}^{\mathcal{I}}$ and $Y \in \mathcal{D}^{\mathcal{I}}$, by explicitly constructing the natural isomorphism and its inverse. Let $\eta : \mathbf{1}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbf{1}_{\mathcal{D}}$ be the unit and the counit of the adjunction. Let $\alpha : F \circ X \Rightarrow Y$, $\beta : X \Rightarrow G \circ Y$ be natural transformations and define:

$$\alpha^\sharp = (G \circ \alpha) \cdot (\eta \circ X) : X \Rightarrow G \circ Y \text{ and } \beta^\flat = (\varepsilon \circ Y) \cdot (F \circ \beta) : F \circ X \Rightarrow Y$$

Then,

$$\begin{aligned}
(\alpha^\sharp)^\flat &= ((G\circ\alpha)\cdot(\eta\circ X))^\flat \\
&= (\varepsilon\circ Y)\cdot(F\circ((G\circ\alpha)\cdot(\eta\circ X))) \\
&= (\varepsilon\circ Y)\cdot(F\circ G\circ\alpha)\cdot(F\circ\eta\circ X) && \text{(by functoriality of } \circ) \\
&= (\varepsilon\circ\alpha)\cdot(F\circ\eta\circ X) && \text{(by the interchange law)} \\
&= (\mathbf{1}_{\mathcal{D}}\circ\alpha)\cdot(\varepsilon\circ F\circ X)\cdot(F\circ\eta\circ X) && \text{(by the interchange law)} \\
&= \alpha\cdot(((\varepsilon\circ F)\cdot(F\circ\eta))\circ X) && \text{(by functoriality of } \circ) \\
&= \alpha\cdot(F\circ X) && \text{(by the triangle law for } \varepsilon) \\
&= \alpha
\end{aligned}$$

and similarly $(\beta^\flat)^\sharp = \beta$. Thus we have the desired isomorphism which furthermore is natural as it is comprised of vertical and horizontal compositions of natural transformations. □

We can use the above proposition and the fact that $F\circ\Delta_X = \Delta_{FX}$, for every functor $F:\mathcal{C}\to\mathcal{D}$ and every object $X\in\mathcal{D}_0$, to prove that *right adjoints preserve limits* and *left adjoints preserve colimits*.

Proposition 1.5.40. *Let $\mathcal{C},\mathcal{D}$ be categories, $\mathcal{I}$ be an index category and let $F:\mathcal{C}\to\mathcal{D}$ be left adjoint to $G:\mathcal{D}\to\mathcal{C}$. If $D:\mathcal{I}\to\mathcal{D}$ has a limit in $\mathcal{D}$ then*

$$\lim(G\circ D)\cong G\lim D.$$

If $S:\mathcal{I}\to\mathcal{C}$ has a colimit in $\mathcal{C}$, then

$$\mathrm{colim(F\circ S)\cong FcolimS.}$$

Proof. Let $\lim D\in\mathcal{D}_0$ be a limit of D. Then for every $X\in\mathcal{C}_0$:

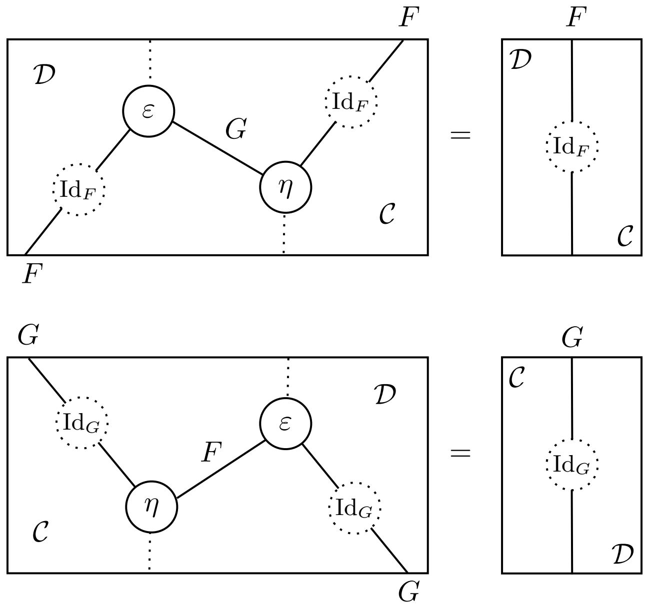
$$\begin{aligned}
\mathcal{C}(X, G\lim D) &\cong \mathcal{D}(FX,\lim D) \\
&\cong \mathcal{D}^{\mathcal{I}}(\Delta_{FX}, D) && \text{(by 1.4.15)} \\
&= \mathcal{D}^{\mathcal{I}}(F\circ\Delta_X, D) \\
&\cong \mathcal{C}^{\mathcal{I}}(\Delta_X, G\circ D) && \text{(by proposition 1.5.39)} \\
&\cong \mathcal{C}(X,\lim(G\circ D)) && \text{(by 1.4.15)}
\end{aligned}$$

so by the Yoneda embedding being fully faithful (see corollary 1.8), $\lim(G\circ D)\cong G\lim D$. Dually, we get that $\mathrm{colim(F\circ S)\cong FcolimS}$. □

One of the many uses of units and counits occurs in cartesian closed categories. The evaluation natural transformation, ε, is the counit of the adjunction $-\times X\dashv -^X$. Another natural transformation which comes naturally with this adjunction is its unit $\eta:\mathbb{1}_{\mathcal{C}}\to(-\times X)^X$, called **coevaluation**. This coevaluation in **Set** is the function which takes a $y\in Y$ and sends it to the function $(y,-):X\to(Y\times X)$, such that $x\mapsto(y,x)$.

Summarizing, we see that an adjunction is defined either by a homset isomorphism or equivalently by the unit and the counit. The homset isomorphism definition takes place inside **Set**, while the unit and counit definition takes place inside **CAT**. The difference to be noted is that the first definition is formulated in terms of morphisms inside the categories, while the second one is in terms of functors and natural transformations, outside the categories involved. This fact allows us to use a tool already developed for **CAT**, the string diagram notation.

The defining properties of an adjunction, in terms of units and counits, are the triangle identities. String diagrams capture these triangle identities as follows.



Conjugate pairs and closed categories

In this part we focus on answering the following question. What is the nature of the family of adjunctions required to define exponential objects and internal hom-functors. In the following, we will denote adjunctions between categories $\mathcal{C}, \mathcal{D}$ as triples $(F, G, \overline{(-)})$. We choose to explicitly state the homset isomorphism of every adjunction involved, in order to distinguish different adjoint pairs between $\mathcal{C}$ and $\mathcal{D}$.

Definition 1.5.41. *Let $\mathcal{C}, \mathcal{D}$ be categories and $(F, G, \overline{(-)})$, $(F', G', \widetilde{(-)})$ be adjunctions between $\mathcal{C}$ and $\mathcal{D}$. Two natural transformations $\sigma : F \Rightarrow F'$ and $\tau : G' \Rightarrow G$ are called **conjugate** if the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D}(F'A, B) & \xrightarrow{\widetilde{(-)}} & \mathcal{C}(A, G'B) \\ {\scriptstyle - \circ \sigma_A} \downarrow & & \downarrow {\scriptstyle \tau_B \circ -} \\ \mathcal{D}(FA, B) & \xrightarrow[\overline{(\ )}]{} & \mathcal{C}(A, GB) \end{array}$$

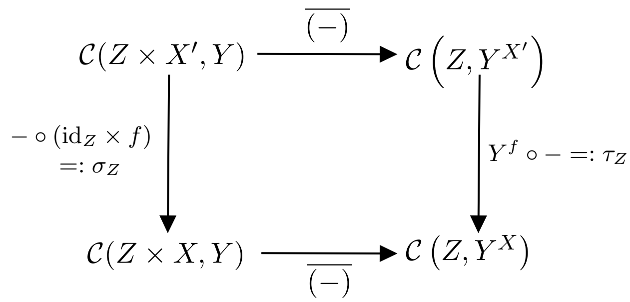
*for every $A \in \mathcal{C}$, $B \in \mathcal{D}$. The pair (σ, τ) is then called a **conjugate pair**.*

A conjugate pair should be seen as a transformation of adjunctions. As a natural transformation transforms a functor to another, so a conjugate pair transforms an adjunction to another.

Example 1.5.42. Let $\mathcal{C}$ be a cartesian closed category. For every object $X \in \mathcal{C}_0$ we have an adjunction $- \times X \dashv -^X$. Functoriality of the product and the exponential induces, for every $f : X \to X'$, natural transformations with components

$$\mathrm{id}_A \times f = \sigma_A : A \times X \to A \times X' \text{ and } A^f = \tau_A : A^{X'} \to A^X.$$

Commutativity of the following diagram



implies that the natural transformations (σ, τ) form a conjugate pair.

Notice that for every $X, X' \in \mathcal{C}_0$ we have an adjunction. Also, notice that every $f : X \to X'$ induces a conjugate pair.

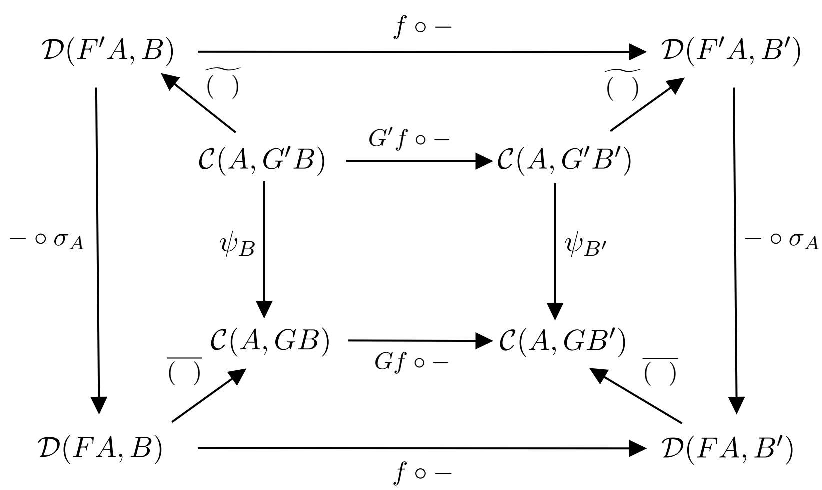
To capture the essence of the above example, we introduce adjunctions with a parameter.

Definition 1.5.43. *Let $\mathcal{C}, \mathcal{D}, \mathcal{P}$ be categories and let $F : \mathcal{C} \times \mathcal{P} \to \mathcal{D}$, $G : \mathcal{D} \times \mathcal{P}^{\mathrm{op}} \to \mathcal{C}$ be functors. If for every $P \in \mathcal{P}$ there exists an adjunction $F(-, P) \dashv G(P, -)$, such that the isomorphism $\bar{(-)} : \mathcal{D}(F(A, P), B) \to \mathcal{C}(A, G(P, B))$ is also natural in P, then we call the triple $(F, G, \bar{(\ )})$ an* ***adjunction with a parameter****.*

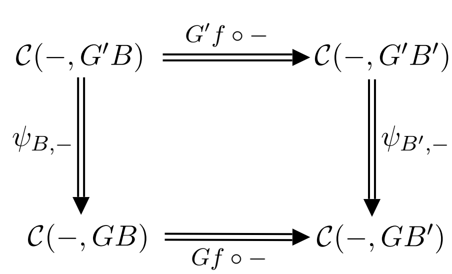
It is obvious that the product and the internal hom functors of a cartesian closed category form an adjunction with a parameter. Now let us return to conjugate pairs.

Proposition 1.5.44. *Let $(F, G, \bar{(\ )})$, $(F', G', \tilde{(-)})$ be adjunctions between $\mathcal{C}, \mathcal{D}$. Then for every natural transformation $\sigma : F \Rightarrow F'$ there exists unique $\tau : G' \Rightarrow G$ such that (σ, τ) is a conjugate pair and vice versa.*

Proof. Let $\sigma : F \Rightarrow F'$. For every $A \in \mathcal{C}_0, B \in \mathcal{D}$, define $\psi_{B,A} : \mathcal{C}(A, G'B) \to \mathcal{C}(A, GB)$ by $\psi_{B,A}(h) := \overline{\tilde{h} \circ \sigma_A}$, for $h \in \mathcal{C}(A, G'B)$. Then in the following diagram, for every $f : \mathcal{D}(B, B')$ and $A \in \mathcal{C}_0$ the lower and upper trapezoids commute by naturality of $\tilde{(-)}$ and $\bar{(\ )}$, and the left and right ones commute by the definition of ψ.



Therefore, the middle diagram also commutes, which shows that $\psi_{-,A}$ is a natural transformation $\psi_{-,A} : \mathcal{C}(A, G'-) \Rightarrow \mathcal{C}(A, G-)$ for every A. Since ψ is natural in A we get the following commutative square inside $[\mathcal{C}, \mathbf{Set}]$



Now since the Yoneda embedding, $GB \mapsto \mathcal{C}(-, GB)$, is fully faithful, for every $B \in \mathcal{D}_0$ there exists unique $\tau_B : G'B \to GB$ such that $- \circ \tau_B = \mathcal{C}(-, \tau_B) = \psi_{B,-}$. Furthermore, the preimage under the Yoneda embedding of the commutative square above also commutes inside $\mathcal{C}$, which shows that $\tau : G' \Rightarrow G$ is a natural transformation.

Note that the definition of $\psi_{B,A}$ shows existence, uniqueness and conjugacy of τ, while the Yoneda embedding being fully faithfull implies uniquess of τ. The other way, τ defining σ uniquely, is similar. □

We will use the above proposition to prove the following theorem.

Theorem 1.5.45. *Let $\mathcal{C}, \mathcal{D}, \mathcal{P}$ be categories, and $F : \mathcal{C} \times \mathcal{P} \to \mathcal{D}$ a functor. If for every $P \in \mathcal{P}$ there exists a functor $G_P : \mathcal{D} \to \mathcal{C}$, such that $F(-, P) \dashv G_P$, then there exists a unique functor $G : \mathcal{P}^{\text{op}} \times \mathcal{D} \to \mathcal{C}$, such that $(F, G, \overline{(\ )})$ forms an adjunction with a parameter.*

Proof. Let $f \in \mathcal{P}(P, P')$. Observe that $F(f, -) : F(P, -) \Rightarrow F(P', -)$ is a natural transformation, so by 1.5.44 there exists a unique $\tau_f : G_{P'} \Rightarrow G_P$ conjugate to $F(f, -)$. Define $G(P, B) := G_P(B)$ for every $P \in \mathcal{P}_0$, $B \in \mathcal{D}_0$ and $G(f, \text{id}_B) := (\tau_f)_B$ for every $f \in \mathcal{P}(P, P')$. By the uniqueness of conjugates we get that G is a contravariant functor in the first variable, therefore a bifunctor $G : \mathcal{P}^{\text{op}} \times \mathcal{D} \to \mathcal{C}$. Bifunctoriality of G implies that for every A, B, the following diagram commutes,

$$\begin{array}{ccc}
\mathcal{D}(F(A, P'), B) & \xrightarrow{\overline{(\ )}} & \mathcal{C}(A, G(P', B)) \\
\Big\downarrow {\scriptstyle - \circ F(\text{id}_A, f)} & & \Big\downarrow {\scriptstyle G(f, \text{id}_B) \circ -} \\
\mathcal{D}(F(A, P), B) & \xrightarrow[\overline{(\ )}]{} & \mathcal{C}(A, G(P, B))
\end{array}$$

which shows that $\overline{(\ )}$ is natural in A, B and P, therefore an adjunction with a parameter.

□

Remark 1.5.46. It is easy to check that the above functor gives, for every $P \in \mathcal{P}_0$, an adjunction $F(-, P) \dashv G(P, -)$. This theorem also holds for the inverse case, where G is a given functor and F is constructed to be a functor. Therefore, adjunction with a parameter is equivalent to a family of adjunctions parametrised by a category $\mathcal{P}$.

Example 1.5.47. Let $\mathcal{C}$ be a cartesian category. By the above theorem and the example 1.5.42 the internal hom functor and the product functor form an adjunction with a parameter. Therefore, the reconstruction of the internal hom functor from the product functor is in exact accordance with the above theorem. Of interest is also the case of reconstructing the product functor from the internal hom functor, which is more involved than one might expect at a first glance[8]

We conclude this chapter by giving the definition of a closed category.

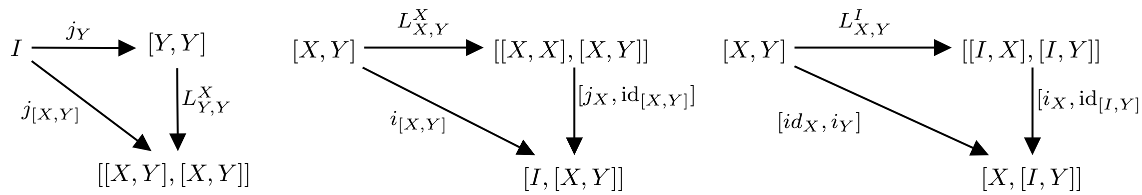
Definition 1.5.48. *A category $\mathcal{C}$ is called **closed** if it is equipped with:*

- *A functor $[-, -] : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ called the **internal hom**,*
- *An object I called the **unit***
- *A natural isomorphism $i : \mathbb{1}_{\mathcal{C}} \Rightarrow [I, -]$*
- *A dinatural transformation with components $j_X : I \to [X, X]$*
- *A family of morphisms $L^X_{Y,Z} : [Y, Z] \to [[X, Y], [X, Z]]$, which is natural in Y, Z and dinatural in X*

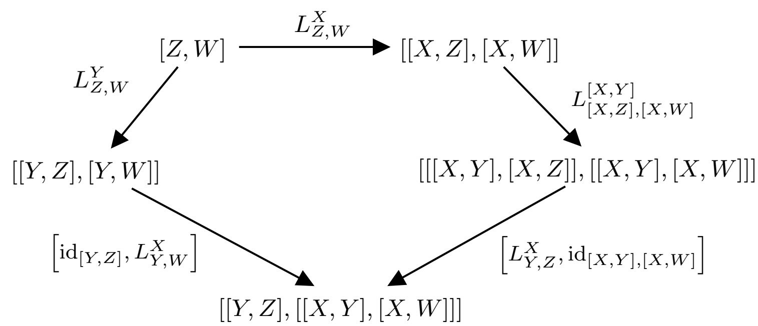
[8]In general a left adjoint to an internal hom functor, leads to a (pro)monoidal product, which is not necessarily cartesian. A general treatment of closed categories is given in [KML71] and [Lap77].

such that:

1. *The following diagrams commute for every $X, Y \in \mathcal{C}_0$:*



2. *The following diagram commutes for every $X, Y, Z, W \in \mathcal{C}_0$:*



3. *For every $X, Y \in \mathcal{C}_0$ the function defined as $[\mathrm{id}_X, -] \circ j_X : \mathcal{C}(X, Y) \to \mathcal{C}(I, [X, Y])$ is a bijection.*

Example 1.5.49. The best example to break down the above definition is **Set**. For every sets A, B there is a set $B^A = \mathbf{Set}(A, B)$ corresponding to the internal hom. The unit object is the terminal object which also represents the identity functor, thus giving the natural isomorphism i. We can pick a global element of $\mathbf{Set}(A, A)$ which corresponds to the identity morphism on A, $j_A(*) = \mathrm{id}_A$. Moreover, every set of functions between two objects A and B corresponds to a subset of functions between the generalized elements of A and B and this correspondence is natural in A and B and dinatural in the shape of these generalized elements, thus providing L. The axioms for i, j and L are easily seen to be satisfied. Finally, the third axiom is just another expression of the representability of the identity functor.

A final note on the above definition is that it is a formalisation of post-composition internally in a category.

1.6 Biproducts and semi-additive categories

In this section we prove some propositions about zero objects, we introduce superposition rules (enrichements in commutative monoids) and semi-additive categories. We also show how biproducts and superposition rules interrelate.

Zero objects, kernels and superposition rules

As we have already seen a **zero object** in a category $\mathcal{C}$ is an object which is both initial and terminal. Furthermore, for every two objects $A, B \in \mathcal{C}_0$ we call the composite of the unique morphisms e_A and u_B the **zero morphism** $0_{A,B} := u_B \circ e_A$. Initiality and terminality of 0 imply that for every two objects A, B there exists a unique morphism factoring through 0, the zero morphism. Finally, although zero objects are unique up to unique isomorphism, one can easily see that zero morphisms are unique.

Before giving some examples we prove a characterisation of a zero object.

Proposition 1.6.1. *In a category, $\mathcal{C}$, with a zero object, an object $A \in \mathcal{C}_0$ is a zero object if and only if* $\mathrm{id}_A = 0_{A,A}$.

Proof. The first direction being trivial, assume that $\mathrm{id}_A = 0_{A,A}$. Then $\mathrm{id}_A = u_A \circ e_A$ and of course $e_A \circ u_A = \mathrm{id}_0$. Thus, u_A and e_A are isomorphisms, rendering A a zero object. □

Example 1.6.2. In the category **Grp** of groups and group homomorphisms, the single element group 0 is a zero object. That is because it is obviously terminal, but also due to homomorphisms preserving units, the image of 0 under any homomorphism is the trivial subgroup 0, so there is only one homomorphism from 0 to every group G.

The case is similiar for the category of monoids and the category of abelian groups. Also the category of vector spaces and the category of Hilbert spaces has a 0 object, the 0-dimensional vector/Hilbert space.

Example 1.6.3. The category **Rel**, i.e. the underlying category of the 2-category **Rel**, also has a zero object, the empty set. The unique morphism from any set to the empty set is the empty relation. Also the unique morphism from the empty set to any set is again the empty relation.

A counterexample is **Set**, where there is no morphism to the empty set therefore, there cannot exist a zero object.

Proposition 1.6.4. *Let $\mathcal{C}$ be a category with a zero object. Then precomposing or postcomposing with a zero morphism yields a zero morphism.*

Proof. Let $A, B, C \in \mathcal{C}_0$, $f : A \to B$ and $g : B \to C$. Then $0_{B,C} \circ f$ and $g \circ 0_{A,B}$ are morphisms in $\mathcal{C}(A, C)$ factoring through 0, therefore uniqueness implies

$$0_{B,C} \circ f = 0_{A,C} = g \circ 0_{A,B}.$$

□

Remark 1.6.5. The above proposition shows that the homsets of a category with a zero object are pointed and the morphisms acting on homsets as precomposition or postcomposition preserve the base points (zero morphisms) of homsets. Thus one could say that categories with zero objects are pointed.

There is no general converse to the above proposition but under stronger assumptions there is a restricted one.

Proposition 1.6.6. *Let $\mathcal{C}$ be a category with a zero object and let $g : B \to C$ be a monomorphism. Then, $g \circ f = 0_{A,C}$ implies $f = 0_{A,B}$, for any $f : A \to B$.*

Proof. Let g be a monomorphism and observe that

$$g \circ f = 0_{A,C} = g \circ 0_{A,B}.$$

Since g is a monomorphism we get that $0_{A,B} = f$. □

In a category with zero objects we can define the kernel of a morphism as follows. Note that the existence of a zero object does not guarantee the existence of a kernel[9].

Definition 1.6.7. *Let $\mathcal{C}$ be a category with a zero object and let $f : A \to B$ be a morphism. The* ***kernel*** *of f is the equalizer of f with the zero morphism $0_{A,B}$.*

Example 1.6.8. In the categories **Vect**, **Hilb**, **Ab**, **Grp**, **Mon** the kernel of a morphism is the usual kernel of a homomorphism.

Remark 1.6.9. Observe that according to **??** the kernel of a morphism is a monomorphism. In the above examples we see that the kernel of a homomorphism is an injection/insertion.

Another remark is that if $(S, \ker f)$ is a kernel of $f : A \to B$, then

$$f \circ \ker f = 0_{A,B} \circ \ker f,$$

but $0_{A,B} \circ \ker f = 0_{S,B}$, so $f \circ \ker f = 0_{S,B}$.

[9] see [Bor94b]

A category with superposition rules is a category enriched in commutative monoids. Since, enriched category theory is beyond the scope of this thesis, we present a concrete definition. To present enrichement in commutative monoids in an enriched-independent way, we follow [HV19] in calling such a structure a superposition rule. Such terminology is borrowed from quantum mechanics, or physics in general.

Definition 1.6.10. *Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ has a* ***superposition rule****, denoted by "$(+, u)$", if for every $A, B, C, D \in \mathcal{C}_0$, $(\mathcal{C}(B, C), +, u_{B,C})$ is a commutative monoid and for every morphism $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(C, D)$ the functions $- \circ f$ and $g \circ -$ are monoid homomorphisms. We say that $\mathcal{C}$ has an invertible superposition rule if instead of commutative hom-monoids and monoid homomorphisms, for composition, we have abelian groups and group homomorphisms.*

Remark 1.6.11. The use of the symbol "+" has been chosen to indicate the commutativity of the "hom-monoids". Moreover, post-composition and pre-composition defining monoid homomorphisms is equivalent to the following two conditions for every $g_1, g_2 \in \mathcal{C}(B, C)$, $f \in \mathcal{C}(A, B)$ and $h \in \mathcal{C}(C, D)$:

1. $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ and $h \circ (g_1 + g_2) = h \circ g_1 + h \circ g_2$
2. $h \circ u_{B,C} = u_{B,D}$ and $u_{B,C} \circ f = u_{A,C}$.

Example 1.6.12. The primary motivation for a superposition rule comes fom **Vect** and **Hilb**, where the commutative monoid structure on hom-sets comes from pointwise addition of operators. The units of the hom-sets are the 0 operators.

Defining a superposition rule pointwise, works in many other categories such as the category of commutative monoids, **Ab**, the category of rings, the category of modules over a ring, some of whose superposition rules are invertible.

Example 1.6.13. Another interesting example is the category **Rel**, with ordinary union of relations. In this case the units are just the empty relations and it is easy to check that pre-composing or post-composing with another relation yields a monoid homomorphism. So $(\cup, \emptyset)$ is a superposition rule for **Rel**.

A counter example is **Set** where the emptiness of $\mathbf{Set}(T, \emptyset)$ makes this hom-set unable to bear a monoid structure, since there is no unit. Thus there is no superposition rule for **Set**.

All the above examples seem to hint at a harmonic coexistence between zero objects and superposition rules. This is due to the following two propositions.

Proposition 1.6.14. *Let $\mathcal{C}$ be a category with a zero object and a superposition rule. Then the units of the hom-monoids are the zero morphisms.*

Proof. Let $A, B \in \mathcal{C}_0$ and $u_{A,B}$ be the unit of $\mathcal{C}(A, B)$. Observe that $\mathcal{C}(A, 0)$ and $\mathcal{C}(0, B)$ are singleton sets, which makes

$$u_{A,0} = 0_{A,0} \text{ and } u_{0,B} = 0_{0,B}.$$

Units being preserved by composition means that

$$u_{0,B} \circ u_{A,0} = u_{A,B},$$

which means that $u_{A,B}$ is a zero morphism and hence

$$u_{A,B} = 0_{A,B}.$$

□

Proposition 1.6.15. *Let $\mathcal{C}$ be a category with a superposition rule. Then the following are equivalent.*

1. *$\mathcal{C}$ has a terminal object T*
2. *$\mathcal{C}$ has an initial object I*

3. $\mathcal{C}$ has a zero object 0.

Proof. Obviously, 3) implies both 1) and 2) by definition. So it suffices to prove 1) implies 3), since duality gives 2) $\Rightarrow$ 3) for free. So let $T \in \mathcal{C}_0$ be terminal. This immediately tells us that

$$u_{T,T} = \mathrm{id}_T.$$

Note that for every $A \in \mathcal{C}_0$, $\mathcal{C}(T, A)$ is a monoid and thus, it is not empty. For $f \in \mathcal{C}(T, A)$, preservation of units by post-composition implies that

$$\begin{aligned} f =& f \circ \mathrm{id}_T \\ =& f \circ u_{T,T} \\ =& u_{T,A} \end{aligned}$$

Thus $\mathcal{C}(T, A)$ is a one element set which implies that T is also inital and thus a zero object. □

Remark 1.6.16. In the proof of the above proposition, we also showed that zero morphisms coincide with the units of the superposition rule. This is actually an extra condition superposition rules have to obey, Thus, the choices of superposition rules a category can be equipped with are limited, given that one of the above three conditions hold.

Since terminal objects are nullary products and initial objects are nullary coproducts, one might wonder what kind of nullary operation zero objects are. We answer this in the following section.

Biproducts and matrix malculus

Definition 1.6.17. *Let $\mathcal{C}$ be a category with a zero object and a superposition rule and let $A_1, A_2 \in \mathcal{C}_0$. A tuple $(A_1 \oplus A_2, p_1, p_2, i_1, i_2)$, where $p_i : A_1 \oplus A_2 \to A_i$ and $i_i : A_i \to A_1 \oplus A_2$ for $i, j = 1, 2$, is called the **biproduct** of A_1 and A_2, if:*

1. $p_i \circ i_j = \begin{cases} \mathrm{id}_{A_i}, & \text{if } i = j \\ 0_{A_i, A_j}, & \text{if } i \neq j \end{cases}$ *, for $i, j = 1, 2$,*
2. $\mathrm{id}_{A_1 \oplus A_2} = i_1 \circ p_1 + i_2 \circ p_2$.

Example 1.6.18. The archetypal example of a biproduct is the direct product of vector spaces. Obviously, given vector spaces V, W every element of $V \oplus W$ is written in the form (v, w) for unique $v \in V$ and $w \in W$. For the projections and the coprojections we have

$$p_V(v, w) = v, \; p_W(v, w) = w, \; i_V(v) = (v, 0) \text{ and } i_W(w) = (0, w),$$

which obviously satisfy 1. but also 2. since

$$(v, w) = (v, 0) + (0, w).$$

The case is also similar for **Hilb**, **Ab** and the category of representations of a group, or in general the category of modules over a ring.

Observe, that all the categories in the example above have superposition rules. In general, the existence of a biproduct depends on the existence of either a product or a coproduct. Furthermore, biproducts are a combination of a product and a coproduct structure on an object. This is the content of the following proposition.

Proposition 1.6.19. *Let $\mathcal{C}$ be a category with zero objects and a superposition rule and let $A, B \in \mathcal{C}_0$. Then the following are equivalent:*

1. $(A \times B, p_A, p_B)$ is a product of A and B

2. $(A+B, i_A, i_B)$ *is a corpoduct of* A *and* B

3. $(A \oplus B, p_A, p_B, i_A, i_B)$ *is a biproduct of* A *and* B*.*

Proof. We will prove that 1. is equivalent to 3. since duality implies the equivalence of 2. and 3. thus providing the equivalence of 1. and 2. So for 1. ⇒ 3. let $(A \times B, p_A, p_B)$ be a product. Define i_A by the universal property of the product as the unique morphism satistying:

$$p_A \circ i_A = \mathrm{id}_A \text{ and } p_B \circ i_A = 0_{A,B}.$$

Similarly, define i_B as the unique morphism satistying:

$$p_B \circ i_B = \mathrm{id}_B \text{ and } p_A \circ i_B = 0_{B,A}.$$

Observe that, by the fact that p_A, p_B distribute over "+", for $i_A \circ p_A + i_B \circ p_B : A \oplus B \to A \oplus B$ we have

$$\begin{aligned} p_A \circ (i_A \circ p_A + i_B \circ p_B) &= p_A \circ i_A \circ p_A + p_A \circ i_B \circ p_B \\ &= \mathrm{id}_A \circ p_A + 0_{B,A} \circ p_B \\ &= p_A + 0_{A \oplus B, A} \\ &= p_A \\ &= p_A \circ \mathrm{id}_{A \times B} \end{aligned}$$

and similarly $p_B \circ (i_A \circ p_A + i_B \circ p_B) = p_B$, so by 1.4.16 we get that

$$i_A \circ p_A + i_B \circ p_B = \mathrm{id}_{A \times B}.$$

So $(A \times B, p_A, p_B, i_A, i_B)$ is a biproduct.

To prove 3. ⇒ 1. observe that given $f : X \to A$ and $g : X \to B$ we can define the morphism $i_A \circ f + i_B \circ g : X \to A \oplus B$, which satisfies

$$\begin{aligned} p_A \circ (i_A \circ f + i_B \circ g) &= \mathrm{id}_A \circ f + 0_{B,A} \circ g \\ &= f + 0_{C,A} \\ &= f \end{aligned}$$

and similarly $p_B \circ (i_A \circ f + i_B \circ g) = g$. Furthermore, if $h : C \to A \oplus B$ satisfies $p_A \circ h = f$ and $p_B \circ h = g$, then

$$\begin{aligned} h =& \mathrm{id}_{A \oplus B} \circ h \\ =& (i_A \circ p_A + i_B \circ p_B) \circ h \\ =& i_A \circ p_A \circ h + i_B \circ p_B \circ h \\ =& i_A \circ f + i_B \circ g. \end{aligned}$$

Thus $i_A \circ f + i_B \circ g$ is unique and therefore $(A \oplus B, p_A, p_B)$ is a product. □

Remark 1.6.20. According to the above proposition in a category with a superposition rule a product is automatically a biproduct and so is a coproduct. Although products and coproducts are a limit and a colimit, to show that a biproduct is unique up to unique isomrphism is a little more subtle. This is because one needs to show that the unique isomorphism between two biproducts as products is the same isomorphism with the one between them as coproducts. To this end, let (P, π, ι) and $(A \oplus B, p, i)$ be biproducts of A, B in $\mathcal{C}$. Then there exists a unique isomorphism $f : P \to A \oplus B$

$$p_A \circ f = \pi_A \text{ and } p_B \circ f = \pi_B,$$

since $(A \oplus B, \pi)$ is a product, and there exists a unique isomorphism $g : P \to A \oplus B$ such that

$$\iota_A \circ g = i_A \text{ and } \iota_B \circ g = i_B,$$

since (P, ι) is a coproduct. $(A \oplus B, p, i)$ and (P, π, ι) being biproducts implies that

$$\begin{aligned} g^{-1} \circ f &= g^{-1} \circ (i_A \circ p_A + i_B \circ p_B) \circ f \\ &= g^{-1} \circ i_A \circ p_A \circ f + g^{-1} \circ i_B \circ p_B \circ f \\ &= \iota_A \circ \pi_A + \iota_B \circ \pi_B \\ &= \mathrm{id}_P \\ \Leftrightarrow f &= g \end{aligned}$$

This can be used to define n-ary biproducts, as was the case with n-ary products or n-ary coproducts. This is achieved via induction, so the resulting biproducts are finite. Note that the definition for an n-ary biproduct involves a "finite sum" to describe the morphism:

$$\mathrm{id}_{\bigoplus_{j=1}^{n} A_j} = \sum_{j=1}^{n} i_j \circ p_j, \ n \in \mathbb{N}.$$

We can also think of 0-ary biproducts as the zero object.

Definition 1.6.21. *A category $\mathcal{C}$ with a zero object and a superposition rule is called* ***semi-additive****, if every finite collection of objects has a biproduct.*

Example 1.6.22. The categories of example 1.6.18 are all semi-additive.

Remark 1.6.23. Uniqueness of biproducts up to unique isomorphism and a choice of biproducts in a semi-additive category allow us to see that there is a biproduct funtor, which will be denoted by $f \oplus g : A \oplus B \to C \oplus D$ when applied to morphisms $f : A \to B$ and $g : C \to \mathcal{D}$.

Remark 1.6.24. Observe that taking products in a semi-additive category, $\mathcal{C}$, corresponds to taking coproducts in $\mathcal{C}^{\mathrm{op}}$. Thus, by proposition 1.6.19, a semi-additive category has the same biproducts with its opposite category.

To conclude with the coexistence of superposition rules and birpoducts in a semi-additive category, we show that a biproduct structure reduces the possible choices of a superposition rule to a single one.

Proposition 1.6.25. *Let $\mathcal{C}$ be a semi-additive category and let $(+, u)$ and $(+', v)$ be superposition rules giving rise to the same biproduct structure. Then $(+, u) = (+', v)$.*

Proof. Firstly, since zero morphisms are unique, we get that $u = v = 0$. Let $f, g \in \mathcal{C}(A, B)$ and $p_1,\ p_2,\ i_1$ and i_2 denote the projections and the coprojections of $B \oplus B$. Since, $p_n \circ i_n + p_n \circ i_m = \mathrm{id}_B + 0_{n,m} = \mathrm{id}_B$, for $n \neq n$, we get

$$\begin{aligned} f +' g =& (p_1 \circ i_1 + p_2 \circ i_1) \circ f +' (p_1 \circ i_2 + p_2 \circ i_2) \circ g \\ =& (p_1 + p_2) \circ i_1 \circ f +' (p_1 + p_2) \circ i_2 \circ g \\ =& (p_1 + p_2) \circ (i_1 \circ f +' i_2 \circ g) \\ =& (p_1 \circ i_1 \circ f +' p_1 \circ i_2 \circ g) + (p_2 \circ i_1 \circ f +' p_2 \circ i_2 \circ g) \\ =& (\mathrm{id}_B \circ f +' 0_{B,B} \circ g) + (0_{B,B} \circ f +' \mathrm{id}_B \circ g) \\ =& f + g \end{aligned}$$

Thus, for every $A, B \in \mathcal{C}_0$, $(\mathcal{C}(A, B), +, 0)$ and $(\mathcal{C}(A, B), +', 0)$ are the same commutative monoid. ☐

Matrix calculus for semi-additive categories

In a semi-additive, given the above proposition, there is a unique way to create a matrix calculus. This makes use of all the tools provided by finite biproducts and the unique superposition rule.

Definition 1.6.26. *Let $\mathcal{C}$ be a semi-additive category and let $(\bigoplus_{j=1}^m A_j, p, \iota)$ and $(\bigoplus_{i=1}^n B_i, \pi, i)$ be biproducts of $(A_j)_{j=1}^m \subset \mathcal{C}_0$ and $(B_i)_{i=1}^n \subset \mathcal{C}_0$, for $n, m \in \mathbb{N}$, respectively. The **matrix** of the morphisms $f_{i,j} : A_j \to B_i$, $1 \leq j \leq m,\ 1 \leq i \leq n$, is defined as:*

$$\begin{pmatrix} f_{1,1} & \dots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \dots & f_{n,m} \end{pmatrix} := \sum_{i,j} i_i \circ f_{i,j} \circ p_j.$$

An interesting thing about the matrix notation is that any morphism $f : \bigoplus_{j=1}^m A_j \to \bigoplus_{i=1}^n B_i$ can be given as a matrix.

Proposition 1.6.27. *Let $\mathcal{C}$ be a semi-additive category and let $f : \bigoplus_{j=1}^m A_j \to \bigoplus_{i=1}^n B_i$ be a morphism between finite biproducts. Then the matrix representation of f is:*

$$\begin{pmatrix} \pi_1 \circ f \circ \iota_1 & \dots & \pi_1 \circ f \circ \iota_m \\ \vdots & \ddots & \vdots \\ \pi_n \circ f \circ \iota_1 & \dots & \pi_n \circ f \circ \iota_m \end{pmatrix}$$

Proof. Observe that

$$\begin{aligned} \begin{pmatrix} \pi_1 \circ f \circ \iota_1 & \dots & \pi_1 \circ f \circ \iota_m \\ \vdots & \ddots & \vdots \\ \pi_n \circ f \circ \iota_1 & \dots & \pi_n \circ f \circ \iota_m \end{pmatrix} &= \sum_{i,j} i_i \circ \pi_i \circ f \circ \iota_j \circ p_j \\ &= (\sum_i i_i \circ \pi_i) \circ f \circ (\sum_j \iota_j \circ p_j) \\ &= \mathrm{id}_{\bigoplus_{i=1}^n B_i} \circ f \circ \mathrm{id}_{\bigoplus_{j=1}^m A_j} \\ &= f \end{aligned}$$

which concludes the proof. □

A thing to note on matrix notation, which guarantees it is a useful tool for calculations, is that it is somehow faithful, to be precise in the following proposition.

Proposition 1.6.28. *Let $\mathcal{C}$ be a semi-additive category and let $g, f : \bigoplus_{j=1}^m A_j \to \bigoplus_{i=1}^n B_i$ be morphisms between biproducts, with matrices*

$$f = \begin{pmatrix} f_{1,1} & \dots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \dots & f_{n,m} \end{pmatrix} \text{ and } g = \begin{pmatrix} g_{1,1} & \dots & g_{1,m} \\ \vdots & \ddots & \vdots \\ g_{n,1} & \dots & g_{n,m} \end{pmatrix}.$$

Then $f = g$ if and only if their matrices are equal entrywise.

Proof. Observe that by the above proposition for every $1 \leq j \leq m,\ 1 \leq i \leq n$,

$$\begin{aligned} \pi_i \circ f \circ \iota_j =& \pi_i \circ g \circ \iota_j \Leftrightarrow \\ f_{i,j} =& g_{i,j} \end{aligned}$$

So the two matrices being equal entrywise, means

$$\pi_i \circ f \circ \iota_j = \pi_i \circ g \circ \iota_j,$$

which by proposition 1.4.16 is not only implied by but equivalent to $f = g$. □

According to the above, every morphism between biproducts in a semi-additive category can be expressed as a matrix. To this end we show how the biproduct of morphisms is given as a sum and consequently as a matrix.

Proposition 1.6.29. *Let $f_j : A_j \to B_j$, $1 \leq j \leq n$ be morphisms in a semi-additive category $\mathcal{C}$ and let $\bigoplus_{m=1}^n f_m : \bigoplus_{j=1}^n A_j \to \bigoplus_{l=1}^n B_l$ be their biproduct between $(\bigoplus_{j=1}^n A_j, p, \iota)$ and $(\bigoplus_{l=1}^n B_l, \pi, i)$. Then*

$$\bigoplus_{m=1}^n f_m = \sum_{m=1}^n i_m \circ f_m \circ p_m = \begin{pmatrix} f_1 & 0_{2,1} & \dots & 0_{n,1} \\ 0_{1,2} & f_2 & \dots & 0_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1,n} & \dots & \dots & f_n \end{pmatrix}$$

Proof. Observe that by the universal property of products, which biproducts are, we get that

$$\pi_l \circ \bigoplus_{m=1}^n f_m = f_l \circ p_l$$

and that the second biproduct property takes the following form

$$\sum_{l=1}^n i_l \circ \pi_l = \mathrm{id}_{\bigoplus_{l=1}^n B_l}.$$

So we get:

$$\begin{aligned} \bigoplus_{m=1}^n f_m &= \mathrm{id}_{\bigoplus_{l=1}^n B_l} \circ \bigoplus_{m=1}^n f_m \\ &= \sum_{l=1}^n i_l \circ \pi_l \circ \bigoplus_{m=1}^n f_m \\ &= \sum_{l=1}^n i_l \circ f_l \circ p_l \end{aligned}$$

and thus adding the zero morphisms $i_j \circ 0_{l,j} \circ p_l$ for every $1 \leq l, j \leq n$, we get the diagonal matrix

$$\bigoplus_{m=1}^n f_m = \begin{pmatrix} f_1 & 0_{2,1} & \dots & 0_{n,1} \\ 0_{1,2} & f_2 & \dots & 0_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1,n} & \dots & \dots & f_n \end{pmatrix}.$$

□

Remark 1.6.30. According to the above the identity matrix on a biproduct is the diagonal matrix with the identities on all the constituents of the biproduct on the diagonal.

A matrix notation implies some sort of matrix composition. This composition is exactly what someone might expect. To compose two matrices their dimensions have to match. This demand is equivalent to composability of the corresponding morphisms. So given

$$f = \begin{pmatrix} f_{1,1} & \dots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \dots & f_{n,m} \end{pmatrix} \text{ and } g = \begin{pmatrix} g_{1,1} & \dots & g_{1,k} \\ \vdots & \ddots & \vdots \\ g_{m,1} & \dots & g_{m,k} \end{pmatrix}$$

such that $\mathrm{dom}(f) = \mathrm{cod}(g)$ their composite is:

$$f \circ g = \begin{pmatrix} f_{1,1} & \dots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \dots & f_{n,m} \end{pmatrix} \circ \begin{pmatrix} g_{1,1} & \dots & g_{1,k} \\ \vdots & \ddots & \vdots \\ g_{m,1} & \dots & g_{m,k} \end{pmatrix} = \sum_{i,j} i_i \circ f_{i,j} \circ p_j \circ \sum_{r,l} i_l \circ g_{r,l} \circ p_l$$
$$= \sum_{i,j,r,l} i_i \circ f_{i,j} \circ p_j \circ i_r \circ g_{r,l} \circ p_l$$
$$= \sum_{i,j,l} i_i \circ f_{i,j} \circ g_{j,l} \circ p_l$$
$$= \sum_{i,l} i_i \circ (\sum_{j} f_{i,j} \circ g_{j,l}) \circ p_l$$
$$= \begin{pmatrix} \sum_j f_{1,j} \circ g_{j,1} & \dots & \sum_j f_{1,j} \circ g_{j,k} \\ \vdots & \ddots & \vdots \\ \sum_j f_{n,j} \circ g_{j,1} & \dots & \sum_j f_{n,j} \circ g_{j,k} \end{pmatrix}.$$

Remark 1.6.31. Since biproducts are both products and coproducts, we may form diagonal and co-diagonal morphisms in every semi-additive category $\mathcal{C}$. That is for every $A \in \mathcal{C}_0$ there exist uniquely defined

$$d_A : A \to A \oplus A \text{ and } m_A : A \oplus A \to A,$$

such that

$$\pi_n \circ d_A = \mathrm{id}_A \text{ and } i_n \circ m_A = \mathrm{id}_A,$$

for $n = 1, 2$. Using the fact that unary biproducts are of the form $(A, \mathrm{id}_A, \mathrm{id}_A)$, we can easily see that these two special morphisms have the following matrix representations:

$$d_A = \sum_{n=1}^{2} i_n \circ (\pi_n \circ d_A \circ \mathrm{id}_A) \circ \mathrm{id}_A = \sum_{n=1}^{2} i_n \circ \mathrm{id}_A = \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix}$$

and similarly

$$m_A = \begin{pmatrix} \mathrm{id}_A & \mathrm{id}_A \end{pmatrix}.$$

Using this idea we may "reconstruct" the unique superposition rule of $f, g \in \mathcal{C}(A, B)$ in one of the following ways:

$$f + g = \begin{pmatrix} \mathrm{id}_A & \mathrm{id}_A \end{pmatrix} \circ \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mathrm{id}_A & \mathrm{id}_A \end{pmatrix} \circ \begin{pmatrix} f & 0_{A,B} \\ 0_{A,B} & g \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix} = \begin{pmatrix} f & g \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix}.$$

Functoriality and naturality in semi-additive categories

In general a functor between ordinary categories is defined as a function sending objects to objects and a family of functions from hom-sets to hom-sets. Functors between categories with superposition rules may not be enough to preserve such a structure, so we have to pick only the ones that preserve some or all of it. A general case is to choose only the ones whose families of functions between hom-sets are monoid homomorphisms. In what follows, though, preserving unit objects can be dropped. So we introduce linear functors.

Definition 1.6.32. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with superposition rules. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **linear** if for every $A, B \in \mathcal{C}_0$ and $f, g \in \mathcal{C}(A, B)$,*

$$F(f + g) = Ff + Fg.$$

Example 1.6.33. Hilbert spaces, having a superposition rule which is the same superposition rule as the one in vector spaces, make the forgetful functor to **Vect** a linear functor. Similarly, the forgetful functor from **Vect** to **Ab** is a linear functor, and so is the forgetful functor from **Ab** to the category of commutative monoids.

In a semi-additive category the unique superposition rule and the biproduct are interrelated. So there should be a notion of biproduct preservation by a functor and it should harmonically coexist with the notion of linearity of a functor.

Definition 1.6.34. *Let $\mathcal{C}$ and $\mathcal{D}$ be semi-additive categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F* ***preserves biproducts****, if for every $A, B \in \mathcal{C}$ such that $(A \oplus B, p, i)$ is their biproduct, $(F(A \oplus B), Fp, Fi)$ is a biproduct of FA and FB.*

Remark 1.6.35. Note that by uniqueness up to unique isomorphism of biproducts, $F(A) \oplus F(B) \cong F(A \oplus B)$ is the case for every every pair of objects A, B in the domain of F.

Remark 1.6.36. The above definition instantly extends to finite biproducts, but not to zero objects. This extension is achieved by induction on the number of the constituents of the finite biproduct.

To include zero objects in the above definition one needs to include the requirement that given a zero object $0 \in \mathcal{C}_0$, $F0$ is a zero object in $\mathcal{D}$. This requirement, can be weakened, according to proposition 1.6.15, by demanding either that F preserves terminal or initial objects.

Proposition 1.6.37. *Let $\mathcal{C}, \mathcal{D}$ be semi-additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ preserving zero objects, preserves biproducts if and only if it is linear.*

Proof. Let F preserve biproducts and let $(A \oplus B, p, i)$ be a biproduct of $A, B \in \mathcal{C}_0$. Then F being a functor implies

$$\begin{aligned}
\mathrm{id}_{F(A_1 \oplus A_2)} &= F\mathrm{id}_{A_1 \oplus A_2} \Leftrightarrow \\
Fi_1 \circ Fp_1 + Fi_2 \circ Fp_2 &= F(i_1 \circ p_1 + i_2 \circ p_2) \Leftrightarrow \\
F(i_1 \circ p_1) + F(i_2 \circ p_2) &= F(i_1 \circ p_1 + i_2 \circ p_2).
\end{aligned}$$

Now let $f, g \in \mathcal{C}(A, B)$ and let the projections and the coprojections of $B \oplus B$ be denoted by p_1, p_2 and i_1, i_2, respectively. Then we have:

$$\begin{aligned}
F(f + g) =& F[(p_1 \circ i_1 + p_2 \circ i_1) \circ f + (p_2 \circ i_2 + p_1 \circ i_2) \circ g] \\
=& F[(p_1 + p_2) \circ (i_1 \circ f + i_2 \circ g)] \\
=& F[(p_1 + p_2) \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ (i_1 \circ f + i_2 \circ g)] \\
=& F(p_1 + p_2) \circ F(i_1 \circ p_1 + i_2 \circ p_2) \circ F(i_1 \circ f + i_2 \circ g) \\
=& F(p_1 + p_2) \circ (F(i_1 \circ p_1) + F(i_2 \circ p_2)) \circ F(i_1 \circ f + i_2 \circ g) \\
=& F(p_1 + p_2) \circ (F(i_1 \circ p_1) \circ F(i_1 \circ f + i_2 \circ g) + F(i_2 \circ p_2) \circ F(i_1 \circ f + i_2 \circ g)) \\
=& F(p_1 + p_2) \circ (F(i_1 \circ p_1 \circ i_1 \circ f + i_1 \circ p_1 \circ i_2 \circ g) + F(i_2 \circ p_2 \circ i_1 \circ f + i_2 \circ p_2 \circ i_2 \circ g)) \\
=& F(p_1 + p_2) \circ (F(i_1 \circ f) + F(i_2 \circ g)) \\
=& F((p_1 + p_2) \circ i_1 \circ f) + F((p_1 + p_2) \circ i_2 \circ g) \\
=& F(p_1 \circ i_1 \circ f + p_2 \circ i_1 \circ f) + F(p_1 \circ i_2 \circ g + p_2 \circ i_2 \circ g) \\
=& Ff + Fg
\end{aligned}$$

On the other hand, observe that since F preserves zero objects it must also strictly preserve zero morphisms. That is because $F0$ is a zero object in $\mathcal{D}$ and given $A, B \in \mathcal{C}_0$, the image under F of the composite of the unique morphisms $Fu_B \circ Fe_A$ factors through $F0$, thus it is the unique zero morphism in $\mathcal{D}(FA, FB)$.

Furthermore, if F is linear, then given a biproduct $(A_1 \oplus A_2, p, i)$ in $\mathcal{C}$, functoriality and preservation of zero objects by F implies that

$$F(p_n) \circ F(i_m) = F(p_n \circ i_m) = \begin{cases} \mathrm{id}_{FA_n}, & \text{if } n = m, \\ 0_{FA_m, FA_n}, & \text{if } n \neq m \end{cases} .$$

Functoriality and linearity also imply that

$$Fi_1 \circ Fp_1 + Fi_2 \circ Fp_2 = F(i_1 \circ p_1 + i_2 \circ p_2) = F\mathrm{id}_{A_1 \oplus A_2} = \mathrm{id}_{F(A_1 \oplus A_2)},$$

thus $(F(A_1 \oplus A_2), Fp, Fi)$ is a biproduct in $\mathcal{D}$. □

Remark 1.6.38. Observe that preservation of the zero object was crucial in the above proof. This condition is actually equivalent to having a linear functor which is also a monoid homomorphism on homsets.

To conclude we focus on natural transformation between linear functors between semi-additive categories. What is remarkable is that if we consider semi-additive categories as having the biproduct functor as a monoidal functor and the 0 object as a monoidal unit, then linear functors preserving zero objects are strong monoidal[10]. But every natural transformation between such functors is automatically monoidal, which we will prove using the following proposition.

Proposition 1.6.39. *Let $\mathcal{C}, \mathcal{D}$ be semi-additive categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be linear functors preserving zero objects. If $\eta : F \Rightarrow G$ is a natural transformation and $(A \oplus B, p, i)$ is a biproduct in $\mathcal{C}$, then*

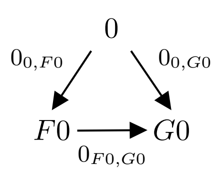
$$\mu_{A \oplus B} = Gi_A \circ \mu_A \circ Fp_A + Gi_B \circ \mu_B \circ Fp_B.$$

Proof. According to proposition 1.6.37, G makes $(G(A \oplus B), Gp, Gi)$ into a biproduct in $\mathcal{D}$. So

$$\begin{aligned} \mu_{A \oplus B} =& (Gi_A \circ Gp_A + Gi_B \circ Gp_B) \circ \mu_{A \oplus B} \\ =& Gi_A \circ Gp_A \circ \mu_{A \oplus B} + Gi_B \circ Gp_B \circ \mu_{A \oplus B} \\ =& Gi_A \circ \mu_A \circ Fp_A + Gi_B \circ \mu_B \circ Fp_B \quad \text{(by naturality of } \mu). \end{aligned}$$

□

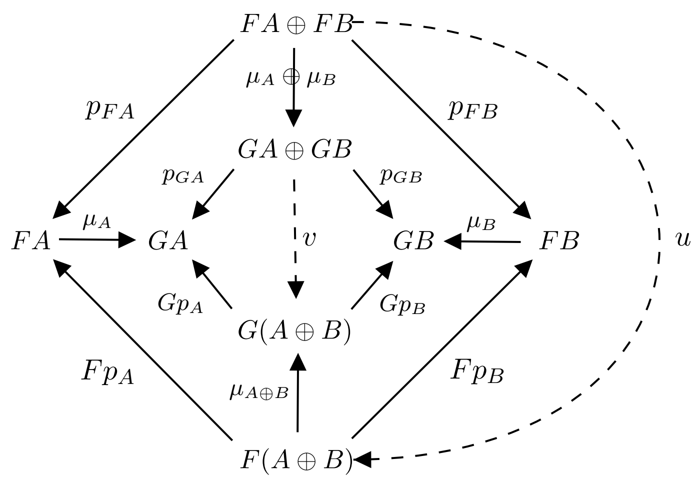
Since, $(F(A \oplus B), Fp, Fi)$ and $(G(A \oplus B), Gp, Gi)$ are biproducts, therefore also products and coproducts, of FA, FB and GA, GB accordingly, there exist unique isomorphisms $u : FA \oplus FB \to F(A \oplus B)$ and $v : GA \oplus GB \to G(A \oplus B)$ providing the multiplicators of the strong monoidal functors F and G. Furthermore, since $F0$ and $G0$ are zero objects in $\mathcal{D}$ there exist unique (zero) morphisms from and to the zero object of $\mathcal{D}$, providing the unitor coherent isomorphisms of F and G. Now given a natural transformation $\mu : F \Rightarrow G$, observe that $\mu_0 : F0 \to G0$ is a zero morphism, thus making the following triangle commute:



thus providing unitality for μ.

Furthermore, proposition 1.6.39, the definition of the biproduct functor of $\mathcal{D}$ and naturality of μ, make the following diagram commute:

[10] with coherence isomorphisms given by the universal properties of product and coproduct



which gives the commutativity of the following multiplicativity diagram for μ.

$$\begin{array}{ccc} FA \oplus FB & \xrightarrow{\;u\;} & F(A \oplus B) \\ {\scriptstyle \mu_A \oplus \mu_B}\big\downarrow & & \big\downarrow{\scriptstyle \mu_{A\oplus B}} \\ GA \oplus GB & \xrightarrow[\;v\;]{} & G(A \oplus B) \end{array}$$

Thus we have proven the following proposition, which will make sense after we introduce the concept of monoidal functors.

Proposition 1.6.40. *Every natural transformation between zero object preserving linear functors among semi-additive categories, as monoidal categories, is automatically monoidal.*

Chapter 2

Monoidal Categories and Bicategories

In this chapter we start by discussing monoidal categories while also giving examples originating in topology and physics. We continue by presenting the appropriate notions of monoidal functors and monoidal natural transformations, and we also show that every monoidal category is monoidally equivalent to a strict monoidal category - as a consequence we prove a variant of Mac Lane's coherence theorem. Strictification, as it is called, allows us to make string diagram notation compatible with monoidal categories, providing a powerful tool to study all sorts of monoidal categories.

We continue by describing 2-categories and bicategories, which are very reasonable generalisations of ordinary categories and provide an alternative view of monoidal categories. That is, bicategories are a categorification of ordinary categories, but also constitute a "many-object" version of monoidal categories. Bicategories, being a higher structure than ordinary categories, allow for morphisms of higher order, which we present and relate to monoidal functors and monoidal natural transformations.

Key Sources: [BS10], [Bén67], [BS00], [Bra18], [?], [EGNO15], [Fox76], [HV19], [JY21], [Koc04], [JS88], [JS91], [Koc04], [Lei98], [ML98], [SR72], [Sel10], [Str04] and various nlab entries.

2.1 Basic definitions

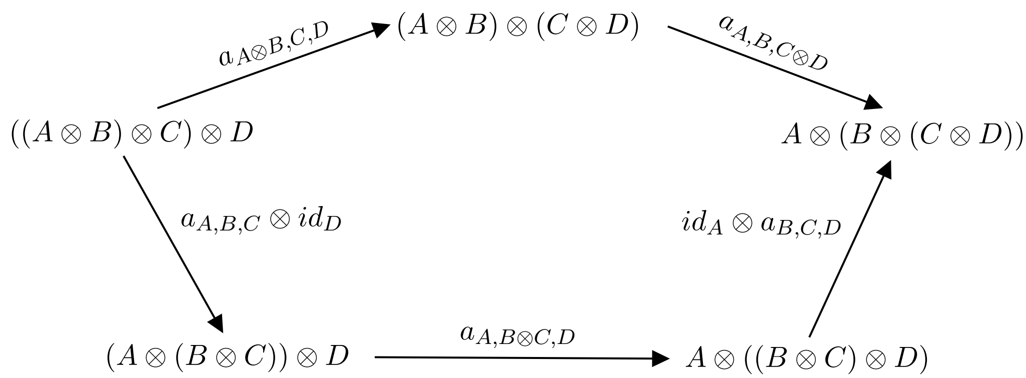
Monoidal categories are categories equipped with an abstract "multiplication" operation. This means that both objects and morphisms can be multiplied, in a functorial way. Such a functor resembles the multiplication of a monoid, hence the name monoidal.

Definition 2.1.1. *A monoidal category* $(\mathcal{M}, \otimes, I, a, l, r)$*, or just* $\mathcal{M}$*, consists of the following data:*

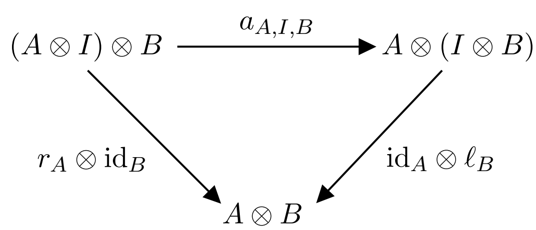
- *a category* $\mathcal{M}$*,*
- *a functor* $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$*, called* ***tensor product*** *and denoted by* $- \otimes -$*,*
- *a distinguished object* $I \in \mathcal{M}_0$ *called* ***unit****,*
- *a natural isomorphism* $\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$ *called* ***associator****,*
- *two natural isomorphisms* $\ell : I \otimes - \Rightarrow \mathrm{Id}_{\mathcal{M}},\ r : - \otimes I \Rightarrow \mathrm{Id}_{\mathcal{M}}$ *called* ***left*** *and* ***right unitors*** *respectively,*

obeying the following conditions:

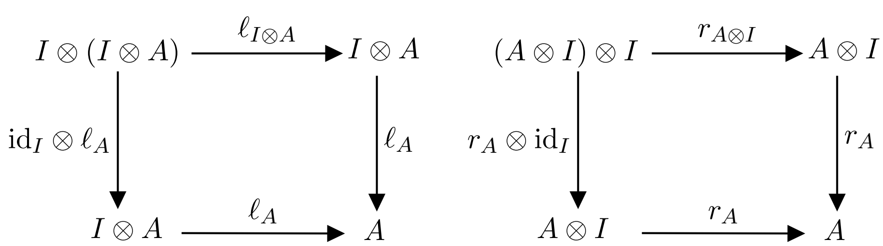
- *for every* $A, B, C, D \in \mathcal{M}$ *the following, so called* ***pentagon identity****, holds:*



- *for every $A \in \mathcal{M}_0$ the* ***triangle identity*** *holds:*



Remark 2.1.2. In a monoidal category we have that $\ell_{I\otimes A} = \mathrm{id}_I \otimes \ell_A$ and $r_{A\otimes I} = r_A \otimes \mathrm{id}_I$. This follows directly from the two naturality squares below and the fact that unitors are isomorphisms.



Remark 2.1.3. Functoriality of the monoidal product is equivalent to the statement that

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g),$$

for $f \in \mathcal{M}(A, A'), f' \in \mathcal{M}(A', A''), g \in \mathcal{M}(B, B'), g' \in \mathcal{M}(B', B'')$. The above equation is sometimes called the **interchange law** and is similar to the horizontal composition of natural transformations. This condition allows for the interpretation of morphisms as processes that can be composed sequentially or in parallel. Generally, though, in a monoidal category not every morphism between products of objects needs to occur as a product of morphisms. This fact will manifest when we examine the notions of entangled and product states in definition 2.1.17.

Remark 2.1.4. The notion of a monoidal category, $\mathcal{M}$, occurs as a generalisation of that of a monoid. An ordinary monoid is a set M equipped with a multiplication function and a unit element such that associativity and unit laws hold. In a monoidal category the above structure is updated as follows. The carrier set becomes a category, the multiplication function becomes the tensor functor and the unit element becomes the unit object. Note that the elements of a set can either be pairwise identical/equal or not, while in a category two objects can also be isomorphic. So in this direction, instead of requiring threefold tensor products to obey strict associativity one demands the existence of an associator natural isomorphism. Unitors appear from similar concerns. In case the associator and the unitors are identities, the monoidal category is called **strict** and its set of objects is a monoid. That is because we have the equalities:

$$I \otimes A = A = A \otimes I \text{ and } (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

for all $A, B, C \in \mathcal{M}$.

Given a non strict monoidal category $\mathcal{M}$, one might consider the equivalence classes of isomorphic objects or the skeleton of $\mathcal{M}$. The class/set of equivalence classes and the objects of the skeleton are again a monoid.[1]. Considering equivalence classes of isomorphic objects is a well defined process called **decategorification**. The "opposite" process is called **categorification** but it is generally not well defined. In broad terms the idea of categorification is upgrading a set theoretic structure to a categorical one, by replacing functions with functors and axioms/laws/identities by (natural) isomorphisms. In this sense a monoidal category is a categorification of a monoid.

Remark 2.1.5. Consider a category equipped with a functor, $\otimes$, and an associator, a, obeying the pentagon law. This, in turn, is the categorification of a semigroup, i.e. an algebraic structure like a monoid not necessarily with a unit. Following the standard name giving, such a category should be called **semigroupal**[2].

Categorification is followed by some standard terminology. This is neccesary because when categorifying an algebraic structure properties need only hold up to isomorphism. The arising isomorphisms are called **coherent**.

Example 2.1.6.

1. A trivial example of a monoidal category is the posetal category **1**. If 1 is its only object then $1 \otimes 1 = 1$, which is also the unit object. Due to the lack of morphisms other than id_1 the associator and the unitors are equal to id_1. Therefore, this is a strict monoidal category.

   In general a monoidal category, $\mathcal{M}$, with at most one morphism between any of its objects is called a **monoidal preorder**. If in addition $\mathcal{M}$ is posetal then it is called a **monoidal poset**.

2. The category **Set** equipped with the cartesian product and a terminal object is a monoidal category. The associator and the unitors are those in 1.1.14 and 1.1.15. The pentagon and the triangle identities are indeed satisfied.

   Similarly, **Set** equipped with the coproduct as a monoidal product and an initial object as a unit is a monoidal category.

3. Generally any cartesian category can be viewed as a monoidal category specifically equipped with the cartesian product and the terminal object as a unit. In this case we may speak of a **cartesian monoidal category**. Similarly, a co-cartesian category, equipped with the coproduct as the monoidal product and the initial object as a unit is a **co-cartesian monoidal category**.

4. The category $\mathbf{Vect}_{\mathbb{F}}$, equipped with the usual tensor product of vector spaces as a monoidal product and the base field $\mathbb{F}$ as a monoidal unit, is a monoidal category. The left and right unitors amount to scalar multiplication by the following rule: $\ell_V : \mathbb{F} \otimes V \to V$ such that for $\lambda \in \mathbb{F}$ and $v \in V$, $\lambda \otimes v \mapsto \lambda v$ and similarly $r_V(v \otimes \lambda) = v\lambda = \lambda v$. Associators in this category come from the universal property of the tensor product of vector spaces, as in 1.4.18.

   Again, the category $\mathbf{Vect}_{\mathbb{F}}$ equipped with the direct sum as a monoidal product and a 0-dimensional vector space as the monoidal unit is a monoidal category. This view of $\mathbf{Vect}_{\mathbb{F}}$ is essentially viewing it as a semi-additive category.

5. The category **Hilb**, whose objects are complex (or real) Hilbert spaces and whose morphisms are bounded linear maps, can be viewed as a monoidal category, equipped with the usual tensor product (the completion of the tensor product vector space) as a monoidal product and $\mathbb{C}$ (or $\mathbb{R}$) as the monoidal unit.

   Of course, as is the case with $\mathbf{Vect}_{\mathbb{F}}$, **Hilb** becomes a monoidal category using the direct sum of Hilbert spaces and a $0-$dimensional Hilbert space to provide the biproduct monoidal structure.

[1] note that this does not imply that the associators and the unitors become identities in the skeleton of $\mathcal{M}$

[2] for more on this subject and a reformulation of the definition of monoidal category one might look at [EGNO15]

6. Let $\mathcal{C}$ be a category. Denote $End(\mathcal{C})$ the category of endofunctors of $\mathcal{C}$, i.e. $[\mathcal{C}, \mathcal{C}]$. The vertical composition of natural transformations provides the composition of the morphisms of this category. The horizontal composition of natural transformations provides a functor

$$\circ : End(\mathcal{C}) \times End(\mathcal{C}) \to End(\mathcal{C}),$$

satisfying associativity and unit laws. Thus we can identify this functor with a monoidal product in $End(\mathcal{C})$ and the identity endofunctor $\mathbb{1}_{\mathcal{C}}$ with a unit.

7. There exists a family of categories, topological in nature, whose members are denoted by $\mathbf{Tang}_k^n$, for $n, k \in \mathbb{N}$ with $n > k$. The objects of the category $\mathbf{Tang}_k^n$ are smooth embeddings of compact $(k-1)$-dimensional manifolds M into $\mathbb{R}^{n-1}$. The morphisms of this category, denoted $W : M_0 \to M_1$, are smooth embeddings (considered up to ambient isotopy) of compact k-dimensional manifolds with boundary into $\mathbb{R}^{n-1} \times [0, 1]$, such that

$$\partial W = M_0 \coprod M_1,$$

where the components of the boundary are mapped into $\mathbb{R}^{n-1} \times \{0\}$ and $\mathbb{R}^{n-1} \times \{1\}$, respectively. Composition in $\mathbf{Tang}_k^n$ is defined by gluing two k-tangles along their common boundary, followed by a smooth reparametrisation of the interval $[0, 1]$ to restore the standard form. The identity morphism for an object M is given by the embedding

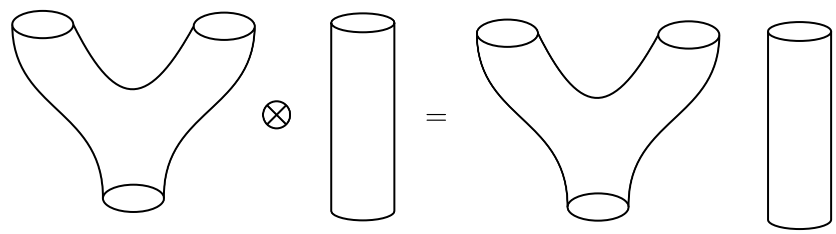
$$M \times [0, 1] \hookrightarrow \mathbb{R}^{n-1} \times [0, 1],$$

that is, the trivial cylinder over M. A monoidal structure on $\mathbf{Tang}_k^n$ is provided by the disjoint union of manifold embeddings, with the empty manifold serving as the unit object. Finally, note that a smooth reparametrisation $s : [0, 1] \to [0, 1]$ satisfying $s(0) = 1$ and $s(1) = 0$ induces an operation on k-tangles, which reverses their orientation as morphisms and can be easily seen to define a contravariant functor on $\mathbf{Tang}_k^n$. The special case where $n = 3$ and $k = 1$ yields the classical **category of tangles**, on which several of the string diagrammatic calculi presented in this thesis are based.

8. There is a subcategory of $\mathbf{Tang}_1^3$ which will be of interest when we study coherence for braided monoidal categories, which is the category of braids, denoted by $\mathbb{B}$. Since $\mathbf{Tang}_3$ is strict monoidal, so is $\mathbb{B}$. The objects in this category are again sets of n colinear points in $\mathbb{R}^3$, denoted by $n \in \mathbf{N}$ and their tensor product is their disjoint union i.e. their sum $n \otimes m = n + m$. The unit object is the empty set, denoted by 0. The homsets $\mathbb{B}(n, m)$ are empty when $n \neq m$ and $\mathbb{B}(n, n)$ forms the braid group, $\mathbb{B}_n$ under composition, the elements of which are called braids. The tensor product of morphisms is the disjoint union of braids.

9. There is another family of categories resembling $\mathbf{Tang}_k^n$. A member of this family is called $n\mathbf{Cob}$ and is defined for positive $n \in \mathbb{N}$. An object in n**Cob** is the diffeomorphism class of a closed oriented $(n-1)$-dimensional manifold. A morphism is the diffeomorphism class of an n-dimensional compact oriented manifold whose boundary is the disjoint union of two $(n-1)$-dimensional manifolds, i.e. the domain and the codomain. Such a (diffeomorphism class of a) manifold is called a **cobordism**.

   Composition in this family of categories amounts to the gluing of two cobordisms along their common boundary. The tensor product is the disjoint union of manifolds, which is trivially associative, thus satisfying the pentagon law. The unit of n**Cob** is the empty manifold viewed as a cobordism between empty manifolds. Of course, the disjoint union of a cobordism with the empty cobordism is the former cobordism itself, therefore n**Cob** satisfies the triangle laws trivially. This shows that the category n**Cob** is a strict monoidal category. Below we see an example of the disjoint union of a cobordism from one to two circles and a cobordism from one circle to another inside 2-**Cob**. The convention used is that the sources of the morphisms are at the bottom and the targets at the top.



An interesting fact about 2-**Cob** is that when we only consider closed 1-manifolds as objects, every object is a disjoint union of circles with possibly different orientations. If on the other hand we consider compact 1-manifolds as objects then objects are disjoint unions of circles and closed 1-dimensional intervals.

The **oppposite monoidal category**, $\mathcal{M}^{\text{op}}$, of a monoidal category $(\mathcal{M}, \otimes, I, a, l, r)$, is the opposite category of $\mathcal{M}$, the tensor product functor is the same on objects and for every two morphisms f, g in $\mathcal{M}$, $f^{\text{op}} \otimes g^{\text{op}} := (f \otimes g)^{\text{op}}$. The monoidal unit is defined as $I^{\text{op}} := I$ and the associator and unitor isomorphisms correspond to the inverses of a, l, r. The pentagon and triangle laws hold, since they reduce to the ones holding in $\mathcal{M}$. Thus, $(\mathcal{M}^{\text{op}}, \otimes, I, (a^{-1})^{\text{op}}, (l^{-1})^{\text{op}}, (r^{-1})^{\text{op}})$ is a monoidal category.

There is another construction which has a flavour of duality. For every monoidal category $\mathcal{M}$, there is the **reverse monoidal category** $\mathcal{M}^{\text{rev}}$, defined as follows. $\mathcal{M}^{\text{rev}}$ is $\mathcal{M}$ as a category and has the same monoidal unit. The monoidal product of two objects $A, B \in \mathcal{M}_0$ is $A \otimes^{\text{rev}} B = B \otimes A$ and similarly the order is reversed in the tensor product of morphisms. The associator and the unitor isomorphisms are given componentwise as $a^{\text{rev}}_{C,B,A} := a^{-1}_{A,B,C}$, $\ell^{\text{rev}}_A := r_A$ and $r^{\text{rev}}_A := \ell_A$. Again the pentagon and triangle laws hold since they trace back to the corresponding ones in $\mathcal{M}$. So $(\mathcal{M}^{\text{rev}}, \otimes^{\text{rev}}, I, a^{\text{rev}}, l^{\text{rev}}, r^{\text{rev}})$ is a monoidal category.

Since the category **Cat** is cartesian monoidal, and since monoidal categories are categories, one can form the product of two monoidal categories. This cartesian product is a monoidal category on its own. Concretely:

Definition 2.1.7. *Let $(\mathcal{M}, \otimes, I, a, l, r)$ and $(\mathcal{M}', \otimes', I', a', l', r')$ be monoidal categories. We define the* ***product monoidal category*** *$\mathcal{M} \times \mathcal{M}'$ as follows:*

- *The tensor product of objects $(X, X'), (Y, Y') \in \mathcal{M} \times \mathcal{M}'$ is*

$$(X, X') \bigotimes (Y, Y') := (X \otimes Y, X' \otimes' Y')$$

  and the tensor product of morphisms is

$$(f, f') \bigotimes (g, g') := (f \otimes g, f' \otimes' g').$$

- *The unit object is (I, I').*
- *The associator is given by (a, a') and the unitors by (l, l') and (r, r').*

This is easily seen to be a monoidal category since the pentagon and the triangle laws for the product category are equivalent to the ones in the constituent categories.

Definition 2.1.8. *A subcategory $\mathcal{N}$ of a monoidal category $(\mathcal{M}, \otimes, I, a, l, r)$ is a* ***monoidal subcategory*** *of $\mathcal{M}$, when $I \in \mathcal{N}_0$ and it is closed under the tensor.*

Remark 2.1.9. Some of the previous examples, such as **Vect** and **Hilb** are monoidal categories under their usual tensor product, and they also are semi-additive under the direct product. Thus, in general, a monoidal structure can coexist with a semi-additive one. This results in some interesting properties. For example, since in a semi-additive monoidal category $I \otimes 0$ is isomorphic, through the left unitor, to 0 and $0_{I,0} \circ 0_{0,I} = 0_{0,0} = \text{id}_0$, we get that

$$\ell_0 \circ (0_{0,I} \otimes \text{id}_0) : 0 \otimes 0 \to 0 \text{ and } (0_{I,0} \otimes \text{id}_0) \circ \ell_0^{-1} : 0 \to 0 \otimes 0$$

are mutual inverses. So tensoring the monoidal unit or the zero object with a zero object results in a zero object.

On the other hand a superposition rule or a semi-additive structure need not co-operate with the tensor product. There are cases where the tensor respects/distributes over the biproduct, including closed monoidal categories with biproducts. In such cases we may speak of **distributive** or **rig categories.**

Remark 2.1.10. In **Hilb** and **Vect** equipped with the tensor product, both left and right unitors' components at the identity represent the commutative multiplication of the base field. The fact that $\ell_{\mathbb{C}} = r_{\mathbb{C}}$ is not a coincidence, as this is the case in any monoidal category. To prove this we proceed as follows.

Proposition 2.1.11. *Let $\mathcal{M}$ be a monoidal category. Then*

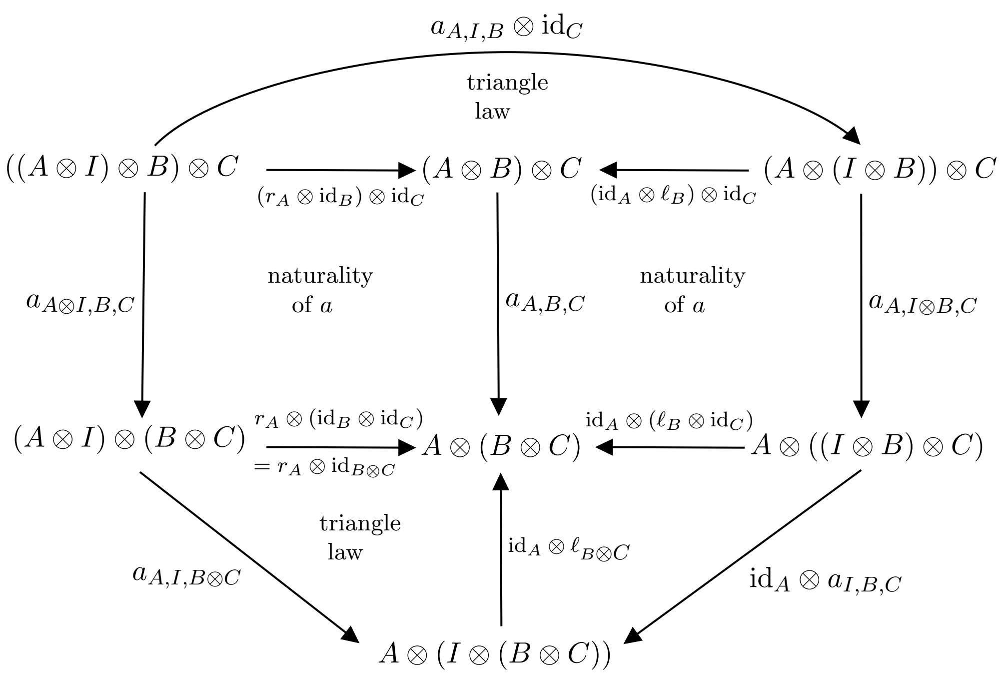
1. *For every $A, B \in \mathcal{M}_0$,*

$$(\ell_A \otimes \mathrm{id}_B) = \ell_{A\otimes B} \circ a_{I,A,B} \text{ and}$$
$$r_{A\otimes B} = (\mathrm{id}_A \otimes r_B) \circ a_{A,B,I}$$

2. *The right and left unitors coincide at the identity, i.e.*

$$\ell_I = r_I.$$

Proof. We will prove 1. by diagram chasing. Let $A, B, C \in \mathcal{M}_0$. In the following diagram, the "curvy" outside pentagon commutes by the pentagon law. The other subdiagrams except the bottom right one commute either as instances of the triangle law and its image under a functor $(- \otimes \mathrm{id}_C)$, or as naturality squares for the associator.



Therefore, the bottom right diagram commutes or equivalently

$$\mathrm{id}_A \otimes (\ell_{B\otimes C} \circ a_{I,B,C}) = \mathrm{id}_A \otimes (\ell_B \otimes \mathrm{id}_C).$$

Picking A to be the unit I and renaming the objects, we get that

$$\ell_{A\otimes B} \circ a_{I,A,B} = \ell_A \otimes \mathrm{id}_B.$$

Proceeding similarly

$$r_{A\otimes B} = \mathrm{id}_A \otimes r_B.$$

To prove 2. we will use 1. and the triangle law. Since l is a natural isomorphism we get the following equivalent equalities.

$$\begin{aligned} \ell_I \circ \ell_{I\otimes I} &= \ell_I \circ (\mathrm{id}_I \otimes \ell_I) \Leftrightarrow \\ \ell_{I\otimes I} &= \mathrm{id}_I \otimes \ell_I \end{aligned} \tag{2.1}$$

According to 1. we get that

$$\ell_I \otimes \mathrm{id}_I = \ell_{I\otimes I} \circ a_{I,I,I}$$

which, using 2.1 and the triangle law, gives:

$$\begin{aligned} \ell_I \otimes \mathrm{id}_I &= (\mathrm{id}_I \otimes \ell_I) \circ a_{I,I,I} \\ &= r_I \otimes \mathrm{id}_I \end{aligned}$$

so

$$\ell_I = r_I.$$

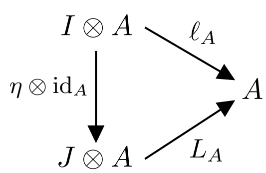
□

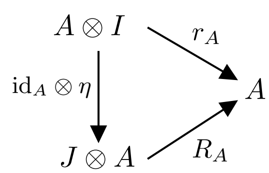
Remark 2.1.12. The significance of $r_I = \ell_I := i$, is that we can relate every component of the unitors to i. Speciafically, by the triangle law and the above proposition:

$$\begin{aligned} \mathrm{id}_I \otimes \ell_A &= (i \otimes \mathrm{id}_A) \circ a^{-1}_{I,I,A} \\ r_A \otimes \mathrm{id}_I &= (\mathrm{id}_A \otimes i) \circ a_{A,I,I}. \end{aligned}$$

The above proposition allows us to prove that in a semigroupal category that is also monoidal, there is a unique monoidal structure up to unique isomorphism. For this to be precise, we define a way to compare two such structures.

Definition 2.1.13. *Let $(\mathcal{M}, \otimes, a)$ be a semigroupal category (as in 2.1.5) and let (I, r, l), (J, R, L) be such that $(\mathcal{M}, \otimes, I, a, r, l)$ and $(\mathcal{M}, \otimes, J, a, R, L)$ are monoidal categories. We say that the units I and J are* ***compatible****, if there exists an isomorphism $\eta : I \to J$, such that for every $A \in \mathcal{M}_0$ the following diagrams commute.*

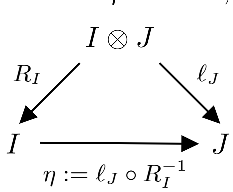




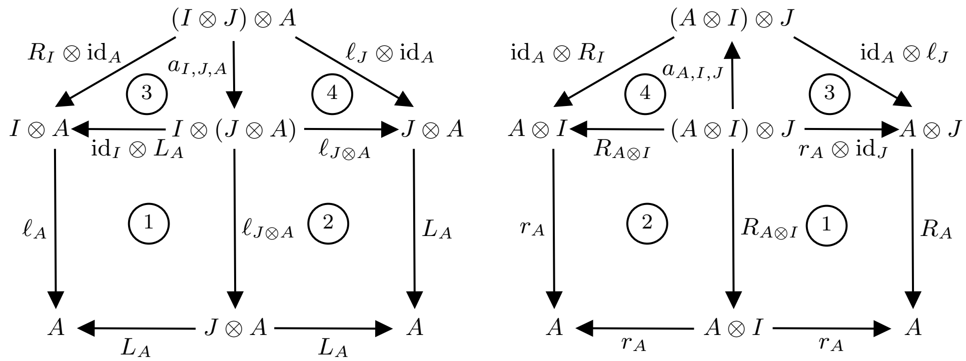
A comparison between units in a monoidal category has to ensure that all unit properties, expressed by the unitor isomorphisms, need to be preserved as well.

Proposition 2.1.14. *Let $(\mathcal{M}, \otimes, I, a, r, l)$ and $(\mathcal{M}, \otimes, J, a, R, L)$ be two monoidal structures on a semigroupal category $(\mathcal{M}, \otimes, a)$. Then there is a unique isomorphism $\eta : I \to J$ that makes I and J compatible.*

Proof. For the existence part of the proof note that $\eta : I \to J$, defined as follows, is an isomorphism.



For the compatibility condition of η, we draw the following two diagrams.



These diagrams commute since 1's are naturality squares for l and R, 2's are trivial, 3's are the triangle laws for both pairs (r, l) and (R, L) and 4's commute by proposition 2.1.11. Therefore, from the first one we get,

$$L_A \circ (\ell_J \otimes \mathrm{id}_A) \circ (R_I^{-1} \otimes \mathrm{id}_A) = L_A \circ L_A^{-1} \circ \ell_A \Leftrightarrow$$
$$L_A \circ (\ell_J \circ R_I^{-1} \otimes \mathrm{id}_A) = \ell_A \Leftrightarrow$$
$$L_A \circ (\eta \otimes \mathrm{id}_A) = \ell_A,$$

and similarly from the second one,

$$R_A \circ (\mathrm{id}_A \otimes \eta) = r_A.$$

Finally, to prove uniqueness note that the compatibility condition only involves isomorphisms. Therefore, any isomorphism $\phi : I \to J$ would have to satisfy

$$\phi \otimes \mathrm{id}_A = L_A^{-1} \circ \ell_A = \eta \otimes \mathrm{id}_A,$$

for every $A \in \mathcal{M}_0$, so $\phi = \eta$. □

Remark 2.1.15. Obviously, given a monoidal category $(\mathcal{M}, \otimes, I, a, l, r)$ any isomorphic object to the unit can be equipped with unitors that make it compatible with (I, l, r), given a choice of an isomorphism. So the units of a monoidal category form a "clique" (see 1.5.8). There is a bigger theory on the role of units in monoidal categories, which is omitted in this presentation. For more one can look at [Koc08].

A monoidal structure in functor categories

Now an interesting thing about a monoidal category $\mathcal{M}$ is that for every category $\mathcal{C}$, the functor category $[\mathcal{C}, \mathcal{M}]$ can be equipped with a monoidal structure coming from $\mathcal{M}$. This goes as follows.

Define the tensor product of functors $G, F \in \mathbf{Cat}(\mathcal{C}, \mathcal{M})$ as

$$(G \otimes F)(X) := GX \otimes_{\mathcal{M}} FX,$$

for every $X \in \mathcal{C}$ and given $f : X \to Y$

$$(G \otimes F)(f) := Gf \otimes_{\mathcal{M}} Ff.$$

To define the tensor product of natural transformations let $\eta : F \Rightarrow F'$ and $\theta : G \Rightarrow G'$. Then $\theta \otimes \eta : G \otimes F \Rightarrow G' \otimes F'$ is the natural transformation whose components are

$$(\theta \otimes \eta)_X := \theta_X \otimes_{\mathcal{M}} \eta_X.$$

It is an easy task, working componentwise, to check that the above assignment is functorial. Therefore, $\otimes : [\mathcal{C}, \mathcal{M}] \times [\mathcal{C}, \mathcal{M}] \to [\mathcal{C}, \mathcal{M}]$ is a bifunctor. The role of the unit in $[\mathcal{C}, \mathcal{M}]$ is played by the constant functor $\Delta_I : \mathcal{C} \to \mathcal{M}$. To make $[\mathcal{C}, \mathcal{M}]$ into a monoidal category we also need an associator and two unitors. These will be defined componentwise in terms of the associator and the unitors in $\mathcal{M}$ as well. The associator's components $\mathbf{a}_{H,F,G} : (H \otimes G) \otimes F \Rightarrow H \otimes (G \otimes F)$ are the natural isomorphisms

$$\mathbf{a}_{H,F,G} := a_{H-,F-,G-},$$

whose component at $X \in \mathcal{C}$ is $(\mathbf{a}_{H,F,G})_X = a_{HX,FX,GX}$. Checking that the pentagon law holds is easy working with the components of the components of $\mathbf{a}$, since they obey the pentagon law in $\mathcal{M}$. Similarly, the triangle law holds in $\mathbf{Cat}(\mathcal{C}, \mathcal{M})$ because it holds in $\mathcal{M}$, defining the unitors as

$$\mathbf{l}_F := \ell_{F-} \text{ and } \mathbf{r}_F := r_{F-},$$

for every $F : \mathcal{C} \to \mathcal{M}$.

The above construction, doesn't actually depend on the fact that the elements of $[\mathcal{C}, \mathcal{M}]$ are covariant functors. So defining the tensor product of contravariant functors appropriately, leads to contravariant functor categories.

Example 2.1.16. A good example of a (contravariant) monoidal functor category is the case where $\mathcal{M}$ is **Set**. This is the case of presheaves and copresheaves. Since **Set** is cartesian monoidal, the resulting tensor product of (co)presheaves is given as the "pointwise" product, the unit is the constant (co)presheaf on the terminal object of **Set** and the associators and unitors are provided by the universal properties of products and the terminal object. One can fairly easily check that the categories of (co)presheaves are cartesian monoidal with such a tensor product. Dually, such (co)presheaf categories are also cocartesian monoidal, with pointwise disjoint union as a tensor and the constant (co)presheaf Δ as a unit.

States, Effects and Scalar Multiplication

Since cartesian categories are monoidal, one might say that monoidal categories resemble cartesian ones. Following this resemblance we present the analog of global elements for monoidal categories. The fact that in a cartesian category a terminal object is also the unit of the product, while generally a unit object is unique up to unique isomorphism, justifies the following definition.

Definition 2.1.17. *Let $\mathcal{M}$ be a monoidal category. A* ***state*** *of an object $A \in \mathcal{M}_0$ is a morphism:*

$$\psi : I \to A,$$

where I is the monoidal unit. A ***joint state*** *of objects A and B is a state ψ of $A \otimes B$. A* ***product state*** *ψ of A and B is a joint state of the form:*

$$f = (a \otimes b) \circ i^{-1},$$

where a, b are states of A and B respectively. An ***entangled state*** *of A and B is a joint state which is not a product state.*

Remark 2.1.18. Using generalised elements, as in definition 1.2.9, a state $\psi : I \to A$ may be denoted by $\psi \in_I A$. Furthermore, given a morphism $f : A \to B$, the composite morphism $f \circ \psi : I \to B$ may be denoted by $f(\psi) \in_I B$.

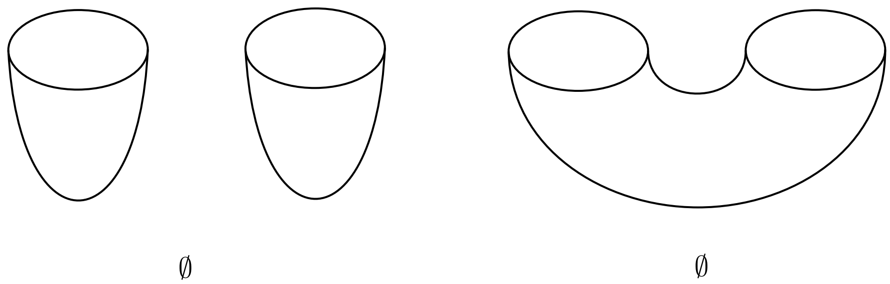
Example 2.1.19. The above terminology has its roots in quantum theory, where the states of a quantum system correspond to rays in its associated Hilbert space. Rays in a Hilbert space, $\mathcal{H}$, are naturally identified with continuous linear maps of unit length in $\mathbf{Hilb}(\mathbb{C}, \mathcal{H})$. Since, rays in a Hilbert space are in bijection with unit vectors, a state is actually a "ket" in Dirac notation. Formally, given a vector $a \in \mathcal{H}$ we have $|a\rangle : \mathbb{C} \to \mathcal{H}$, such that for every $z \in \mathbb{C}$,

$$|a\rangle (z) = za.$$

Example 2.1.20. In **Set** but also in cartesian categories, every joint state amounts to a pair of (global) elements of a product, so there are no entangled states.

Example 2.1.21. A state of an endofunctor, F, in $End(\mathcal{C})$ is a natural transformation $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow F$. A product state of the composite of two endofunctors, $G \circ F$, is the horizontal composition of natural transformations $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow F$ and $\theta : \mathbb{1}_{\mathcal{C}} \Rightarrow G$.

Example 2.1.22. In n**Cob** a state amounts to the "creation" of a manifold/boundary out of nowhere. A product state and an entangled state are shown below where we see how entanglement arises in n**Cob** for $n = 2$:



With the notion of states we can generalise well pointedness to a monoidal category.

Definition 2.1.23. *Let $\mathcal{M}$ be a monoidal category. We say that $\mathcal{M}$ is **well-pointed** if for every $f, g : A \to B$*

$$f = g \Leftrightarrow f(a) = g(a),$$

for all states $a \in_I A$.

*We say that $\mathcal{M}$ is **monoidally well-pointed**, if for every $B, A_i \in \mathcal{M}_0$, for $n \in \mathbb{N}$ and $1 \leq i \leq n$, and every $f, g : \bigotimes_{i=1}^{n} A_i \to B$,*

$$f = g \Leftrightarrow f(a) = g(a),$$

for all product states $a = (a_1 \otimes \cdots \otimes a_n) \circ \ell_I^{-1} \in_I \bigotimes_{i=1}^{n} A_i$.

The concept of a state is defined in terms of morphisms. Therefore, there is also a dual notion which is usually not called a costate. So making use of the quantum theoretic terminology we get the following definition.

Definition 2.1.24. *Let $\mathcal{M}$ be a monoidal category. An **effect** of an object $A \in \mathcal{M}_0$ is a morphism:*

$$\phi : A \to I,$$

where I is the monoidal unit.

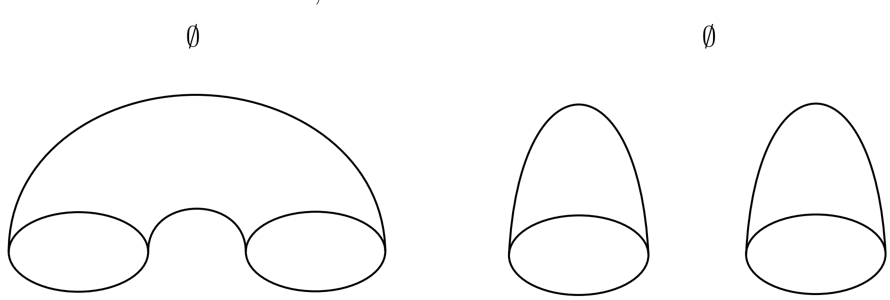
Example 2.1.25. In **Hilb** an effect of a space $\mathcal{H}$ amounts to a bounded linear functional on $\mathcal{H}$. So the effects form the dual space $\mathbf{Hilb}(\mathcal{H}, \mathbb{C}) = \mathcal{H}^{\dagger}$ of $\mathcal{H}$ whose elements in Dirac notation are the "bra"s. Formally, given $b \in \mathcal{H}$, we have

$$\langle b| (a) = \langle b|a \rangle,$$

for every $a \in \mathcal{H}$.

Example 2.1.26. In **Set** with the cartesian monoidal product, every object has a unique effect, since the monoidal unit is terminal.

Example 2.1.27. In n**Cob** an effect amounts to the "destruction" of a "pre-existing" manifold/boundary. Two such effects in n**Cob** are shown below, for $n = 2$:



In a semicartesian category, i.e. a category with monoidal uniform deleting, we see that there is a unique state for every object. This also holds for the unit, whose single state/effect is in this case the identity morphism. Terminality of the unit implies that composing any state with an effect yields the identity morphism of the unit. Such behaviour is called **deleting a state**.

If there is a zero object, which is not the monoidal unit, there is a possibility that an effect pre-composed with a morphism will result in a zero morphism. In addition, more than one states might yield the same morphism when post-composed with the same morphism.

Definition 2.1.28. *In a monoidal category $\mathcal{M}$, with a zero object, we call a set of effects $\Psi = \{\psi_i : B \to I\}_{i \in \mathcal{I}}$* ***complete****, if for every morphism $f \in \mathcal{C}(A, B)$, $f \neq 0_{A,B}$, there exists an $i_f \in \mathcal{I}$, such that*

$$\psi_{i_f} \circ f \neq 0_{A,I}.$$

Remark 2.1.29. Obviously, if in the above case the monoidal unit is terminal, thus a zero object, then there is no complete set of effects.

In the presence of a semi-additive structure, there is a characterization of finite complete sets of effects.

Proposition 2.1.30. *Let $\mathcal{M}$ be a monoidal category with a semi-additive structure. A finite set of effects $\Psi = \{\psi_i : B \to I\}_{i=1}^n$ is complete if and only if*

$$\psi := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

has $(0, 0_{0,B})$ as a kernel.

Proof. We firstly assume that Ψ is complete. Assume there is an $f : A \to B$ such that $\psi \circ f = 0$. Then

$$\psi_i \circ f = 0$$

for every $i = 1, \ldots, n$. Completeness of Ψ implies that $f = 0_{A,B}$, so f factors uniquely through the zero object. Thus $(0, 0_{0,B})$ is a kernel of ψ.

For the opposite direction, let ψ have $(0, 0_{0,B})$ as a kernel and assume that Ψ is not complete. Then there exists an $f : A \to B$, $f \neq 0_{A,B}$ such that

$$\psi_i \circ f = 0,$$

for every $i = 1, \ldots, n$. So

$$\psi \circ f = 0$$

and by the universal property of the kernel there exists a unique $e : A \to 0$, such that $f = 0_{0,B} \circ e$. But then f is a zero morphism which is a contradiction, thus Ψ is complete. □

A valid question at this point would be about the states of the unit. One can easily observe that every such state is also an effect of the unit and, conversely, every effect of the unit is also a state.

Definition 2.1.31. *In a monoidal category $\mathcal{M}$ every endomorphism of the unit $s \in \mathcal{M}(I, I)$ is called a* ***scalar****.*

Remark 2.1.32. Since the endomorphisms of an object in any category form a monoid, this is also the case for scalars. Thus the multiplication of the monoid corresponds to the composition of scalars. Furthermore, observe that any scalar in a monoidal category can be written as a composite of a state and an effect (considering the elements of $\mathcal{M}(I, I)$ as either states or effects).

Example 2.1.33. We have already seen that in both **Hilb** and $\mathbf{Vect}_{\mathbb{F}}$ the states correspond to the base field, where the name scalars has its roots.

Example 2.1.34. In the cartesian category **Set** the set of scalars is actually the set only containing the identity morphism of the one element set.

Example 2.1.35. In $End(\mathcal{C})$, where $\mathcal{C}$ is a category, scalars amount to the natural transformations $\eta : \mathbb{1}_{\mathcal{C}} \Rightarrow \mathbb{1}_{\mathcal{C}}$.

In the case of vector spaces and Hilbert spaces, the multiplication of the base field is commutative. This is actually a general fact for the scalars in any monoidal category.

Proposition 2.1.36 (Eckmann-Hilton argument)**.** *Let $\mathcal{M}$ be a monoidal category and $\mathcal{M}(I,I)$ its set of scalars. Then $\mathcal{M}(I,I)$ is a commutative monoid.*

Proof. Firstly note that, by the coincidence $\ell_I = r_I = i$ and the naturality of l and r, we get that for every scalar f,

$$\begin{aligned} f &= i \circ (f \otimes \mathrm{id}_I) \circ i^{-1} \\ &= i \circ (\mathrm{id}_I \otimes f) \circ i^{-1}. \end{aligned}$$

Now let $s, t \in \mathcal{M}(I,I)$. Then

$$\begin{aligned} s \circ t &= i \circ (\mathrm{id}_I \otimes s) \circ i^{-1} \circ i \circ (t \otimes \mathrm{id}_I) \circ i^{-1} \\ &= i \circ (t \otimes s) \circ i^{-1} \\ &= i \circ (t \otimes \mathrm{id}_I) \circ i^{-1} \circ i \circ (\mathrm{id}_I \otimes s) \circ i^{-1} \\ &= t \circ s, \end{aligned}$$

where we have also used the interchange law. □

Remark 2.1.37. In the presence of a superposition rule and a zero object the monoid of scalars is actually a rig. This follows from the fact that composition of scalars distributes over the superposition rule.

Continuing the analogy between **Hilb**, $\mathbf{Vect}_{\mathbb{F}}$ and a general monoidal category, observe that there is a scalar multiplication for linear or bounded linear maps. Accordingly, we define scalar multiplication for morphisms of a general monoidal category.

Definition 2.1.38. *Let $\mathcal{M}$ be a monoidal category. For an $s \in \mathcal{M}(I,I)$ and an $f \in \mathcal{M}(A,B)$, where $A, B \in \mathcal{M}_0$ we define the left scalar multiplication as*

$$s \cdot f := \ell_B \circ (s \otimes f) \circ \ell_A^{-1}$$

and the right scalar multiplication as

$$f \cdot s := r_B \circ (f \otimes s) \circ r_A^{-1}.$$

From now on, since left and right scalar multiplication are defined analogously, we will focus only on left scalar multiplication, which will be referred to just as "scalar multiplication".

Proposition 2.1.39 (Properties of scalar multiplication)**.** *Let $\mathcal{M}$ be a monoidal category, and $s, t \in \mathcal{M}(I,I)$, $f \in \mathcal{M}(A,B)$, $g \in \mathcal{M}(B,C)$, $\psi \in \mathcal{M}(I,A)$, $\phi \in \mathcal{M}(A,I)$ be scalars, morphisms, a state and an effect, respectively. The following properties hold.*

1. $\mathrm{id}_I \cdot f = f$,
2. $s \cdot t = s \circ t$,
3. $s \cdot \psi = \psi \cdot s$ *and* $s \cdot \phi = \phi \cdot s$
4. $(t \cdot g) \circ (s \cdot f) = (t \circ s) \cdot (g \circ f) = (t \cdot s) \cdot (g \circ f)$
5. $s \cdot (t \cdot f) = (s \cdot t) \cdot g$.

Proof. For 1. note that naturality of l and the definition of scalar multiplication imply

$$f = \ell_B \circ (\mathrm{id}_I \otimes f) \circ \ell_A^{-1} = \mathrm{id}_I \cdot f.$$

For 2. observe that

$$\begin{aligned} s \cdot t &= i \circ (s \otimes t) \circ i^{-1} \\ &= i \circ (s \otimes \mathrm{id}_I) \circ i^{-1} \circ i \circ (\mathrm{id}_I \otimes t) \circ i^{-1} \\ &= s \circ t. \end{aligned}$$

For 3. observe that the definition of scalar multiplication, naturality of the unitors and functoriality of the tensor "⊗", imply

$$\begin{aligned} s \cdot \psi &= \ell_A \circ (s \otimes \psi) \circ i^{-1} \\ &= \ell_A \circ (\mathrm{id}_I \otimes \psi) \circ (s \otimes \mathrm{id}_I) \circ i^{-1} \\ &= \psi \circ i \circ i^{-1} \circ s \\ &= \psi \circ s \end{aligned} \tag{2.2}$$

and

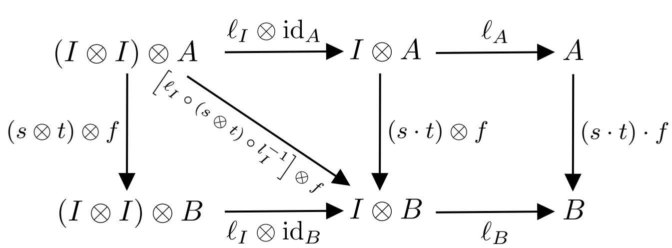
$$\begin{aligned} \psi \cdot s &= r_A \circ (\psi \otimes s) \circ i^{-1} \\ &= r_A \circ (\psi \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes s) \circ i^{-1} \\ &= \psi \circ i \circ i^{-1} \circ s \\ &= \psi \circ s \end{aligned} \tag{2.3}$$

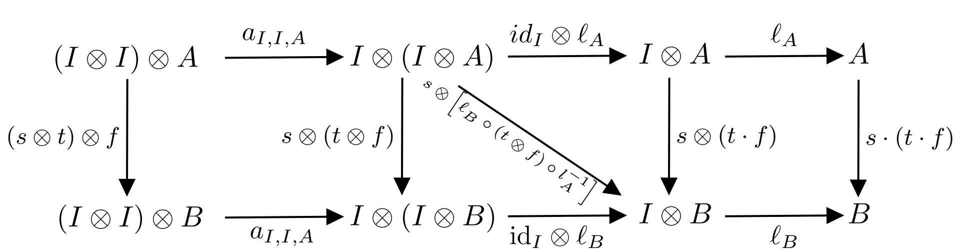
Therefore, by 1.2 and 1.3 we get the desired equality. The proof is similar for scalar multiplication of effects.

For 4. the second equality is an application of 2. For the first equation notice that the similarity with the interchange law is not a coincidence. In precise terms we have that

$$\begin{aligned} (t \cdot g) \circ (s \cdot f) &= \ell_C \circ (t \otimes g) \circ \ell_B^{-1} \circ \ell_B \circ (s \otimes f) \circ \ell_A^{-1} \text{ (interchange law)} \\ &= \ell_C \circ [(t \circ s) \otimes (g \circ f)] \circ \ell_A^{-1} \\ &= (t \circ s) \cdot (g \circ f). \end{aligned}$$

For 5. observe that the two following diagrams are commutative by naturality of the associator, by the definition of scalar multiplication and by the interchange law.





Therefore, from the first one we get that

$$(s \cdot t) \cdot f = \ell_B \circ (\ell_I \otimes \mathrm{id}_B) \circ [(s \otimes t) \otimes f] \circ (\ell_I \otimes \mathrm{id}_A)^{-1} \circ \ell_A^{-1}$$

and from the second one

$$\begin{aligned} s \cdot (t \cdot f) &= \ell_B \circ [(\mathrm{id}_I \otimes \ell_B) \circ a_{I,I,B}] \circ [(s \otimes t) \otimes f] \circ [(\mathrm{id}_I \otimes \ell_A) \circ a_{I,I,A}]^{-1} \circ \ell_A^{-1} \\ &= \ell_B \circ (\ell_I \otimes \mathrm{id}_B) \circ [(s \otimes t) \otimes f] \circ (\ell_I \otimes \mathrm{id}_A)^{-1} \circ \ell_A^{-1} \\ &= (s \cdot t) \cdot f \end{aligned}$$

□

States, effects and scalars can be combined to create what in quantum theory is known as an amplitude. This is nothing more than the creation of a scalar out of a state and an effect of an object.

Definition 2.1.40. *Let $\mathcal{M}$ be a monoidal category, $A \in \mathcal{M}_0$ an object, $\psi \in \mathcal{M}(I, A)$ be a state and $\phi \in \mathcal{M}(A, I)$ and effect. We call the scalar $\phi \circ \psi \in \mathcal{M}(I, I)$ the* ***amplitude*** *for ψ to evolve into ϕ.*

Remark 2.1.41. There is an obvious generalisation of amplitudes as defined above. Given a morphism $f \in \mathcal{M}(A, B)$, a state $\psi \in \mathcal{M}(I, A)$ and an effect $\phi \in \mathcal{M}(b, I)$, we can call $\psi \circ f \circ \phi$ the amplitude for the transition of ψ into ϕ, but since $\phi \circ f$ is also an effect and $f \circ \psi$ is a scalar, this definition, while distinguishing the notions of evolution and transition, seems redundant.

Example 2.1.42. In $\mathbf{Vect}_{\mathbb{F}}$ and **Hilb**, forming an amplitude corresponds to the pairing operation between vectors and functionals. This is being used in quantum mechanics to form the transition and evolution amplitudes, the source of the bra-ket notation. The only exception is that in quantum mechanics only unit vectors are used, but such things only exist in Hilbert spaces, or normed spaces in general. There is a way to generalise the inner product of Hilbert spaces to a bigger class of categories, but this requires the notion of a dagger, to be defined in the section about dagger monoidal categories.

Semi-(co-)cartesian monoidal categories and uniform deleting

A good question about monoidal categories is "how much (co)cartesian are they?". To investigate possible answers to this question we introduce the notions of semi-cartesian and semi-co-cartesian categories as bridging a gap between "full fledged" monoidal categories and (co)cartesian ones.

Definition 2.1.43. *Let $\mathcal{M}$ be a monoidal category and let $P_1, P_2 : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ stand for the usual projections of the categorical product of $\mathcal{M}$ with itself. $\mathcal{M}$ is called a*

1. ***monoidal category with semi-projections*** *if there exist natural transformations $\pi^1 : - \otimes - \Rightarrow P_1$ and $\pi^2 : - \otimes - \Rightarrow P_2$. If*
$$\pi^1_{I,I} = \pi^2_{I,I} = \ell_I = r_I = \iota$$
then $\mathcal{M}$ is called a ***monoidal category with projections***

2. ***monoidal category with semi-coprojections*** *if there exist natural transformations $i^1 : P_1 \Rightarrow - \otimes -$ and $i^2 : P_2 \Rightarrow - \otimes -$. If*
$$i^1_{I,I} = i^2_{I,I} = \iota^{-1}$$
then $\mathcal{M}$ is called a ***monoidal category with coprojections****.*

Remark 2.1.44. Obviously any cartesian category is a monoidal category with projections and any co-cartesian category is a monoidal category with co-projections.

Example 2.1.45. An example is the category **Vect** with the direct sum as a monoidal product and the 0-dimensional vector space as a unit. This category is actually semi-additive, as defined in 1.6, so it is a monoidal category with both projections and coprojections, as it is both cartesian and cocartesian.

The interesting thing about it, though, is that it can be a monoidal category with projections or injections in more than one ways. This is achieved by keeping the monoidal structure as it is, but taking the projections or the coprojections as a fixed scalar multiple of the original ones that make it (co)cartesian. Then naturality of the (co)projections would still hold and, furthermore, since the unit object is the zero vector space we would have the same components for the (co)projections at $0 \otimes 0$.

Definition 2.1.46. *Let $(M, \otimes, I, a, l, r, \pi^1, \pi^2)$ be a monoidal category with projections. We say that π^1 and π^2 are* ***compatible with the unitors*** *if for every $X \in \mathcal{M}_0$:*

$$\pi^1_{X,I} = r_X \text{ and } \pi^2_{I,X} = \ell_X.$$

Dually, let $(M, \otimes, I, a, l, r, i^1, i^2)$ be a monoidal category with co-projections. We say that i^1 and i^2 are ***compatible with the unitors*** *if for every $X \in \mathcal{M}_0$:*

$$i^1_{X,I} = r_X^{-1} \text{ and } i^2_{I,X} = \ell_X^{-1}.$$

Remark 2.1.47. Observe that compatibility with the unitors implies that we already have a category with projections or coprojections, depending on the context. On the other hand having a category with projections does not imply compatibility with the unitors. This can be seen clearly in the case of **Vect** considered above, since scalar multiples of the projections are not compatible with the unitors. Duality implies also that having a category with certain co-projections does not imply that they are compatible with the unitors. As we will see below this can be tamed, i.e. we can always pick (co)projections that are compatible with the unitors.

Theorem 2.1.48. *Let $(\mathcal{M}, \otimes, I, a, l, r)$ be a monoidal category. $\mathcal{M}$ is a monoidal category with projections if and only if I is terminal. Furthermore, if $\mathcal{M}$ is a monoidal category with projections compatible with the unitors, then the projections are given by*

$$\pi^1_{X,Y} = r_X \circ (\mathrm{id}_X \otimes e_Y) \text{ and } \pi^2_{X,Y} = \ell_Y \circ (e_X \otimes \mathrm{id}_Y),$$

where e_X, e_Y are the unique maps to the terminal object for every $X, Y \in \mathcal{M}_0$.

Proof. Firstly, let I be terminal. Define for every $X, Y \in \mathcal{M}_0$

$$\pi^1_{X,Y} = r_X \circ (\mathrm{id}_X \otimes e_Y) \text{ and } \pi^2_{X,Y} = \ell_Y \circ (e_X \otimes \mathrm{id}_Y).$$

These are natural transformations as vertical compositions of natural transformations. In addition, since $e_I = \mathrm{id}_I$ by terminality, we get that

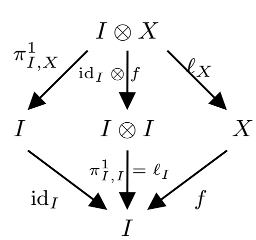
$$\pi^1_{X,I} = r_X \circ (\mathrm{id}_X \otimes \mathrm{id}_I) = r_X \circ (\mathrm{id}_{X \otimes I}) = r_X$$

and similarly

$$\pi^2_{I,Y} = \ell_Y$$

so the projections are compatible with the unitors.

Conversely, if $\mathcal{M}$ has projections, then for every $f : X \to I$, naturality of π^1 and the left unitor makes the following diagram commute:



thus, every morphism, f, to the unit has to be

$$f = \pi^1_{I,X} \circ \ell^{-1}_X,$$

which proves existence and uniqueness. Therefore, I is terminal.

Finally, if $\mathcal{M}$ has projections compatible with the unitors, then for every $X, Y \in \mathcal{M}_0$ compatibility with the unitors and terminality of I implies:

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\mathrm{id}_X \otimes e_Y} & X \otimes I \\ {\scriptstyle \pi^1_{X,Y}} \downarrow & & \downarrow {\scriptstyle \pi^1_{X,I} = r_X} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X \otimes Y & \xrightarrow{e_X \otimes \mathrm{id}_Y} & I \otimes Y \\ {\scriptstyle \pi^2_{X,Y}} \downarrow & & \downarrow {\scriptstyle \pi^2_{I,Y} = \ell_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

□

The above theorem can be dualized by substituting "terminal" with "initial" and "projections" with "coprojections" given by:

$$i^1_{X,Y} = (\mathrm{id}_X \otimes u_Y) \circ r^{-1}_X \text{ and } i^2_{X,Y} = (u_X \otimes \mathrm{id}_Y) \circ \ell^{-1}_Y.$$

Remark 2.1.49. The above theorem essentially tells us that a monoidal category with projections/coprojections can equivalently be defined as a monoidal category whose unit is terminal/initial. Moreover, in every such category there is a unique way of picking the (co)projections, utilizing the unique morphisms to/from the unit and the unitors, such that compatibility with the unitors holds. so the following definition is a manifestation of these facts.

Definition 2.1.50. *Let $(\mathcal{M}, \otimes, I, a, l, r)$ be a monoidal category. $\mathcal{M}$ is called:*

1. ***semi-cartesian** if the unit I is terminal and*
2. ***semi-co-cartesian** if the unit I is initial.*

Terminality of the unit object hints at a notion already mentioned previously, namely uniform deleting. To make this precise we give the following general definition.

Definition 2.1.51. *Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ has **uniform deleting** if there exists an object $T \in \mathcal{C}_0$ and a natural trasnformation $e : \mathbb{1}_{\mathcal{C}} \to \Delta_T$, called **uniform deleting**, such that $e_T = \mathrm{id}_T$.*

Proposition 2.1.52. *Let $\mathcal{C}$ be a category. $\mathcal{C}$ has uniform deleting $e : \mathrm{Id}_{\mathcal{C}} \to \Delta_T$ if and only if T is terminal.*

Proof. Obviously, if T is terminal then for every object $A \in \mathcal{C}_0$ there exists a unique morphism $e_A : A \to T$, so $e_T = \mathrm{id}_T$ and naturality follows from the uniqueness of e_A, thus $(e_A)_{A \in \mathcal{C}_0}$ provides uniform deleting.

Conversely, given uniform deleting $(e_A)_{A \in \mathcal{C}_0}$ and a morphism $f : A \to T$, we get that naturality of e gives,

$$e_T \circ f = \mathrm{id}_T \circ e_A,$$

but $e_T = \mathrm{id}_T$, so $f = e_A$. Thus T is terminal.

□

Every semi-cartesian category has uniform deleting, but the converse does not hold. As a counter-example take **Vect** with the usual tensor product of vector spaces as monoidal product and the base field as a unit. Then the zero vector space is terminal, but it is not isomorphic or equal to the base field. Thus we conclude that not every monoidal category having a terminal object is semi-cartesian, but it has uniform deleting. To make those notions compatible we say that *a monoidal category has **monoidal uniform deleting** if the uniform deleting natural transformation is a co-cone under I, the monoidal unit.* Equivalently, a monoidal category has monoidal uniform deleting if I is terminal.

There is also a subtler notion of deletion available for monoidal categories which is expressed only in terms of states.

Definition 2.1.53. *Let M be a monoidal category, $\psi : I \to A$ a state and $e_A : A \to I$ a morphism in $\mathcal{M}$. We say that e_A **deletes the state** ψ, if*

$$e_A \circ \psi = \mathrm{id}_I.$$

The set of deletable states by e_A will be denoted by $\mathbf{DS}(e_A)$.

Example 2.1.54. In **Vect** and **Hilb**, for every state $\psi \in_I V$ there exists an effect $e_V : V \to I$, such that

$$e_V(\psi(1)) = 1.$$

But this state is guaranteed to not delete all states of V, since every linear functional has co-dimension 1 and therefore a non-trivial kernel. This, however, does not say that every other monoidal category has such a property, but to characterise it we need the following proposition.

Proposition 2.1.55. *Let $\mathcal{M}$ be a monoidal category and $(e_A : A \to I)_{A\in\mathcal{M}_0}$ a family of maps. If $(e_A)_{A\in\mathcal{M}_0}$ provides monoidal uniform deleting, then this family deletes every state.*

Proof. Since monoidal uniform deleting implies that the unit, I, is terminal we get that for every $A \in \mathcal{M}_0$ and every state $\psi \in_I A$, $e_A \circ \psi$ is the unique morphism id_I. □

What is interesting about deleting all states is actually the converse to the above proposition, which only holds under certain assumptions.

Proposition 2.1.56. *Let $\mathcal{M}$ be a well-pointed monoidal category and $(e_A : A \to I)_{A\in\mathcal{M}_0}$ a family of maps. If $(e_A)_{A\in\mathcal{M}_0}$ deletes every state then, this family provides monoidal uniform deleting.*

Proof. To show this we only need to prove that I is terminal. For every $A \in \mathcal{M}_0$, $\mathcal{M}(A, I)$ is non-empty since $e_A \in \mathcal{M}(A, I)$, so let $f, g : A \to I$ be two morphisms. Since e_I is a state of I and it also deletes any state of I we have $e_I \circ e_I = \mathrm{id}_I$. Then for every state $\psi \in_I A$,

$$e_I \circ f \circ \psi = \mathrm{id}_I = e_I \circ g \circ \psi,$$

which implies, by well-pointedness, that $e_I \circ f = e_I \circ g$, so

$$e_I \circ e_I \circ f = e_I \circ e_I \circ g \Leftrightarrow f = g$$

and thus I is terminal. □

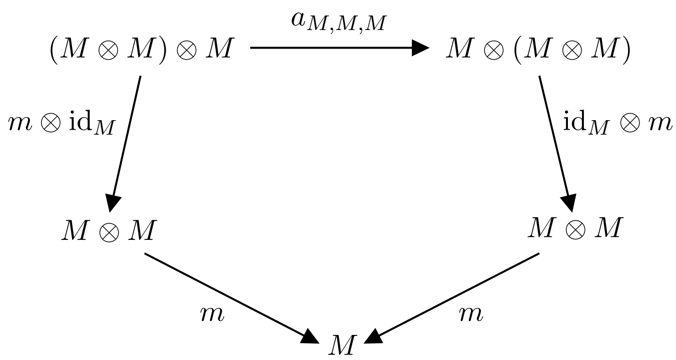
Internal (co)monoids and Frobenius objects

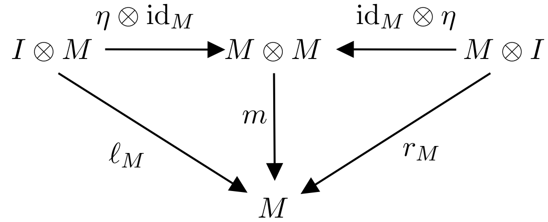
The ordinary definition of a monoid depends on the existence of a terminal object and a cartesian product in **Set**. Replacing cartesian product with monoidal and terminal object with monoidal unit we get the following definition.

Definition 2.1.57. *An **internal monoid** (M, m, η) in a monoidal category $\mathcal{M}$ is an object $M \in \mathcal{M}_0$ equipped with morphisms:*

- *$m : M \otimes M \to M$ called **multiplication** and*
- *$\eta : I \to M$ called **unit**,*

*such that the following diagrams, called **associativity** and **unit** laws commute:*





Example 2.1.58. Obviously, when such a construction is carried out inside **Set** we get an ordinary monoid, while an internal monoid in $\mathbf{Vect}_{\mathbb{F}}$ is a unital associative algebra. Similarly, a monoid inside **Ab**[3] is a ring.

Example 2.1.59. The category of preorders/posets equipped with the cartesian product of preorders/posets as a monoidal product and the terminal poset (the one object preorder category) as a unit is a monoidal category. An internal monoid in the category of preorders/posets is a monoidal preorder/poset.

Example 2.1.60. A very interesting example is an internal monoid in the category of endofunctors, $[\mathcal{C}, \mathcal{C}]$ of a category $\mathcal{C}$. In this case the most commonly used multiplication symbol is μ, the unit map symbol is η and the monoid symbol is T. This particular example is widely known as a **Monad**. The associativity law satisfied by a monad is actually a square, since $[\mathcal{C}, \mathcal{C}]$ is strict monoidal for every category $\mathcal{C}$. Similarly the left and right unitors involved in the unit laws are actually identity morphisms.

Furthermore, to every internal monoid (M, m, u) in a monoidal category $\mathcal{M}$, there is an associated monad $(M \otimes -, (m \otimes \mathrm{id}_-) \circ a^{-1}_{M,M,-}, (u \otimes \mathrm{id}_-) \circ \ell^{-1}_-)$. The monad axioms (associativity and unitality of the monad) boil down to the monoid axioms of M. Dually, there is another monad associated to the monoid M, given by tensoring with the monoid on the right.

Example 2.1.61. Since **CAT** is a cartesian category, one might wonder what is a monoid in this category. The answer is that a monoid in the category of categories is a strict monoidal category, $(\mathcal{M}, \otimes, I)$. Its multiplication morphism is the tensor functor, the unit morphism is a generalised element of $\mathcal{M}$ corresponding to the monoidal unit. The associativity and unit laws hold trivially since associators and unitors are identity morphisms, due to the pentagon and triangle laws of the internal monoid holding strictly. Notice that to form a monoidal category that is not strict as an internal monoid in some category requires more structure than a category can give. Actually, the associator and the unitor isomorphisms can be captured only by the notion of 2-cells in a 2-Category or a Bicategory.

Example 2.1.62. An abstract example of an internal monoid, is the identity I, of a monoidal category $\mathcal{M}$. In clear terms $(I, \ell_I, \mathrm{id}_I)$ is an internal monoid in $\mathcal{M}$, because the unit laws are trivially true and the associativity law is implied by the triangle law for $\mathcal{M}$ and the fact that the unitors coincide at the identity.

Another abstract example is the one object of the one object posetal and monoidal category **1**. This is trivially a monoid since every map for the monoid structure is provided by the identity morphism of 1.

Definition 2.1.63. *Let $\mathcal{M}$ be a monoidal category and M, N be monoids in $\mathcal{M}$. An internal monoid homomorphism h is a morphism $h \in \mathcal{M}(M, N)$ such that*

$$f \circ m_M = m_N \circ (f \otimes f)$$

and

$$\eta_N = f \circ \eta_M,$$

where m_M, m_N and η_M, η_N are the multiplications and units of M and N respectively.

[3] as a full subcategory of **Grp**

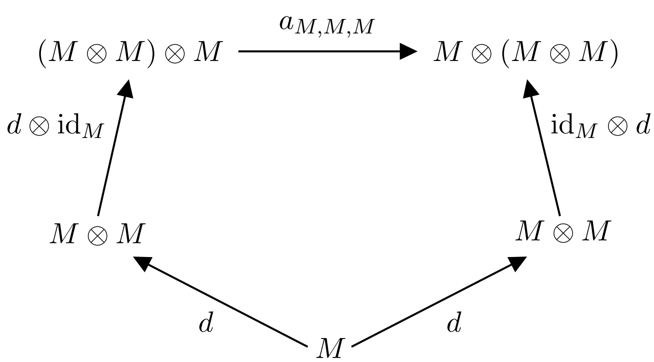
What is captured by the above definition is that the unit and multiplication of a monoid are preserved by a monoid homomorphism. Again, this definition both in **Set** and in $\mathbf{Vect}_{\mathbb{F}}$ gives the usual notions of monoid homomorphism and unital associative algebra homomorphism, respectively.

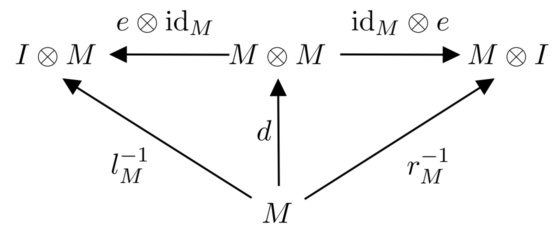
The internalisation of an algebraic structure achieved by the above definition of internal monoid allows us to easily define the dual notion of a monoid.

Definition 2.1.64. *An **internal comonoid** (M, d, e) in a monoidal category $\mathcal{M}$ is an object $M \in \mathcal{M}_0$ equipped with morphisms:*

- *$d : M \to M \otimes M$ called **comultiplication** and*
- *$e : M \to I$ called **counit**,*

*such that the following diagrams, called **coassociativity** and **counitality**, commute:*





Remark 2.1.65. Notice that an internal comonoid in $\mathcal{M}$ is actually an internal monoid in the opposite category, $\mathcal{M}^{\mathrm{op}}$.

Example 2.1.66. An internal comonoid in **Set** is a usual comonoid, while an internal comonoid in $\mathbf{Vect}_{\mathbb{F}}$ is a coalgebra.

Example 2.1.67. An internal comonoid in $End(\mathcal{C})$, where $\mathcal{C}$ is a category, is what is called a **comonad**. Since $End(\mathcal{C})$ is a strict monoidal category the coassociativity pentagon reduces to a square and the left and right unitors involved in counitality are identity arrows.

Similarly to monoids, to every comonoid (M, d, e) in a monoidal category $\mathcal{M}$, there is an associated comonad $(M \otimes -, a_{M,M,-} \circ (d \otimes \mathrm{id}_-), \ell_- \circ (e \otimes \mathrm{id}_-))$ given by tensoring with the comonoid on the left. Dually, there is another comonad associated with M, given by tensoring on the right.

Example 2.1.68. The unique object of the posetal monoidal category $\mathbf{1}$, can be given a comonoid structure, since the axioms of a comonoid hold trivially. Therefore, the object $1 \in \mathbf{1}_0$ is both a monoid and a comonoid inside $\mathbf{1}$.

Remark 2.1.69. In a general cartesian monoidal category $\mathcal{C}$, every object A can be naturally equipped with a comonoid structure. The comultiplication is given by the diagonal morphism d_A and the counit by the unique morphism, e_A, to the terminal object. This will be expanded in the section about braided monoidal categories.

In exact accordance with the monoid case, we define a comonoid homomorphism.

Definition 2.1.70. *Let $\mathcal{M}$ be a monoidal category and M, N be internal comonoids in $\mathcal{M}$. A comonoid homomorphism $h : (M, d_M, e_M) \to (N, d_N, e_N)$ internal to $\mathcal{M}$ is a morphism $h \in \mathcal{M}(M, N)$ such that*

$$f \otimes f \circ d_M = d_N \circ f$$

and

$$e_N \circ f = e_M.$$

Remark 2.1.71. Interestingly, the set of states of a monoid (M, m, u) in $\mathcal{M}$, is actually a monoid, in **Set**. This obviously requires a multiplication and a unit. So let μ be equal to the following composite:

$$\mathcal{M}(I, A) \times \mathcal{M}(I, A) \xrightarrow{-\otimes-} \mathcal{M}(I \otimes I, A \otimes A) \xrightarrow{m \circ - \circ \iota^{-1}} \mathcal{M}(I, A)$$

and $\eta = u$. Then $(\mathcal{M}(I, M), \mu, \eta)$ is a monoid in **Set**.

Duality in this case is somehow tricky. That is, the set of effects of a comonoid (A, d, e) is again a monoid, under the multiplication, μ, given as the following composite:

$$\mathcal{M}(A, I) \times \mathcal{M}(A, I) \xrightarrow{-\otimes-} \mathcal{M}(A \otimes A, I \otimes I) \xrightarrow{\iota \circ - \circ d} \mathcal{M}(A, I).$$

So $(\mathcal{M}(A, I), \mu, e)$ is a monoid and not a comonoid.

These two cases become instances of a general fact, by considering the unit as a monoid or a comonoid, accordingly.

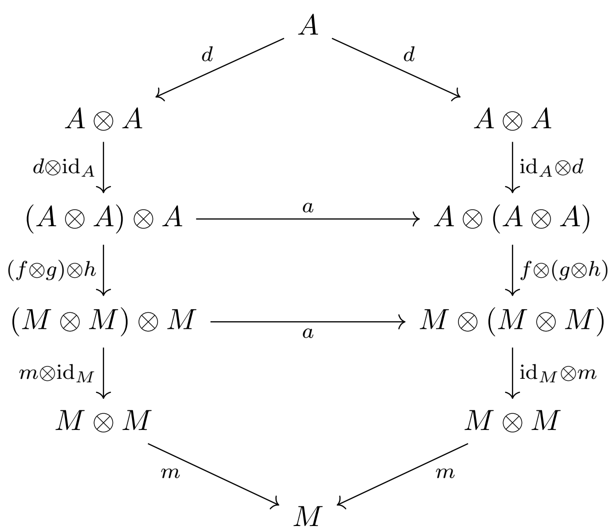
Definition 2.1.72. *Let $\mathcal{M}$ be a monoidal category, (M, m, u) a monoid and (A, d, e) a comonoid in $\mathcal{M}$. The function defined as the composite:*

$$\star : \mathcal{M}(A, M) \times \mathcal{M}(A, M) \xrightarrow{\otimes} \mathcal{M}(A \otimes A, M \otimes M) \xrightarrow{m \circ - \circ d} \mathcal{M}(A, M)$$

is called the ***convolution product****.*

Proposition 2.1.73. *Given a monoid (M, m, u) and a comonoid (A, d, e), the set of morphisms $\mathcal{M}(A, M)$, becomes a monoid under the convolution product, with $u \circ e$ as a unit.*

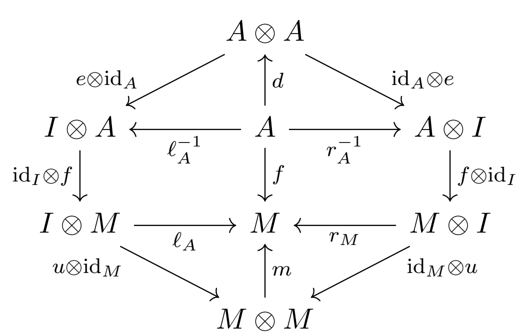
Proof. Let $f, g, h \in \mathcal{M}(A, M)$ be morphisms and consider the following diagram.



The above diagram commutes by associativity of M, naturality of the associator and coassociativity of A. Thus, $m \circ ((m \circ (f \otimes g) \circ d) \otimes h) \circ d = m \circ (f \otimes (m \circ (g \otimes h) \circ d)) \circ d$, or equivalently

$$(f \star g) \star h = f \star (g \star h)$$

Furthermore, observe that the following diagram commutes by unitality of M, naturality of the unitors and counitality of A:



Therefore, $m \circ ((u \circ e) \otimes f) \circ d = f = m \circ (f \otimes (u \circ e)) \circ d$, which is equivalent to:

$$(u \circ e) \star f = f = f \star (u \circ e).$$

□

Corollary 2.1.74. *Let (M, m, u), be a monoid inside a monoidal category $\mathcal{M}$. Then $(\mathcal{M}(I, A), \mu, u)$ is a monoid in* **Set**.

There are cases where an internal monoid is also a comonoid. More precisely, both a monoid and a comonoid structure can be imposed on the same carrying object. Such objects in **Vect** are both algebras and coalgebras. In general, we will refer to them as **monoid-comonoid pairs.** Imposing some compatibility condition between the monoid and the comonoid structure, e.g. having the monoid multiplication be a comonoid homomorphism and vice versa, results in objects whose consideration is very fruitful, such as bialgebras and Hopf algebras. Such objects can only exist in braided monoidal categories, so in this section we present another class of objects, called Frobenius objects, which may exist in any monoidal category.

Definition 2.1.75. *Let $\mathcal{M}$ be a monoidal category and let $M \in \mathcal{M}_0$ be such that (M, m, u) is a monoid and (M, d, e) is a comonoid internal to $\mathcal{M}$. We call (M, m, u, d, e) a* ***Frobenius object*** *if*

$$(\mathrm{id}_M \otimes m) \circ a_{M,M,M} \circ (d \otimes \mathrm{id}_M) = d \circ m = (m \circ \mathrm{id}_M) \circ a^{-1}_{M,M,M} \circ (\mathrm{id}_M \otimes d).$$

If (M, m, u, d, e) satisfies:

$$m \circ d = \mathrm{id}_M,$$

then it is called a ***special Frobenius object****. A morphism $f : M \to N$ between Frobenius objects, is a* ***Frobenius object homomorphism*** *if it is simultaneously a monoid and a comonoid homomorphism between the underlying monoids and comonoids of M and N.*

Example 2.1.76. A Frobenius object inside **Vect** is called a **Frobenius algebra**. The interesting thing about Frobenius algebras is that they are all finite dimensional. So every Frobenius object of **Vect** lives inside **FdVect**.

Example 2.1.77. Any Hilbert space $H \in$ **FdHilb** can be equipped with a Frobenius structure, given an orthogonal basis, $\mathcal{B} = \{e_i\}_{i=1}^n$, where $n = \dim H$. Firstly, we define $d : H \to H \otimes H$, $e : H \to I$, $m : H \otimes H \to H$ and $u : I \to H$ as the unique linear maps such that

$$d(e_i) = e_i \otimes e_i, \quad e(e_i) = 1$$

$$m(e_i \otimes e_j) = < e_j, e_i > e_j \text{ and } u(1) = \sum_{i=1}^n e_i,$$

for every $1 \leq i, j \leq n$. It is easy to see that orthogonality implies (H, d, e) is a comonoid and (H, m, u) is a monoid. Secondly and rather informally[4], observe that for every $1 \leq i, j \leq n$,

$$\begin{aligned}(\mathrm{id}_H \otimes m) \circ (d \otimes \mathrm{id}_H)(e_i \otimes e_j) &= e_i \otimes m(e_i \otimes e_j) \\ &= < e_j, e_i > e_i \otimes e_j \\ &= \delta_{i,j} < e_i, e_i > e_i \otimes e_i\end{aligned}$$

$$\begin{aligned}(m \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes d)(e_i \otimes e_j) &= m(e_i \otimes e_j) \otimes e_j \\ &= < e_j, e_i > e_j \otimes e_j \\ &= \delta_{i,j} < e_i, e_i > e_i \otimes e_i\end{aligned}$$

[4]disregarding the coherent isomorphisms in **FdHilb**

$$d \circ m(e_i \otimes e_j) = < e_j, e_i > d(e_j)$$
$$= \delta_{i,j} < e_i, e_i > e_i \otimes e_i$$

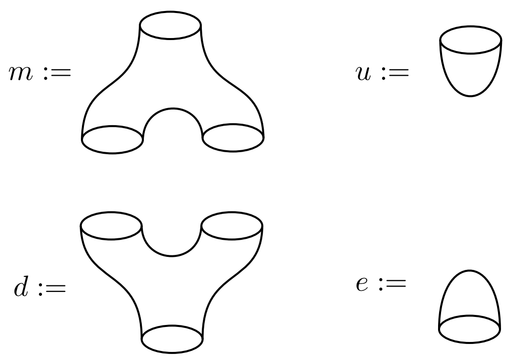
are all equal, so the Frobenius law holds. Finally, observe that for $1 \leq i \leq n$,

$$m \circ d(e_i) = < e_i, e_i > e_i,$$

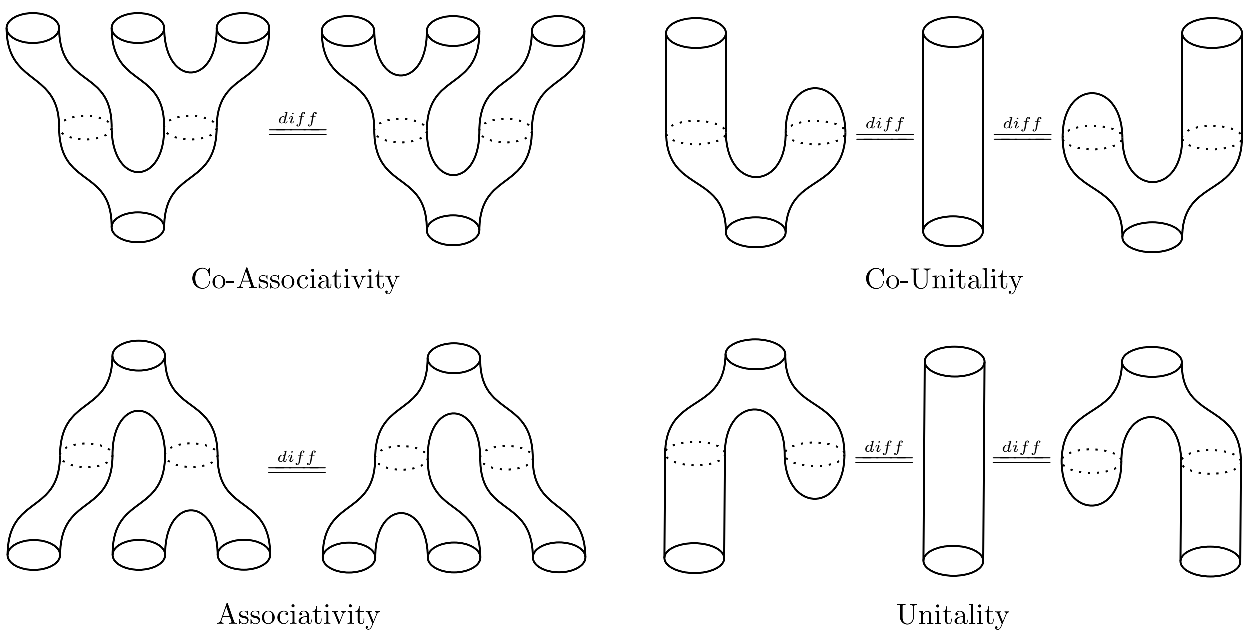
so $m \circ d = \mathrm{id}_H$ if and only if $\mathcal{B}$ is orthonormal. Thus, orthogonal bases correspond to Frobenius structures and orthonormal bases to special Frobenius structures.

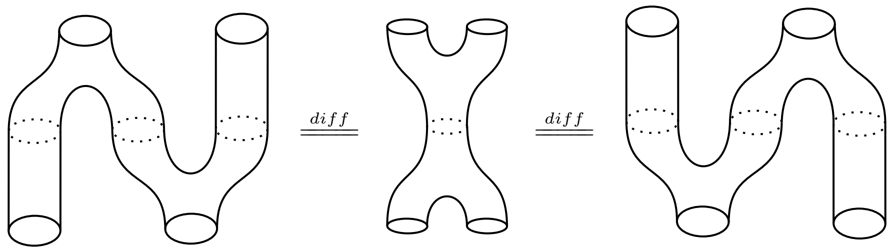
Example 2.1.78. The unique object of the trivial monoidal category **1** is both a monoid and a comonoid. The Frobenius condition also holds in this case, since every morphism is the identity, and so does the speciality condition. Therefore 1 is a special Frobenius object inside **1**.

Example 2.1.79. In 2-**Cob**, the circle S^1 can be equipped with a Frobenius object structure. Define the multiplication, the unit, the co-multiplication and the co-unit as:



The monoid and comonoid axioms, but also the Frobenius conditions are satisfied, as it is evident from the following:





Frobenius law

It is an immediate fact, both for internal monoids and comonoids, but also for Frobenius objects, that the identity morphism is a homomorphism in each case and that composition of homomorphisms is again a homomorphism. Therefore, we can associate to any monoidal category, $\mathcal{M}$, its category of monoids, $\mathbf{Mon}(\mathcal{M})$, its category of comonoids, $\mathbf{Com}(\mathcal{M})$ and its category of Frobenius objects $\mathbf{Frob}(\mathcal{M})$. The problem here would be that we won't be able to identify the commutative ones in each case respectively. For this to be possible we need a symmetry or a braiding in the ambient monoidal category.

In any case we can state the following two results conserning the unit object and its role in the category of internal (co)monoids.

Proposition 2.1.80. *Let (M, m, u) be a monoid inside a monoidal category $\mathcal{M}$. Then the unit $u : I \to M$ is a monoid homomorphsim.*

Proof. We only need to show that

$$m \circ (u \otimes u) = u \circ \ell_I.$$

This follows from naturality of the left unitor ℓ and the unit law for M, which provide the following commutative diagram:

$$\begin{array}{ccccc} I \otimes I & \xrightarrow{\mathrm{id}_I \otimes u} & I \otimes M & \xrightarrow{u \otimes \mathrm{id}_M} & M \otimes M \\ \downarrow{\scriptstyle \ell_I} & & \downarrow{\scriptstyle \ell_M} & \swarrow{\scriptstyle m} & \\ I & \xrightarrow[u]{} & M & & \end{array}$$

□

Proposition 2.1.81. *Let $\mathcal{M}$ be a monoidal category and $\mathbf{Mon}(\mathcal{M})$ be the category of internal monoids. Then $(I, \ell_I, \mathrm{id}_I)$ is initial in $\mathbf{Mon}(\mathcal{M})$.*

Proof. We need to show that for every monoid (M, m, u), there exists a unique monoid homomorphism $f : I \to M$. Uniqueness is covered by the previous proposition, so we only need to show that the unit is unique as a monoid homomorphism. Let $f : I \to M$ be a monoid homomorphism. From the fact that monoid homomorphisms preserve the unit, we have the following commutative triangle:

$$\begin{array}{ccc} I & \xrightarrow{f} & M \\ {\scriptstyle \mathrm{id}_I}\nwarrow & & \nearrow{\scriptstyle u} \\ & I & \end{array}$$

which asserts that $u = f$ and thus u is unique. □

Remark 2.1.82. Dually, the unit is a terminal object in the category of comonoids inside a monoidal category.

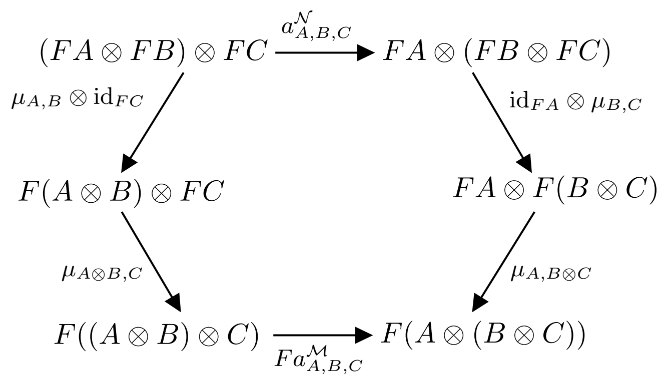
Remark 2.1.83. According to the above the category of internal monoids is semicocartesian and the category of internal comonoids is semicartesian.

2.2 Monoidal functors and monoidal natural transformations

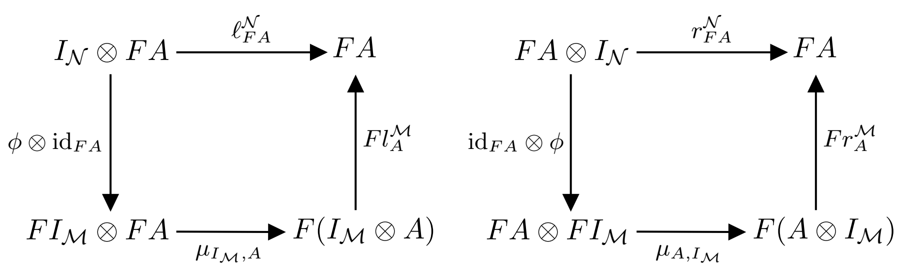
In this section we present the morphisms between monoidal categories and the morphisms between them.

Definition 2.2.1. *Let* $(\mathcal{M}, \otimes_{\mathcal{M}}, I_{\mathcal{M}}, a^{\mathcal{M}}, l^{\mathcal{M}}, r^{\mathcal{M}}), (\mathcal{N}, \otimes_{\mathcal{N}}, I_{\mathcal{N}}, a^{\mathcal{N}}, l^{\mathcal{N}}, r^{\mathcal{N}})$ *be monoidal categories. A triple* (F, μ, ϕ)*, where* $F : \mathcal{M} \to \mathcal{N}$ *is a functor,* $\mu : \otimes_{\mathcal{N}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{M}}$ *is a natural transformation and* $\phi : I_{\mathcal{N}} \to FI_{\mathcal{M}}$ *is a morphism, is called a*

i) ***lax monoidal functor*** *if for every* $A, B, C \in \mathcal{M}_0$ *the following diagram commutes*



called ***associativiy*** *and for every* $A \in \mathcal{M}_0$ *the following diagrams commute*



called ***unitality****.*

ii) ***strong monoidal functor*** *if in addition to unitality and associativity* μ *and* ϕ *are isomorphisms.*

iii) ***strict monoidal functor*** *if* μ *and* ϕ *are identities.*

Remark 2.2.2. Obviously every strict monoidal functor is a strong monoidal functor and every strong monoidal functor is lax. Since monoidal categories are a categorification of monoids, monoidal functors are a categorification of monoid homomorphisms. It is obvious that considering discrete monoidal categories, a.k.a monoids, one gets that any monoidal functor between discrete monoidal categories is a monoid homomorphism. Therefore μ and ϕ should also be called **coherence maps**.

Furthermore, at first glance there seems to be a redundancy in the definition of lax and strong monoidal functors. Actually, it turns out that there is an equivalent definition which only seems weaker. The first thing to notice in this direction is that unitality implies

$$\phi \otimes \mathrm{id}_{FI_{\mathcal{M}}} = (\ell^{\mathcal{N}}_{FI})^{-1} \circ Fi^{\mathcal{M}} \circ \mu_{I,I},$$

where i denotes the unitors component at the identity and I is used instead of $I_{\mathcal{M}}$. A similar condition holds for the right unitors, which is also compatible with the left unitor one. It is easy to check that this condition implies unitality. Secondly, this condition is enough to determine a unique ϕ, given that it exists. This is due to the fact that the functor $- \otimes I_{\mathcal{N}}$, and consequently $- \otimes FI_{\mathcal{M}}$ being naturally isomorphic to the first one, is an equivalence. Therefore, instead of unitality one might only demand that the above condition holds. In the case of strong monoidal functors, one should also demand that $FI_{\mathcal{M}} \cong I_{\mathcal{N}}$, without necessarily specifying the isomorphism.

Remark 2.2.3. An obvious remark is that if $\mathcal{C}$ is a category and $\mathcal{M}$ is a strict monoidal category, such that $F : \mathcal{C} \to \mathcal{M}$ is an isomorphism, then $\mathcal{C}$ can be equipped with a strict monoidal structure such that F is a strict monoidal functor.

Example 2.2.4. An example of a strong monoidal functor is provided by the representable functor $\mathcal{M}(I,-) : \mathcal{M} \to \mathbf{Set}$, for every monoidal category $\mathcal{M}$. Here $\mu : \mathcal{M}(I,-) \times \mathcal{M}(I,-) \Rightarrow \mathcal{M}(I,- \otimes -)$ is such that $\mu_{A,B}(f,g) = (f \otimes g) \circ i^{-1}$ and $\varepsilon : T \to \mathcal{M}(I,I)$ is defined as $\varepsilon(s) = \mathrm{id}_I$, where s is the single element of T. Associativity is satisfied since

$$\begin{aligned} a_{A,B,C} \circ (((f \otimes g) \circ i^{-1}) \otimes h) \circ i^{-1} &= a_{A,B,C} \circ ((f \otimes g) \otimes h) \circ (i^{-1} \otimes \mathrm{id}_I) \circ i^{-1} \\ &= (f \otimes (g \otimes h)) \circ a_{I,I,I} \circ (i^{-1} \otimes \mathrm{id}_I) \circ i^{-1} \text{ (by naturality of } a) \\ &= (f \otimes (g \otimes h)) \circ (\mathrm{id}_I \otimes i^{-1}) \circ i^{-1} \text{ (by the triangle law)} \\ &= (f \otimes ((g \otimes h) \circ i^{-1})) \circ i^{-1} \end{aligned}$$

for every $f \in \mathcal{M}(I,A), g \in \mathcal{M}(I,B), h \in \mathcal{M}(I,C)$. Similarly, unitality is implied by naturality of the left and right unitors.

Example 2.2.5. Forgetful functors provide another example of monoidal functors. The forgetful functor from **Hilb** to **Vect**, which "forgets" the inner product, is obviously a strict monoidal functor. Similarly, the forgetful functor from **Vect** to **Ab**, which forgets scalar multipllication is again a strict monoidal functor.

Example 2.2.6. Given a monoidal category $\mathcal{M}$ we already know that $\mathcal{M} \times \mathcal{M}$ is a monoidal category. There is also the uniquely determined diagonal functor $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ given by the universal property of the product in **Cat**. This diagonal functor is actually a strict monoidal functor since

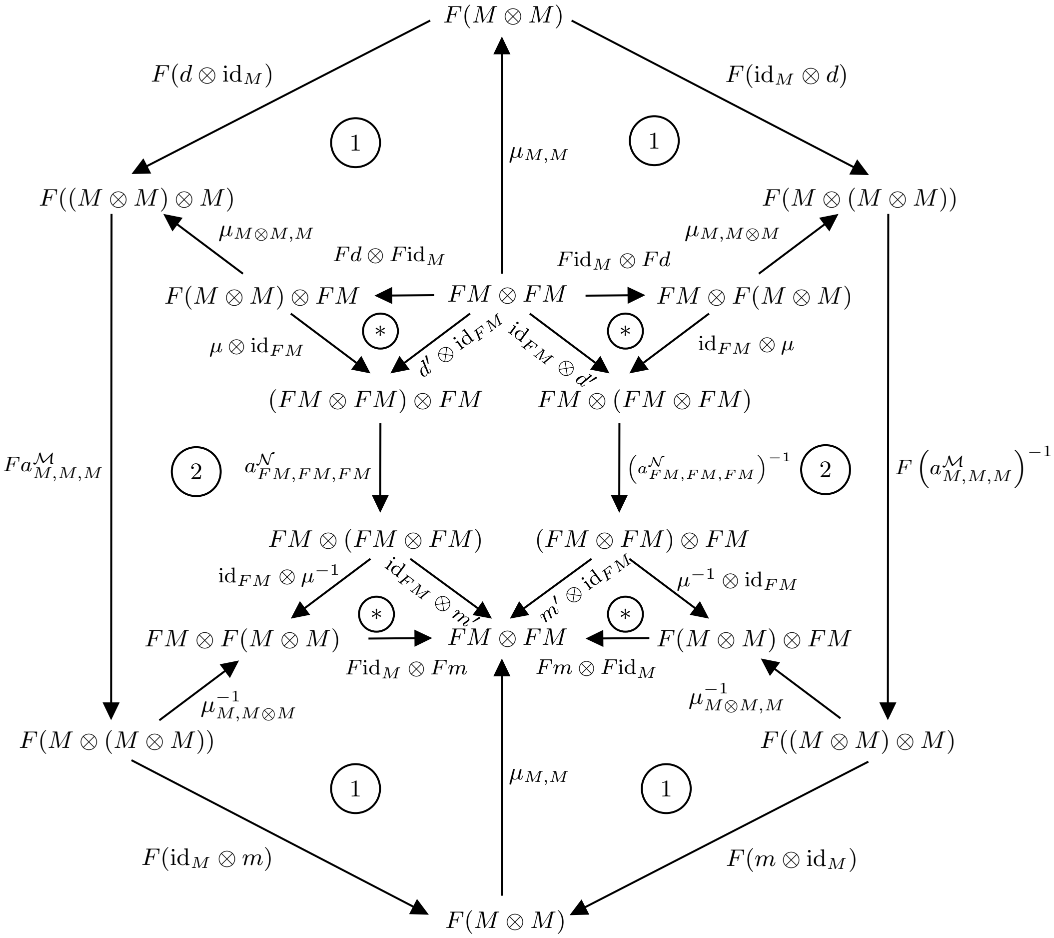
$$\Delta_{\mathcal{M}}(X) \otimes \Delta_{\mathcal{M}}(Y) = (X,X) \otimes (Y,Y) = (X \otimes Y, X \otimes Y) = \Delta_{\mathcal{M}}(X \otimes Y),$$

for every objects $X, Y \in \mathcal{M}_0$ and $\Delta_{\mathcal{M}}(I) = (I,I)$.

Remark 2.2.7. An interesting fact about lax monoidal functors is that they preserve internal monoids. Indeed, let $\mathcal{M}, \mathcal{N}$ be monoidal categories, $(F,\mu,\phi) : \mathcal{M} \to \mathcal{N}$ be a lax monoidal functor and (M,m,η) an internal monoid in $\mathcal{M}$. Then $(F(M), Fm \circ \mu_{M,M}, F\eta \circ \phi)$ is an internal monoid in $\mathcal{N}$. In addition, if $f : M \to M'$ is an internal monoid homomorphism in $\mathcal{M}$, then Ff is an internal monoid homormophism in $\mathcal{N}$. Therefore F can be assigned to a functor $\mathbf{Mon}(F) : \mathbf{Mon}(\mathcal{M}) \to \mathbf{Mon}(\mathcal{N})$, and this assignment is functorial. Obviously all this is true for strong and strict monoidal functors.

Interestingly lax monoidal functors don't preserve comonoids, but strong and strict ones do. This is due to the fact that μ and ϕ are not invertible, thus providing no way to create a map $FM \to FM \otimes FM$. To remedy this there is another notion of monoidal functor called **colax** monoidal. Such a functor is defined as a lax monoidal functor $F : \mathcal{M}^{op} \to \mathcal{N}^{op}$. This formal trick is capturing the reversal of the coherent maps. Therefore, comonoids are preserved by colax monoidal functors, which also can be assigned to functors $\mathbf{Com}(\mathrm{F}) : \mathbf{Com}(\mathcal{M}) \to \mathbf{Com}(\mathcal{N})$. Finally, there is no way to preserve monoid objects that are also comonoids other than a monoidal functor that is both lax and colax.

Thus, a strong monoidal functor preserves both monoid and comonoid structures, but furthermore, it also preserves Frobenius objects. To see this we only prove that one of the Frobenius laws holds for the image of a Frobenius object, since in the section about string diagrams we will prove that the other law is redundant. So let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories, $(F,\mu,\phi) : \mathcal{M} \to \mathcal{N}$ be a strong monoidal functor and $(M,m,\eta,,d,\varepsilon)$ be a Frobenius object internal to $\mathcal{M}$. Denote $m' = Fm \circ \mu$ and $d' = \mu^{-1} \circ Fd$ and proceed by diagram chasing. Observe that in the following diagram the 1 subdiagrams commute by naturality of μ, the 2 subdiagrams commute by associativity of (F,μ,ϕ) and the * subdiagrams commute by the definitions of d' and m'.



Therefore, the inner hexagon commutes, which is the Frobenius law for FM.

A last thing to note is that strong monoidal functors preserve special Frobenius objects. This is easy to check given that the Frobenius object which is an image, under the strong monoidal functor $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$, of the special Frobenius object $(M, m, \eta, d, \varepsilon)$, has as multiplication $m' = Fm \circ \mu$ and as comultiplication $d' = \mu^{-1} \circ Fd$. Therefore,

$$
\begin{aligned}
m' \circ d' &= Fm \circ \mu \circ \mu^{-1} \circ Fd \\
&= Fm \circ Fd \\
&= F(m \circ d) \\
&= F\mathrm{id}_M \\
&= \mathrm{id}_{FM},
\end{aligned}
$$

which is the speciality condition for $(FM, m', \eta', d', \varepsilon')$.

Remark 2.2.8. Another connection between monoidal functors, monoids and comonoids is that any constant functor $\Delta_M : \mathcal{M} \to \mathcal{M}'$, where $\mathcal{M}, \mathcal{M}'$ are monoidal categories and M is a (co)monoid inside $\mathcal{M}'$, can automatically be equipped with a (co)lax monoidal structure. In the case of monoids, (Δ_M, m, η) is a lax monoidal structure and in the case of comonoids (Δ_M, d, ϵ) is a colax monoidal structure. This is easy to check since associativity and unitality of Δ_M are given by the (co)associativity and (co)unitality of M. If in addition the (co)monoid under consideration is the unit, I, of the monoidal category, its multiplication and comultiplication maps are mutual inverses and also its unit and counit maps are equal to id_I. So the functor Δ_I is actually strong monoidal. One can easily see that this is an instance of the previous remark since every constant functor is defined as factoring through the terminal category.

Since monoidal functors are morphisms between monoidal categories, one might wonder if they all live inside a bigger category. Firstly, observe that the identity functor $\mathbf{1}_{\mathcal{M}}$, together with $\mathrm{id}_{-\otimes-}$ and id_I, is a strict, therefore a lax, monoidal endofunctor for every monoidal category $\mathcal{M}$. Furthermore, given monoidal categories $\mathcal{M}, \mathcal{N}, \mathcal{L}$ and lax monoidal functors $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$, $(G, \nu, \psi) : \mathcal{N} \to \mathcal{L}$, we can define a new lax monoidal functor $G \circ F$, whose coherence maps are

$$\begin{aligned} \nu_{FA,FB} \circ G\mu_{A,B} &: GF(A \otimes B) \to G(FA \otimes FB) \to GFA \otimes GFB \\ G\phi \circ \psi &: I_{\mathcal{L}} \to GI_{\mathcal{N}} \to GFI_{\mathcal{M}}. \end{aligned}$$

Similarly we can define the composition of colax monoidal functors. Obviously if we restrict to strong monoidal functors, or strict monoidal functors, their composition will be again a strong monoidal functor, or a strict one respectively.

Therefore we can form a category $\mathbf{MonCat}_{\mathrm{l}}$, whose objects are monoidal categories and its morphisms are lax monoidal functors. There are also two evident subcategories $\mathbf{MonCat}$ and $\mathbf{MonCat}_{\mathrm{s}}$, whose morphisms are strong and strict monoidal functors respectively. A non-trivial thing to note is that to compose a lax and a colax monoidal functor it is obligatory that one of them is already strong.

An interesting fact about $\mathbf{MonCat}_{\mathrm{l}}$ is that global elements of monoidal categories are monoids. This is due to the fact that $\mathbf{1}$ is terminal and its only object is trivially a monoid. Therefore, we get that the functor $\mathbf{MonCat}_{\mathrm{l}}(\mathbf{1}, -)$ is naturally isomorphic to $\mathbf{Mon}(-)$. But for this to be precise we need the notion of monoidal natural transformation to be defined in the following.

Continuing this line of thought, the category created by colax monoidal functors and monoidal categories is called $\mathbf{MonCat}_{\mathrm{co}}$. Given a monoidal category $\mathcal{M} \in \mathbf{MonCat}_{\mathrm{co}0}$, we can see that its global elements are comonoids, since colax monoidal functors preserve comonoids. Therefore, $\mathbf{MonCat}_{\mathrm{co}}(\mathbf{1}, -)$, in the same sense as above, is isomorphic to $\mathbf{Com}(-)$.

Finally, since the single object of the monoidal category $\mathbf{1}$ is a trivial special Frobenius object, we see that the global elements of monoidal categories in $\mathbf{MonCat}$ are Frobenius objects. Furthermore, monoidal natural transformations between global elements of monoidal categories are both homomorphisms between the underlying monoids and comonoids of the Frobenius objects, but $\mathbf{MonCat}(\mathbf{1}, -)$ is not isomorphic to $\mathbf{Frob}(-)$. Actually, since strong monoidal functors preserve the speciality condition, the correct thing to say would be that there is also a category of special Frobenius objects, which is a subcategory of $\mathbf{Frob}(\mathrm{M})$. So this category of special Frobenius objects, denoted by $\mathbf{SpFrob}(\mathrm{M})$ is a functorial construction and so $\mathbf{MonCat}(\mathbf{1}, -)$ is monoidally isomorphic to a subcategory $\mathbf{SpFrob}(-)$. But for all of this to be precise we need to introduce monoidal natural transformations.

Example 2.2.9. Let $\mathcal{M}$ be a monoidal category. Since $\mathcal{M}(I, -)$ is a strong monoidal functor, it preserves monoids. Therefore, given a monoid M in $\mathcal{M}$, there is an associated monoid, $\mathcal{M}(I, M)$, in $\mathbf{Set}$. If $\mathcal{M} = \mathbf{Vect}_{\mathbb{F}}$, then for every vector space V, $\mathcal{M}(\mathbb{F}, V) \cong V$[5]. Therefore, an internal monoid in $\mathbf{Vect}_{\mathbb{F}}$,(a.k.a. unital associative algebra) is sent by $\mathcal{M}(I, -)$ to its underlying set.

Now let us turn our attention to transformations of monoidal functors.

Definition 2.2.10. *Let $\mathcal{M}, \mathcal{N}$ be monoidal categories and $(F, \mu, \phi), (G, \nu, \psi) : \mathcal{M} \to \mathcal{N}$ lax monoidal functors. A natural transformation $\eta : F \to G$ is called a* ***monoidal natural transformation*** *if the following two diagrams, called* ***multiplicativity*** *and* ***unitality****, commute*

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\eta_A \otimes \eta_B} & GA \otimes GB \\ \downarrow{\scriptstyle \mu_{A,B}} & & \downarrow{\scriptstyle \nu_{A,B}} \\ F(A \otimes B) & \xrightarrow{\eta_{A \otimes B}} & G(A \otimes B) \end{array} \qquad \begin{array}{ccc} FI_{\mathcal{M}} & \xrightarrow{\eta_I} & GI_{\mathcal{M}} \\ {\scriptstyle \phi}\nwarrow & & \nearrow{\scriptstyle \psi} \\ & I_{\mathcal{N}} & \end{array}$$

for every $A, B \in \mathcal{M}_0$. If in addition η is a natural isomorphism then it is called a ***monoidal natural isomorphism****.*

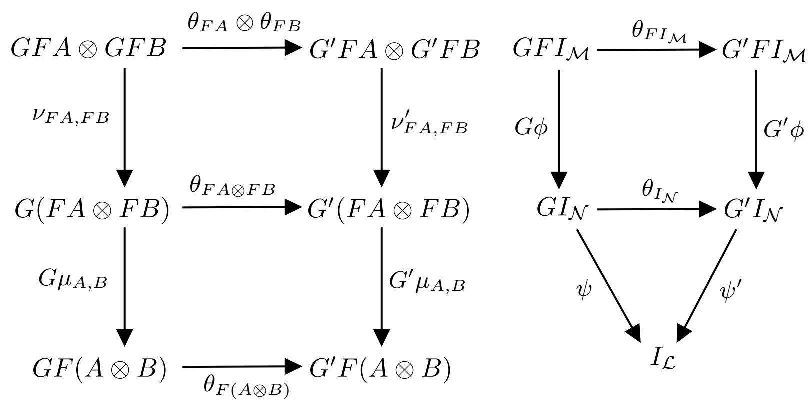
[5]Note that this isomorphism is just a bijective function. Formally, $\mathcal{M}(\mathbb{F}, V) \cong UV$, where $U : \mathbf{Vect}_{\mathbb{F}} \to \mathbf{Set}$ is the forgetful functor.

Remark 2.2.11. Monoidal natural transformations are those that preserve the monoidal structure of the functors. One might even say that they respect the preservation of monoidal structure. Having this in mind we also get monoidal natural transformations between colax monoidal functors, by reversing the arrows of the coherence maps of the functors, in the above diagrams. Obviously, the above definition also works for strong and strict monoidal functors.

Example 2.2.12. For every monoidal category $\mathcal{M}$, $\mathbf{1}_{\mathcal{M}}$ and $\Delta_I : \mathcal{M} \to \mathcal{M}$ are strong monoidal functors. It is easy to check that if $\mathcal{M}$ has uniform deleting $e : \mathbf{1}_{\mathcal{M}} \Rightarrow \Delta_I$, then e is a monoidal natural transformation.

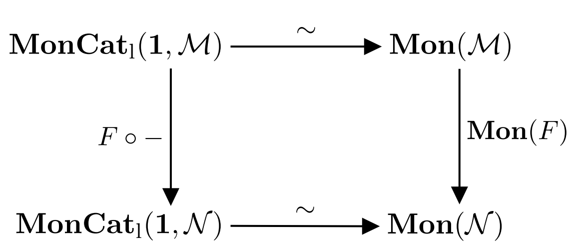
Remark 2.2.13. We can vertically compose monoidal natural transformations and this goes as follows. Let $F, G, H \in \mathbf{MonCat}_{\mathrm{l}}(\mathcal{M}, \mathcal{N})$, $\eta : F \Rightarrow G$ and $\theta : G \Rightarrow H$. Then it is easy to check that $\theta \cdot \eta$ satisfies both conditions required to be a monoidal natural transformation. Since composition of natural transformations are associative and the identity natural transformation is trivially monoidal, we deduce that $\mathbf{MonCat}_{\mathrm{l}}(\mathcal{M}, \mathcal{N})$ is a category, for every $\mathcal{M}, \mathcal{N} \in \mathbf{MonCat}_{\mathrm{l}}$.

Furthermore, whiskering a monoidal natural transformation with a lax monoidal functor results in a monoidal natural transformation. This is easily seen by the commutativity of the outer diagrams, implied by naturality and monoidality of a given monoidal natural tranformation $\theta : (G, \nu, \psi) \Rightarrow (G', \nu', \psi')$



where $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$ and $G, G' : \mathcal{N} \to \mathcal{L}$ are lax monoidal. So whiskering a monoidal natural transformation on the right by a monoidal functor results in a monoidal natural transformation. Similar results are obtained by whiskering on the right. Finally, since horizontal composition of natural transformations can be obtained by vertically composing whiskerings and vertical composotition preserves the monoidal structure of monoidal natural transformations, we derive that horizontal composition of monoidal natural transformations preserves the monoidal structure.

Remark 2.2.14. It is easy to check that a monoidal natural transformation η, between two lax monoidal functors $(F, \mu, \phi), (G, \nu, \psi) : \mathbf{1} \to \mathcal{M}$ provides a homomorphism of the internal monoids defined by the unique component $\eta_1 : (F1, \mu_{1,1}, \phi) \to (G1, \nu_{1,1}, \psi)$. This sheds some light to the similarity between $\mathbf{MonCat}_{\mathrm{l}}(\mathbf{1}, -)$ and $\mathbf{Mon}(-)$. Since every internal monoid in $\mathcal{M}$ can be used to construct a functor $F_M : \mathbf{1} \to \mathcal{M}$ and every monoid homomorphism $h : M \to N$ is provided by a monoidal natural transformation $\eta^h : F_M \to F_N$, and vice versa, we arrive at $\mathbf{MonCat}_{\mathrm{l}}(\mathbf{1}, \mathcal{M}) \cong \mathbf{Mon}(\mathcal{M})$. Thus $\mathbf{MonCat}_{\mathrm{l}}(\mathbf{1}, -)$ is naturally isomorphic to $\mathbf{Mon}(-)$ and this is because the following diagram commutes:



since every lax monoidal functor $F : \mathcal{M} \to \mathcal{N}$ preserves internal monoids. Similarly, we get that $\mathbf{MonCat}_{\mathrm{co}}(\mathbf{1}, -) \cong \mathbf{Com}(-)$. For the strong monoidal case, we may only show that monoidal natural transformations also preserve the speciality condition of frobenius objects. This is actually the case since every Frobenius object homomorphism between special Frobenius objects preserves the speciality condition, but to prove this we will need the help of string diagrams.

Remark 2.2.15. In accordance with remark 2.2.8, having a homomorphism between internal (co)monoids gives a monoidal natural transformation between constant (co)lax monoidal functors. Multiplicativity and unitality follow from the (co)monoid homomorphism defining properties.

Example 2.2.16. A monoidal category, $\mathcal{M}$, with monoidal uniform deleting, i.e. a semi-cartesian category, is defined as having a natural transformation $e : \mathrm{Id}_{\mathcal{M}} \Rightarrow \Delta_I$. Since both of these endofunctors are strong monoidal, it makes sense to ask whether e is a monoidal natural transformation. The answer to this question is positive, and this follows from terminality of I. That is for every $A, B \in \mathcal{M}_0$,

$$\ell_I \circ (e_A \otimes e_B) = e_{A\otimes B} \circ \mathrm{id}_{A\otimes B} \text{ and } \mathrm{id}_I = \mathrm{id}_I.$$

Monoidal Equivalences

Monoidal natural transformations can be used to provide a refinement of the notion of equivalence between categories. As is the case with ordinary equivalences, monoidal equivalences have two equivalent definitions. In the following we give a proof of this equivalence, after providing the definitions of monoidal equivalence.

Definition 2.2.17. *Let $\mathcal{M}, \mathcal{N}$ be monoidal categories, $F : \mathcal{M} \to \mathcal{N}$ a lax monoidal functor and $G : \mathcal{N} \to \mathcal{M}$ a colax monoidal functor. A* ***monoidal equivalence*** *between $\mathcal{M}$ and $\mathcal{N}$ is a quadruple $(F, G, \eta, \varepsilon)$, such that $\varepsilon : F \circ G \Rightarrow \mathbf{1}_{\mathcal{N}}$ and $\eta : \mathbf{1}_{\mathcal{M}} \Rightarrow G \circ F$ are natural isomorphisms.*

Remark 2.2.18. A quick remark is that a monoidal equivalence is also an ordinary equivalence. This, of course, is true since an equivalence is about ordinary functors and ordinary natural isomorphisms, which monoidal ones already are.

Remark 2.2.19. This definition, already carries the information that both FG and GF are actually strong monoidal functors. This can easily be deduced from the fact that η and ε are monoidal isomorphisms. Therefore, whenever a pair of a lax and a colax monoidal functor define an equivalence, then their composites are strong monoidal. This is going to be very usefull as we continue, so we give a slightly more general lemma.

Lemma 2.2.20. *Let $\mathcal{M}$ be a monoidal category and $H : \mathcal{M} \to \mathcal{M}$ a functor such that $\varepsilon : H \Rightarrow \mathbf{1}_{\mathcal{M}}$ is a natural isomorphism. Then H is strong monoidal and ε is a monoidal natural isomorphism.*

Proof. For every $A, B \in \mathcal{M}_0$ set

$$\kappa_{A,B} := \varepsilon^{-1}_{A\otimes B} \circ (\varepsilon_A \otimes \varepsilon_B) : HA \otimes HB \to H(A \otimes B)$$

and

$$\chi := \varepsilon_I^{-1} : I \to HI.$$

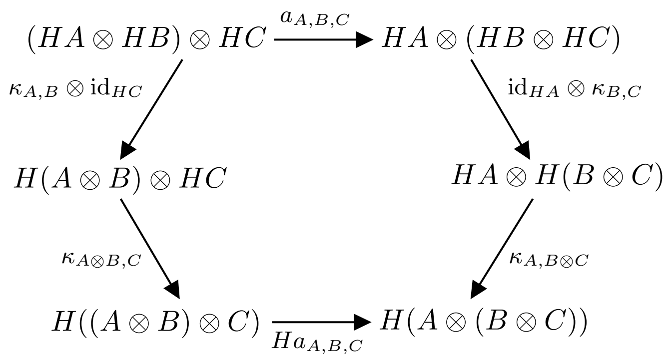
Then κ is a natural isomorphism as a composition of natural isomorphisms and χ is an isomorphism since ε_I is. Now we prove associativity and unitality for (H, κ, χ). By naturality of associator we get

$$a_{A,B,C} \circ [(\varepsilon_A \otimes \varepsilon_B) \otimes \varepsilon_C] = [\varepsilon_A \otimes (\varepsilon_B \otimes \varepsilon_C)] \circ a_{HA,HB,HC} \tag{2.4}$$

and by naturality of ε we get

$$\varepsilon_{A\otimes(B\otimes C)} \circ Ha_{A,B,C} = a_{A,B,C} \circ \varepsilon_{(A\otimes B)\otimes C}, \tag{2.5}$$

for every $A, B, C \in \mathcal{M}_0$. So the following associativity diagram commutes



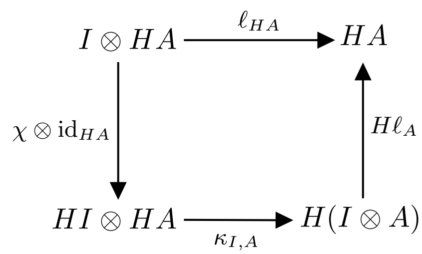
since the bottom left morphism is

$$
\begin{aligned}
&Ha_{A,B,C} \circ {\varepsilon^{-1}}_{(A \otimes B) \otimes C} \circ (\varepsilon_{A \otimes B} \otimes \varepsilon_C) \circ [(\varepsilon^{-1}_{A \otimes B} \circ (\varepsilon_A \otimes \varepsilon_B)) \otimes \mathrm{id}_{HC}] = \\
&Ha_{A,B,C} \circ {\varepsilon^{-1}}_{(A \otimes B) \otimes C} \circ [(\varepsilon_A \otimes \varepsilon_B) \otimes \varepsilon_C] && \text{(by the interchange law)}
\end{aligned}
$$

while the top right one is also

$$
\begin{aligned}
&\varepsilon_{A \otimes (B \otimes C)} \circ (\varepsilon_A \otimes \varepsilon_{B \otimes C}) \circ [\mathrm{id}_{HA} \otimes (\varepsilon^{-1}_{B \otimes C} \circ (\varepsilon_B \otimes \varepsilon_C))] \circ a_{HA,HB,HC} = \\
&\varepsilon_{A \otimes (B \otimes C)} \circ (\varepsilon_A \otimes (\varepsilon_B \otimes \varepsilon_C)) \circ a_{HA,HB,HC} = && \text{(by the interchange law)} \\
&\varepsilon^{-1}_{A \otimes (B \otimes C)} \circ a_{A,B,C} \circ [(\varepsilon_A \otimes \varepsilon_B) \otimes \varepsilon_C] = && \text{(by naturality of a)} \\
&Ha_{A,B,C} \circ {\varepsilon^{-1}}_{(A \otimes B) \otimes C} \circ [(\varepsilon_A \otimes \varepsilon_B) \otimes \varepsilon_C]. && \text{(by naturality of } \varepsilon)
\end{aligned}
$$

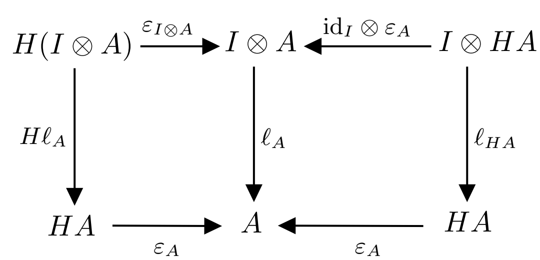
To prove unitality, we need to show that the following diagram commutes



which is equivalent to

$$
\begin{aligned}
&\ell_{HA} = H\ell_A \circ \varepsilon^{-1}_{I \otimes A} \circ (\varepsilon_I \otimes \varepsilon_A) \circ (\varepsilon_I^{-1} \otimes \mathrm{id}_{HA}) \Leftrightarrow \\
&\ell_{HA} = H\ell_A \circ \varepsilon^{-1}_{I \otimes A} \circ (\mathrm{id}_I \otimes \varepsilon_A) && \text{(by the interchange law)}
\end{aligned}
$$

but this holds since in the following the left and right diagrams commute

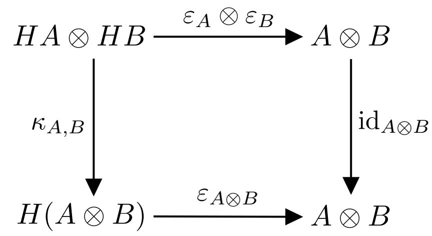


by naturality of ε and l, respectively. Similarly, the right unitor version of unitality holds. Therefore, H is a strong monoidal functor.

To prove that ε is monoidal we firstly have to show that

$$\varepsilon_I \circ \chi = \mathrm{id}_I,$$

which holds by definition, and secondly that the following diagram commutes,



which is equivalent to

$$\varepsilon_{A\otimes B} \circ \kappa_{A,B} = (\varepsilon_A \otimes \varepsilon_B)$$

but this is also the case by the definition of κ. Therefore, ε is a monoidal natural isomorphism.

□

The previous definition, while conceptually clear, requires a lot of work to check. This makes the following characterisation useful.

Definition 2.2.21. *Let $\mathcal{M}, \mathcal{N}$ be monoidal categories and $F : \mathcal{M} \to \mathcal{N}$ a lax monoidal functor. We call F a **monoidal equivalence** if it is an equivalence as a functor.*

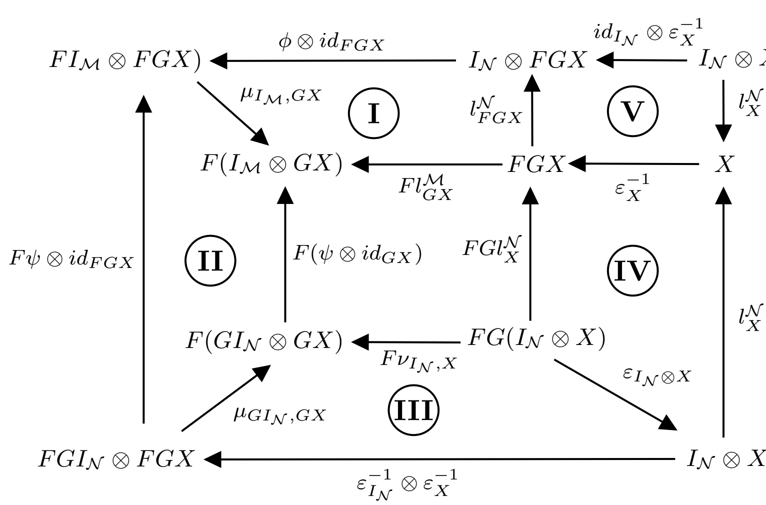
Proposition 2.2.22. *The definitions 2.2.17 and 2.2.21 are equivalent.*

Proof. A functor (F, μ, ϕ) is a monoidal equivalence according to definition 2.2.21 given that it satisfies definition 2.2.17 by remark 2.2.18. Proving the converse will take the following steps. Firstly, we will define a functor $G : \mathcal{N} \to \mathcal{M}$ and natural isomorphisms $\eta : \mathbb{1}_{\mathcal{M}} \Rightarrow GF$, $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{N}}$ such that FG and GF are strong monoidal and ε, η are monoidal. Secondly, we show that G can be equipped with a colax monoidal structure.

Since F is an equivalence, there exist a functor $G : \mathcal{N} \to \mathcal{M}$, and two natural isomorphisms η, ε such that $\eta : \mathbb{1}_{\mathcal{M}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{N}}$. By the lemma 2.2.20 we automatically get that both FG and GF are strong monoidal functors, being naturally isomorphic to the identity, but also that η and ε are monoidal natural isomorphisms. To be clear we get that η^{-1} is monoidal, which easily implies that η is monoidal.

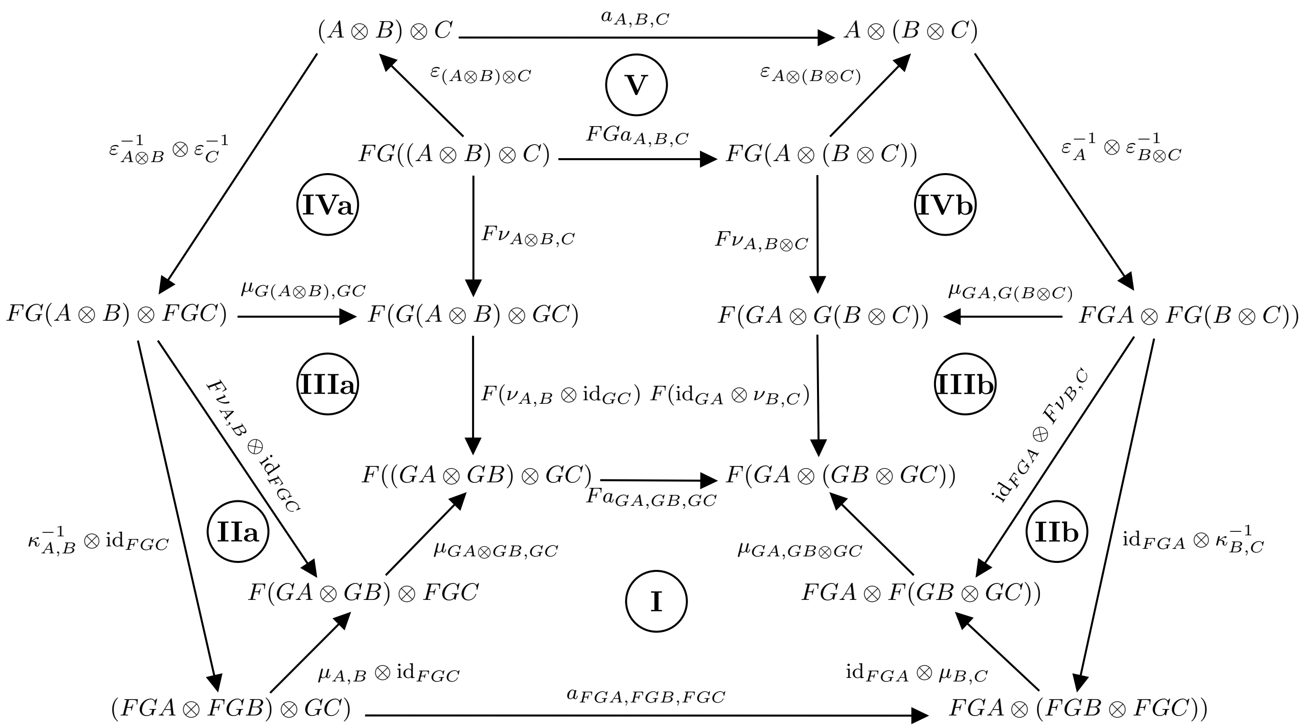
Now for every $A, B \in \mathcal{N}_0$ we define $\nu_{A,B}$ and ψ as the inverse images of $\mu_{GA,GB} \circ \kappa^{-1}_{A,B}$ and $\phi \circ \chi^{-1}$ under F respectively, where $\kappa_{A,B} = \varepsilon^{-1}_{A\otimes B} \circ (\varepsilon_A \otimes \varepsilon_B)$ and $\chi = \varepsilon^{-1}_I$ as in 2.2.20. These are well defined and $\nu : G(- \otimes -) \Rightarrow (G-) \otimes (G-)$ is natural since F is an equivalence and because $F\nu$ is natural as a composite.

To prove associativity and unitality for ν and ψ it is enough to show that the image of every associativity and unitality diagram under F commutes. So let $A, B, C \in \mathcal{M}_0$, $X \in \mathcal{N}_0$ and observe that the image under F of a unitality diagram for the left unitor is in the middle of the following diagram:



This center diagram is commutative because the outer one commutes by the definition of ψ, **I** commutes by unitality of ϕ, **II** commutes by naturality of μ, **III** commutes by the definition of ν, **IV** commutes by naturality of ε and **V** commutes by naturality of $l^{\mathcal{N}}$. Similarly unitality holds for the right unitor.

Now observe that in the following diagram **I** commutes by associativiy of (F, μ, ϕ), **II**'s commute by the definition of ν, **III**'s commute by naturality of μ, **IV**'s commute by the definitions of ν and κ, **V** commutes by naturality of the associator and the outer diagram commutes by naturality of associator, the interchange law and the definition of κ. Thus the middle diagram commutes, which is the image of an associativity diagram for (G, ν, ψ).



Therefore, (G, ν, ψ) is a colax monoidal functor and definition 2.2.17 is equivalent to definition 2.2.21. $\square$

Remark 2.2.23. It is worth noting that in the case where F is strong monoidal both of the above definitions imply that G is also strong monoidal. This is derived from the definition of the colax monoidal structure of G in the proof above. Therefore weak inverses of strong monoidal functors are strong monoidal.

Example 2.2.24. In the example 1.5.9 of the section about equivalences between categories we saw that $\mathbf{Mat}_{\mathbb{C}} \simeq \mathbf{Vect}$. Actually, this is a monoidal equivalence, if we define the usual multiplication of natural numbers as the monoidal product of objects and the kronecker product of matrices as the monoidal product of morphisms in $\mathbf{Mat}_{\mathbb{C}}$. It is easy to verify that $\mathbf{Mat}_{\mathbb{C}}$ equipped with such a tensor product is a strict monoidal category.

Furthermore, the equivalence $F : \mathbf{Mat}_{\mathbb{C}} \to \mathbf{Vect}$ sending $n \mapsto \mathbb{C}^n$ and $A \mapsto f_A$ also satisfies $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$ and $\mathbb{C} = \mathbb{C}^1$. Since $\mathbf{Mat}_{\mathbb{C}}$ is strict, associativity for F reduces to a pentagon in $\mathbf{Vect}$, which commutes since

$$(\mathbb{C}^n \otimes \mathbb{C}^m) \otimes \mathbb{C}^k \cong \mathbb{C}^{(nmk)} \cong \mathbb{C}^n \otimes (\mathbb{C}^m \otimes \mathbb{C}^k),$$

and unitality reduces to the fact that the isomorphism $\mathbb{C} \otimes \mathbb{C}^n \cong \mathbb{C}^n$ can be chosen to be the left unitor $\ell_{\mathbb{C}^n}$. Therefore, F is strong monoidal and an equivalence, thus a monoidal equivalence.

Example 2.2.25. Another example is the monoidal equivalence between **FdVect** and **FdHilb**. This is almost trivially true, since every linear map between finite dimensional topological vector spaces is automatically continuous and also that every finite dimensional vector space is complete (therefore the tensor products of vector spaces and hilbert spaces coincide).

Strictification and coherence

Since monoidal categories are a categorification of the notion of monoid, one might wonder if they can be embedded inside a category of endofunctors of some kind of category. This question is based on the fact that there is a version of Cayley's theorem for monoids. This goes as follows.

If M is an ordinary monoid (a monoid internal to **Set**), we define $\lambda_a := a \cdot - : M \to M$, for every $a \in M$. Note that this is not a monoid homomorphism. Then define

$$\Lambda := \{\lambda_a |\ a \in M\}$$

and observe that this is a monoid with composition as multiplication and the action of the unit as a unit. Furthermore, and due to associativity, $\lambda_a \circ \lambda_b = \lambda_{ab}$, which shows that $\lambda_- : M \to \Lambda$ is a monoid homomorphism. Finally, this homomorphism is actually an isomorphism, since it is onto by the definition of Λ and "1-1" because $\lambda_a = \lambda_b$ implies $\lambda_a(e) = \lambda_b(e)$ or equivalently $a = b$. Therefore, M is embedded in the endomorphisms of M as a set.

A very useful property, which will serve as a starting point for what follows in monoidal categories, is that

$$f(m) \cdot n = f(m \cdot n)$$

for every $m, n \in M$ and $f \in \Lambda$. This is actually very similar to a monoid endomorphism, but it isn't. Now the categorified version of the above property is as follows.

Let $\mathcal{M}$ be a monoidal category and (F, γ) such that F is an endofunctor $F : \mathcal{M} \to \mathcal{M}$ and γ is a natural isomorphism such that

$$\gamma_{A,B} : F(A) \otimes B \to F(A \otimes B),$$

for every $A, B \in \mathcal{M}_0$. We can create a category $\mathcal{L}$ with such endofunctors of $\mathcal{M}$ as objects. The morphisms are natural transformations $\theta : F \Rightarrow G$, such that

$$\begin{array}{ccc} FA \otimes B & \xrightarrow{\theta_A \otimes \mathrm{id}_B} & GA \otimes B \\ {\scriptstyle \gamma_{A,B}} \downarrow & & \downarrow {\scriptstyle \delta_{A,B}} \\ F(A \otimes B) & \xrightarrow[\theta_{A \otimes B}]{} & G(A \otimes B) \end{array} \tag{2.6}$$

commutes for every $A, B \in \mathcal{M}_0$. Composition of morphisms is the usual vertical composition of natural transformations, which obviously preserves property 2.6, while the identity morphism for every object is the identity natural transformation.

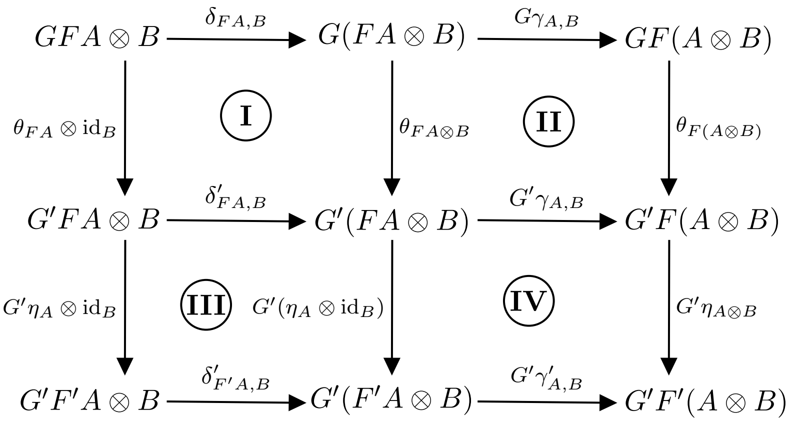
Furthermore, $\mathcal{L}$ can be equipped with a monoidal structure. The tensor product of the objects (F, γ), (G, δ) is defined as

$$(G, \delta) \otimes (F, \gamma) := (G \circ F, (G\gamma) \cdot (\delta(F \times \mathbb{1}_{\mathcal{M}}))) = (GF, \epsilon),$$

where the "$\circ$", used to denote the whiskerings, is omited. To spell this out clearly

$$G\gamma_{A,B} \circ \delta_{FA,B} =: \epsilon_{A,B} : GFA \otimes B \to GF(A \otimes B),$$

for every $A, B \in \mathcal{M}_0$. The tensor product of morphisms is the horizontal composition of natural transformations. This horizontal composite preserves property 2.6 since the outer square, in the following diagram, commutes



because **I** commutes by the definition of θ, **II** commutes by naturality of θ, **III** commutes by naturality of δ' and **IV** commutes by the definition of η. The unit object is the identity endofunctor with the identity natural transformation, $(\mathbb{1}_{\mathcal{M}}, \mathrm{Id}_{\mathbb{1}_{\mathcal{M}} \otimes \mathbb{1}_{\mathcal{M}}})$. Finally, $\mathcal{L}$ is a strict monoidal category since $\mathcal{L}$ is isomorphic to a monoidal subcategory of the strict monoidal category $End(\mathcal{M})$ by the functor that sends $(F, \gamma) \mapsto F$ and $\eta \mapsto \eta$ "forgetting" property 2.6, which is obviously strict monoidal.

To prove the categorified version of Cayley's theorem, we need to embed $\mathcal{M}$ inside $\mathcal{L}$. Observe that for every $A \in \mathcal{M}$, the pair $(A \otimes -, a_{A,-,-})$ is an object of $\mathcal{L}$. Obviously, $A \otimes -$ is an endofunctor of $\mathcal{M}$ such that

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

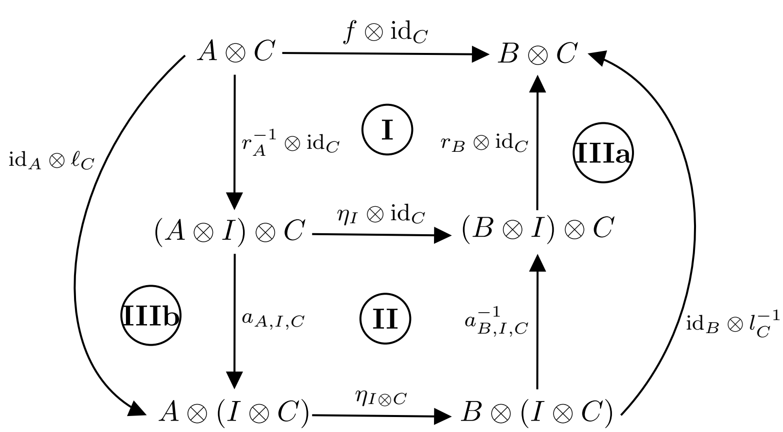
is an isomorphism, for every $B, C \in \mathcal{M}_0$. Furthermore, given $f \in \mathcal{M}(A, A')$, there is a unique natural transformation $f \otimes - : A \otimes - \Rightarrow A' \otimes -$ satisfying 2.6 by naturality of the associator. It is straight forward to check that sending A to $(A \otimes -, a_{A,-,-})$ and f to $f \otimes -$ is functorial, thus we have a functor $L = (F_-, \gamma^-) : \mathcal{M} \to \mathcal{L}$ such that

$$F_A = A \otimes -,\ \gamma^A := a_{A,-,-} \text{ and } F_f = f \otimes \mathrm{id}_-.$$

In addition this functor is fully faithful. For faithfulness let $g, f : A \to B$, such that $F_f = F_g$. Then $f \otimes \mathrm{id}_I = g \otimes \mathrm{id}_I$, which by **??** implies that $f = g$. For fullness let $\eta : A \otimes - \Rightarrow B \otimes -$ satisfy property 2.6. Then $\eta_I : A \otimes I \to B \otimes I$, so we define $f : A \to B$ as the following composite:

$$A \xrightarrow{r_A^{-1}} A \otimes I \xrightarrow{\eta_I} B \otimes I \xrightarrow{r_B} B,$$

so we need to prove that $Ff = \eta$, or that for every $C \in \mathcal{M}_0$, $(Ff)_C = \eta_C$. Consider the following diagram.



I commutes by definition, **II** commutes by property 2.6, and the **III**'s commute by the triangle law. Therefore, the outer diagram commutes, which is equivalent to

$$(\mathrm{id}_B \otimes l_C^{-1}) \circ \eta_{I \otimes C} \circ (\mathrm{id}_A \otimes \ell_C) = f \otimes \mathrm{id}_C,$$

but naturality of η implies that $(\mathrm{id}_B \otimes l_C^{-1}) \circ \eta_{I\otimes C} \circ (\mathrm{id}_A \otimes \ell_C) = \eta_C$, so $\eta_C = f \otimes \mathrm{id}_C$, which proves that F is full.

Finally we need to equip F with a monoidal structure. So we need a natural transformation with components

$$\theta_{A,B} : (A \otimes (B \otimes -), (\mathrm{id}_A \otimes a_{B,-,-}) \circ a_{A,B\otimes -,-}) \Rightarrow ((A \otimes B) \otimes -, a_{A\otimes B,-,-}),$$

satisfying property 2.6. One can easily check that by picking $\theta_{A,B} := a_{A,B,-}$, 2.6 becomes equivalent to the pentagon law in $\mathcal{M}$, which holds, and that naturality of the associator in the first two variables implies naturality of θ. To prove associativity, observe that, since $\mathcal{L}$ is strict, the associativity hexagon reduces to a pentagon which, componentwise, is equivalent to the pentagon law in $\mathcal{M}$.

We also need a morphism, $\phi : (\mathbb{1}_{\mathcal{M}}, \mathrm{Id}_{\mathbb{1}_{\mathcal{M}}\otimes \mathbb{1}_{\mathcal{M}}}) \to (I \otimes -, a_{I,-,-})$ in $\mathcal{L}$, that satisfies unitality. Picking $\phi := \ell_{-}^{-1}$, we see that property 2.6 is actually a naturality square for l and that the unitality axioms are equivalent, working with components, to the conditions of 2.1.12 and the triangle law for $\mathcal{M}$. Since a and l are natural isomorphisms, (F, θ, ϕ), as defined above, is a strong monoidal functor. It is also fully faithful, so $\mathcal{M}$ is equivalent to a monoidal subcategory, $\mathcal{L}_{\mathcal{M}}$, of $\mathcal{L}$, which is strict, therefore $\mathcal{M}$ is monoidally equivalent to a strict monoidal category. Thus we have proven the following **strictification** theorem.

Theorem 2.2.26. *Every monoidal category, $\mathcal{M}$, is monoidally equivalent to a strict monoidal category $\mathcal{L}_{\mathcal{M}}$.*

A powerful corolary of the strictification theorem is that every diagram created by associators, unitors and identity morphisms is commutative. This corolary is so crucial for monoidal categories that it has a special name, *the coherence theorem*, sometimes it is also called the Kelly-McLane coherence theorem. To state this theorem clearly we need to define what a bracketing, v, is and we will do so recursively.

So a bracketing of objects $A_1, ..., A_n \in \mathcal{M}_0$ will be denoted by $v(A_1, ..., A_n)$. The empty bracketing is ( ), for any object $A \in \mathcal{M}_0$ there exists a bracketing $v(A) := (A)$ and if bracketings v, w are defined for n and m objects respectively, then the bracketing $v \otimes w$ is defined as

$$(v \otimes w)(A_1, ..., A_n, B_1, ..., B_m) := ((v(A_1, ..., A_n) \otimes w(B_1, ..., B_m)).$$

At this point it should be obvious that the empty bracketing acts as a monoidal unit on the bracketings. We can also define morphisms, $\beta : v \to w$, between the bracketings that are independent of the arguments. These morphisms are of course either the identities on every bracketing or associators between different bracketings or unitors between bracketings that have the empty bracketing as a factor. Furthermore, every finite composition of the above three types of morphisms, but also every finite tensor product of those, is also permitted. Such an example is the following composite:

$$(((- \otimes -) \otimes I) \otimes -) \xrightarrow{r_{-\otimes -}\otimes \mathrm{id}_{-}} ((- \otimes -) \otimes -) \xrightarrow{a_{-,-,-}} (-(- \otimes -)).$$

This definition of morphisms originating in coherence maps and bracketings provides a category-independent construction. Thus, such bracketings can also be applied to strict monoidal categories, with trivial outcomes of course. This gives the following coherence theorem its generality.

Theorem 2.2.27. *Let $\mathcal{M}$ be a monoidal category. Every diagram constructed by coherence maps and identities, solely by repeated application of composition and tensoring, is commutative.*

Proof. Let $\mathcal{M}$ be a monoidal category and $\mathcal{L}$ its equivalent strictified version via the monoidal equivalence $(F, \theta, \phi) : \mathcal{M} \to \mathcal{L}$ as defined in the strictification theorem. Using θ and ϕ we can define inductively an isomorphism $L_v : v(F-, ..., F-) \to Fv(-, ..., -)$, for every bracketing v as follows. Set

$$L_{(\ )} := \phi : I_{\mathcal{L}} \to F(I_{\mathcal{M}}),\ L_{(-)} := \mathrm{id}_{F-}$$

$$\text{and } L_{(v\otimes w)} := \theta_{v(-,...,-),w(-,...,-)} \circ (L_v \otimes L_w).$$

Now let $\beta, \beta' : v \to w$ be morphisms between bracketings. Obviously, these correspond to distinct paths from $v(A_1, ..., A_n)$ to $w(A_1, ..., A_n)$, exclusively built from coherence maps inside $\mathcal{M}$. Since $\mathcal{L}$ is strict, we have that $\beta_{FA_1,...,FA_n} = \beta'_{FA_1,...,FA_n}$. Furthermore, by the definition of L_v and L_w the following diagram commutes:

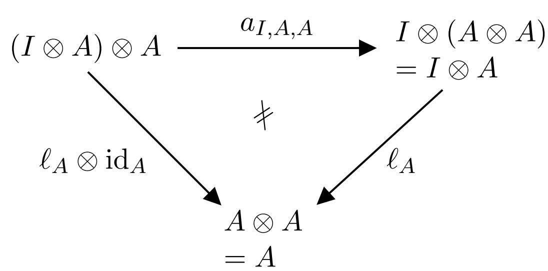
$$\begin{array}{ccc} v(FA_1,\ldots,FA_n) & \xrightarrow{L_v} & Fv(A_1,\ldots,A_n) \\ \downarrow{\scriptstyle \beta_{FA_1,\ldots,FA_n}} & & \downarrow{\scriptstyle F\beta_{A_1,\ldots,A_n}} \\ w(FA_1,\ldots,FA_n) & \xrightarrow[L_w]{} & Fw(A_1,\ldots,A_n) \end{array}$$

and similarly for β'. But, L_v, L_w are isomorphisms, so we have that $F\beta_{A_1,\ldots,A_n} = F\beta'_{A_1,\ldots,A_n}$ and since F is an equivalence we get $\beta_{A_1,\ldots,A_n} = \beta'_{A_1,\ldots,A_n}$. This proves that two paths between bracketings built from coherence maps are equal, therefore the statement of the theorem is true. □

Remark 2.2.28. The application of the coherence theorem to "non-well formed" diagrams leads to fallacies. An example of this sort is the following, where the coincidental equality

$$A \otimes A = A,$$

holding for a specific object A in a particular monoidal category $\mathcal{M}$, forbids the application of the coherence theorem to any diagram formed using this equality, such as the following:

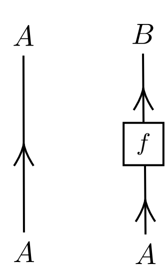


since these morphisms do not constitute morphisms bewtween bracketings and $\ell_{(A\otimes A)}$ need not equal ℓ_A.

Remark 2.2.29. The prototype of a monoidal category is a cartesian category. In a cartesian category all coherence diagrams commute due to the uniqueness implied by the universal property of the product and the terminal object. The coherence theorem allows us to extend this property to any monoidal category.

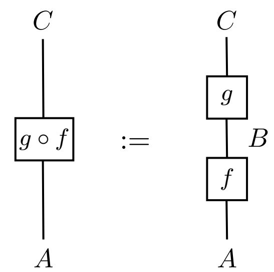
String diagrams for monoidal categories

The previously presented coherence and strictification theorems allow us to extend the so called *string diagram calculus* to monoidal categories. String diagrams for monoidal categories, sometimes also called *graphical calculus*, give a process status to both objects and morphisms of a monoidal category. This is because an object A of a monoidal category $\mathcal{M}$ is depicted by a directed line and a morphism $f \in \mathcal{M}(A, B)$ by a box as follows:



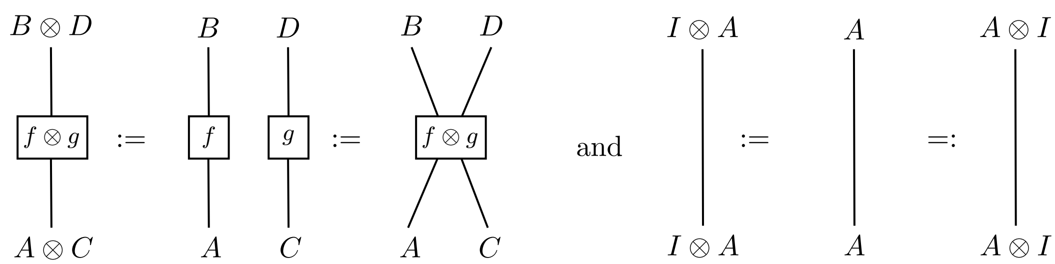
The identity morphism of an object A is not depicted at all, which makes the object as a line, a process of creating itself by doing nothing. In a sense the line is actually the identity morphism on an object, which makes the graphical calculus of a category only about morphisms. Any non-identity morphism can be interpreted as a process transforming its input into its output. There are different conventions on how to read such string diagrams, but the one we choose here is bottom to top. This way the arrows inserted on the strings are redundant.

To represent composition of morphisms one inserts the output of the first one to the input of the second one. So, given $f : A \to B,\ g : B \to C$, $g \circ f$ is denoted by:

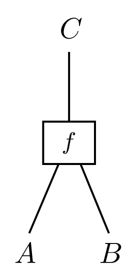


The composition of morphisms is interpreted as two processes composing *sequentially*. This way both axioms, unitality and associativity, of a category hold trivially, since parentheses are (most of the times) implicit in the string diagrams calculus.

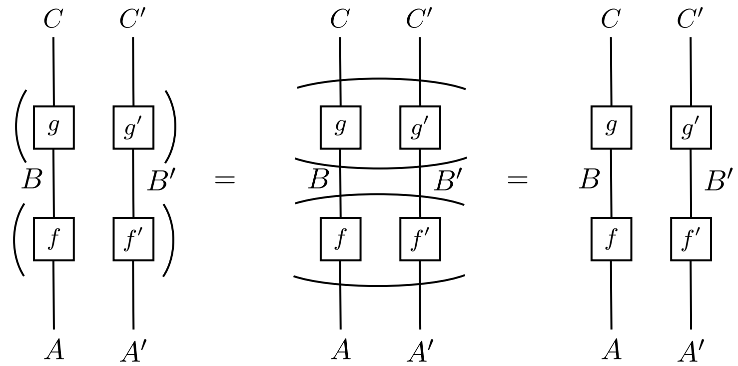
Up until this point the graphical notation is just string diagrams for a general category. To represent the tensor product of a monoidal category, we put strings and boxes side by side, which is interpreted as *parallel* composition of processes. The unit of the monoidal category is depicted by an empty string, so implicitly empty space acts as a unit. Thus, given $f \in \mathcal{M}(A, B)$, $g \in \mathcal{M}(C, D)$, we depict $f \otimes g : A \otimes C \to B \otimes D$, $I \otimes A$ and $A \otimes I$ as:



where the empty space on the left and/or on the right of the identity morphism of A, on the second equation of diagrams above, is considered as the identity morphism of the unit $I \in \mathcal{M}_0$. Something that should be pointed out at this point is that a morphism need not be a tensor product of morphisms, so the numbers of input and output lines need not match, as seen in the following diagram.



Functoriality of the tensor product in $\mathcal{M}$ is also implicit, but if parentheses were to be used it would look like the following:

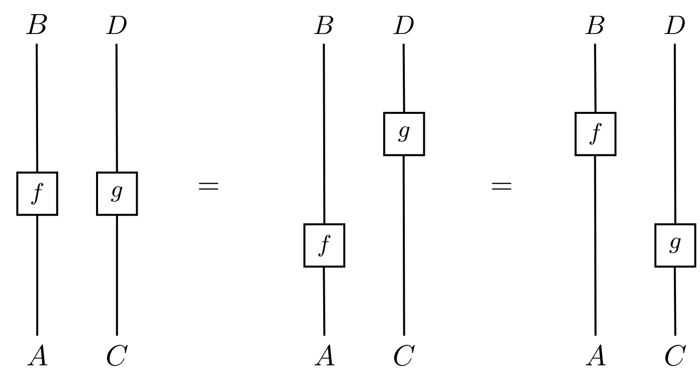


Planar isotopic string diagrams represent the same morphisms. Therefore, manipulations involving isotopies of diagrams yield valid equations between morphisms, provided these equations are well formed, in the sense of remark 2.2.28. So, non-well formed equations that might hold in particular monoidal categories have to be imposed as extra conditions on the string diagram calculus.

According to the above we can get the following equation for morphisms $f \in \mathcal{M}(A, B)$ and $g \in \mathcal{M}(C, D)$:

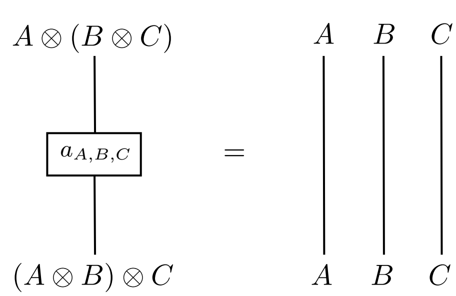
$$f \otimes g = (\mathrm{id}_B \otimes g) \circ (f \otimes \mathrm{id}_C) = (f \otimes \mathrm{id}_D) \circ (\mathrm{id}_A \otimes g),$$

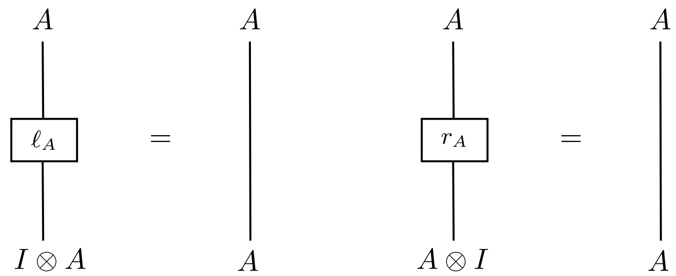
which holds either by functoriality of the tensor or by the following planar isotopy:



We can also take this equality as saying that the graphical calculus allows us to slide parallel boxes past each other.

The associator and the unitor isomorphisms are not depicted since due to the coherence theorem everything that holds in the category $\mathcal{M}$ can be deduced by claims about its strictified version. Therefore, there is no need for the graphical calculus to depict coherence isomorphisms. So the graphical calculus of a monoidal category is actually about its strictification. Thus the associator and the unitors, if depicted, would look like the following in the string diagram calculus:

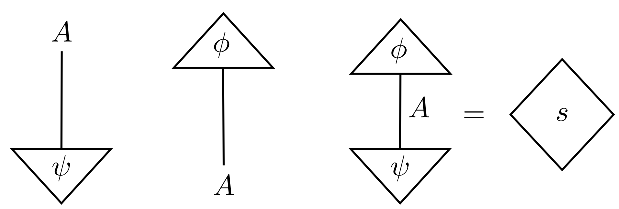




Since string diagrams are a tool for strict monoidal categories, the pentagon and triangle laws lead to trivial string diagrams, which could be considered as evidence that they hold.

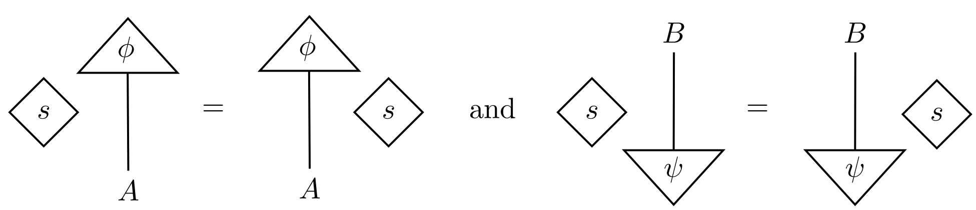
String diagrams are a very intuitive tool for monoidal categories, but they have also been formalized. There are also versions for braided, symmetric, rigid and compact monoidal categories, which have also been formalized in the [JS91]. These will be presented in the next chapter.

Using the graphical notation, we can translate the notions of states, effects and scalars, but also the laws for internal monoids, comonoids and Frobenius objects. So, since a state is a morphism $\psi : I \to A$ for an object $A \in \mathcal{M}_0$ and an effect is of the form $\phi : A \to I$, we use a triangle instead of a square to depict them. Furthermore, since any scalar, $s \in \mathcal{M}(I, I)$, is the composition of a state and an effect we can depict it in two different ways as follows:

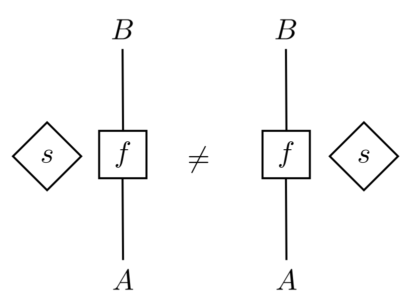


State Effect Scalar

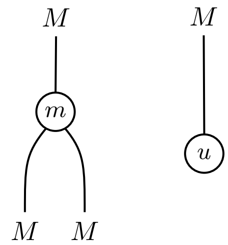
The string diagram calculus allows us to see why the left and right scalar multiplications of arbitray morphisms do not coincide, while in the case of states and effect they do, as proven in 2.1.39. Given a scalar s a state ψ, an effect ϕ and a morphism f in string diagram terms as below, we see that a scalar passing over an effect and under a state yields planar isotopic diagrams, while there is no planar isotopy transporting a scalar from the left to the right of a morphism in general.



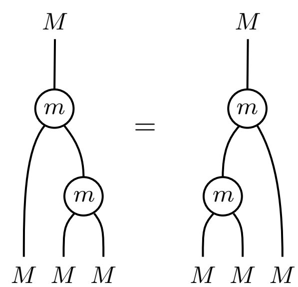
while



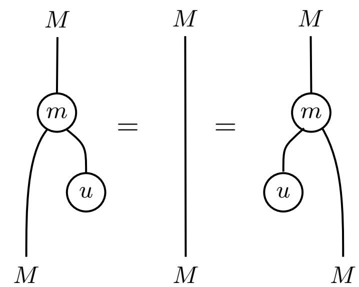
For an internal monoid (M, m, u) the multiplication and the unit are depicted as circles instead of boxes, although the unit is a state:



So the associativity and unit laws for an internal monoid look as follows:

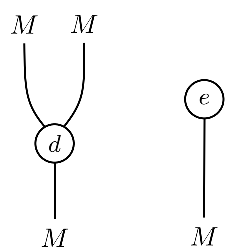


Associativity

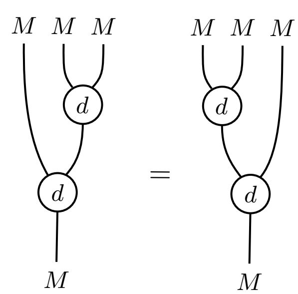


Unitality

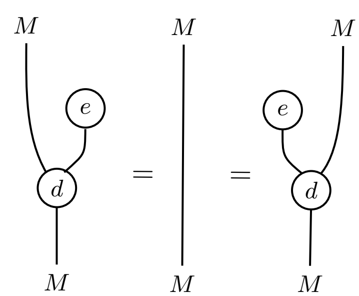
Similarly, if (M, d, e) is an internal comonoid in $\mathcal{M}$, then the monoid operations take the following form in the graphical calculus:



while coassociativity and counitality are given as the following conditions:

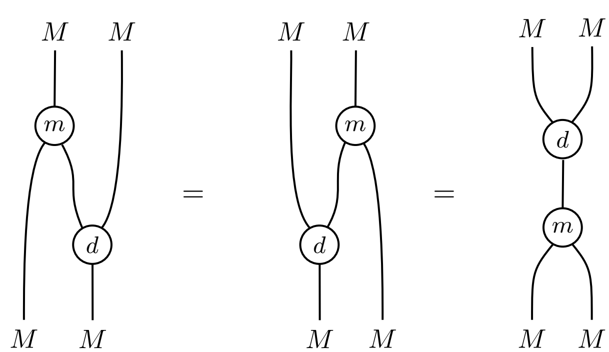


Coassociativity

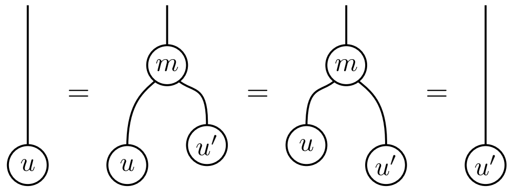


Counitality

Finally, if (F, m, u, d, u) is a Frobenius object, then obviously the above (co)associativity and (co)unitality conditions are again depicted as above. Furthermore, the Frobenius condition is presented as:

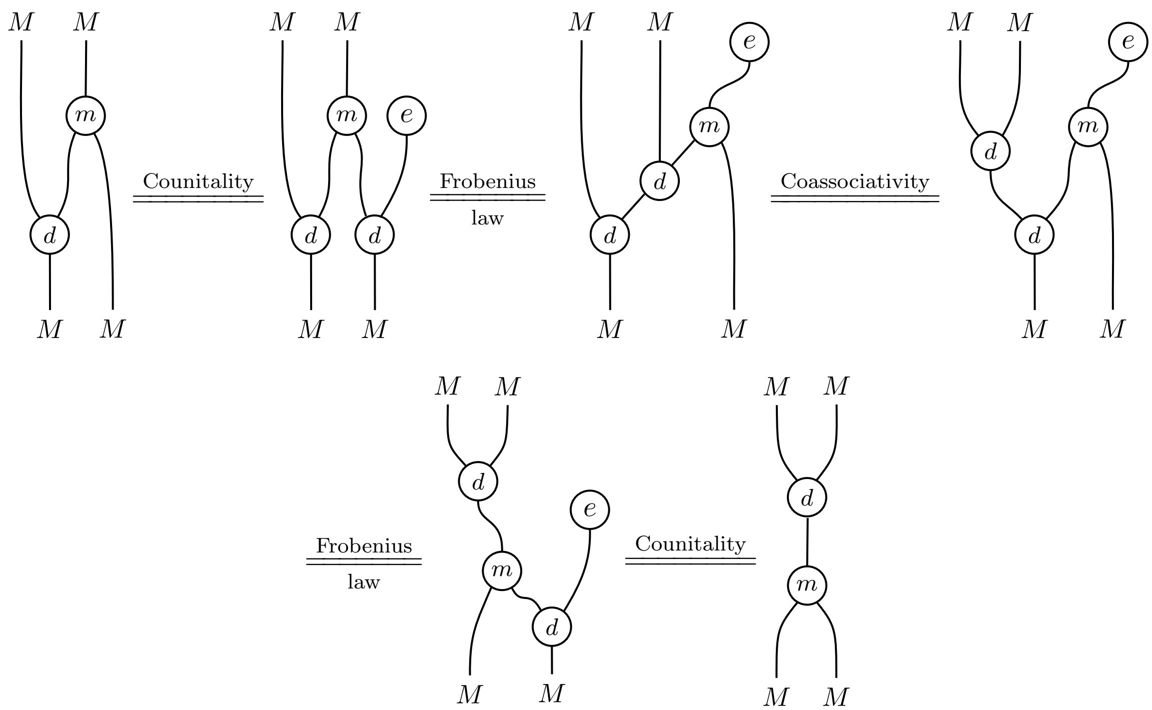


Using the graphical calculus we can prove the uniqueness of units for monoids and consequently of co-units for comonoids. The proof is actually a diagrammatic version of the original proof for monoids internal to **Set** and goes as follows. Let (M, m, u) and (M, m, u') be internal monoids in a monoidal category $\mathcal{M}$. Then unitality for u and u' imply:

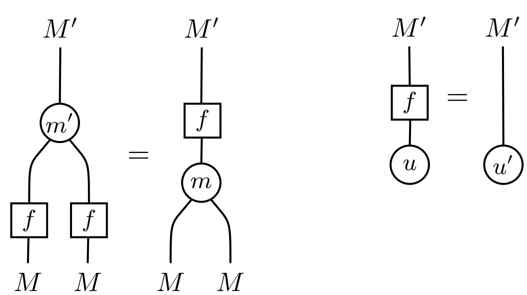


Thus, one might be justified to say that a unit (and dually a co-unit) are properties of "internal semigroups" rather than structures.

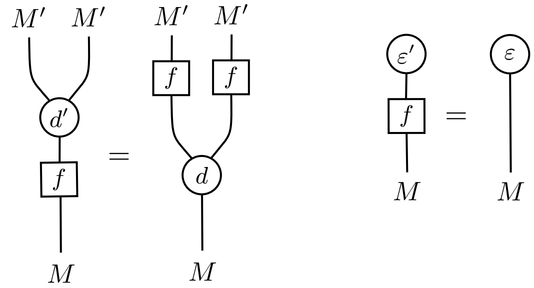
The graphical calculus tool reveals a redundancy in the definition of a Frobenius object. More concretely the last equation in the above two diagrammatic equations is a consequence of the first and the comonoid laws. This goes as follows, where the first of the above equations will be called the "Frobenius law":



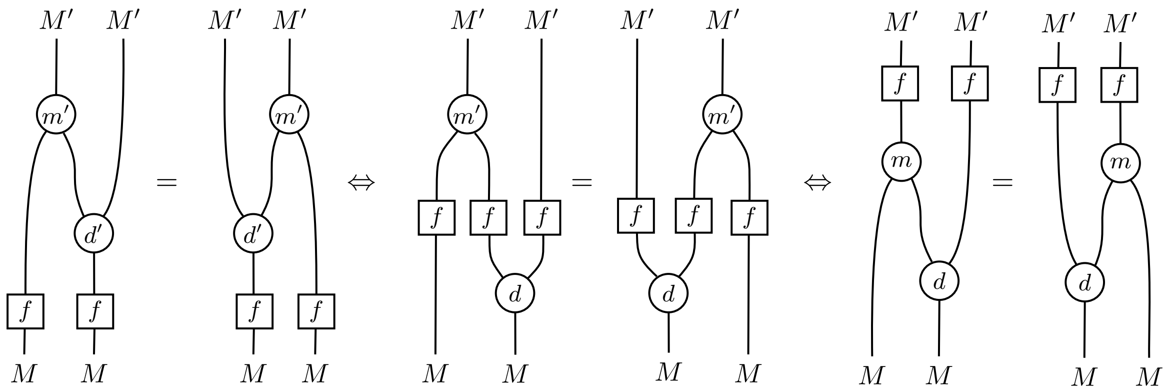
In the graphical calculus a monoid homomorphism $f : (M, m, u) \to (M', m', u')$ is a morphism satisfying the following properties:



while a comonoid homomorphism $f : (M, d, e) \to (M', d', e')$ satisfies:



Finally, using string diagrams we see that the Frobenius condition is preserved by a simultaneous monoid-comonoid homomorphism $f : M \to M'$, or rather the Frobenius condition holds for (M, m, d, u, e) if and only if it holds for (M', m', d', u', e'), since:

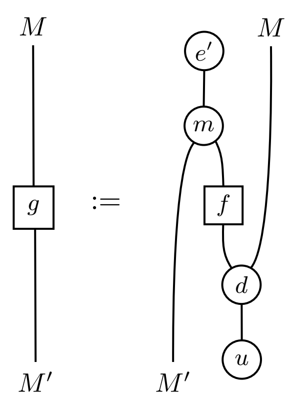


This actually shows that if a monoid which is also a comonoid is homomorphic to a Frobenius object, then it already is a Frobenius object. There is a deeper issue here, which is the following.

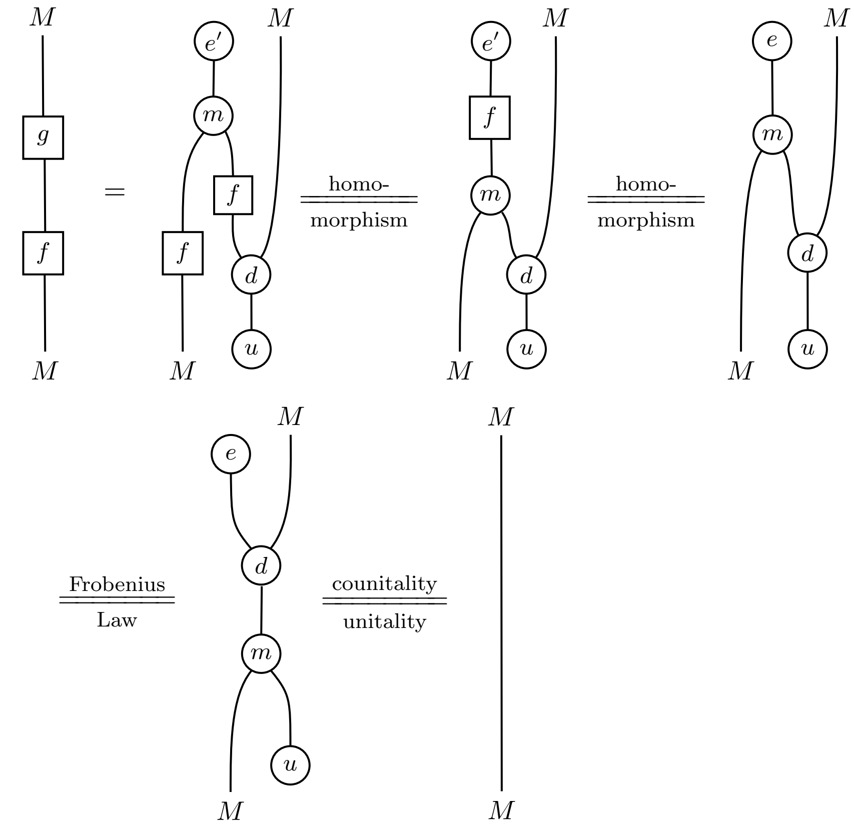
Proposition 2.2.30. *Let* (M, m, d, u, e) *and* (M', m', d', u', e') *be Frobenius objects in a monoidal category* $\mathcal{M}$*. Then they are isomorphic as Frobenius objects, if and only if they are homomorphic.*

Proof. Obviously if $M \cong M'$ as Frobenius objects, then the isomorphism between them is also a homomorphism. For the other direction, we will prove an even stronger result. *Every homomorphism between* M *and* M' *is an isomorphism.*

Let $f : M \to M'$ be a homomorphism. Define $g : M' \to M$ as follows:

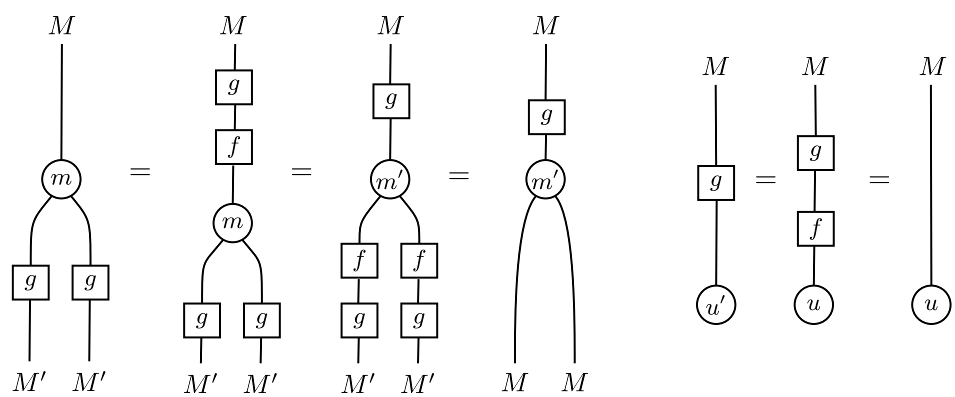


Then precomposing with f yields:



And similarly post composition with f gives $f \circ g = \mathrm{id}_{M'}$. Thus g is an inverse of f.

Finally, we need to prove that g is a Frobenius homomorphism, which easily follows from its invertibility. Since the monoid and comonoid axioms amount to reflected string diagrams, we only show that g is a monoid homomorphism and the rest follows by similar arguements.

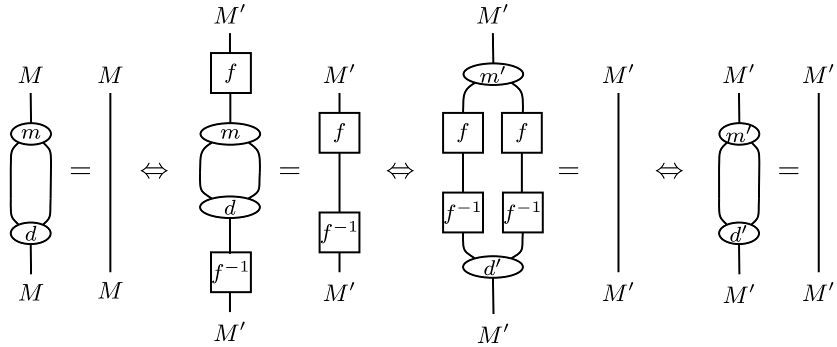


□

With the help of the above proposition we can easily prove the following corollary which, combined with the results of the remarks 2.2.7 and 2.2.14, result in $\mathbf{MonCat}(\mathbf{1}, -)$ being a full subcategory of $\mathbf{SpFrob}(-)$.

Corollary 2.2.31. *Let $f : (M, m, \eta, d, \varepsilon) \to (M', m', \eta', d', \varepsilon')$ be a homomorphism of Frobenius object in a monoidal category $\mathcal{M}$. Then M is special if and only if M' is special.*

Proof. Since any homomorphism of Frobenius objects is also an isomorphism of Frobenius objects, we have:



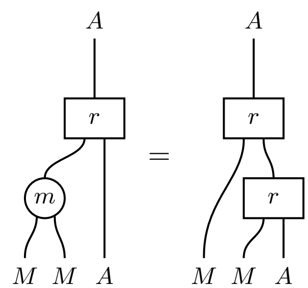
and this concludes the proof. □

Modules and comodules

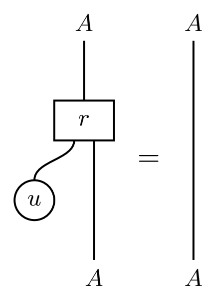
String diagrams allow for a really intuitive representation of the notions of modules and co-modules inside a monoidal category.

Definition 2.2.32. *Let $\mathcal{M}$ be a monoidal category, (M, m, u) be an internal monoid in $\mathcal{M}$ and $A \in \mathcal{M}_0$. We call $r : M \otimes A \to A$ a* ***left action*** *of $\mathcal{M}$ on A if r is*

1. ***associative***, *meaning that:*



2. ***unital***, *meaning that:*



*The triple (M, A, r) is called a (left) M-**module**.*

Remark 2.2.33. In non string diagram terms, the above conditions are given as

$$r \circ (r \otimes M) = r \circ (\mathrm{id}_A \otimes m) \circ a^{-1} \text{ and } r \circ (u \otimes \mathrm{id}_M) = \ell_A.$$

Example 2.2.34. Internal monoids in **Set** are monoids. A monoid action on a set is exacty a left module.

Example 2.2.35. An internal monoid, R, in **Ab** is a unital ring. An R-module in **Ab** is exactly a module.

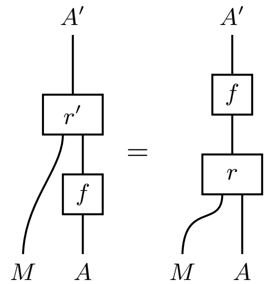
Example 2.2.36. A monoid A internal to **Vect** is an associative unital algebra. Such an algebra action determines an A-module. This justifies the construction of the group algebra, so as to capture a linear group action internally to **Vect**.

Evidently, there are two modules that can be constructed in any monoidal category. Given a monoid M, (M, M, m) is trivially an M-module due to associativity and unitality. The second one is a module for the same reasons but it gets a special name.

Definition 2.2.37. *Let $\mathcal{M}$ be a monoidal category and (M, m, u) be an internal monoid. For every object A there exists a module $(M, M \otimes A, m \otimes \mathrm{id}_A)$ called a* ***free module****.*

In a monoidal category, M-modules form a category. For this to be sensible, a notion of morphism needs to be defined.

Definition 2.2.38. *Let $\mathcal{M}$ be a monoidal category, (M, m, u) an internal monoid and let (M, A, r), (M, A', r') be M-modules. A morphism $f : A \to A'$ is a* ***module homomorphism*** *if:*

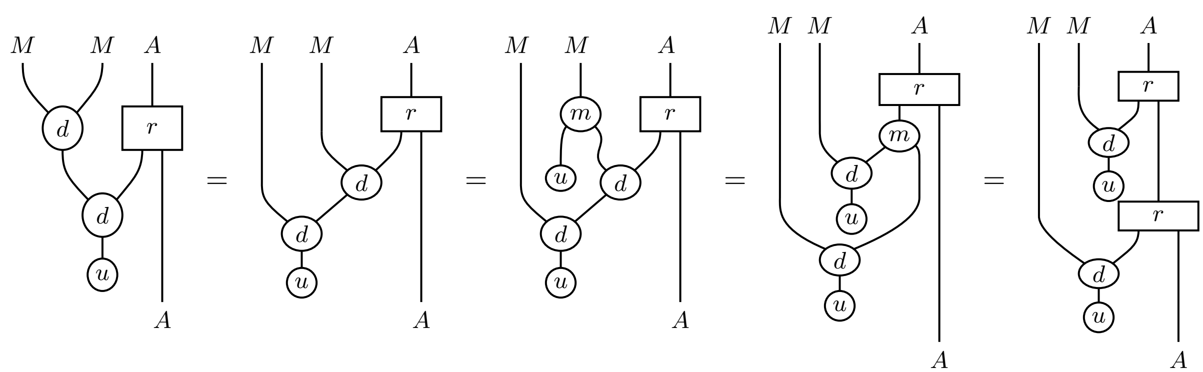


Remark 2.2.39. Obviously composition of module homomorphisms results in a module homomorphism, but also the identity map is a module homomorphism. So associated to every internal monoid, (M, m, u) is its category of modules $Mod_{\mathcal{M}}(M)$. Especially, the category of modules of the unit object monoid with the the action given by the left unitor[6] is the whole monoidal category since naturality of the left unitor implies that every morphism is an I-module homomorphism. Thus, $Mod_{\mathcal{M}}(I) \simeq \mathcal{M}$.

One can define a right action and right modules as left actions and left modules in $\mathcal{M}^{\mathrm{rev}}$, the reverse monoidal category. Also similarly, one may define co-actions of internal comonoids and co-modules, as monoid actions and modules in $\mathcal{M}^{\mathrm{op}}$, the opposite monoidal category. Obviously, given any (co)monoid, both of these notions admit categories of (co)-modules using the respective notions of homomorphism. Given the above and since a monoid and a comonoid structure may coexist, both actions and co-actions of the same object may be defined. These need not be compatible in any way in general, but Frobenius objects and their modules and co-modules exhibit some interesting properties.

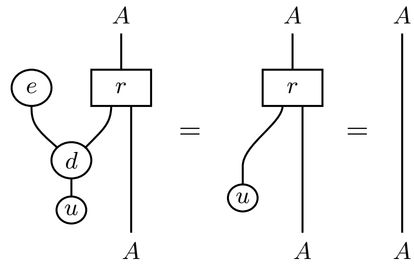
Proposition 2.2.40. *Let (M, m, u, d, e) be a Frobenius object in a monoidal category $\mathcal{M}$ and let $r : M \otimes A \to A$ be an (M, m, u)-action. Then $(\mathrm{id}_M \otimes r) \circ a_{M,M,A} \circ ((d \circ u) \otimes \mathrm{id}_A) \circ \ell_A^{-1} : A \to M \otimes A$ is a co-action of (M, d, e).*

Proof. We will prove this diagrammatically. For co-associativity we have:



and for co-unitality:

[6]The triangle law and the coincidence of the unitors components at the unit object allow us to define the action (I, A, ℓ_A).



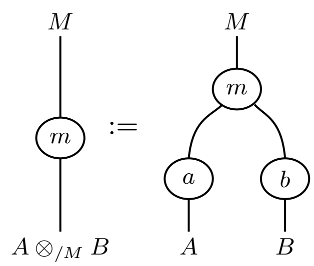
□

Remark 2.2.41. Dualising, every co-action induces an action. Thus, Frobenius object actions and co-actions are defined simultaneously.

Monoidal (co)slice categories

In any monoidal category, $\mathcal{M}$, one can form its slice and coslice categories over and under any object. There are two cases where the new categories are also monoidal, slicing over a monoid and coslicing under a comonoid. This topic could have been presented earlier in this thesis but the string diagram calculus makes it easier and more intuitive. We will present the case for slice categories over a monoid (M, m, u) since the other case is similar by duality.

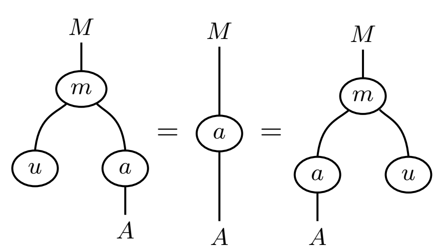
So let (M, m, u) be a monoid inside a monoidal category $\mathcal{M}$ and form the slice category $\mathcal{M}/M$. Define for all $a : A \to M$ and $b : B \to M$ the monoidal product as follows:



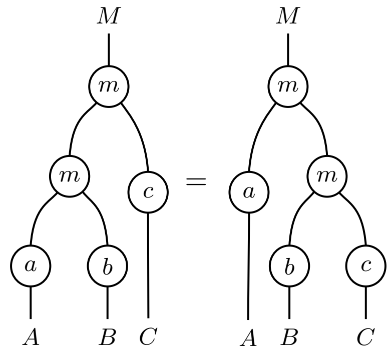
On morphisms $f : (A, a) \to (A', a')$ and $g : (B, b) \to (B', b')$, define

$$f \otimes_{/M} g := f \otimes g$$

and note that functoriality of $\otimes$ implies functoriality of $\otimes_{/M}$. Also note that by defining $I_{/M} := (I, u)$, $\ell_{(A,a)} := \ell_A$ and $r_{(A,a)} := r_A$ for all $(A, a) \in \mathcal{M}/m$, coherence of $\mathcal{M}$ and unitality of M give the following diagrams:



Thus the above are the unit and the unitors. Naturality of the unitors follows immediately from their naturality in $\mathcal{M}$. Similarly, from the following diagrammatic equality, which holds due to associativity of M and coherence of $\mathcal{M}$:



we see that the associator must be defined by $a_{(A,a),(B,b),(C,c)} := a_{A,B,C}$ and naturality follows again from naturality of a in $\mathcal{M}$. Finally, the triangle and pentagon laws hold in $\mathcal{M}/M$ since they hold in $\mathcal{M}$.

Normal forms and diagrammatic calculus for monoidal functors

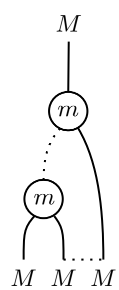
The string diagram calculus yields coherence type results for internal monoids, comonoids and Frobenius objects. In particular, one can show that special kinds of morphisms strictly built from the ingredients provided by either monoids, comonoids or Frobenius objects can be brought to a normal form, to which they are equal. This leads to some intuition behind coherence results for lax and colax monoidal functors as given in [MP15], which in turn allows for a graphical calculus for monoidal functors.

Fix a monoidal category $\mathcal{M}$ and an internal monoid (M, m, η). In what follows we will use the notion of a *connected diagram*. Intuitively, such a diagram has only one connected component.

So, any connected diagram built exclusively out of $m, \eta, \otimes$ and identities corresponds to a morphism with codomain M[7]. This is indeed the case since the only way to increase the number on M's in the output is to insert them using the unit of the monoid, but this, in turn, would create another connected component.

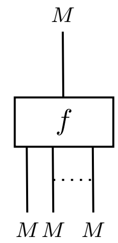
So, concerning connected diagrams involving M and the morphisms m, η, id_M and $\otimes$, we have the following theorem.

Theorem 2.2.42. *Let, (M, m, η) be a monoid in $\mathcal{M}$. Every connected diagram built from m, η, id_M and $\otimes$ can be brought to the following normal form.*

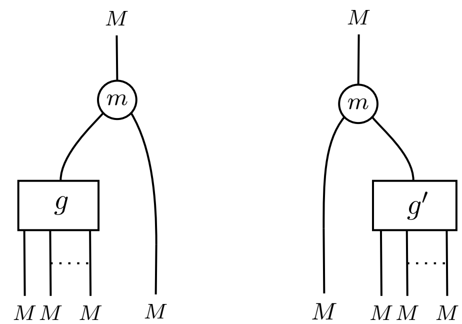


Proof. As a first step in the proof we will restrict to connected diagrams not involving units. We will proceed by induction on the number of copies of M as inputs. As base cases the empty and the one string diagrams are in the required normal form. For the inductive step, assume that all diagrams involving up to n copies of M as inputs can be brought to the above normal form. Denote any such morphism by $D_k : \bigotimes_{i=1}^{k} M \to M$, for $1 \leq k \leq n$, and consider the following diagram depicting the morphism $f : \bigotimes_{i=1}^{n+1} M \to M$.

[7]Here we make non-trivial use of the coherence theorem for monoidal categories identifying M with any tensor product involving M and finite copies of the unit I.

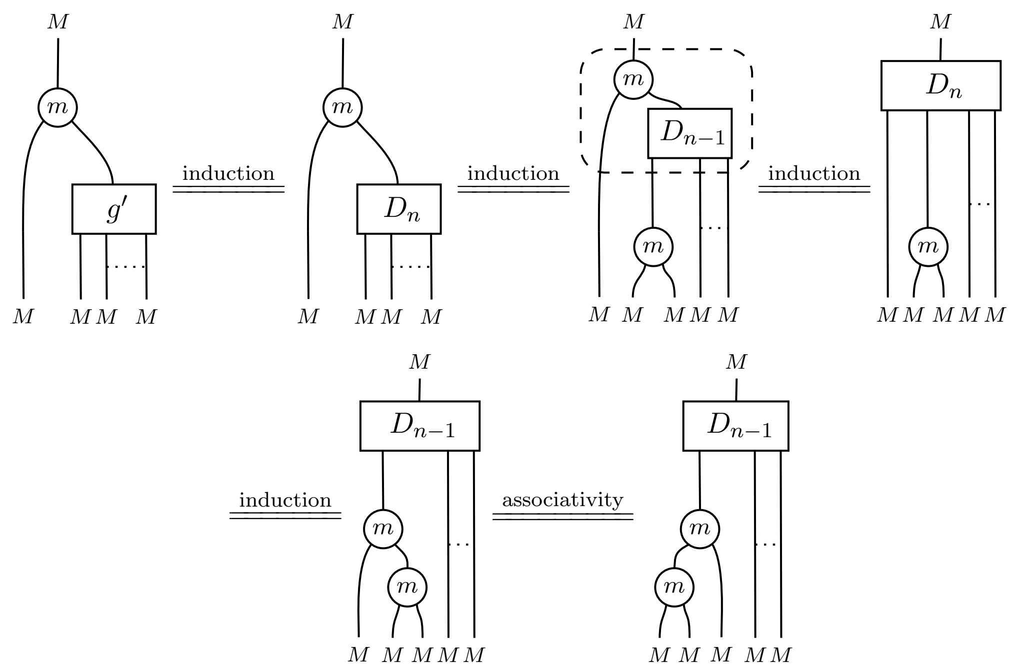


This diagram can only decompose in one of the following two forms:



a) b)

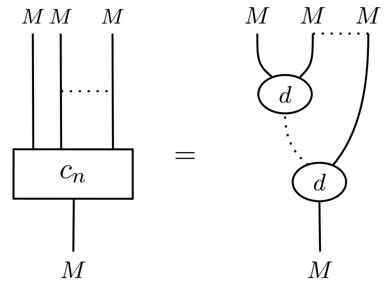
By the inductive hypothesis, $g = D_n$ and $g' = D_n$, so a) can be brought to the required normal form. On the other hand, for b), observe that $D_n = m \circ (D_{n-1} \otimes \mathrm{id}_M)$, so associativity gives:



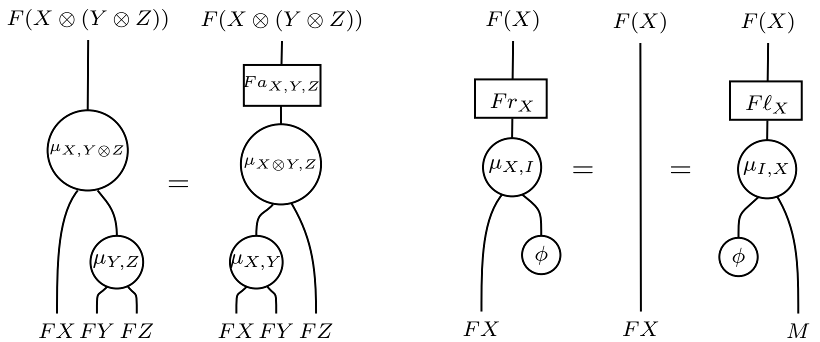
which is again in the required normal form.

Now the only possible way to also have a unit is if it is an input of a D_n. In any other case the diagram would not be connected. So if a unit is an input, then unitality would instantly transform the diagram in the normal form. □

The above theorem can obviously be dualised to comonoids yielding a normal form for connected string diagrams made exclusively out of tensoring identity morphisms with co-multiplications and co-units. Any such diagram can be brought to the following normal form, denoted by c_n for n copies of the comonoid (M, d, ε) as inputs:



A more involved implication of the above theorem concerns lax monoidal functors and, after dualizing, also colax monoidal ones. We will actually use it to construct a more sophisticated version of the string diagram calculus. To construct this clearly, observe that associativity and unitality for a lax monoidal functor $(F, m, \phi) : \mathcal{M} \to \mathcal{N}$ can be translated to string diagrams as follows:

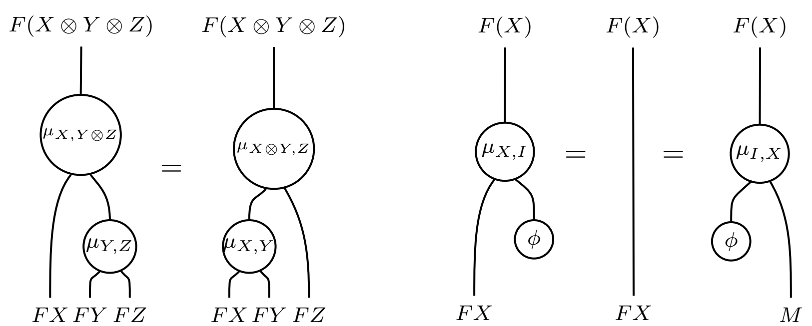


Associativity Unitality

for every $X, Y, Z \in \mathcal{M}_0$.

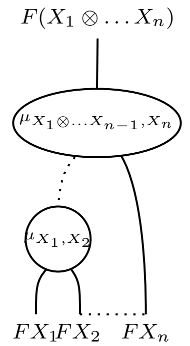
At this point it is easy to observe the relevance with the monoid string diagrams above. Another very useful thing to note is that the coherence isomorphisms of $\mathcal{M}$ enter the picture. Since both their domains and codomains are images of certain bracketed tensor products in $\mathcal{M}$ and due to the coherence theorem for monoidal categories, we can omit them while dropping the brackets. That is because there is a unique way to insert them back in the diagram. Omitting them actually amounts to precomposing F with the strictification functor's weak inverse, a strong monoidal functor. Therefore the composite would again be a lax monoidal functor from which the original functor F can be retrieved. Thus omitting the images of the coherence morphisms of $\mathcal{M}$ is no essential loss of information about F.

So under these conditions, we can redraw associativity and unitality as follows:



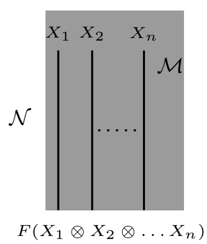
Associativity Unitality

In this kind of graphical notation, one can use similar, if not the same, graphical arguements as in 2.2.42 to give a normal form to certain connected diagrams. These diagrams would only involve input strings labeled by FX_i and one output string labeled by $F(X_1 \otimes \dots X_n)$, for $X_i \in \mathcal{M}_0$ and $i = 1, \dots, n \in \mathbb{N}$, and morphisms formed by tensoring the appropriate components of the multiplicator, the unitor and identity morphisms id_{FX_i}. Such a normal form is depicted as follows:

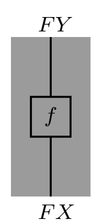


Summing up the above we see that every connected diagram in a category $\mathcal{N}$ made of components of the multiplicator and the unitor, with input wires labeled by images of objects of $\mathcal{M}$ and one output wire labeled by the image under F of the tensor product of the corresponding objects can be brought to a normal form. This implies that every morphism made by composing or tensoring components of the multiplicator, the unitor and Id_F is unique. Therefore, we have a coherence result for lax monoidal functors. Similarly, we have a normal form and a coherence result for colax monoidal functors. Before giving the colax monoidal version of the normal form, we introduce a final refinement of the string diagram calculus for a lax monoidal functor.

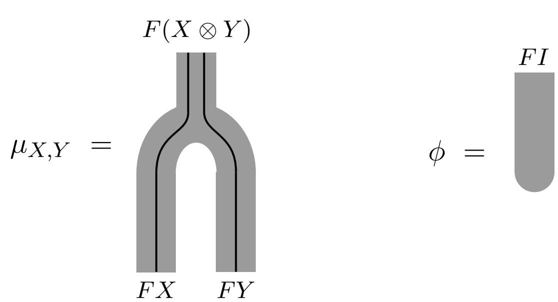
For every $X_i \in \mathcal{M}_0,\ i = 1, \dots n \in \mathbb{N}$, denote $F(X_1 \otimes \dots X_n)$ by:



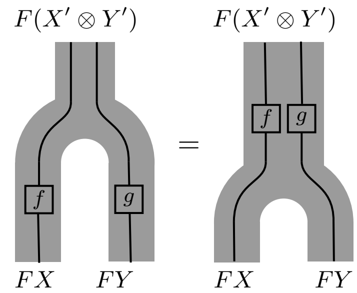
where inside the grey area we draw "what happens inside $\mathcal{M}$" and outside we are inside $\mathcal{N}$. The image under F of a morphism $f \in \mathcal{M}(X, Y)$ is depicted as:



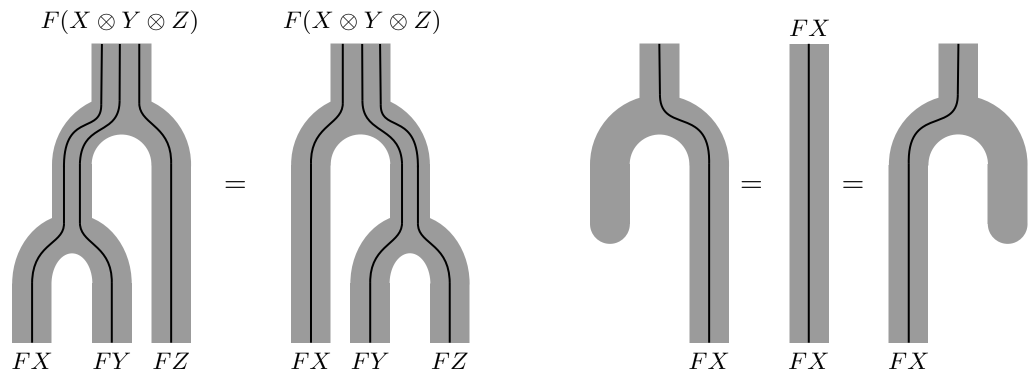
The multiplicator's components, $\mu_{X,Y}$, and the unitor, ϕ, are depicted as:



for $X, Y \in \mathcal{M}_0$. Naturality of μ is the following diagrammatic equality for $f \in \mathcal{M}(X, Y),\ g \in \mathcal{M}(X', Y')$:

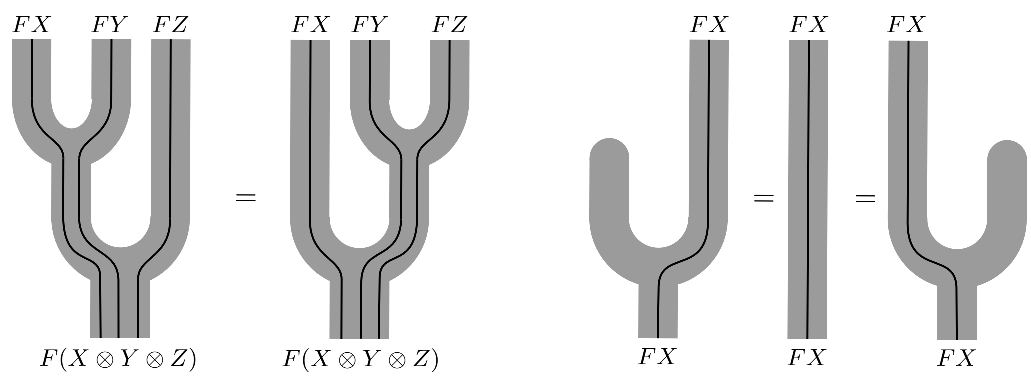


which allows us to think of the multiplicator as merging two $\mathcal{M}$-areas into one. Finally, associativity and unitality take the following diagrammatic form:



Associativity Unitality

Dualising the above to a colax monoidal functor (F, μ, ϕ), we get that the multiplicator tears one $\mathcal{M}-$area creating two branches, while the unitor terminates one such area. So unitality and associativity are depicted in a similar fashion to co-associativity and co-unitality of a comonoid, as follows:

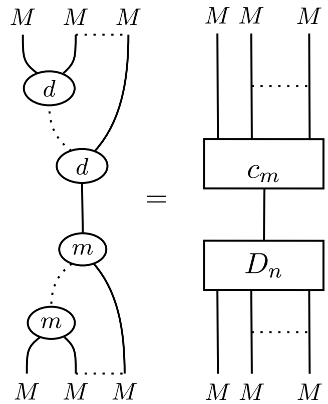


Associativity Unitality

Generalising this to strong monoidal functors is not as easy as it may seem. The difficulty one faces is that there is a greater variety of connected diagrams than in the cases of lax and colax monoidal functors. To tackle this problem we shift our focus to normal forms for special Frobenius objects inside a monoidal category M. From there two paths can be taken towards generalisation. The first one concerns normal forms for non-special Frobenius objects, while the second one is about strong monoidal functors. We will take both paths.

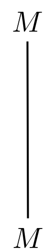
In what follows fix a special Frobenius object $(M, m, \eta, d, \varepsilon)$ inside $\mathcal{M}$.

Theorem 2.2.43. *Let $(M, m, \eta, d, \varepsilon)$ be a special Frobenius object inside a monoidal category $\mathcal{M}$. Any connected diagram formed by composing or tensoring m, η, d, ε and possibly multiple copies of id_M can be brought to the following normal form:*



for $m, n \in \mathbb{N}$.

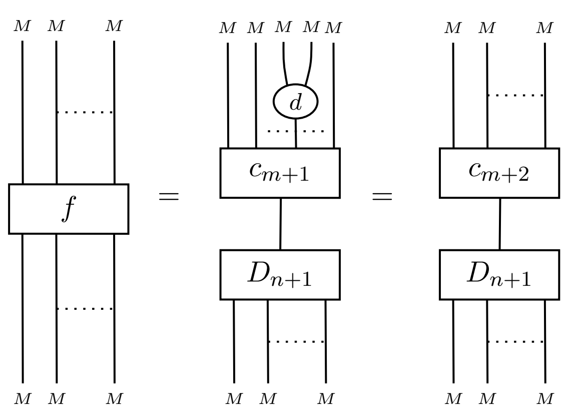
Proof. We will prove this theorem by induction on the number of morphisms/nodes involved. As a base case consider the 0 nodes diagram:



which is trivially in the above normal form. For the inductive hypothesis, assume that every connected diagram with at most $k = n + m$ nodes can be brought to the above normal form. Also note that such a diagram would have at most $n+1$ inputs of type M and $m+1$ outputs of type M, since m nodes correspond to $m + 1$ outputs in the c_{m+1} part and similarly for the D_{n+1} part.

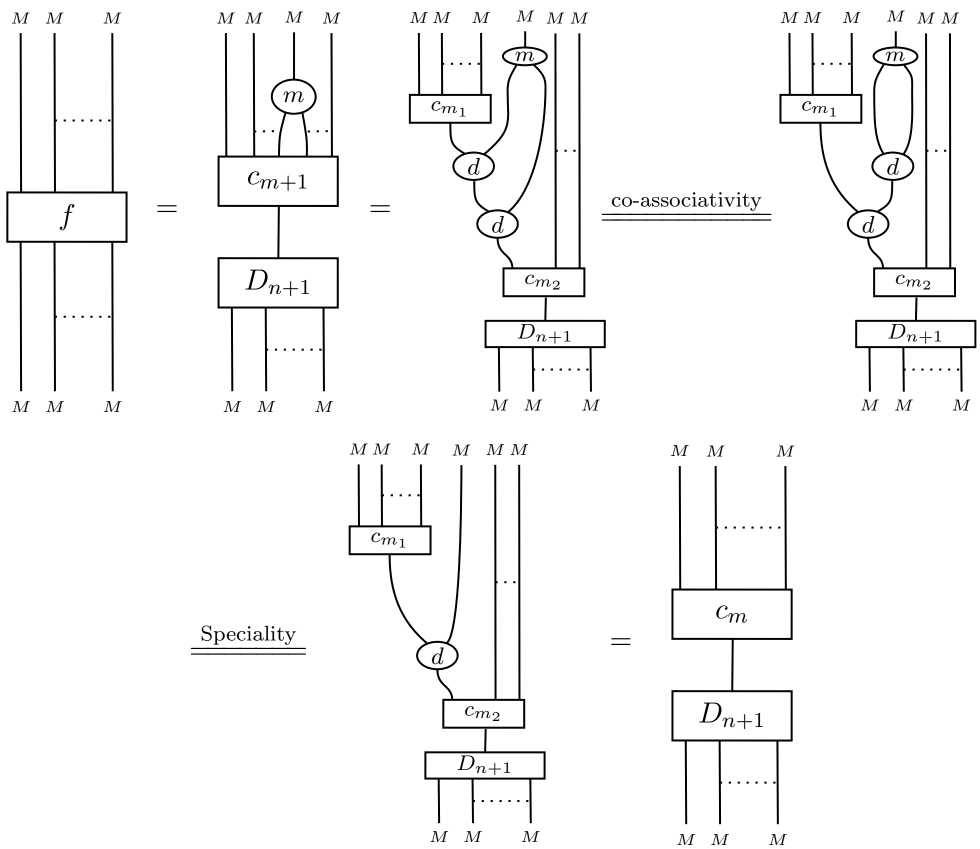
Let f be a morphism represented by a connected diagram with $n+m+1$ nodes, for $m+n = k$. Without loss of generality and due to the interchange law, there is a topmost node so that under it we get a diagram with $n+m$ nodes. Since identity morphisms are not graphically depicted, there are four cases for the topmost node.

1. If the topmost node is a *comultiplication*, then the diagram under it must be connected, since comultiplication, having one input, cannot connect to disconnected components. Thus, applying the inductive hypothesis to the diagram under we have the following graphical equality, which follows from the corresponding normal form theorem for comonoids and gives the desired result.



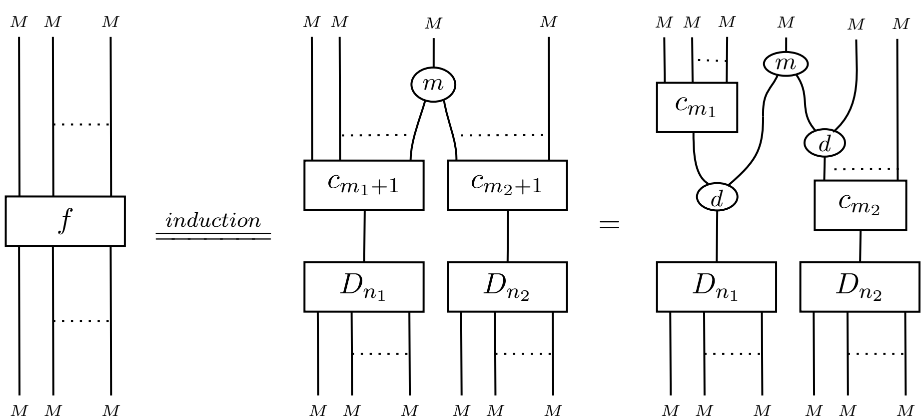
2. If the topmost node is a *counit*, then again the diagram below it is already connected. So applying the inductive hypothesis and the corresponding theorem for comonoids we get the desired normal form.

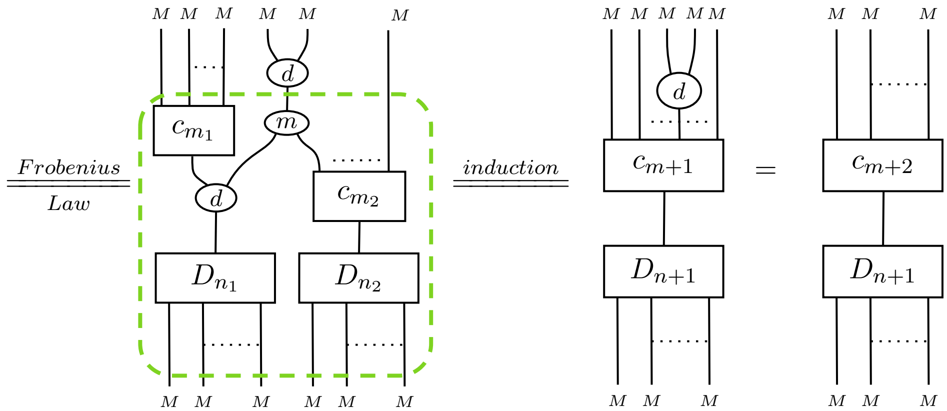
3. If the topmost node is a *unit*, then the diagram is disconnected and thus this is a contradiction.
4. If the topmost node is a *multiplication* then there are again two cases.
    (a) If the diagram below the multiplication is *connected*, then by induction we have for $m + n = k$ and $m_1 + m_2 + 1 = m$:



    which is in the desired form.

    (b) If the diagram below the multiplication is *disconnected*, then for suitable m_1, m_2, n_1, n_2 we get:

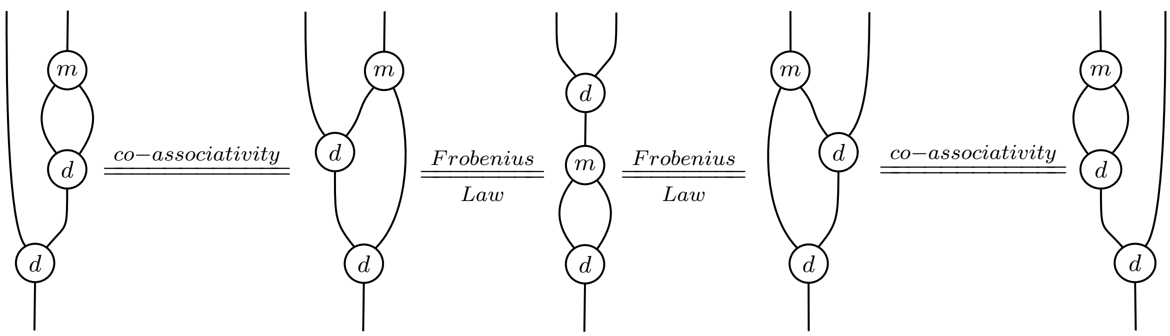




which is in the normal form and the last equality follows from the first case.

□

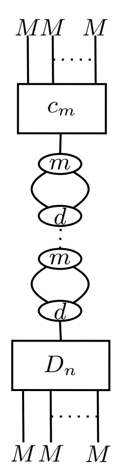
Remark 2.2.44. Observe that the only case where the speciality condition is used is 4(a). With this in mind the generalisation to the non special Frobenius case is fairly straightforward. Given a Frobenius object $(M, m, \eta, d, \varepsilon)$ the following diagrammatic equalities are crucial:



as they show that a morphism of type $m \circ d$ as one of the two outputs of co-multiplication can equivalently be transfered to the input (middle diagram above). Furthermore, this process can be iterated finitely many times resulting in the following equality:

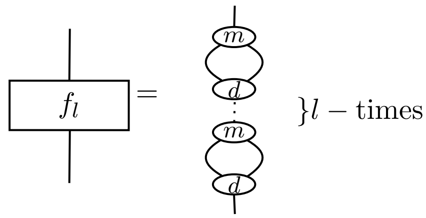
$$((m \circ d) \otimes (\otimes_{i=1}^{n-1} \mathrm{id}_M)) \circ c_n = c_n \circ (m \circ d).$$

Theorem 2.2.45. *Let $(M, m, \eta, d, \varepsilon)$ be a Frobenius object inside a monoidal category $\mathcal{M}$. Any connected diagram formed by composing or tensoring m, η, d, ε and possibly multiple copies of id_M can be brought to the following normal form:*

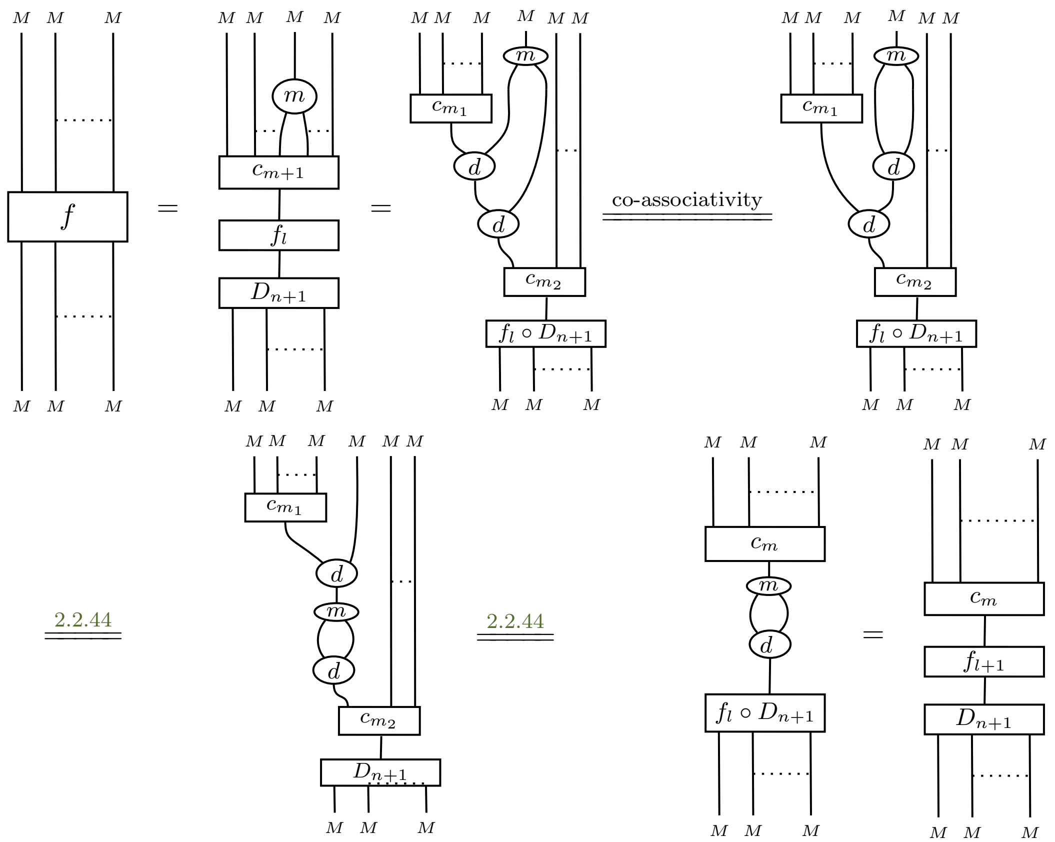


for $m, n \in \mathbb{N}$.

Proof. This proof is terribly similar to the one for special Frobenius objects. We will again procceed by induction on the number of nodes and we will only focus on the cases where the speciality condition played a role. So obviously the base case trivially holds. For the inductive step, assume that every connected diagram with up to k nodes can be brought to a normal form and let $f : \otimes_{i=1}^{n+1} M \to \otimes_{j=1}^{m+1}$ be a morphism represented by a connected string diagram, where $n + m \leq k + 1$. Then the possible cases for the topmost node are the same with the ones in the proof of 2.2.43 and yield the same results except for 4(a), where the speciality condition is used. We will use the following notation:



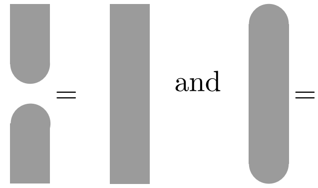
So if the topmost node is a multiplication and the diagram below is connected, then we have the following:



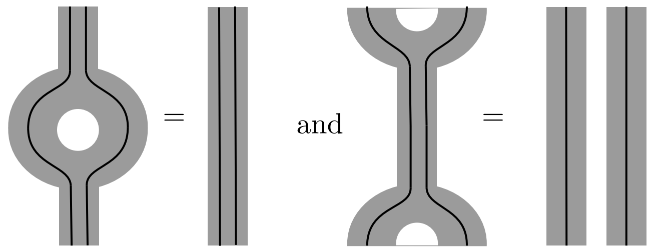
which is the desired normal form. □

The normal forms for Frobenius objects also give results for strong monoidal functors. The connection between Frobenius objects and strong monoidal functors drawn at this point is purely diagrammatic, as was the case for (co)lax monoidal functors and (co-)monoids. We will straight off adopt the convention introduced for string diagrams of monoidal functors.

Let $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$ be a strong monoidal functor. Since this is both lax and colax, we represent everything as in both these cases. There are also, though, some further restrictions having to do with the invertibility of the coherence morphisms. So the invertibility of ϕ poses the following two restrictions:

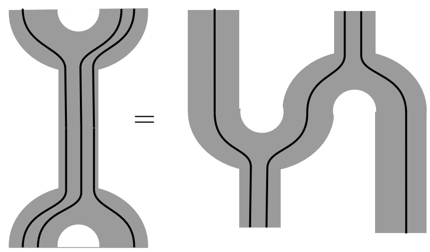


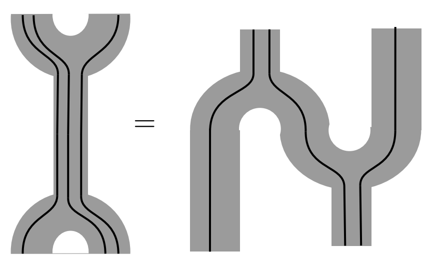
and for μ we have:



where both these equalities correspond to invertibility and the first one above is also similar to the speciality condition for a Frobenius object.

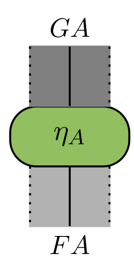
There are also strong monoidal functor string diagrams analogous to the Frobenius law.



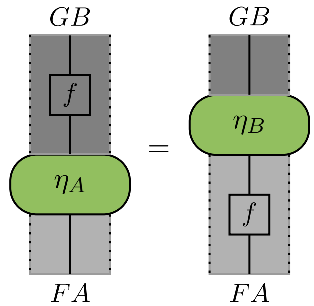


These can be easily seen to hold, being equivalent to the associativity conditions for μ and μ^{-1}.

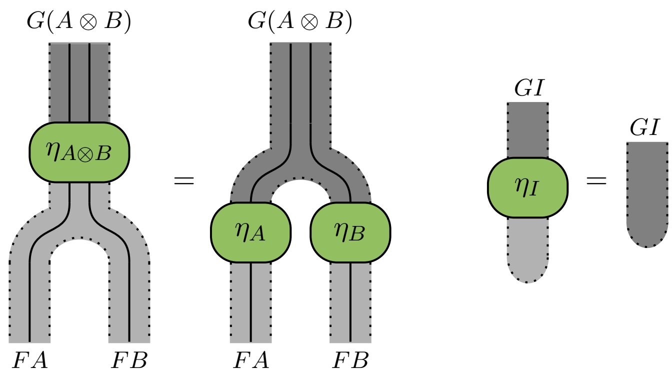
To conclude this subsection, we introduce a graphical calculus for monoidal natural transformations. So let $\mathcal{M}, \mathcal{N}$ be monoidal categories, $(F, \mu, \phi), (G, \nu, \chi) : \mathcal{M} \to \mathcal{N}$ be (lax, colax or strong) monoidal functors, $\eta : F \Rightarrow G$ be a monoidal natural transformation and let $f \in \mathcal{M}(A, B)$. We depict the component η_A as:



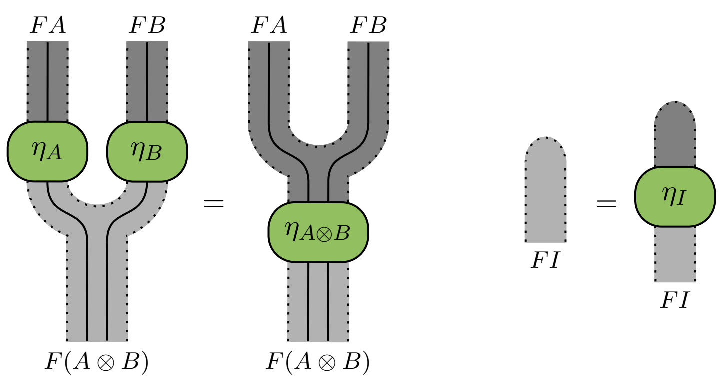
The naturality condition in these terms takes the following graphical form



while the monoidality conditions for η, if F, G are lax, are given as:



and obviously look very similar to the monoid homomorphism axioms. If F, G are colax the monoidality axioms take the following graphical form:



which are similar to the comonoid homomorphism axioms. Finally, if F, G are strong, both of the above are satisfied.

2.3 Aspects of 2-dimensional category theory

Formal category theory is the name attached to the abstract algebraic study of the relations between and the properties of categories, functors and natural transformations. In order for this task to be non-trivial, an abstraction of these concepts, as manifested in **CAT**, is needed. But since in elementary category theory identities and isomorphisms (natural or not) tend to be treated differently, there occur two ways of abstracting the laws of a category. The first one corresponds to equalities and has as its objects of study **2-Categories**, also called **strict 2-Categories**, while the other corresponds to isomorphisms and has as its objects of study **Bicategories**, also called **weak 2-Categories**. Both notions can be a source of monoidal categories, which will be explored in the following.

2-categories and bicategories

Before giving the definition of a bicategory, we restate the fact that the preorder category **1** is terminal in **CAT** and **Cat**, and that for every category $\mathcal{C}$, $\mathcal{C} \times \mathbf{1} \cong \mathcal{C} \cong \mathbf{1} \times \mathcal{C}$, where the isomorphisms are the corresponding projection functors of the products.

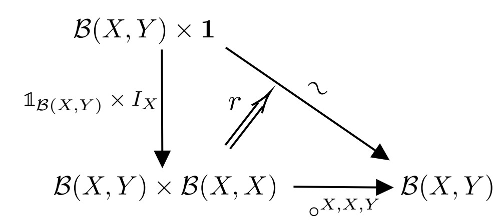
WARNING: Long definition!

Definition 2.3.1. *A bicategory $(\mathcal{B}, \circ, I_-, a, l, r)$, or just $\mathcal{B}$, consists of:*

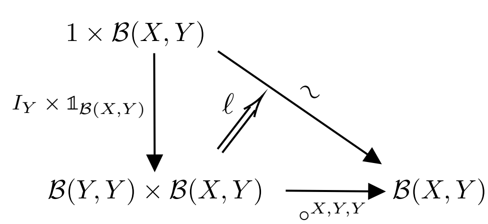
- *a collection/class of objects denoted $\mathcal{B}_0$, also called **0-cells**, signified by capital english letters,*
- *for every $X, Y \in \mathcal{B}_0$, a category $\mathcal{B}(X, Y)$, whose*
  - *i) objects are called **1-cells** of $\mathcal{B}$, are denoted by lower case english letters and the collection/class of all 1-cells of $\mathcal{B}$ is denoted by $\mathcal{B}_1$,*
  - *ii) morphisms are called **2-cells** of $\mathcal{B}$, are denoted by lower case greek letters and the collection/class of all 2-cells of $\mathcal{B}$ is denoted by $\mathcal{B}_2$,*
  - *iii) composition of morphisms/2-cells is called **vertical composition** and is denoted by "·",*
  - *iv) identity morphisms are called **identity 2-cells** and are denoted by $\mathbf{1}_f$, for every $f \in \mathcal{B}(X, Y)_0$,*
- *for every $X \in \mathcal{B}_0$, a functor $I_X : \mathbf{1} \to \mathcal{B}(X, X)$, where $I_X(*)$ is called the **identiy 1-cell** on X, sometimes written as I_X, and $I_X(\mathrm{id}_*)$ is the **identity 2-cell** of the identity 1-cell, denoted by $\mathbf{1}_{I_X}$,*
- *for every $X, Y, Z \in \mathcal{B}_0$, a functor $\circ : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \to \mathcal{B}(X, Z)$, such that*

$$(g \xrightarrow{\beta} g', f \xrightarrow{\alpha} f') \mapsto (g \circ f \xrightarrow{\beta \circ \alpha} g' \circ f'),$$

  *called **horizontal composition**,*
- *for every $X, Y, Z, W \in \mathcal{B}_0$ there exists a natural isomorphism $a^{X,Y,Z,W}$, or more concisely just $a : (- \circ -) \circ - \Rightarrow - \circ (- \circ -)$, called **associator**, such that $\circ_{h,g,f} : (h \circ g) \circ f \xrightarrow{\sim} h \circ (g \circ f)$ inside $\mathcal{B}(X, W)$,*
- *for every $X, Y \in \mathcal{B}_0$ a natural isomorphism $r^{X,Y}$, or just r, of type:*

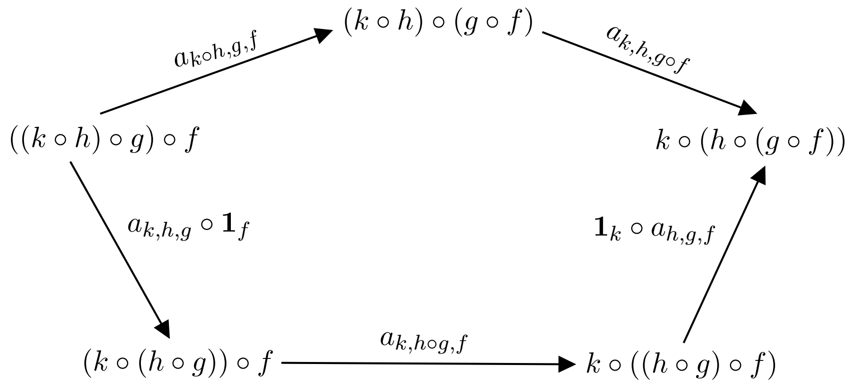


  *called **right unitor**, such that $r_f^{X,Y} : f \circ I_Y \xrightarrow{\sim} f$ is natural in f and*
- *for every $X, Y \in \mathcal{B}_0$ a natural isomorphism $\ell^{X,Y}$, or just ℓ, of type:*

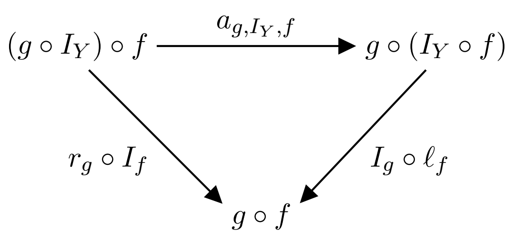


  *called **left unitor**, such that $\ell_f^{X,Y} : I_Y \circ f \xrightarrow{\sim} f$ is natural in f,* *satisfying the following axioms.*

Associativity Axiom: *for every horizontally composable 1-cells $f \in \mathcal{B}(X,Y)$, $g \in \mathcal{B}(Y,Z)$, $h \in \mathcal{B}(Z,W)$ and $k \in \mathcal{B}(W,V)$ the following pentagon, called* ***associativity pentagon****, commutes in $\mathcal{B}(X,V)$:*

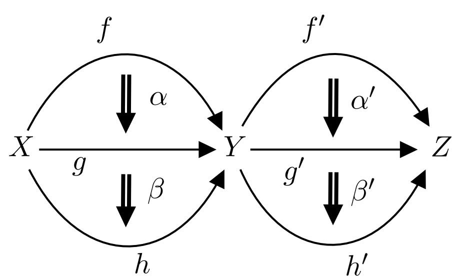


Unitality Axiom: *for every $f \in \mathcal{B}(X,Y)$ and $g \in \mathcal{B}(Y,Z)$ the following triangle, called* ***unitality triangle****, commutes in $\mathcal{B}(X,Z)$:*



Remark 2.3.2. 1. In the above definition we see that for every two 0-cells there exists a hom-category, instead of just a hom-set. This is the reason why the composition of 1-cells can be associative up to isomorphism and not strictly, since two functions in set cannot be isomorphic, while two functors can. Similarly, the identity 1-cells can compose with other 1-cells as weak/lax identities, for the same reason. While vertical composition of 2-cells is strictly associative, taking place exclusively in a hom-category, the horizontal composition of 2-cells is not strictly associative, since a natural isomorphism relates the two different bracketings. All that is said here is that the two possible compositions of three parallel 2-cells are isomorphic, considered as objects of an arrow category. Similarly, identity 2-cells are not strict identities for horizontal composition of 2-cells, but weak ones.

2. Functoriality of the vertical composition, "∘", of a bicategory $\mathcal{B}$ implies an interchange law, similar to that of a monoidal category. Explicitly, given 0-cells, 1-cells and 2-cells as below



we get the equality

$$(\beta' \cdot \alpha') \circ (\beta \cdot \alpha) = (\beta' \circ \beta) \cdot (\alpha' \circ \alpha).$$

Similarities with the category of categories and with monoidal categories should be more than just suggestive at this point. To extend the analogy with **Cat** even further, we can define the *whiskering* of a 2-cell with a 1-cell as

$$\alpha' \circ f := \alpha' \circ \mathbf{1}_f \text{ and } f' \circ \alpha := \mathbf{1}_{f'} \circ \alpha.$$

Functoriality also implies that $\mathbf{1}_{f'} \circ \mathbf{1}_f = \mathbf{1}_{f' \circ f}$.

3. In the case of bicategories, one needs to refine the notions of locality. So, we say that a bicategory is locally small, if every one of its hom-categories is a small category. Similarly, we define locally locally small bicategories and locally large bicategories. There, is also a notion of local discreteness, which means that the hom-categories of a bicategory are discrete categories. Finaly, if the hom-categories of a bicategory $\mathcal{B}$ are preorders or partial orders, then we say that $\mathcal{B}$ is locally (pre)ordered.

4. An interesting case/variant of a bicategory is a 2-category. *A 2-category is a bicategory whose coherence natural transformations, a, l and r are identity maps.* In the light of this variant and similarly to the terminology of monoidal categories, one could call 2-categories, strict bicategories, but the term strict 2-category or just 2-category has prevailed, and accordingly a bicategory is sometimes called a weak 2-category. Anyhow, the horizontal composition of a 2-category is strictly associative. Thus to every 2-category we can assign its underlying category just by not considering 2-cells. This can also be done to a bicategory, but an identification of coherently isomorphic 1-cells has to be made.

Definition 2.3.3. *Let $\mathcal{B}$ be a bicategory. A* ***sub-bicategory*** *$\mathcal{B}'$ of $\mathcal{B}$ consists of a subcollection $\mathcal{B}'_0 \subseteq \mathcal{B}_0$ and for every $X, Y \in \mathcal{B}'_0$ a subcategory $\mathcal{B}'(X,Y) \subseteq \mathcal{B}(X,Y)$ containing all components of associators and unitors and in the case $X = Y$, $I_X \in \mathcal{B}(X,X)$ is contained in $\mathcal{B}'(X,X)$, such that $\mathcal{B}'$ is closed under the composition of 1-cells and the compositions of 2-cells.*

Remark 2.3.4. According to the above definition a 2-category might be a sub-bicategory of a bicategory, but if a bicategory is a sub-bicategory of a 2-category, then it already is a 2-category. The above definition can also be used for 2-categories, giving rise to the notion of a *sub-2-category*.

As was the case in the chapter about ordinary categories we will present both abstract and concrete examples of bicategories. That is we will give structures whose collections give rise to bicategories, but we will also give structures that can be viewed as bicategories.

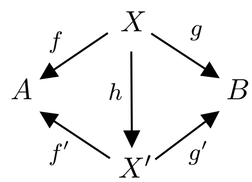
Example 2.3.5. Firstly, we give examples of concrete bicategories.

1. The archetype of a bicategory is the category of small categories **Cat**, which is actually a 2-category. The 0-cells are categories, the 1-cells are functors and the 2-cells are natural transformations. The vertical composition of natural transformations provides the vertical composition of 2-cells while the horizontal composition of functors and natural transformations provides the horizontal composition functor. The identity 2-cells are given by the identity natural transformations on functors and the identity 1-cells by the identity functors on categories.

2. Oddly, **Set** is a 2-category, with sets as 0-cells, functions, taken as subsets of a cartesian product of sets, as 1-cells and inclusions of functions as 2-cells. To be precise, a function $f : A \to B$ is considered as $f \subset A \times B$, so the set $\mathbf{Set}(A,B) \subset \mathcal{P}(A \times B)$ can be orderd by inclusion. Obviously, the composition of functions preserves the inclusion relation, thus defining a horizontal composition of 2-cells. **Set** considered this way is a locally ordered 2-category, since every hom-category is a poset. The identity 2-cells are given by the reflexive property of inclusion and the identity 1-cells are given by the identity functions on a set. Note though, that there are no non-trivial inclusions of total functions between fixed sets. So Even as a locally preordered 2-category, *Set* is locally discrete. The non-trivial cases is the following.

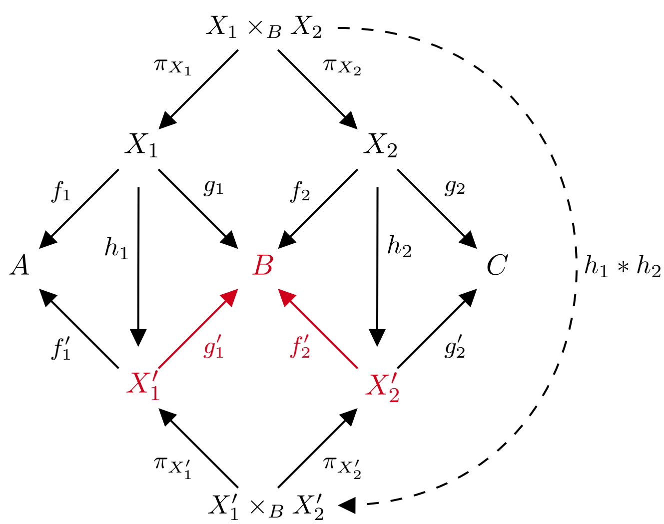
3. A more general view of **Set** as a bicategory is when, instead of taking functions as 1-cells, we take relations $R \subseteq A \times B$. This 2-category, called **Rel**, generalises the above example, while being similar in nature. This is because **Set** is a sub-2-category of **Rel**. One final thing to note, is that **Rel** can be seen as an ordinary category, forgetting the inclusion 2-cells, in which case **Set** is again a subcategory, of **Rel**.

4. A *span*, $s : A \rightsquigarrow B$, between two objects A, B in a category $\mathcal{C}$, is a triple $s = (X, f, g)$, where $X \in \mathcal{C}_0$, $f : X \to A$ and $g : X \to B$. This generalises a morphism in a symmetric way, since if $f = \mathrm{id}_A$, then the span reduces to a morphism $g : A \to B$ and similarly if $g = \mathrm{id}_B$, then a span is just a morphism $f : B \to A$. A morphism of spans $s, s' : A \rightsquigarrow B$, where $s = (X, f, g)$ and $s' = (X', f', g')$ is a morphism $h : X \to X'$ in $\mathcal{C}$, such that the following diagram commutes:



Obviously, morphisms of spans can be composed, there is an identity morphism for every span and the composition is associative. Therefore, for every two objects, A, B of $\mathcal{C}$, there is a category with objects the corresponding spans, and morphisms the morphisms of spans. This, will play the role of the hom-category in a bicategory of spans to be defined subsequently, but firstly note that we may view

For two spans $s_1 = (X_1, f_1, g_1) : A \rightsquigarrow B$ and $s_2 = (X_2, f_2, g_2) : B \rightsquigarrow C$ to be composed, one forms the pullback[8] of f_2 along g_1[9]. Therefore, the composite, denoted by $s_2 * s_1 : A \rightsquigarrow C$, is given by $(X_1 \times_B X_2, f_1 \circ \pi_{X_1}, g_2 \circ \pi_{X_2})$. We can extend this construction to incorporate also a horizontal composition of morphisms of spans as follows. Let $s_1' = (X_1', f_1', g_1') : A \rightsquigarrow B$ and $s_2' = (X_2', f_2', g_2') : B \rightsquigarrow C$ also be spans and let $h_1 : X_1 \to X_1'$, $h_2 : X_2 \to X_2'$ be morphisms of spans. Since $(X_1 \times_B X_2, h_1 \circ \pi_{X_1}, h_2 \circ \pi_{X_2})$ is a cone over the red diagram below, by the universal property of pullbacks, there exists a unique morphism of spans $h_1 * h_2 : s_2 * s_1 \to s_2' * s_1'$ such that everything in the following diagram commutes:



The associator, for the $*$ composition mentioned above, is well defined and satisfies the pentagon law, since pullbacks are universal constructions and thus the components of the associator are uniquely defined. Furthermore, there is an identity span for every object $A \in \mathcal{C}_0$ defined as $I_A := (A, \mathrm{id}_A, \mathrm{id}_A)$. In general, this is not a strict unit for the composition of spans, due to the fact that the pullbacks used to define the compositions $I_A * s$ and $s' * I_A$, where $s = (X, f, g) : C \rightsquigarrow A$ and $s' = (X', f', g') : A \rightsquigarrow B$ need not equal $(X, f \circ \mathrm{id}_X, \mathrm{id}_A \circ g)$ and $(X', \mathrm{id}_A \circ f', g' \circ \mathrm{id}_{X'})$ accordingly. But, by 1.5.26, $I_A * s \cong s$ and $s' * I_A \cong s'$, and these isomorphisms provide the uniquely defined components of the unitors and

[8] Pullbacks are described as examples of limits in the appendix of this thesis.

[9] For this to be well defined we need to pick a specific pullback and the uniqueness up to isomorphism property will take care of everything else.

uniqueness implies that the triangle laws hold. One crucial thing to note is that this construction can be realised only inside a category $\mathcal{C}$ having all pullbacks and the resulting bicategory is denoted by $Span(\mathcal{C})$. The underlying category of such a bicategory is sometimes also called *the category of spans in* $\mathcal{C}$ and is denoted by $Span(\mathcal{C})$. Note that we may view **Rel** as a subcategory of $Span(\mathbf{Set})$.

5. As we already mentioned in remark 2.2.13 the categories $\mathbf{MonCat}_{\mathrm{co}}$, $\mathbf{MonCat}_{\mathrm{l}}$, **MonCat** and $\mathbf{MonCat}_{\mathrm{s}}$ are actually 2-categories, since composition of all the variants of monoidal functors is strictly associative and strictly unital. These are actually sub-2-categories of the 2-category **Cat**.

Example 2.3.6. Now we give examples of abstract bicategories.

1. A bicategory $\mathcal{B}$ which is locally discrete is just a category. Local discreteness forces coherence maps to be identities, so this is also an example of a 2-category.

2. Let $\mathcal{B}$ be a bicategory with only one 0-cell, denoted by $*$. Then, the hom-category $\mathcal{B}(*,*)$ equipped with all the structure of compositions and coherence 2-cells, is a monoidal category $(\mathcal{B}(*,*), \circ, I_*, a, l, r)$. As a category is a many objects version of a monoid, considered as a one object category, so a bicategory is a many 0-cells version of a monoidal category. To support this claim, given a monoidal category $\mathcal{M}$ we can formally construct a one 0-cell bicategory $\mathbf{B}\mathcal{M}$ as follows. For every object $A \in \mathcal{M}_0$ there is a 1-cell $f_A \in \mathbf{B}\mathcal{M}(*,*)$, for every morphism $g \in \mathcal{M}(A,B)$ there is a 2-cell $\alpha_g \in \mathbf{B}\mathcal{M}(*,*)(f_A, f_B)$, such that $f_I := I_*$, $f_A \circ f_B := f_{A\otimes B}$ and $\alpha_g \circ \alpha_h := \alpha_{g\otimes h}$ for A, B objects of $\mathcal{M}$ and g, h morphisms in $\mathcal{M}$. According to the above, the tensor product functor of the monoidal category provides the horizontal composition of 1-cells and 2-cells of the corresponding bicategory and the composition of morphisms in the monoidal category provides the vertical composition of 2-cells in the bicategory. Finally, the coherence maps of the monoidal category, $\mathcal{M}$, already satisfying the pentagon and triangle laws, provide the coherence 2-cells of the bicategory, whose pentagon and triangle law are the ones provided by $\mathcal{M}$. From this construction it is clear that $\mathcal{M}$ and $\mathbf{B}\mathcal{M}(*,*)$ are strictly monoidally isomorphic.

   For an extension of the above, one might associate to every 0-cell of a general bicategory $X \in \mathcal{B}_0$ a monoidal category $\mathcal{B}(X,X)$. Such a monoidal category will, somewhat informally, be called a **hom-endocategory**. Another perspective on this fact is that as categories can be viewed as partial monoids, so bicategories can be viewed as partial monoidal categories.

   Since internal monoids live inside monoidal categories, we could have an internal monoid in a hom-endocategory $\mathcal{B}(A,A)$, for any 0-cell A in a bicategory $\mathcal{B}$. Such a monoid is called a **monad** on A. Similarly, an internal comonoid in $\mathcal{B}(A,A)$ is called a **comonad** on A. Finally, a Frobenius object inside $\mathcal{B}(A,A)$ is called a **Frobenius monad** on A. Such a construction in the 2-category **Cat** yields the ordinary notions for monads, comonads and Frobenius monads.

   Finally, a bicategory with only one 0-cell and one 1-cell is essentially a monoidal category with only one object, the unit. Thus, it is a commutative monoid according to 2.1.36.

3. A very usefull example of an abstract bicategory is the bicategory $\mathbf{1}$. This is a bicategory with one 0-cell, one 1-cell which is an identity and one 2-cell, again an identity. This bicategory is the terminal monoidal category, in the sense of the previous example, and is called the **terminal bicategory** for reasons described in the following subsection about lax functors. Moreover, the only 1-cell of this bicategory is a Frobenius object inside $\mathbf{1}(*,*)$, where $*$ denotes the single 0-cell of $\mathbf{1}$.

Properties of Bicategories and Dual Bicategories

In what follows we give properties of coherence 2-cells for any bicategory $\mathcal{B}$. These properties and their proofs are completely analogous to the ones given for monoidal categories, so, under the identification of monoidal categories with "hom-endocategories" of 0-cells, these properties are also more general.

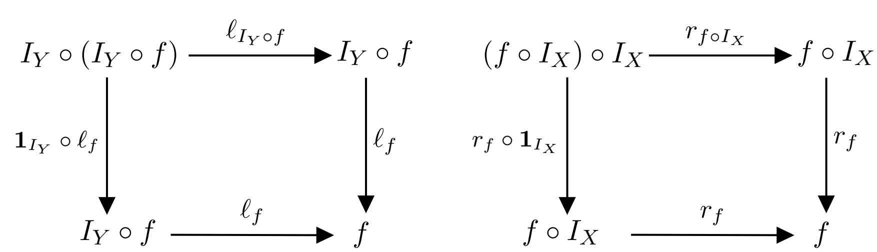
Lemma 2.3.7. *Let X, Y be 0-cells of a bicategory $\mathcal{B}$. Then the functors $-\circ \mathbf{1}_{I_X}, \mathbf{1}_{I_Y} \circ - : \mathcal{B}(X,Y) \to \mathcal{B}(X,Y)$ are equivalences.*

Proof. The right and left unitors in $\mathcal{B}(X,Y)$ are natural isomorphisms, so that $- \circ \mathbf{1}_{I_X} \cong \mathbb{1}_{\mathcal{B}(X,Y)} \cong \mathbf{1}_{I_Y} \circ -$. This implies that $- \circ \mathbf{1}_{I_X}$ and $\mathbf{1}_{I_Y} \circ -$ are both fully faithful and essentially surjective on objects (1-cells), therefore equivalences. □

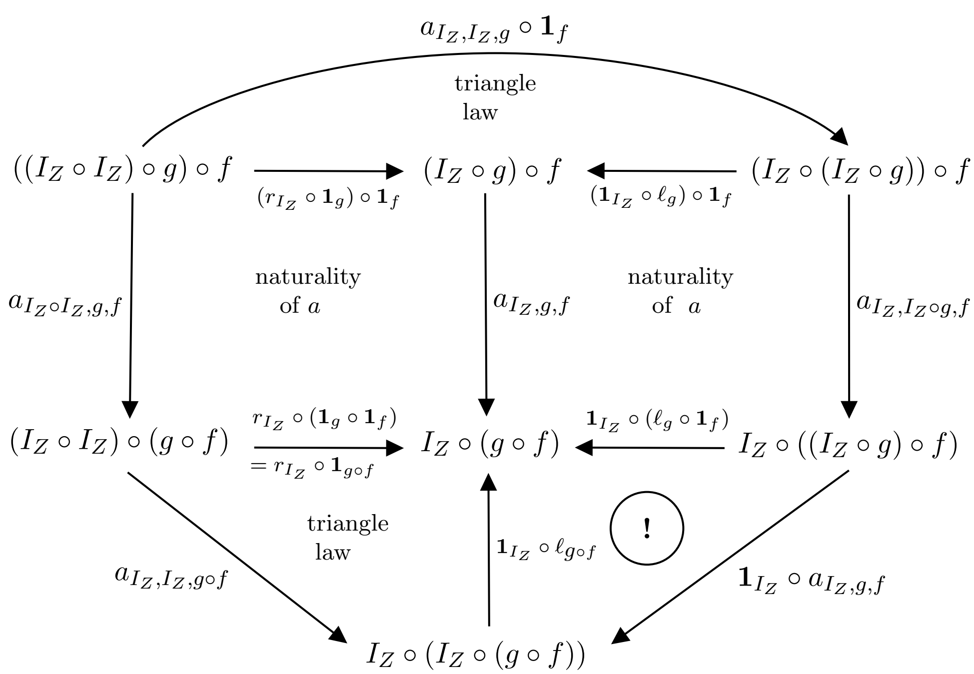
Proposition 2.3.8. *Let $\mathcal{B}$ be a bicategory. For 0-cells $X, Y, Z \in \mathcal{B}_0$ and 1-cells $f : X \to Y$, $g : Y \to Z$ the following equalities of 2-cells hold:*

1. $r_{f \circ I_X} = r_f \circ \mathbf{1}_{I_X}$ *and* $\ell_{I_Y \circ f} = \mathbf{1}_{I_Y} \circ \ell_f$ *inside* $\mathcal{B}(X,Y)$,
2. $\ell_g \circ \mathbf{1}_f = \ell_{g\circ f} \cdot a_{I_Z,g,f}$ *and* $r_{g \circ f} = (\mathbf{1}_g \circ r_f) \cdot a_{g,f,I_X}$ *inside* $\mathcal{B}(X,Z)$.

Proof. 1. Both of these equalities follow from the following naturality squares, shown below, for the right and left unitors and the fact that they are isomorphisms.

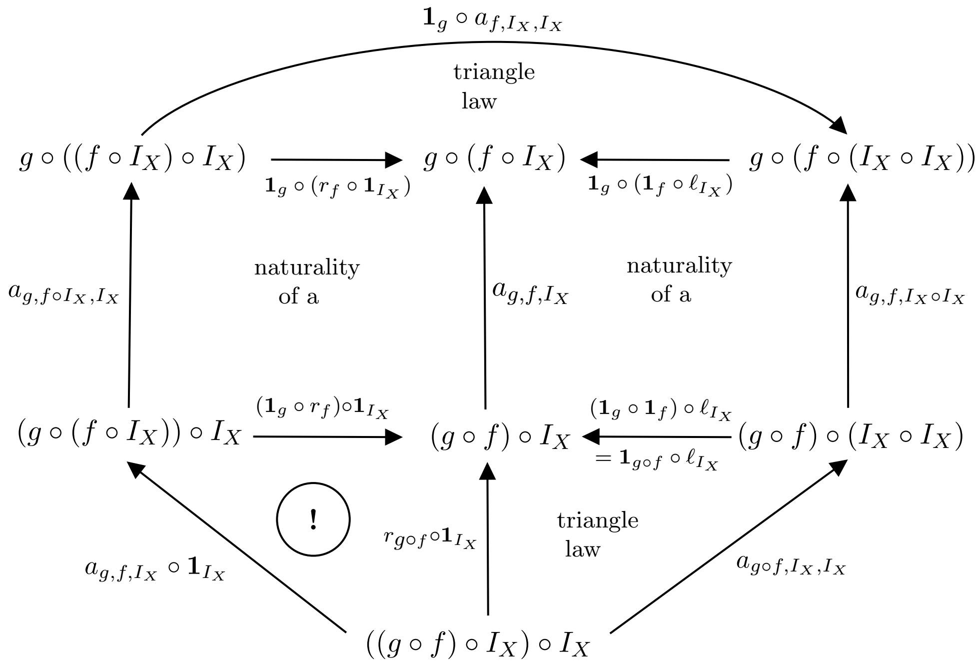


2. To prove the first equation, we will prove that it holds in the image of the equivalence $\mathbf{1}_{I_Z} \circ -$.



The outer diagram above commutes, as it represents the pentagon law, the inner diagrams commute by naturality of the associator and the triangle law, thus the exclamation mark diagram also commutes, therefore the first equation holds in the image of the equivalence $\mathbf{1}_{I_Z} \circ -$ and thus it holds.

Similarly we will prove that the second equation holds in the image of the equivalence $- \circ \mathbf{1}_{I_X}$.



Again, the outer diagram above is an instance of the pentagon law, the inner diagrams are instances of naturality of the associator or the triangle law, thus the exclamation mark diagram commutes, which is the image under $- \circ \mathbf{1}_{I_X}$ of the second equation.

□

Even though the above properties don't follow from the corresponding ones in monoidal categories, there is one that does. Namely that the unitors' components at the identity coincide. This is because every "hom-endocategory" is a monoidal category. Therefore, for every 0-cell X in a bicategory $\mathcal{B}$,

$$r_{I_X} = \ell_{I_X} =: i_{I_X}.$$

In a bicategory there are two types of "arrows", 1-cells and 2-cells. Therefore, there are two independent ways to "reverse" them, thus creating two different types of opposite bicategories called $\mathcal{B}^{\text{co}}$ and $\mathcal{B}^{\text{op}}$. There is also a combination of these notions called $\mathcal{B}^{\text{coop}}$. To proceed with the definitions we denote the symmetric braiding of the cartesian monoidal **CAT** as $c_{\mathcal{C},\mathcal{D}}$.

Definition 2.3.9. *Let $\mathcal{B} = (\mathcal{B}_0, \circ, I_-, a, l, r)$ be a bicategory.*

1. *The **opposite bicategory** $\mathcal{B}^{\text{op}} = ({\mathcal{B}^{\text{op}}}_0, {}^{\text{op}}\circ, {}^{\text{op}}I_-, {}^{\text{op}}a, {}^{\text{op}}l, {}^{\text{op}}r)$ is the bicategory having:*
   - *as 0-cells ${\mathcal{B}^{\text{op}}}_0 := \mathcal{B}_0$*
   - *as hom-categories $\mathcal{B}^{\text{op}}(X,Y) := \mathcal{B}(Y,X)$, for every $X, Y \in \mathcal{B}^{\text{op}}_0$, with the vertical composition of 2-cells given by the one in $\mathcal{B}(Y,X)$,*
   - *horizontal composition functors ${}^{\text{op}}\circ : \mathcal{B}^{\text{op}}(Y,Z) \times \mathcal{B}^{\text{op}}(X,Y) \to \mathcal{B}^{\text{op}}(X,Z)$ given by the composite*

     $${}^{\text{op}}\circ : \mathcal{B}(Z,Y) \times \mathcal{B}(Y,X) \xrightarrow{c_{\mathcal{B}(Z,Y),\mathcal{B}(Y,X)}} \mathcal{B}(Y,X) \times \mathcal{B}(Z,Y) \xrightarrow{\circ} \mathcal{B}(Z,X) \equiv \mathcal{B}^{\text{op}}(X,Z),$$

     for every 0-cells $X, Y, Z \in \mathcal{B}^{\text{op}}_0$,
   - *as identity 1-cells ${}^{\text{op}}I_X := I_X \in \mathcal{B}^{\text{op}}(X,X) \equiv \mathcal{B}(X,X)$, for every $X \in \mathcal{B}^{\text{op}}_0$,*
   - *an associator given componentwise as*

     $${}^{\text{op}}a_{h,g,f} := a^{-1}_{f,g,h} : (h \; {}^{\text{op}}\circ \; g) \; {}^{\text{op}}\circ \; f = f \circ (g \circ h) \to (f \circ g) \circ h = h \; {}^{\text{op}}\circ \; (g \; {}^{\text{op}}\circ \; f),$$

     for every composable 1-cells f, g, h,

- *a left unitor with components*

$$^{\text{op}}l_f := r_Y : {}^{\text{op}}I_Y \; {}^{\text{op}}\!\circ f = f \circ I_Y \to f$$

and a right unitor with components

$$^{\text{op}}r_f := \ell_Y : f \; {}^{\text{op}}\!\circ \; {}^{\text{op}}I_X = I_X \circ f \to f,$$

for every $X, Y \in \mathcal{B}^{\text{op}}{}_0$ *and* $f \in \mathcal{B}^{\text{op}}(X, Y)$.

2. *The* ***co-bicategory*** $(\mathcal{B}^{\text{co}}, {}^{\text{co}}\!\circ, {}^{\text{co}}I_{-}, {}^{\text{co}}a, {}^{\text{co}}l, {}^{\text{co}}r)$ *is the bicategory having*

   - *as 0-cells* $\mathcal{B}^{\text{co}}{}_0 := \mathcal{B}_0$
   - *as hom-categories* $\mathcal{B}^{\text{co}}(X, Y) = \mathcal{B}(X, Y)^{\text{op}}$*, for every* $X, Y \in \mathcal{B}^{\text{co}}$,
   - *horizontal composition functors* ${}^{\text{co}}\!\circ : \mathcal{B}^{\text{co}}(Y, Z) \times \mathcal{B}^{\text{co}}(X, Y) \to \mathcal{B}^{\text{co}}(X, Z)$ *are given by*

$$^{\text{co}}\!\circ := \circ^{\text{op}} : \mathcal{B}^{\text{co}}(Y, Z) \times \mathcal{B}^{\text{co}}(X, Y) \to \mathcal{B}^{\text{co}}(X, Z),$$

   for every $X, Y, Z \in \mathcal{B}^{\text{co}}{}_0$*, where "* $\circ^{\text{op}}$ *" is the opposite covariant functor,*

   - *and coherence isomorphisms given componentwise as*

$$^{\text{co}}a_{h,g,f} := (a^{-1}_{h,g,f})^{\text{op}},$$

$$^{\text{co}}l_f := (\ell^{-1}_f)^{\text{op}} \text{ and } {}^{\text{co}}r_f := (r^{-1}_f)^{\text{op}},$$

   for every composable 2-cells h, g, f*, where the superscript* "op" *denotes the reversed morphism in the opposite category.*

It is almost trivial checking that the above two definitions, do give well defined bicategories. In both cases the pentagon and triangle laws are satisfied because they are satisfied in the original bicategory.

Remark 2.3.10.

1. In broad terms, the opposite bicategory, has inverted 1-cells but not 2-cells. The co-bicategory has inverted 2-cells but not 1-cells. Furthermore, one can easily check that $(\mathcal{B}^{\text{op}})^{\text{op}} = \mathcal{B}$ and $(\mathcal{B}^{\text{co}})^{\text{co}} = \mathcal{B}$. Therefore, there are two distinct notions of duality for bicategories.

2. There is also a third dual bicategory, of a bicategory $\mathcal{B}$, called the **opposite co-bicategory** and as the name suggests it is defined as $\mathcal{B}^{\text{coop}} := (\mathcal{B}^{\text{co}})^{\text{op}} = (\mathcal{B}^{\text{op}})^{\text{co}}$.

3. The above constructions of duals when performed to a 2-category, yields again a 2-category. Therefore, there are also three dual 2-categories to a 2-category.

Example 2.3.11.

1. In the case of a monoidal category, $\mathcal{M}$, seen as a one 0-cell bicategory $\mathbf{B}M$, the co-bicategory $\mathbf{B}M^{\text{co}}$ is the opposite monoidal category $\mathcal{M}^{\text{op}}$.

2. Similarly, the opposite bicategory of a one 0-cell bicategory, is just the reverse monoidal category.

3. Let $\mathcal{C}$ be a category with pullbacks. Then as we already saw above $Span(\mathcal{C})$ is the bicategory of spans in $\mathcal{C}$. The dual notion of a pullback is a pushout, so $\mathcal{C}^{\text{op}}$ is a category with pushouts. Dualizing the construction of a span we get a **cospan**. The horizontal composition of cospans is given by a pushout and the vertical composition of morphisms of cospans is given by the universal property of pushouts. So there is a bicategory $Cospan(\mathcal{C}^{\text{op}})$, which is isomorphic, in a sense described below, to $Span(\mathcal{C})^{\text{co}}$. This is because to every span in $\mathcal{C}$ there is a unique cospan in $\mathcal{C}^{\text{op}}$ and such cospans compose in the same direction as the spans they originate from. Actually, span and cospan categories are dagger categories, so the direction of composition can be chosen freely. Furthermore, morphisms of spans in $\mathcal{C}$ get reversed as morphisms of cospans in $\mathcal{C}^{\text{op}}$, so this is where the source of $Span(\mathcal{C})^{\text{co}} \cong Cospan(\mathcal{C}^{\text{op}})$ lies in.

Internal equivalences and adjunctions

In this subsection, we study what invertibility would mean for a 1-cell in a bicategory. Apart from the isomorphism case already manifest in categories, there is also a notion of weak inverse for a 1-cell, which is possible to define by the existence of 2-cells. This is terribly similar to the case of equivalence between categories, which in turn is similar to adjunctions. All these connections to basic category theory lead to the idea of internal notions inside a bicategory or a 2-category.

Definition 2.3.12. *Let $\mathcal{B}$ be a bicategory and $f, g \in \mathcal{B}(X,Y)$ be 1-cells. We say that f and g are isomorphic as 1-cells and write $f \cong g$, if there exists an invertible 2-cell $\eta : f \Rightarrow g$.*

An example of isomorphic 1-cells is the case of two functors, between ordinary categories, being naturally isomorphic. Obviously, associators and unitors in a bicategory make specific 1-cells isomorphic.

Definition 2.3.13. *Let $\mathcal{B}$ be a bicategory and $f \in \mathcal{B}(X,Y)$ be a 1-cell. We say that f is*

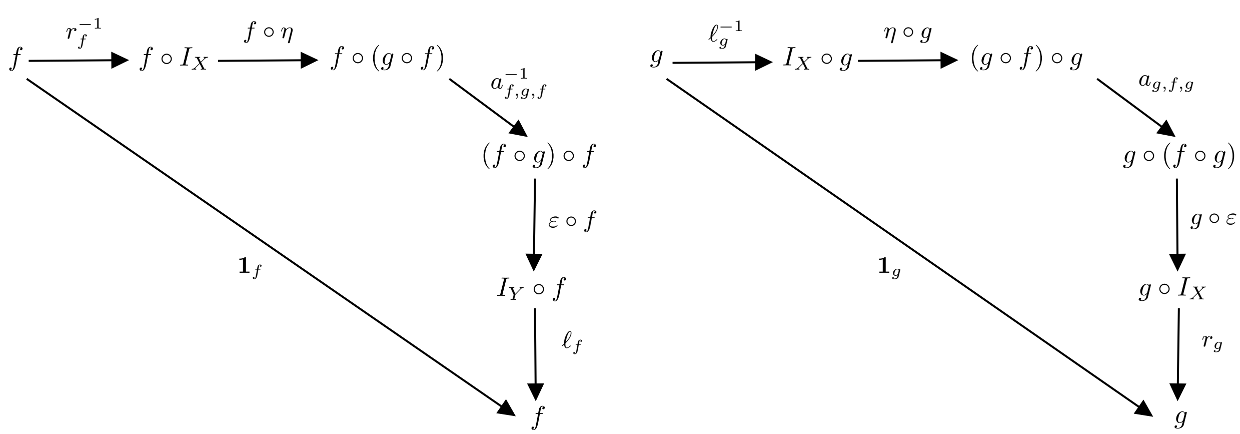
1. *an* ***isomorphism****, if there exists a 1-cell $f^{-1} \in \mathcal{B}(Y,X)$, such that*

$$f \circ f^{-1} = I_Y \text{ and } f^{-1} \circ f = I_X.$$

   In this case f^{-1} is called a ***strict inverse****, or just an* ***inverse*** *of f.*

2. *an* ***internal equivalence*** *between the 0-cells X and Y, if there exists a 1-cell $g \in \mathcal{B}(Y,X)$ and two invertible 2-cells $\eta : I_X \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow I_Y$. In this case f is called a* ***weak inverse*** *of g and vice versa.*

3. *a* ***map****, or a* ***left adjoint****, if there is a 1-cell $g \in \mathcal{B}(Y,X)$ and two 2-cells $\eta : I_X \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow I_Y$, such that the following diagrams, called* ***triangle identities****, commute inside $\mathcal{B}(X,Y)$ and $\mathcal{B}(Y,X)$ respectively.*



   In this case g is called a ***right adjoint****, $(f, g, \eta, \varepsilon)$ is said to form an* ***internal adjunction*** *and η is called the* ***unit of the adjunction****, while ε is called the* ***counit of the adjunction****.*

Remark 2.3.14. In general, given an internal equivalence $f \in \mathcal{B}(X,Y)$ inside a bicategory $\mathcal{B}$, a weak inverse, $g \in \mathcal{B}(Y,X)$, is not unique. Actually, not even unique up to unique isomorphism, but it is just unique up to isomorphism. This is due to the fact that, given a second weak inverse $g' \in \mathcal{B}(Y,X)$, the isomorphisms provided by the following two composites

$$g \xRightarrow{\ell_g^{-1}} I_X \circ g \xRightarrow{\eta' \circ g} (g' \circ f) \circ g \xRightarrow{a_{g',f,g}} g' \circ (f \circ g) \xRightarrow{g' \circ \varepsilon} g' \circ I_Y \xRightarrow{r_{g'}} g'$$

and

$$g' \xRightarrow{\ell_{g'}^{-1}} I_X \circ g' \xRightarrow{\eta \circ g'} (g \circ f) \circ g' \xRightarrow{a_{g,f,g'}} g \circ (f \circ g') \xRightarrow{g \circ \varepsilon'} g \circ I_Y \xRightarrow{r_g} g,$$

where $\eta, \eta', \varepsilon, \varepsilon'$ are the corresponding invertible 2-cells, need not be inverses of each other.

This is also the case in 2-categories, although the above composites take a simpler form due to the lack of non-trivial associators and unitors. In contrast to the non-uniqueness of weak inverses in 2-categories, *strict inverses*, i.e. inverses of isomorphisms, are unique in 2-categories, but unique up to isomorphism in bicategories. To show that uniqueness up to isomorphism holds for strict inverses in bicategories let $f \in \mathcal{B}(X,Y)$ be an isomorphism in a bicategory $\mathcal{B}$ and let $g, g' \in \mathcal{B}(Y,X)$ be two inverses. Then,

$$g' \xRightarrow{\ell_{g'}^{-1}} I_X \circ g' = (g \circ f) \circ g' \xRightarrow{a_{g,f,g'}} g \circ (f \circ g) = g \circ I_Y \xRightarrow{r_g} g,$$

so $g' \cong g$.

Example 2.3.15. Obviously, the most characteristic example of an internal equivalence, is the equivalence between ordinary categories inside **Cat**. Another obvious example is the case of **MonCat**. Similarly, an isomorphism, defined in the above way, covers morphisms being isomorphisms in any (1-)category seen as a locally discrete bicategory, but also functors being isomorphisms in **Cat** and monoidal functors in the variants of the category of monoidal categories.

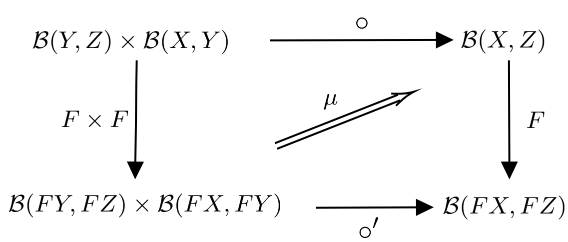
Example 2.3.16. A very interesting example, which differentiates bicategories and 2-categories is witnessed by the identity 1-cell of any object of a bicategory. In 2-categories the identity 1-cell is an isomorphism. On the contrary, identity 1-cells in bicategories are not isomorphisms generally. They rather are internal equivalences since $i_{I_X} : I_X \circ I_X \to I_X$ is only an isomorphism. One object bicategories, a.k.a monoidal categories provide several non trivial examples of this fact.

Lax functors, colax functors and pseudofunctors

Bicategories being structures themselves come with notions of morphisms between them. Since these structures are in a sense higher than basic categories, they allow for different kinds of morphisms between them. Morphisms between monoidal categories, being one 0-cell bicategories, already exhibit such diversity, so the ideas behind monoidal functors are present in the following definitions but the names are slightly different.

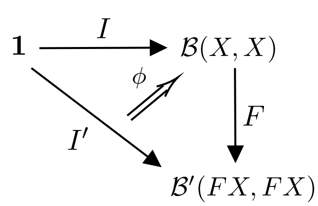
Definition 2.3.17. *Let $(\mathcal{B}, \circ, I_{-}, a, l, r)$ and $(\mathcal{B}', \circ', I'_{-}, a', l', r')$ be bicategories. A **lax functor** from $\mathcal{B}$ to $\mathcal{B}'$ is a triple (F, μ, ϕ) consisting of:*

- *a function $F : \mathcal{B}_0 \to \mathcal{B}'_0$*
- *for every 0-cells $X, Y \in \mathcal{B}_0$ a functor $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX, FY)$ (which strictly preserves vertical composition of 2-cells and identity 2-cells), called a **local functor** of F,*
- *for every $X, Y, Z \in \mathcal{B}_0$ a coherent natural transformation μ of type:*



  *called **compositor** and*

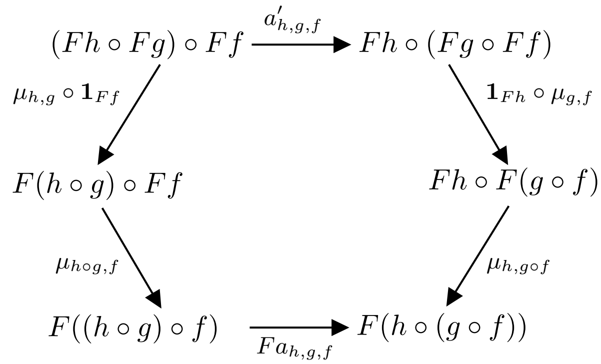
- *for every $X \in \mathcal{B}_0$ a coherent natural transformation ϕ of type:*



called ***unitor,***

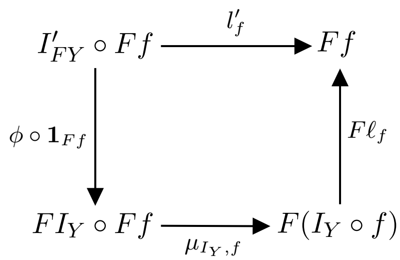
such that:

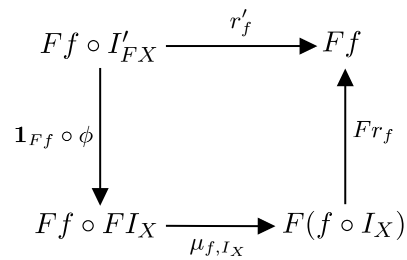
- *for every 1-cells* $f \in \mathcal{B}(X,Y)$, $g \in \mathcal{B}(Y,Z)$, $h \in \mathcal{B}(Z,W)$ *the following diagram commutes in* $\mathcal{B}(FX,FW)$*:*



called ***associativity axiom*** *and*

- *for every 1-cell* $f \in \mathcal{B}(X,Y)$ *the following diagrams commute in* $\mathcal{B}'(FX,FY)$*:*





called ***left*** *and* ***right unitality axioms***

If μ *and* ϕ *are isomorphisms, then* (F,μ,ϕ) *is called a* ***pseudofunctor*** *and if they are identity natural transformations,* (F,μ,ϕ) *is called a* ***strict functor****.*

A lax functor $F : \mathcal{B}^{\text{co}} \to \mathcal{B}'^{\text{co}}$ *is called a* ***colax functor***[10] *from* $\mathcal{B}$ *to* $\mathcal{B}'$*.*

Remark 2.3.18. The above definition, also makes sense in the case of 2-categories, where the coherent isomorphism 2-cells are identities. A strict functor between 2-categories though, is specifically called a **2-functor** or a **strict 2-functor**. Strict 2-functors but also strict functors preserve horizontal composition on the nose, a trait mostly appreciated in the context of 2-categories, hence the special terminology.

Remark 2.3.19. In the above definition there are two redundancies. The first one has to do with the fact that naturality of ϕ is trivial, since it is a natural transformation between constant functors and thus has only one component. The second redundancy is similar to how monoidal functors are defined. The unitality axioms are enough to define ϕ as long as they are compatible. So if the compositor is invertible we may deduce that the (co)lax functor is actually a pseudofunctor.

Remark 2.3.20. It is worth noting that every lax functor $F : \mathcal{B} \to \mathcal{B}'$ can be identified with a colax functor $F^{\text{co}} : \mathcal{B}^{\text{co}} \to \mathcal{B}^{\text{co}\prime}$.

Example 2.3.21. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pullbacks and let $F : \mathcal{C} \to \mathcal{D}$ be a functor that sends pullbacks to pullbacks. Then there is an induced pseudofunctor $F_* : Span(\mathcal{C}) \to Span(\mathcal{D})$, whose coherent 2-cells are provided by the universal property of pullbacks. Furthermore, if the choice of pullbacks is preserved strictly by F, then F_* is a strict functor.

[10]The prefix "op" instead of "co" is sometimes used interchangably, but here we stick to using "co" to distinguish the reversion of 2-cells from that of 1-cells, where we either use "op" as a prefix or "contravariant" as a term.

Example 2.3.22. There is an **identity strict functor** $\mathbb{1}_{\mathcal{B}}$ associated to every bicategory $\mathcal{B}$. This strict functor assigns to every 0-cell, 1-cell and 2-cell itself.

Example 2.3.23. It is easy to check that (co)lax functors between one 0-cell bicategories are actually (co)lax monoidal functors between the corresponding monoidal categories. Similarly, pseudofunctors correspond to strong monoidal transformations and strict functors correspond to strict monoidal transformations.

Interestingly, (co)lax, strong and strict functors between generic bicategories induce a family of (co)lax, strong and strict monoidal functors, parametrized by the 0-cells of the source bicategory, between the respective "hom-endo-categories".

Example 2.3.24. A notion of representable functors is also available for bicategories. To make this precise we firstly need to define post composition functors induced by 1-cells of a bicategory $\mathcal{B}$ and post composition natural transformations induced by 2-cells.

So let $f \in \mathcal{B}(X,Y)$ be a 1-cell and $A \in \mathcal{B}_0$ be a 0-cell. Then $f \circ - : \mathcal{B}(A,X) \to \mathcal{B}(A,Y)$ is a functor. To prove this we need to show that given $g, h, w \in \mathcal{B}(A,X)$, $\theta : h \Rightarrow w$ and $\eta : g \Rightarrow h$, we have

1. $f \circ \mathbf{1}_g = \mathbf{1}_{f \circ g}$ and
2. $f \circ (\theta \cdot \eta) = (f \circ \theta) \cdot (f \circ \eta)$

both of which hold by functoriality of the composition functor " $\circ$ " and the fact that

$$f \circ - := \mathbf{1}_f \circ - = (\mathbf{1}_f \cdot \mathbf{1}_f) \circ -.$$

Furthermore, given a 2-cell $\alpha \in \mathcal{B}(X,Y)(f,g)$, note that $\alpha \circ - : f \circ - \Rightarrow g \circ -$ is a natural transformation. To check this observe that for every 1-cells $h, h' \in \mathcal{B}(A,X)$ and every 2-cell $\eta : h \to h'$ the following naturality square

$$\begin{array}{ccc} f \circ h & \xrightarrow{\alpha \circ \mathbf{1}_h} & g \circ h \\ {\scriptstyle \mathbf{1}_f \circ \eta}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_g \circ \eta} \\ f \circ h' & \xrightarrow{\alpha \circ \mathbf{1}_{h'}} & g \circ h' \end{array}$$

is commutative since

$$\begin{aligned} (\mathbf{1}_g \circ \eta) \cdot (\alpha \circ \mathbf{1}_h) &= (\mathbf{1}_g \cdot \alpha) \circ (\eta \cdot \mathbf{1}_h) \\ &= \alpha \circ \eta \\ &= (\alpha \cdot \mathbf{1}_f) \circ (\mathbf{1}_{h'} \cdot \eta) \\ &= (\alpha \circ \mathbf{1}_{h'}) \cdot (\mathbf{1}_f \circ \eta). \end{aligned}$$

According to the above we can define **corepresentable pseudofunctors**. Let $A \in \mathcal{B}_0$ be a 0-cell, then $(H^A, \mu, \phi) : \mathcal{B} \to \mathbf{CAT}$ is a pseudofunctor defined as follows.

- The image of a 0-cell $X \in \mathcal{B}_0$ is the category $H^A(X) := \mathcal{B}(A,X)$,
- the image of a 1-cell $f \in \mathcal{B}(X,Y)$ is the post composition functor

$$H^A(f) := f \circ - : \mathcal{B}(A,X) \to \mathcal{B}(A,Y),$$

- the image of a 2-cell $\eta : f \to g$ is the post composition natural transformation

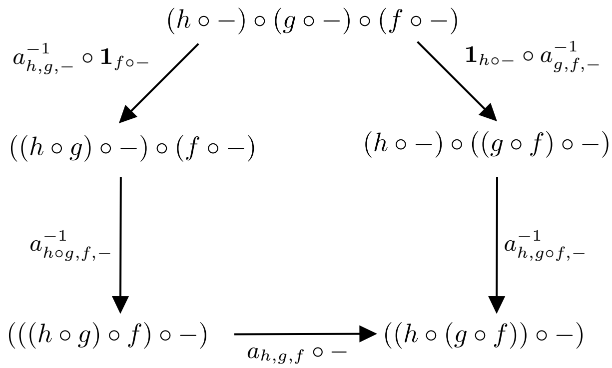
$$H^A(\eta) := \eta \circ - : f \circ - \Rightarrow g \circ -,$$

- for every composable 1-cells $f \in \mathcal{B}(X,Y), g \in \mathcal{B}(Y,Z)$ the compositor's component, $\mu_{g,f}$, in $\mathbf{CAT}$, is given by the natural transformation $a_{g,f,-} : (g \circ -) \circ (f \circ -) \Rightarrow (g \circ f) \circ -$, where a is the associator of $\mathcal{B}$,

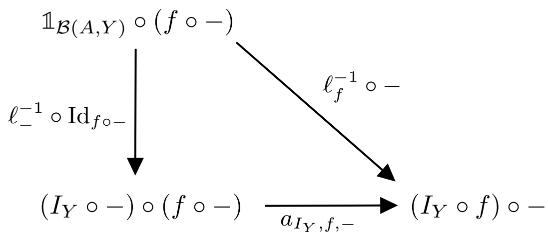
- for every 0-cell $X \in \mathcal{B}_0$, the unitor ϕ, in **CAT**, is given by the natural transformation

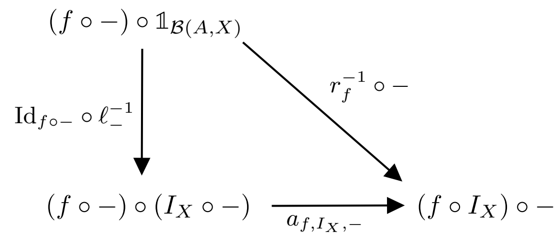
$$\ell_{-}^{-1} : \mathbb{1}_{\mathcal{B}(A,X)} \Rightarrow I_X \circ -.$$

To show that $\mathcal{B}(A,-)$ is a pseudofunctor, observe that **CAT** is a 2-category, so the associativity hexagon is actually a pentagon and the unitality squares are triangles. So for every composable 1-cells f, g, h in $\mathcal{B}$, the following diagrams of natural transformations:



and





commute since their components are equivalent to the pentagon law of $\mathcal{B}$, proposition 2.3.8 and the triangle law, respectively.

Corepresentable pseudofunctors are defined for every 0-cell of a bicategory $\mathcal{B}$. So in the case of a one 0-cell bicategory $\mathbf{B}\mathcal{M}$, a.k.a. a monoidal category $\mathcal{M}$, there is only one corepresentable pseudofunctor, i.e. strong monoidal functor, namely $\mathbf{B}\mathcal{M}(*,-)$. This pseudofunctor is the one defined as a step towards proving the strictification theorem in subsection 2.2, given as $L = (F_{-}, \gamma_{-}) : \mathcal{M} \to \text{Ł}$. This connection shows that there might be a way to generalise the strictification theorem to bicategories, but also that Cayley's theorem and the Yoneda lemma have a common core.

Example 2.3.25. Let $\mathcal{B}$, $\mathcal{B}'$ be locally discrete bicategories. Then a lax functor between them is actually a strict functor, which in turn is a functor between their underlying categories. On the other hand functors between categories induce strict functors between them, considered as locally discrete bicategories. Local discreteness is not the only local property to fit nice with the notion of lax functor.

Definition 2.3.26. *Let $\mathcal{B}, \mathcal{B}'$ be bicategories, and let $F : \mathcal{B} \to \mathcal{B}'$ be a lax functor. Then F is called:*

1. ***locally full on 2-cells*** *or just* ***locally full****, if for every $X, Y \in \mathcal{B}$, the local functor $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX,FY)$ is full*

2. ***locally faithfull on 2-cells*** *or just* ***locally faithful****, if for every $X, Y \in \mathcal{B}$, the local functor $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX,FY)$ is faithful,*

3. *a* ***local isomorphism****, if for every $X, Y \in \mathcal{B}$, the local functor $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX,FY)$ is an isomorphism*

4. ***locally essentially surjective on 1-cells****, if for every $X, Y \in \mathcal{B}$, the local functor $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX,FY)$ is essentially surjective on objects*

5. *a* ***local equivalence****, if for every* $X, Y \in \mathcal{B}$*, the local functor* $F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX,FY)$ *is an equivalence of categories.*

In general, every local property of lax functors has an analogue in ordinary functors. This is obvious by the above definition. A similar example is the case of locally (co)continuous lax functors.

Since different forms of duality are present in the setting of bicategories, there are also different notions of opposite or contravariant functors.

Definition 2.3.27. *Let* $\mathcal{B}, \mathcal{B}'$ *be bicategories. A* ***contravariant lax functor*** *or an* ***oplax functor*** *from* $\mathcal{B}$ *to* $\mathcal{B}'$ *is a lax functor* $\mathcal{B}^{\text{op}} \to \mathcal{B}'$*.*

Remark 2.3.28. Obviously, stronger/stricter versions of colax functors coincide with those of lax functors. That is because colax functors, essentially have reversed coherence 2-cells. On the other hand stronger/stricter versions of contravariant lax functors are just **contravariant pseudofunctors** and **contravariant strict functors**.

Contravariance is a property that doesn't have to do with 2-cells, so there is no restriction in having a **contravariant colax functor** F. Such an $F : \mathcal{B} \to \mathcal{B}'$ should be defined as a lax functor $F : \mathcal{B}^{\text{coop}} \to \mathcal{B}'^{\text{ co}}$, so the inversion of 1-cells wouldn't be interrupted by the inversion of 2-cells captured by returning to the original bicategories from their co-bicategories.

Example 2.3.29. An example of a contravariant pseudofunctor is a **representable pseudofunctor**. This construction is similar to the corepresentable pseudofunctor one. So let $\mathcal{B}$ be a bicategory and $A \in \mathcal{B}_0$. Define $(H_A, \mu, \phi) : \mathcal{B}^{\text{op}} \to \mathbf{CAT}$ as follows:

- for every 0-cell $X \in \mathcal{B}_0$, $H_A(X) := \mathcal{B}(X, A)$,
- the image of a 1-cell $f : Y \to X$ is the obviouslly defined precomposition functor

$$H_A(f) := - \circ f : \mathcal{B}(X, A) \to \mathcal{B}(Y, A),$$

- the image of a 2-cell $\eta \in \mathcal{B}(Y, X)(f, g)$ is the precomposition natural transformation

$$H_A(\eta) := - \circ \eta : - \circ f \Rightarrow - \circ g,$$

- the compositor's components $\mu_{g,f} : H_A(g) \circ H_A(f) \to H_A(f \circ g)$ are defined as:

$$\mu_{g,f} := a_{-,f,g} : (- \circ g) \circ (- \circ f) \to (- \circ (f \circ g)),$$

  where a is the associator of $\mathcal{B}$,
- for every 0-cell $X \in \mathcal{B}_0$, the unitor ϕ is the natural isomorphism

$$\phi := r_-^{-1} : \mathbf{1}_{\mathcal{B}(X,A)} \Rightarrow - \circ I_X.$$

The compositor defined above satisfies the associativity hexagon because **CAT** is a 2-category and because of the pentagon law for $\mathcal{B}$, while the unitor satisfies the unitality laws due to the triangle law of $\mathcal{B}$ and proposition 2.3.8.

Since (co)lax functors and their stricter versions are morphisms of bicategories, there must be a way to compose them. This is the content of the following definition.

Definition 2.3.30. *Let* $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ *be bicategories and let* $(F, \mu, \phi) : \mathcal{B} \to \mathcal{B}'$ *and* $(G, \nu, \psi) : \mathcal{B}' \to \mathcal{B}''$ *be lax functors. Then their* ***composite*** (H, κ, χ) *is defined as:*

- *a function* $H := G \circ F : \mathcal{B}_0 \to \mathcal{B}''_0$*, where* F, G *are the corresponding functions of the lax functors* F *and* G*,*

- *local functors $G \circ F : \mathcal{B}(X,Y) \to \mathcal{B}''(GFX, GFY)$, for every $X, Y \in \mathcal{B}_0$, where F, G are the corresponding local functors of F and G,*
- *2-cells $\kappa_X : I_{GFX} \to GFI_X$ given by the composite:*

$$I_{GFX} \xrightarrow{\psi_{FX}} GI_{FX} \xrightarrow{G\phi_X} GFI_X$$

  as unitors, for every $X \in \mathcal{B}_0$ and
- *2-cells $\chi_{g,f} := G\mu_{g,f} \circ \nu_{Fg,Ff}$, as the components of the compositor, for every $f, g \in \mathcal{B}(X,Y)$.*

Remark 2.3.31. As was the case with lax monoidal functors, the composite of lax functors can easily be seen to be a lax functor. The associativity and unitallity laws hold precisely because they hold for the lax functors being composed.

Remark 2.3.32. Composition of colax functors is very similar to composition of lax functors. The only difference is that the composites defining the coherence 2-cells of a colax functor are exactly the composites defined above with the direction of the arrows reversed.

Pseudofunctors being both lax and colax simultaneously, compose in either of the above two ways, which are in this case equivalent. Obviously, strict functors compose in the above ways trivially.

A thing that is common between morphisms between bicategories and morphisms between monoidal categories, is that to compose a lax (monoidal) and a colax (monoidal) functor, it is necessary for one of the two to be a pseudofunctor (strong monoidal functor).

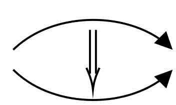
Remark 2.3.33. Since composition of 2-cells in bicategories is strict, as is composition of ordinary functors, it is easy to check that composition of (co)lax functors is strict. That is, every composition needed to define a threefold composite is either a composition of ordinary functors or a composition of 2-cells.

Furthermore, precomposing or postcomposing a (co)lax functor with the identity strict functor of a bicategory yields the (co)lax one we started with on the nose. Therefore, we can create categories of (small) bicategories with either lax or colax functors between them, denoted by $\mathbf{Bicat}_{\mathrm{l}}$ and $\mathbf{Bicat}_{\mathrm{c}}$ respectively. Finally, there are two evident and shared subcategories of the above two categories. Those whose morphisms are pseudofunctors and those that are strict functors, denoted by $\mathbf{Bicat}_{\mathrm{ps}}$ and $\mathbf{Bicat}_{\mathrm{str}}$, respectively.

In exact accordance with the case of categories of monoidal categories, a global element of a bicategory is either a monad, a comonad or a Frobenius monad, depending on which category of bicategories we are considering, $\mathbf{Bicat}_{\mathrm{l}}$, $\mathbf{Bicat}_{\mathrm{c}}$ or $\mathbf{Bicat}_{\mathrm{ps}}$, respectively.

Transformations

Lax functors, colax functors and their stricter versions share the same idea with ordinary functors between categories. That is they produce images of a bicategory in another bicategory, as functors do for categories. The existence of morphisms in ordinary categories gives ways to relate the "one dimensional" images of two parallel functors in the target category. These ways are, of course, the natural transformations. Since, the building blocks of the images of functors are the images of arrows, natural transformations are defined through commutative squares. In the case of bicategories the images of (co)lax functors are "two dimensional", due to the existence of 2-cells. In addition, the building blocks of the images of these higher functors are now "tear-shaped" and look like this:



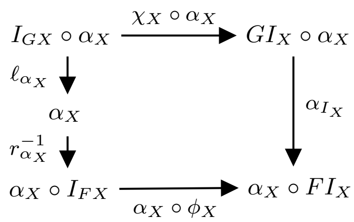
The analogues of commutative squares in this case are "commutative tubes" satisfying further commutativity properties.

Definition 2.3.34. *Let $\mathcal{B}, \mathcal{B}'$ be bicategories and let $(F, \mu, \phi), (G, \nu, \chi) : \mathcal{B} \to \mathcal{B}'$ be lax functors. A **lax transformation**, $\alpha : F \Rightarrow G$, consists of:*

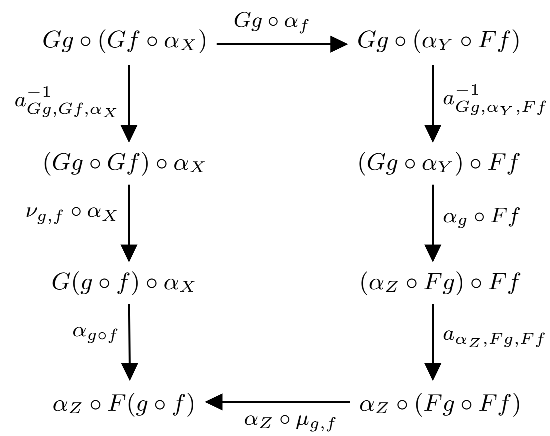
- *a family of 1-cells, $\{\alpha_X : FX \to GX\}_{X\in\mathcal{B}_0}$ parametrized by the 0-cells of $\mathcal{B}$,*
- *a family of 2-cells, $\{\alpha_f : Gf \circ \alpha_X \Rightarrow \alpha_Y \circ Ff\}_{f\in\mathcal{B}(X,Y)}$ natural in f, for every $X, Y \in \mathcal{B}_0$,*

satisfying the following coherence axioms:

1. *for every $X \in \mathcal{B}_0$ the following diagram, called **unital law**, commutes in $\mathcal{B}(FX, GX)$*

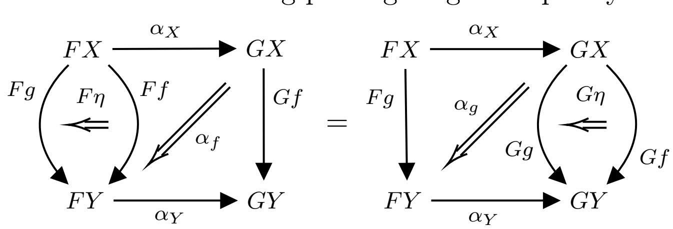


2. *for every $X, Y, Z \in \mathcal{B}_0$, $f \in \mathcal{B}(X, Y)$ and $g \in \mathcal{B}(Y, Z)$ the following diagram, called **composition law**, commutes in $\mathcal{B}'(FX, GZ)$*

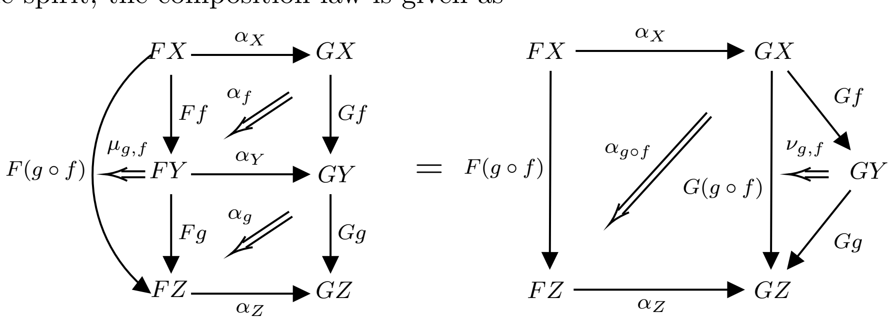


*If the component 2-cells are isomorphisms, the transformation is called **strong**. If the component 2-cells are identity 2-cells the transformation is called **strict**.*

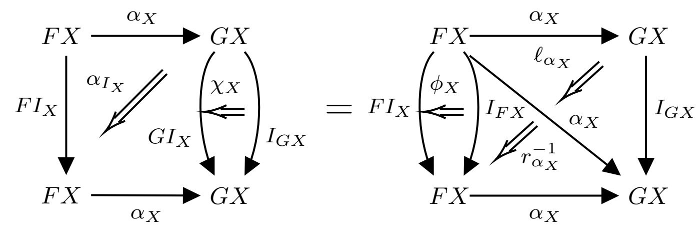
Remark 2.3.35. Since the above definition might seem a little cryptic, we give the pasting diagram version of the above conditions, hoping the reader might be aided in visualising them as embedded in a 3D space. The naturality condition amounts to the following pasting diagram equality:



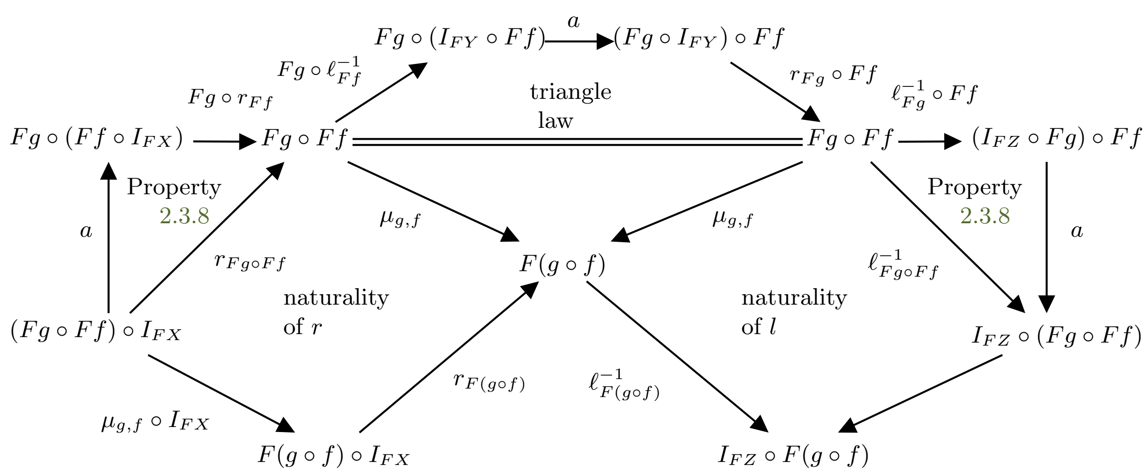
where sides should be considered as those of a "commutative cube", whose edges are glued along the common 1-cells. In the same spirit, the composition law is given as



and the unital law is the equation



Note that for every lax functor $F : \mathcal{B} \to \mathcal{B}'$, there is a strong transformation denoted by I_F whose component 1-cells are $(I_F)_X := I_{FX}$, for every $X \in \mathcal{B}_0$ and whose component 2-cells are $(I_F)_f := l_{Ff}^{-1} \cdot r_{Ff}$, for every $f \in \mathcal{B}(X,Y)$. Indeed, naturality holds since the composition of unitors is natural in f. The unital law holds due to the unitality condition for (F,μ,ϕ), the interchange law and the coincidence of the unitors at every identity 1-cell. The composition law is equivalent to the commutativity of the outer diagram below.

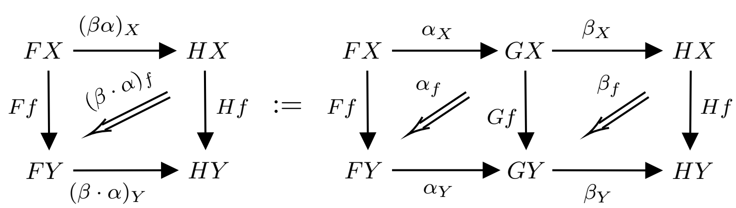


This strong transformation is called the **identity transformation** of F. Note that this transformation is strict in case $\mathcal{B}'$ is a 2-category, but generally it is only guaranteed to be strong due to the nontrivial contribution of the coherence isomorphisms of $\mathcal{B}'$. Note also that the identity transformation of the identity strict endofunctor is the strong transformation $\{(I_X)_{X\in\mathcal{B}_0}, (l_f^{-1} \circ r_f)_{f\in\mathcal{B}_1}\}$.

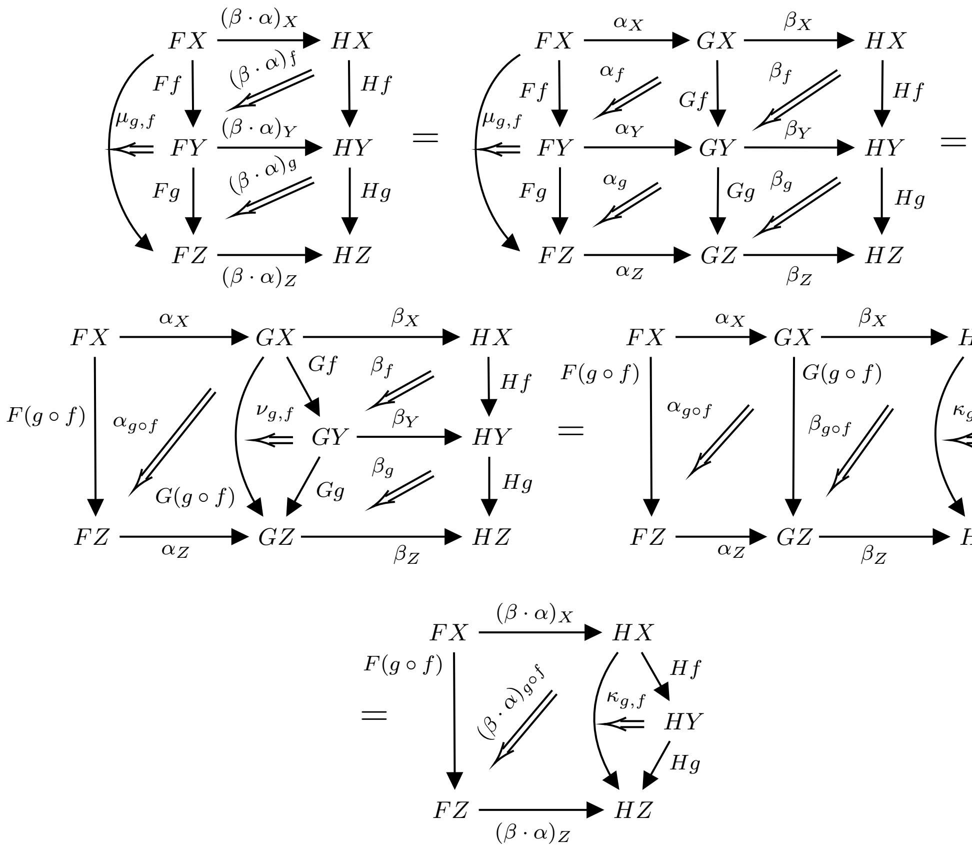
Now, let $(F,\mu,\phi), (G,\nu,\chi), (H,\kappa,\psi) : \mathcal{B} \to \mathcal{B}'$ be parallel lax functors and let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be lax transformations. The family of 1-cells $\{\beta_X \circ \alpha_X\}_{X\in\mathcal{B}_0}$ together with the family of 2-cells of the form

$$Hf \circ (\beta_X \circ \alpha_X) \xrightarrow{a^{-1}} (Hf \circ \beta_X) \circ \alpha_X \xrightarrow{a\cdot(\beta_f \circ \alpha_X)} \beta_Y \circ (Gf \circ \alpha_X) \xrightarrow{\beta_Y \circ \alpha_f} \beta_Y \circ (\alpha_Y \circ Ff) \xrightarrow{a^{-1}} (\beta_Y \circ \alpha_Y) \circ Ff$$

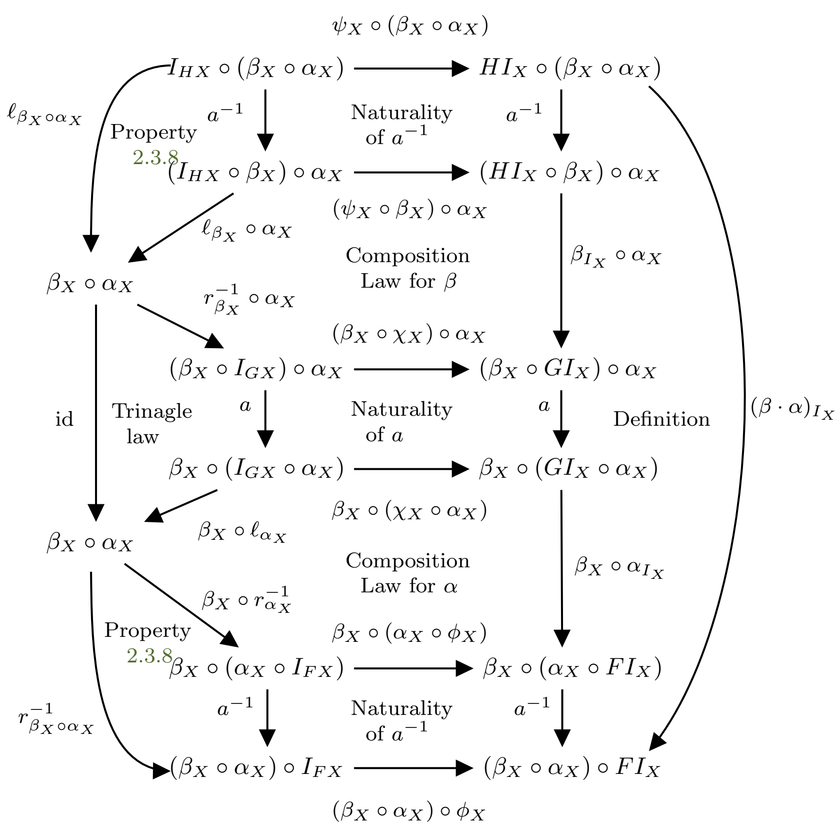
where $X, Y \in \mathcal{B}_0$ and $f \in \mathcal{B}(X,Y)$, constitute a lax transformation called the **vertical composition of** α and β, denoted by $\beta \cdot \alpha$. The component 1-cells and 2-cells look as follows:



for $X, Y \in \mathcal{B}_0$ and $f : X \to Y$. The naturality condition holds since $(\beta \cdot \alpha)_f$ is a vertical composition of the natural transformations α_f, β_f and the associator or its inverse. The composition law follows from the pasting diagram equalities below.



for every $X, Y, Z \in \mathcal{B}_0$, $f : X \to Y, g : Y \to Z$, and the unital law holds since the following diagram commutes for every $X \in \mathcal{B}_0$.



Chapter 3

Flavours of Monoidal Categories and Dagger Structures

In this chapter we investigate in more depth certain types of monoidal categories. These in general fall under two different families. The first family has to do with commutativity, and its different forms, of the tensor product. This family contains braided, balanced and symmetric monoidal categories. The second family is about exponentials and duality, which are different ways an adjunction or an indexed family of adjunctions might present itself. In general all these kinds of categories are closed, some of them are rigid, some of them are pivotal and in all of them an internal version of Cayley's theorem holds. Combining the existence of duals with the existence of braiding, we obtain richer kinds of categories such as ribbon or compact closed categories. Finally, there is a third kind of structure that is independent of the monoidal product and is given by an identity-on-objects contravariant involution called a dagger category. Monoidal categories that combine some or all of the above structures are braided dagger categories, dagger ribbon categories, dagger compact categories and so on.

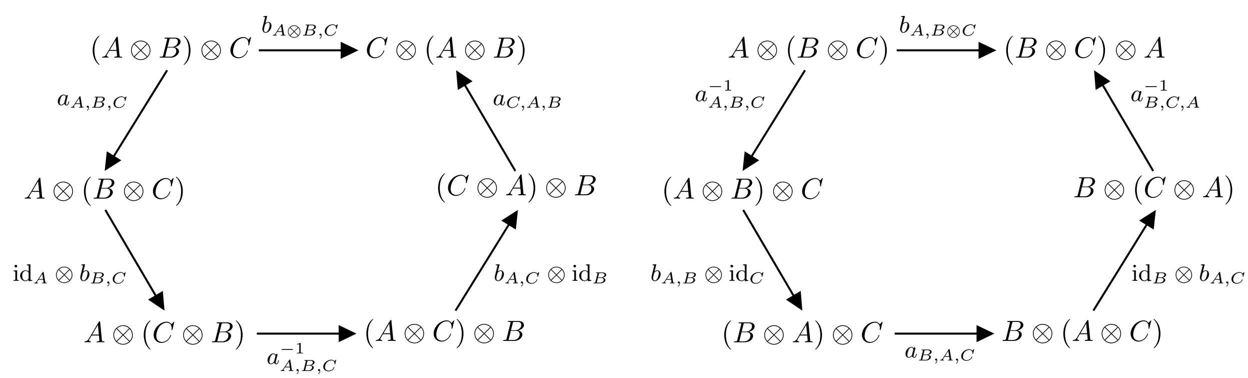
Key Sources: [Abr09], [EGNO15], [TV17], [Fox76], [HV19], [JS86], [JS88], [JS91], [JS95], [ML98], [Sel10], [Str04], [TV17]

3.1 Braided and symmetric monoidal categories

Definition 3.1.1. *Let $(\mathcal{M}, \otimes, I, a, l, r)$ be a monoidal category. A natural isomorphism, b, with components*

$$b_{A,B} : A \otimes B \to B \otimes A$$

is called a ***braiding****, if the following diagrams, called* ***hexagon laws****, commute:*



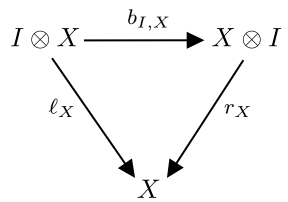
for all $A, B, C \in \mathcal{M}_0$.

Remark 3.1.2. The hexagon laws express $b_{A\otimes B,C}$ and $b_{A,B\otimes C}$ in terms of $b_{A,B}$, $b_{B,C}$ and $b_{A,C}$, for every $A, B, C \in \mathcal{M}_0$. So, if we equip a strict monoidal category with a braiding, the axioms become:

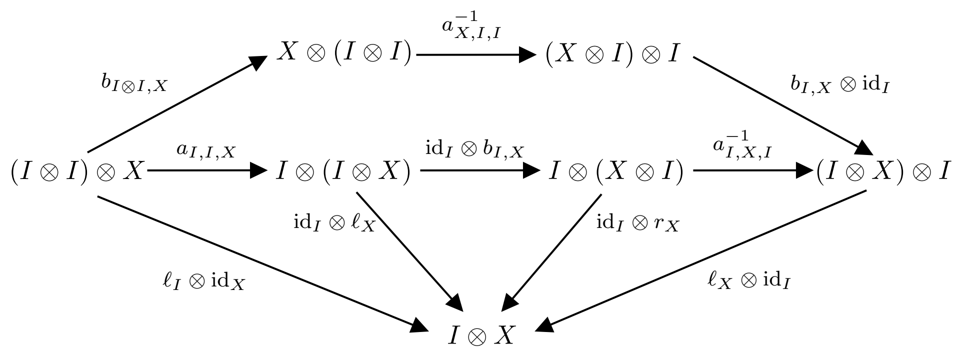
$$b_{A\otimes B,C} = (b_{A,C} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes b_{B,C}) \text{ and } b_{A,B\otimes C} = (b_{A,B} \otimes \mathrm{id}_C) \circ (\mathrm{id}_B \otimes b_{A,C}).$$

The following proposition shows that the data provided by the braiding and one of the unitors is enough to determine the other unitor.

Proposition 3.1.3. *In a braided monoidal category* $(\mathcal{M}, \otimes, I, a, l, r, b)$ *the following diagram commutes for every* $X \in \mathcal{M}_0$*.*



Proof. In the following diagram the upper subdiagram commutes by the hexagon laws, while the bottom left one commutes by 2.1.11 and naturality of l and the bottom right one commutes again by 2.1.11.



Therefore, the bottom middle diagram commutes, which by **??** implies

$$r_X \circ b_{I,X} = \ell_X.$$

□

Remark 3.1.4. The braiding b in a monoidal category $\mathcal{M}$ is a natural isomorphism. So, one can define another braiding by $b'_{A,B} = b^{-1}_{B,A} : A \otimes B \to A \otimes B$, for every $A, B \in \mathcal{M}_0$, and the hexagon laws for b imply the corresponding ones for b'. Since $\beta = b^{-1}$ is a braiding for $\mathcal{M}$, for which proposition 3.1.3 also holds, we get that

$$r_X = \ell_X \circ b_{X,I}.$$

The importance of this observation is that for any braiding of a braided monoidal category we have:

$$b_{X,I} = b^{-1}_{I,X},$$

for all $X \in \mathcal{M}_0$. A direct consequence of this is that given $X, Y \in \mathcal{M}_0$ and a state $\psi : I \to Y$ (Definition 2.1.17) we have:

$$\begin{aligned} b_{X,Y} \circ (\mathrm{id}_X \otimes \psi) &= (\psi \otimes \mathrm{id}_X) \circ b_{X,I} \\ &= (\psi \otimes \mathrm{id}_X) \circ b^{-1}_{I,X} \\ &= b^{-1}_{Y,X} \circ (\mathrm{id}_X \otimes \psi) \end{aligned}$$

Similarly, we could have proven this for a state of X instead of Y, but also for effects of X or Y. As we said earlier, though, the two braidings, b and b', do not coincide at every component. If that happens, we have the following structure.

Definition 3.1.5. *A braided monoidal category $(\mathcal{M}, \otimes, I, a, l, r, b)$ is called* ***symmetric*** *if*

$$b^{-1}_{A,B} = b_{B,A}$$

for every $A, B \in \mathcal{M}_0$. The braiding is then called a ***symmetry****. It is often denoted by c instead of b.*

Remark 3.1.6. The defining condition for a symmetric monoidal category is easily seen to be equivalent to the following condition:

$$c_{A,B} \circ c_{B,A} = \mathrm{id}_{A \otimes B},$$

for every $A, B \in \mathcal{M}_0$.

Example 3.1.7. As we saw in the previous chapter, in every cartesian category the cartesian product is commutative up to canonical isomorphism. This isomorphism provides a braiding. Therefore, both **Set** and **CAT**, being cartesian closed categories, are braided monoidal. This braiding is actually a symmetry and this is evident from the uniqueness up to unique isomorphism part of the universal property of the product. So **CAT** and **Set** are symmetric cartesian monoidal categories.

Example 3.1.8. The categories **Vect** and **Hilb**, equipped with their tensor products, are also braided monoidal categories. Given V, W vector or Hilber spaces and $\{v_i\}_{i \in I}$, $\{w_j\}_{j \in J}$ their corresponding bases, the braiding is the extension of the map defined on the tensor products $v_i \otimes w_j \in V \otimes W$. This map is such that $v_i \otimes w_j \mapsto w_j \otimes v_i \in W \otimes V$. It is easy to check that this map is an isometry between Hilbert spaces and therefore bounded. Now since linear extensions of maps defined on the bases of vector spaces are unique, we deduce that **Vect** and **Hilb** are symmetric monoidal.

With the notion of a symmetry we can speak of commutative monoids and cocommutative comonoids in symmetric monoidal categories.

Definition 3.1.9. *An internal monoid (M, m, u) in a symmetric monoidal category $\mathcal{M}$ is commutative if*

$$m \circ c_{M,M} = m.$$

An internal comonoid (M, d, e) is cocommutative if

$$c_{M,M} \circ d = d.$$

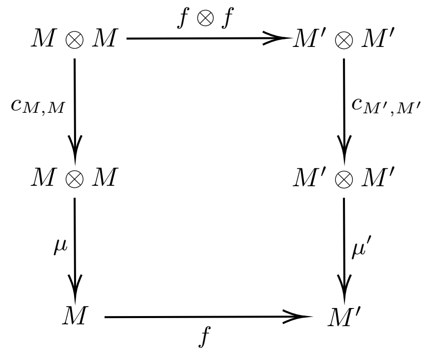
A Frobenius object (M, m, u, d, e) is commutative if its underlying monoid is commutative and its underlying comonoid is cocommutative. A Frobenius object (M, m, u, d, e) is called ***symmetric*** *if the following diagram commutes:*

$$\begin{array}{ccccc}
M \otimes M & & \xrightarrow{c_{M,M}} & & M \otimes M \\
{\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle m} \\
M & & & & M \\
& \searrow^{e} & & {}^{e}\swarrow & \\
& & I & &
\end{array}$$

Remark 3.1.10. A commutative monoid or a cocommutative comonoid can unambiguously be defined also in a braided monoidal category. This is evident from the fact that the (co)commutativity of the (co)multiplication is well defined because of the following equivalences:

$$m \circ b_{M,M} = m \Leftrightarrow m = m \circ b^{-1}_{M,M} \text{ and } b_{M,M} \circ d = d \Leftrightarrow d = b^{-1}_{M,M} \circ d.$$

Here we observe a natural fact, that is (co)commutativity is preserved by homomorphisms. Namely, that a commutative monoid homomorphism $f : (M, \mu, \eta) \to (M', \mu', \eta')$ makes, without any further assumptions, the following diagram commute:



The commutativity of the above diagram is equivalent to:

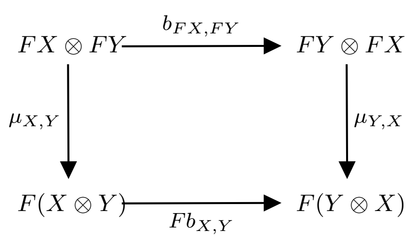
$$f \circ \mu \circ c_{M,M} = \mu' \circ c_{M',M'} \circ (f \otimes f) \Leftrightarrow$$
$$f \circ \mu = \mu' \circ (f \otimes f)$$

which is one of the two conditions f is required to obey in order for it to be a homomorphism. This is dually true for comonoids.

Example 3.1.11. Since **Cat** is a symmetric cartesian monoidal category, one may consider a commutative monoid internal to **Cat**. A monoid, firstly, is a strict monoidal category. Furthermore, the condition for the commutativity of the tensor product, seen as the multiplication of the strict monoidal category, implies that the symmetry coherent isomorphism must be the identity natural transformation. Such categories are called **commutative** or **strictly symmetrtic** strict monoidal categories.

Braided and symmetric monoidal functors

Definition 3.1.12. *Let $\mathcal{M}, \mathcal{M}'$ be braided monoidal categories and $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$ be a lax monoidal functor. We call F a* ***braided lax monoidal functor*** *if the following diagram commutes for every $X, Y \in \mathcal{M}_0$.*



Furthermore, if $\mathcal{M}$ and $\mathcal{M}'$ are symmetric monoidal categories, then (F, μ, ϕ) is called a ***symmetric lax monoidal functor****.*

Remark 3.1.13. It is easy to check that the identity monoidal functor of a braided monoidal category is braided. Also, the composition of two braided lax monoidal functors between braided monoidal categoires is again braided, and similarly for symmetric.

In addition, if a braided/symmetric lax monoidal functor is a strong monoidal functor, we will call it a **braided/symmetric monoidal functor**, and if it is strict we will call it a **braided/symmetric strict monoidal functor**. Obviously, being braided/symmetric can also be a property of a colax monoidal functor between braided/symmetric monoidal categories. The only difference would be that the compositors' direction would be inverted in the colax version of the braiding condition.

Thus, we have created subcategories of $\mathbf{MonCat}_{\mathrm{l}}, \mathbf{MonCat}_{\mathrm{co}}, \mathbf{MonCat}$ and $\mathbf{MonCat}_{\mathrm{s}}$ with objects braided monoidal categories and morphisms

1. braided strict monoidal functors, denoted by $\mathbf{BMonCat}_{\mathrm{s}}$,

2. braided strong monoidal functors, denoted by **BMonCat**,

3. braided lax monoidal functors, denoted by $\mathbf{BMonCat}_\mathrm{l}$,

4. braided colax monoidal functors, denoted by $\mathbf{BMonCat}_\mathrm{co}$,

but also subcategories of the above, respectively, consisting of symmetric monoidal categories and

1. symmetric strict monoidal functors, denoted by $\mathbf{SMonCat}_\mathrm{s}$,

2. symetric strong monoidal functors, denoted by **SMonCat**,

3. symmetric lax monoidal functors, denoted by $\mathbf{SMonCat}_\mathrm{l}$,

4. symmetric colax monoidal functors, denoted by $\mathbf{SMonCat}_\mathrm{co}$.

What is interesting about all the above categories is the way they become 2-categories.

A monoidal natural transformation between any type of braided/symmetric monoidal functor automatically preserves the braided structure. Therefore, there is no need for further definitions.

Remark 3.1.14. In general, cartesian categories are cartesian monoidal categories, so according to the choice of functors between them, we can view their collection as a full subcategory of the corresponding category of monoidal categories. Furthermore, since cartesian categories are symmetric, they form a full subcategory of the corresponding category of braided monoidal categories and similarly of the category of symmetric monoidal categories.

Remark 3.1.15. As we have already seen, lax monoidal functors preserve internal monoids, colax monoidal functors preserve internal comonoids and strong monoidal functors preserve Frobenius objects. Interestingly, symmetric lax monoidal functors preserve commutative monoids as can be witnessed by the following equalities: given a commutative monoid (M, m, η) in a symmetric monoidal category $\mathcal{M}$ and a symmetric lax monoidal functor $(F, \mu, \phi) : \mathcal{M} \to \mathcal{M}'$,

$$\begin{aligned}(Fm \circ \mu_{M,M}) \circ c_{FM,FM} &= Fm \circ Fc_{M,M} \circ \mu_{M,M} \text{ (by the braid condition)}\\ &= F(m \circ c_{M,M}) \circ \mu_{M,M}\\ &= Fm \circ \mu_{M,M} \text{ (since } M \text{ is commutative)}\end{aligned}$$

Similarly, symmetric colax monoidal functors preserve cocommutative comonoids and symmetric strong monoidal functors preserve cocommutative Frobenius objects.

Example 3.1.16. The categories **Hilb** and **Vect** are symmetric monoidal categories. Furthermore, there is a forgetful functor U : **Hilb** $\to$ **Vect** which is strict monoidal. This functor is also symmetric since the components of the symmetric braiding in **Hilb** are mapped to themselves, forgetting they are isometries. Thus the braid condition is trivially satisfied.

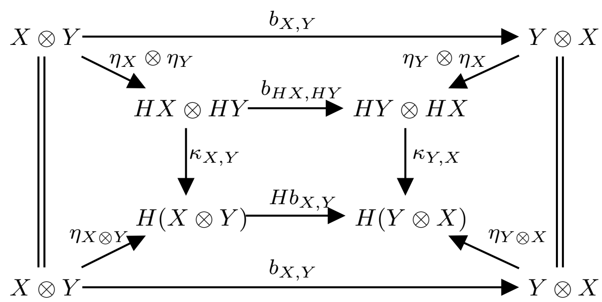
Example 3.1.17. There is a forgetful functor ob : **Cat** $\to$ **Set** taking any small category to its set of objects. Furthermore, depending on the choice of the cartesian product in **Set**, this forgetful functor might either be strong or strict monoidal. In any case though, it is also symmetric, which is easily shown by the universal property of the product in **Set**.

Example 3.1.18. Since n**Cob** and **Vect** are symmetric monoidal categories, there can be symmetric strong monoidal functors between them. Such functors define **topological quantum field theories** and are denoted by Z : n**Cob** $\to$ **Vect**. In broad terms, this is an assignement of state spaces to $n-1$ dimensional manifolds, considered as spaces, and evolution operators to cobordisms between spaces, considered as spacetimes. From the viewpoint of physics, these theories shed light to the connection between topology change and non-unitary evolution of quantum fields. Both spaces and spacetimes in topological quantum field theories do not need to carry a geometry, i.e. a metric, this being the root of the word "topological".

To conclude this section, we proceed by examining the notion of a braided/symmetric monoidal equivalence. Before giving the definition, we present two propositions. These two propositions are direct braided analogues of Lemma 2.2.20. They formalise the idea that additional structure on a monoidal category is translated along monoidal natural isomorphisms. Since we have not seen these facts stated explicitly, we provide self-contained proofs.

Proposition 3.1.19. *Let $\mathcal{M}$ be a braided monoidal category and let $(H, \kappa, \chi) : \mathcal{M} \to \mathcal{M}$ be a lax monoidal endofunctor. If there exists a monoidal natural isomorphism $\eta : \mathbb{1}_{\mathcal{M}} \Rightarrow H$, then H is braided.*

Proof. Since the identity endofunctor is trivially a braided strict monoidal functor, for every $X, Y \in \mathcal{M}_0$ we have:



Here the top rectangle commutes by naturality of the braiding, the bottom one commutes by naturality of η, and the left and right rectangles by monoidality of η. The outer square commutes by definition and, since η is an isomorphism, the inner square, representing the braid condition for H, can be verified to commute. $\square$

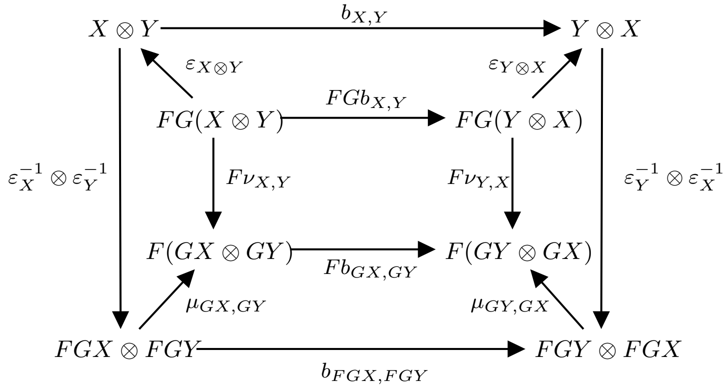
Remark 3.1.20. Duality implies a similar result for colax monoidal H. Furthermore, the above result holds both for strong and stict monoidal functors. Furthermore, if the category $\mathcal{M}$ is symmetric, then H is symmetric.

Proposition 3.1.21. *Let $(F, \mu, \phi) : \mathcal{M} \to \mathcal{N}$ be a lax monoidal functor that is a monoidal equivalence between braided monoidal categories $\mathcal{M}, \mathcal{N}$. If F is braided then its weak inverse is also braided.*

Proof. Let F be a braided monoidal functor and let (G, ν, ψ) be a colax monoidal functor such that G is the weak inverse of F. According to definition 2.2.21, G is colax monoidal and there exist natural isomorphisms $\eta : \mathbb{1}_{\mathcal{M}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbb{1}_{\mathcal{N}}$ such that for every $X, Y \in \mathcal{N}_0$

$$F\nu = \mu_{GX,GY} \circ (\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}) \circ \varepsilon_{X\otimes Y},$$

which uniquely determines the multiplicator ν of G. Furthermore, by the previous proposition we get that FG and GF are braided monoidal functors. Then in the following diagram the outer square is the naturality of the braiding, the bottom square is the braid condition for F, the left and right squares are the definition of ν and the top square depicts naturality of ε.



Thus the middle diagram commutes, giving the image under F of the braid condition for G. Since F is fully faithful, G is braided. □

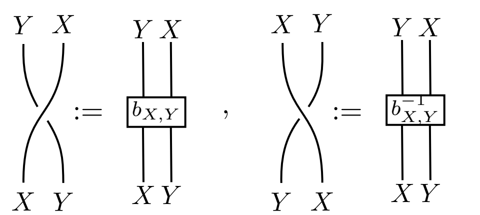
Remark 3.1.22. It is worth mentioning that if the categories $\mathcal{M}$ and $\mathcal{N}$ of the above proposition were symmetric, then G would be symmetric. Furthermore, duality implies that if G is braided, then F would be braided too. So the weak inverse of a braided lax monoidal functor is a braided colax monoidal functor and vice versa.

Definition 3.1.23. *Let $\mathcal{M}, \mathcal{N}$ be braided monoidal categories. A* ***braided monoidal equivalence*** *between $\mathcal{M}$ and $\mathcal{N}$ is a braided monoidal functor $F : \mathcal{M} \to \mathcal{N}$ that is an equivalence as a functor. If $\mathcal{M}, \mathcal{N}$ are symmetric monoidal categories, then F is called a* ***symmetric monoidal equivalence****.*

Note that we haven't used the characterisations "lax, colax, strong or strict" in the above definition, although it is possible. These actually provide different notions of braided monoidal equivalence, but since a lax equivalence would also give a colax equivalence, we let these characterisations be inferred from the context.

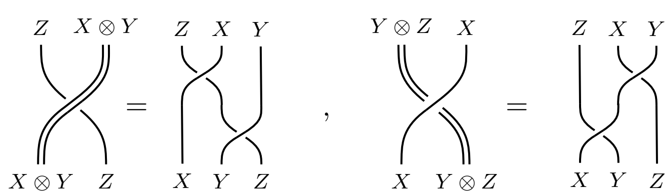
String diagrams for braided and symmetric monoidal categories

We can extend the string diagram calculus to braided and symmetric monoidal categories. This extension is achieved by depicting the braiding's and its inverse's components as follows:

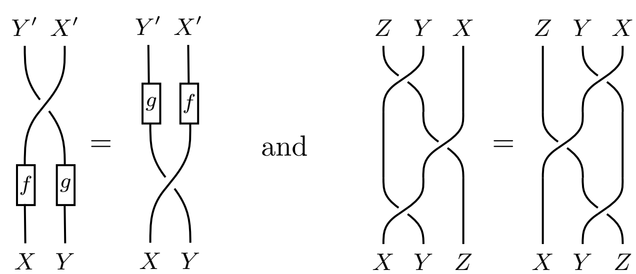


It is clear that if we didn't label the strings, we would have a depiction of braids. Therefore, the implicit topological data in such a calculus is that deforming braids while respecting the braid group laws yields the same braid. This is why the coherence theorem for braided monoidal categories is so useful in establishing such a notation. That is, *any two isotopic diagrams yield the same braid and any morphisms involving braidings commute if they have the same underlying braid.*

The hexagon laws for the braiding are depicted as follows:



while the following two diagrams depict the naturality of the braiding and the so called "**Yang-Baxter equation**", which can easily be seen to hold diagrammaticaly:

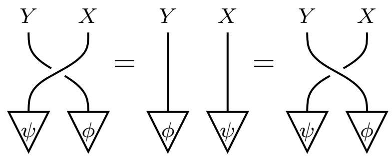


It is worth noting that the Yang-Baxter equation above is actually one of the equations imposed to the generators of the braid group. The second diagrammatic equation holds either using one reidemeister **III**-move and two **II**-moves or by observing that the underlying braid of the two diagrams are the same.

The string diagram calculus allows to easily prove the following proposition, which should be seen as accompanying remark 3.1.4, about the braiding of a monoidal category:

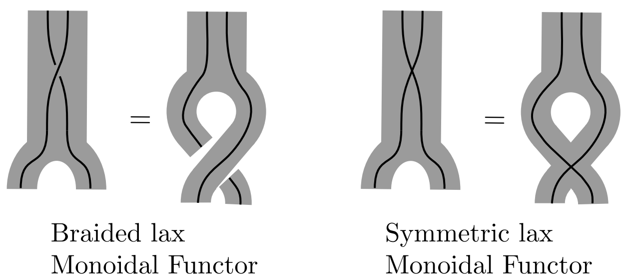
Proposition 3.1.24. *Let $(\mathcal{M}, \otimes, I, a, l, r, b)$ be a braided monoidal category. If $\mathcal{M}$ is monoidally well-pointed, then b is a symmetry and $\mathcal{M}$ is symmetric.*

Proof. Since $\mathcal{M}$ is monoidally well-pointed (Definition 2.1.23), we only need to show that pre-composing the braiding and its inverse with a product state yields the same result. So, let $X, Y \in \mathcal{M}_0$ and let $\psi \in_I X, \phi \in_I Y$ and observe that:



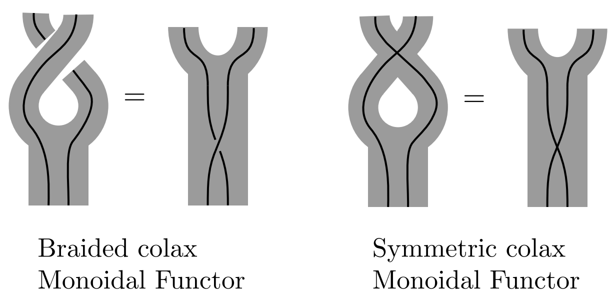
where the above equalities follow from naturality of the braiding. □

Having at our disposal a graphical calculus for braided and symmetric monoidal categories but also for monoidal functors, we can express the conditions for braided/symmetric monoidal functors in such a manner. So the braid condition for a braided and a symmetric lax monoidal functor looks as follows:



Braided lax Monoidal Functor — Symmetric lax Monoidal Functor

The braid condition for a braided or symmetric colax monoidal functor is depicted as:



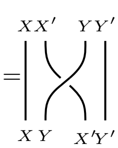
Braided colax Monoidal Functor — Symmetric colax Monoidal Functor

Finally, a braided or symmetric strong monoidal functor satisfies both the above conditions.

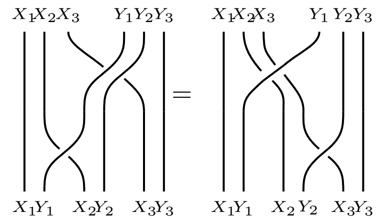
Uniform copying, (co)diagonals and the tensor

In a braided monoidal category the tensor product functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ can actually be equipped with a strong monoidal structure, in two possible ways. This only becomes possible due to the braiding being critically involved in order to define the multiplicator. So, given two pairs $(X, Y), (X', Y') \in \mathcal{M} \times \mathcal{M}$ define the multiplicator as follows:

$\mu_{(X,Y),(X',Y')} =$



which satisfies associativity since for every $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3) \in \mathcal{M} \times \mathcal{M}$:



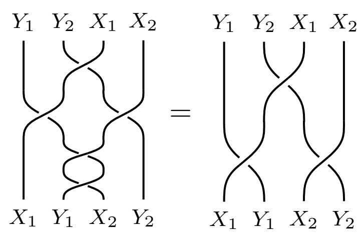
Also define the unitor as $\ell_I^{-1} : I \to I \otimes I$, which obviously satisfies unitality, either because the associated string diagram is just a line or due to the coherence for monoidal categories and the compatibility of the braiding with the monoidal structure 3.1.3.

The second way to equip the tensor product functor with a strong monoidal structure is achieved by replacing the braiding with its inverse and following the exact same construction as above. Of interest is the case where $\mathcal{M}$ is a symmetric monoidal category, where both ways to make $\otimes$ into a strong monoidal functor coincide.

The category of categories is symmetric monoidal under the cartesian product. So for every pair of categories $(\mathcal{C}, \mathcal{D})$, there is a functor $c_{\mathcal{C},\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$ namely the component of the symmetry of **Cat**. The categories of monoidal categories are all subcategories of **Cat** and they are closed under taking cartesian product (definition 2.1.7). Furthermore the components of the above symmetry are strict monoidal functors. To demonstrate this, take monoidal categories $\mathcal{M}, \mathcal{N}$. Then:

$$c_{\mathcal{M},\mathcal{N}}(X_1, Y_1) \otimes c_{\mathcal{M},\mathcal{N}}(X_2, Y_2) = (Y_1, X_1) \otimes (Y_2, X_2) = (Y_1 \otimes Y_2, X_1 \otimes X_2) = c_{\mathcal{M},\mathcal{N}}(X_1 \otimes X_2, Y_1 \otimes Y_2),$$

for every $X_1, X_2 \in \mathcal{M}$ and $Y_1, Y_2 \in \mathcal{N}$. If, in addition, $\mathcal{M} = \mathcal{N}$, we can post-compose the component $c_{\mathcal{M},\mathcal{M}}$ of the symmetry with the tensor product functor. When $\mathcal{M}$ is braided, the composite $\otimes \circ c_{\mathcal{M},\mathcal{M}}$ is a strong monoidal functor. Therefore it is meaningful to ask whether the braiding natural isomorphism $b : \otimes \Rightarrow \otimes \circ c_{\mathcal{M},\mathcal{M}}$ is monoidal. In string diagram terms, monoidality of b amounts to the following equation:



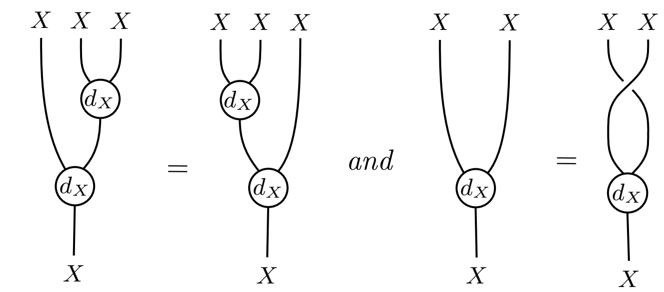
which only holds if and only if b is a symmetry. Concluding this discussion, we have shown the following proposition.

Proposition 3.1.25. *In a braided monoidal category, the tensor product functor is strong monoidal. In a symmetric monoidal category, the tensor product functor is canonically a strong monoidal functor and the symmetry is a monoidal natural transformation.*

Although uniform deleting can be formulated in an arbitrary category and then be specialised to monoidal categories, uniform copying is crucially dependent on a monoidal product and a braiding. To make a step-by-step analysis we firstly introduce categories with (co)diagonals. Note that since **Cat** is cartesian, for every category $\mathcal{C}$ there exists a unique functor $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, such that $\Delta_{\mathcal{C}} = \langle \mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}} \rangle$, given by the universal property of the product.

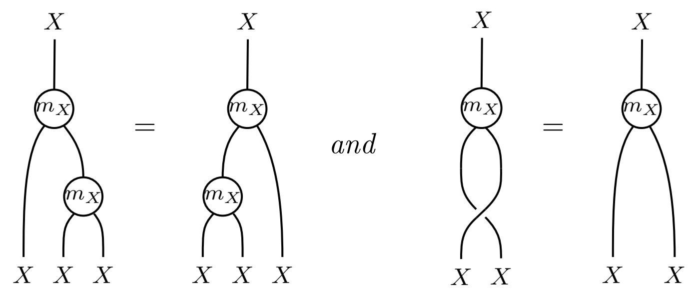
Definition 3.1.26. *A braided monoidal category $\mathcal{M}$ is a category with*

1. ***diagonals*** *if there exists a monoidal natural transformation $d : \mathrm{Id}_{\mathcal{M}} \Rightarrow \otimes \circ \Delta_{\mathcal{M}}$ such that $d_I = \ell_I^{-1}$ and*



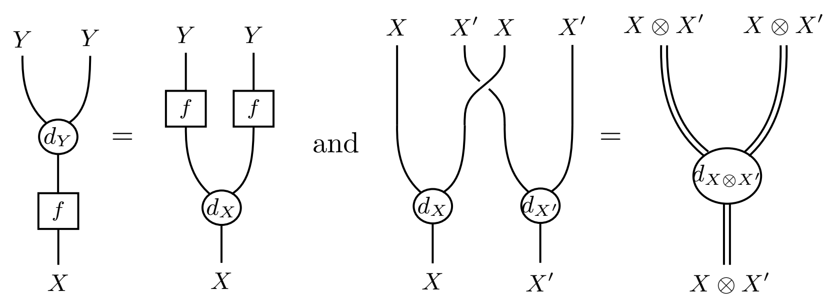
for every $X \in \mathcal{M}$,

2. ***codiagonals** if there exists a monoidal natural transformation $m : \otimes \circ \Delta_{\mathcal{M}} \Rightarrow \mathrm{Id}_{\mathcal{M}}$ such that $m_I = \ell_I$ and for every $X \in \mathcal{M}$*



*If a symmetric monoidal category has diagonals, then it is called a **relevance category** and if it has codiagonals it is called a **corelevance** category.*

Remark 3.1.27. It is part of their definition that diagonals form a natural transforamtion which is moreover monoidal. These two conditions in terms of string diagrams look as follows:



for every $X, X', Y \in \mathcal{M}_0$ and $f : X \to Y$. Intuitively, naturality says that copying the transformed version of an object is like transforming two copies of that object. The second one assures us that duplicating a composite system is like duplicating its ingredients and then forming the composite system. From this point of view, categories with diagonals are equivalently said to have **uniform copying**.

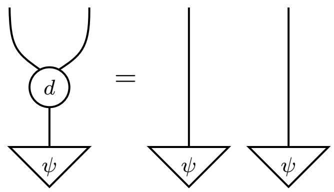
Remark 3.1.28. Observe that in a category with diagonals, every object is an internal cocommutative cosemigroup[1], since the conditions d has to satisfy amount to coassociativity and cocommutativity. Furthermore, naturality of d, implies that every morphism in such a category is a cosemigroup homomorphism. Duality implies that in a category with codiagonals, every object is a commutative semigroup and every morphism a semigroup homomorphism.

Example 3.1.29. Every cartesian monoidal category is automatically equipped with diagonals, which are natural, coassociative and cocommutative, due to the universal property of the product. Dually, every cocartesian monoidal category is a category with codiagonals.

[1]An internal cosemigroup is like an internal comonoid that doesn't necessarily have units and thus there is no counit law to be satisfied. Dually, an internal semigroup is like an internal monoid without a unit or unit law.

In order to discuss the notion of copying, we need the following definition.

Definition 3.1.30. *Let $\mathcal{M}$ be a braided monoidal category, $A \in \mathcal{M}_0$ and $d : A \to A \otimes A$ a morphism*[2]*. A state $\psi \in_I A$ is* ***copyable*** *by d, if*



The set of states which are copyable by d is denoted by $\mathbf{CS}(d)$*. If d is the comultiplication of a comonoid (M, d, e),* $\mathbf{CS}(d)$ *is also denoted by* $\mathbf{CS}(M)$*.*

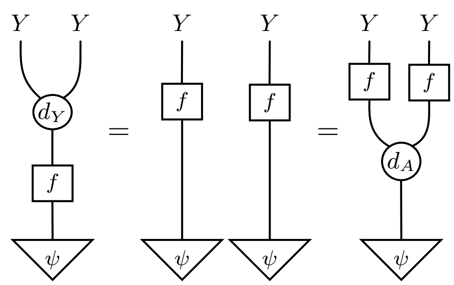
Remark 3.1.31. Observe that copyability of the state depends on the morphism that duplicates it. Generally, there is no reason for all states of an object to be copyable by the same morphism.

Proposition 3.1.32. *Let $\mathcal{M}$ be a braided monoidal category and let $(d_A)_{A\in\mathcal{M}_0}$ be a family of morphisms $d_A : A \to A \otimes A$. Every state is copyable by this family, if $(d_A)_{A\in\mathcal{M}_0}$ provides $\mathcal{M}$ with uniform copying. If $\mathcal{M}$ is monoidally well-pointed (definition 2.1.23), then the converse also holds.*

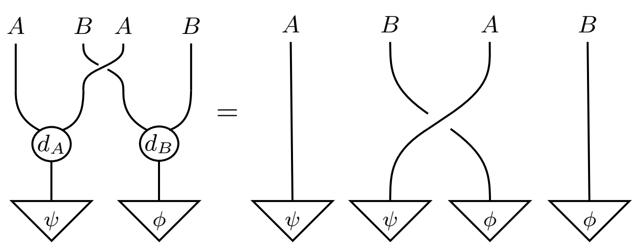
Proof. Firstly, observe that if $(d_A)_{A\in\mathcal{M}_0}$ provides $\mathcal{M}$ with uniform copying then naturality of d implies that for every object A and state $\psi \in_I A$

$$d_A \circ \psi = \psi \otimes \psi \circ \ell_I^{-1}.$$

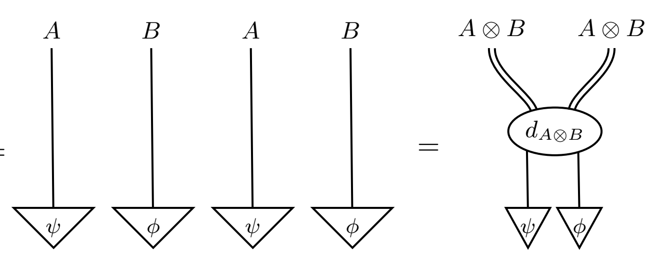
On the other hand if every state is copyable by $(d_A)_{A\in\mathcal{M}}$ and since $\mathcal{M}$ is monoidally well-pointed then for every states $\psi \in_I A$ and $\phi \in_I B$ and every morphism $f : A \to Y$



so d is natural and

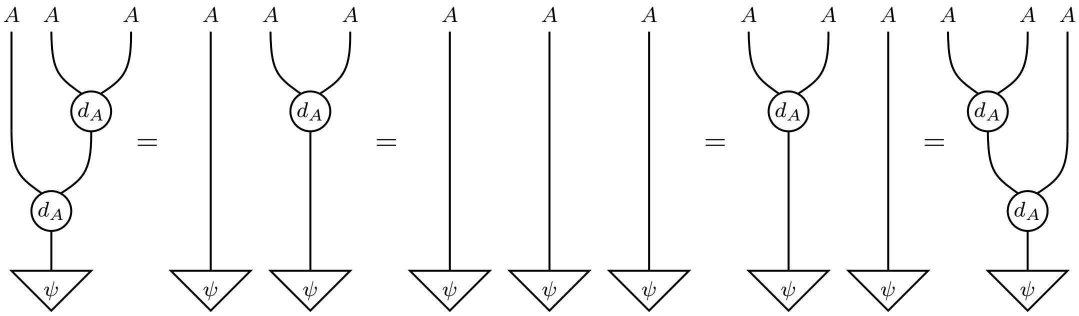


naturality of braiding

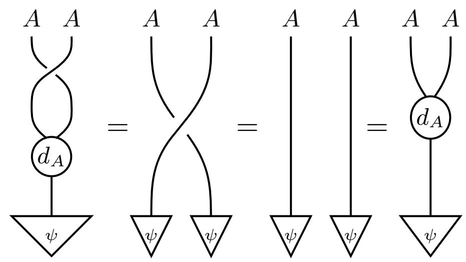


thus d is monoidal. We also have:

[2]A way to think of such a morphsim is as *duplicating* an object, hence the name d.



and



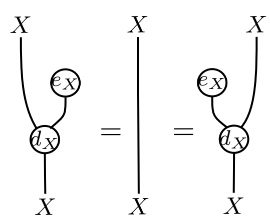
so $(d_A)_{A\in\mathcal{M}_0}$ provides $\mathcal{M}$ with uniform copying indeed. ☐

3.2 Characterisation of cartesian monoidal categories via comonoids

Up to now we have already seen that cartesian monoidal categories have both uniform copying (example 3.1.29) and uniform deleting (definition 2.1.51), since the terminal object is the monoidal unit. Dually, cocartesian categories have codiagonals and their unit object is initial. We will focus on cartesian categories. Cartesian categories have another property, connecting deleting and copying, which intuitively is necessary to interpret diagonals as copying and unique morphisms to the terminal object as deleting. Additionally, this property together with uniform deleting and uniform copying are sufficient to make a monoidal category cartesian.

Proposition 3.2.1. *Let $\mathcal{M}$ be a braided monoidal category. The following are equivalent:*

1. *M is a cartesian monoidal category.*
2. *M has uniform copying, uniform deleting and for every $X \in \mathcal{M}_0$:*



We refer to this condition as ***deleting-copying compatibility****.*

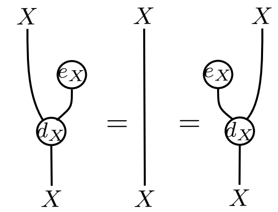
Proof. For the first direction, let $\mathcal{M}$ be cartesian. Then, the unit object of $\mathcal{M}$ is terminal, so $\mathcal{M}$ is semi-cartesian and its projections are given by

$$\pi_1 = r_X \circ (\mathrm{id}_X \otimes e_Y) \text{ and } \pi_2 = \ell_X \circ (e_X \circ \mathrm{id}_Y),$$

for every $X, Y \in \mathcal{M}_0$. The diagonal morphisms are uniquely defined by the universal property of the product as satisfying the conditions

$$\pi_1 \circ d_X = \mathrm{id}_X = \pi_2 \circ d_X$$

which are equivalent to

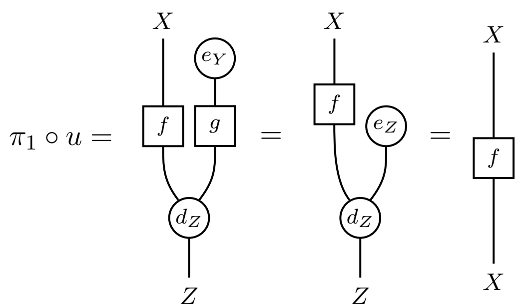


for every $X \in \mathcal{M}_0$. Lastly, by the universal property of the product, the diagonal morphisms are natural, monoidal and satisfy coassociativity and cocomuttativity by uniqueness.

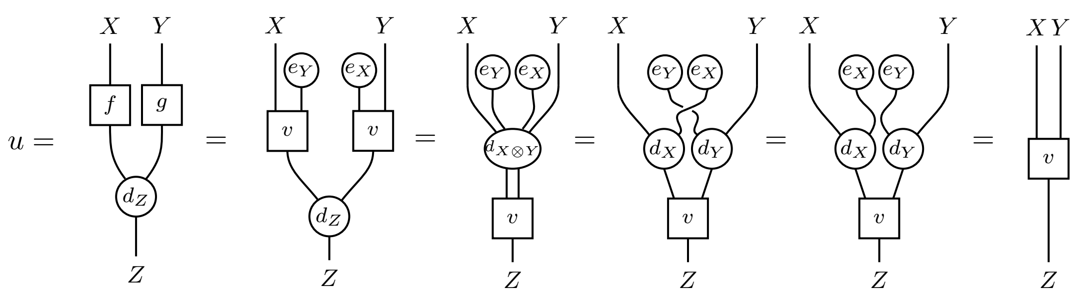
For the opposite direction, let $X, Y \in \mathcal{M}_0$. We need to show that $(X \otimes Y, \pi_1, \pi_2)$ is a cartesian product of X, Y, where the projections are the ones given by the fact that $\mathcal{M}$ has projections compatible with the unitors. Let $Z \in \mathcal{M}_0$ come with two morphisms $f : Z \to X$ and $g : Z \to Y$. We need to show that there exists a unique morphism $u : Z \to X \otimes Y$ such that $\pi_1 \circ u = f$ and $\pi_2 \circ u = g$. So define

$$u = (f \otimes g) \circ d_Z$$

and observe that



and similarly $\pi_2 \circ u = g$, so this covers existence of u. For uniqueness let $v : Z \to X \otimes Y$ such that $\pi_1 \circ v = f$ and $\pi_2 \circ v = g$ and observe that:



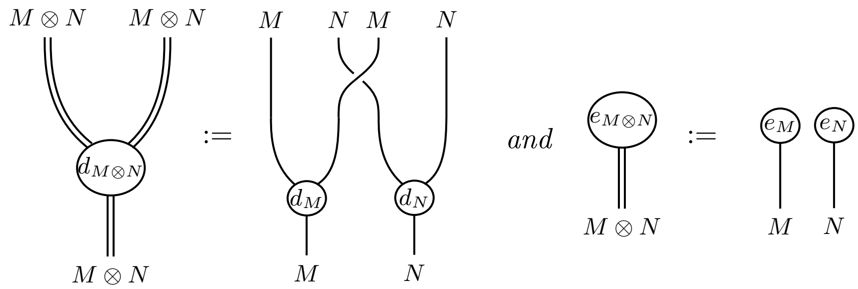
which gives uniqueness and concludes the proof. □

The above result is a characterisation of cartesian monoidal categories, but we can reinterpret it as showing which part of a general braided monoidal category is cartesian. The first thing to notice is that the combination of compatible uniform copying and deleting makes every object in a category naturally a cocommutative comonoid. The dual case, of course, would imply that every object of a cocartesian category is naturally a commutative monoid. So the above proposition can take the following equivalent form, given as a corrolary:

Corollary 3.2.2. *A braided monoidal category is cartesian monoidal if and only if every object is a cocommutative comonoid and every morphism is a comonoid homomorphism.*

If one is handed a generic braided monoidal category, the category of internal comonoids comes for free. Furthermore, and only in the case of braided monoidal categories, we can define the tensor product of comonoids as follows.

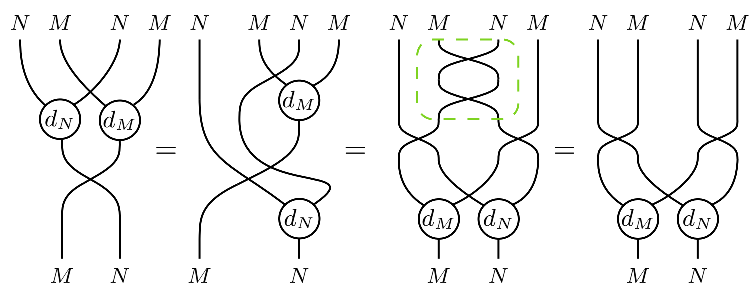
Definition 3.2.3. *Let $\mathcal{M}$ be a braided monoidal category and let $(M, d_M, e_M), (N, d_N, e_N)$ be internal comonoids. The tensor product of M and N is the object $M \otimes N$ with multiplication and unit given by:*



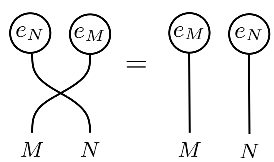
The tensor product of comonoids inside a braided monoidal category $\mathcal{M}$ is a comonoid, since the tensor is a strong monoidal functor (proposition 3.1.25) and thus preserves internal comonoids. Furthermore, this tensor product of comonoids is functorial due to the naturality of the braiding and the coherence of $\mathcal{M}$. In addition, the unit object is an internal comonoid, which is also terminal (remark 2.1.82), and acts as a unit for $\mathbf{Com}(\mathcal{M})$. So the category of internal comonoids is a semicartesian monoidal category, with coherence isomorphisms given by the components of the ones from $\mathcal{M}$. Dually, $\mathbf{Mon}(\mathcal{M})$ is semicocartesian. This is actually the closest one can get to a cartesian structure inside a non-symmetric braided monoidal category.

The situation is different for symmetric monoidal categories. That is because the category of internal comonoids is symmetric monoidal and thus uniform copying may be defined. All this has to do with the braiding in $\mathbf{Com}(\mathcal{M})$ being a symmetry, as we see below.

Let $\mathcal{M}$ be a symmetric monoidal category and let $M, N \in \mathbf{Com}(\mathrm{M})$. Observe that:



and also that:

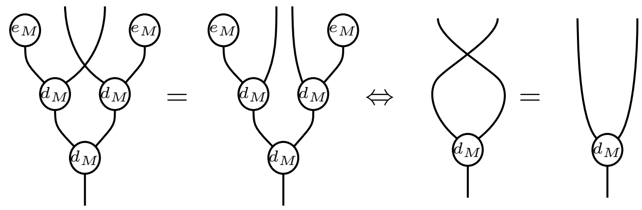


So the symmetry of a symmetric monoidal category forms a comonoid homomorphism between the tensor of two comonoids. This is not the case in (non-symmetric) braided monoidal categories, since the green part of the diagrams above would not disentangle. A symmetry obviously satisfies the hexagon laws when restricted to the comonoid objects, thus making $\mathbf{Com}(\mathcal{M})$ into a symmetric monoidal category. Similarly, $\mathbf{Mon}(\mathcal{M})$ is symmetric monoidal when $\mathcal{M}$ is.

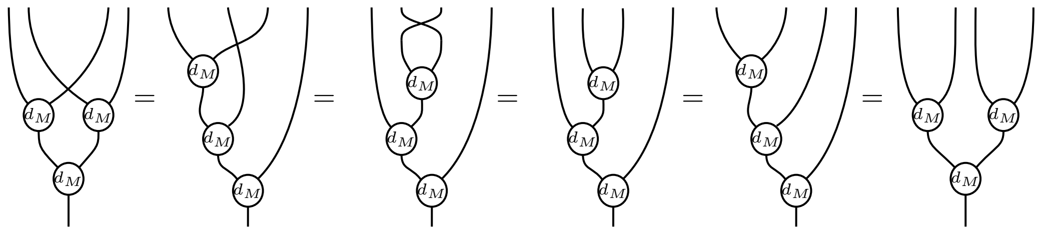
If we were to find a "piece of cartesian structure" inside a symmetric monoidal category, we would have to focus on cocommutative comonoids. This is due to proposition 3.2.1. This necessity exhibits itself in the form of the following proposition.

Proposition 3.2.4. *Let $\mathcal{M}$ be a symmetric monoidal category and (M, d_M, e_M) be an internal comonoid. The comultiplication d_M is a comonoid homomorphism if and only if M is cocommutative.*

Proof. For the first direction, let d_M be a comonoid homomorphism. Then,



so M is cocommutative. For the other direction, let M be cocommutative. Then,



so d_M is a comonoid homomorphism. □

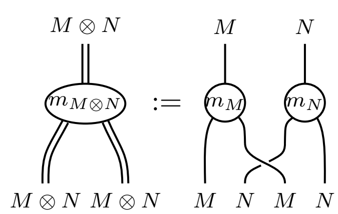
Finally an important proposition is the following. We omit the proof of it, since it is similar to the proof of proposition 3.2.1.

Proposition 3.2.5. *Let $\mathcal{M}$ be a symmetric monoidal category and let $(M, d_M, e_M), (N, d_N, e_N)$ be internal comonoids. The tensor product comonoid $M \otimes N$ is the categorical product of M and N, inside $\mathcal{M}$.*

To summarize, monoidal categories in general might not "enclose" any non-trivial cartesian structure. In the case they are braided, their category of internal comonoids is semi-cartesian and not cartesian. The comultiplication/copying morphism is a comonoid homomorphism, but the braiding is not. In the case the monoidal category is symmetric, the category of internal comonoids is cartesian monoidal, since both the symmetry and the comultiplication is a comonoid homomorphism. There is also a sense in which, the category of internal cocommutative comonoids of a symmetric monoidal category is "the best cartesian approximation" of it. For more on this, we refer to [Fox76].

We can dualise all the above, by substituting comonoids with monoids, cocommutativity with commutativity and (semi-)cartesian categories with (semi)-cocartesian categories. A final thing to note is that, for reasons regarding Frobenius objects and their categories, we will rigorously distinguish between the two different ways $\mathbf{Mon}(M)$ and $\mathbf{Com}(\mathcal{M})$ can be equipped with a monoidal structure in braided monoidal categories.

So, when we adopt the following convention for the tensor product of monoids:



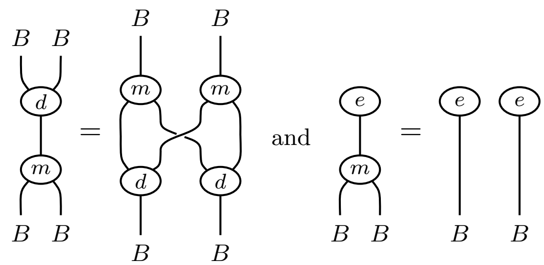
and the one we have already used for comonoids, we say that we are in $\mathbf{Mon}^r(\mathcal{M})$ and $\mathbf{Com}^{\mathrm{r}}(\mathcal{M})$. For the conventions involving the inverse of the braiding in $\mathbf{Com}(\mathcal{M})$ and not the inverse of the braiding in $\mathbf{Mon}(\mathcal{M})$, we say we are in $\mathbf{Com}^{\mathrm{l}}(\mathcal{M})$ and $\mathbf{Mon}^l(\mathcal{M})$, respectively.

Bimonoids

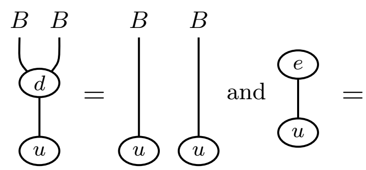
We have already seen that in a braided monoidal category, $\mathcal{M}$, we can form the tensor product of monoids and comonoids. This tensor product makes the categories $\mathbf{Com}(\mathcal{M})$, $\mathbf{Mon}(\mathcal{M})$, but also the commutative and cocommutative cases, monoidal. Moreover, we can consider the following structure on an object

Definition 3.2.6. *Let (B, m, u, d, e) be an object equipped with a monoid and a comonoid structure in a braided monoidal category $\mathcal{M}$. We call B a* ***bimonoid*** *if $m : B \otimes B \to B$ and $u : I \to B$ are comonoid homomorphisms. A morphism $f : B \to B'$, where B, B' are bimonoids, is a* ***bimonoid homomorphism*** *if it is a monoid and a comonoid homomorphism.*

Remark 3.2.7. Note that in string diagram terms, the conditions m should satisfy are given as:



and the ones for u are given as:



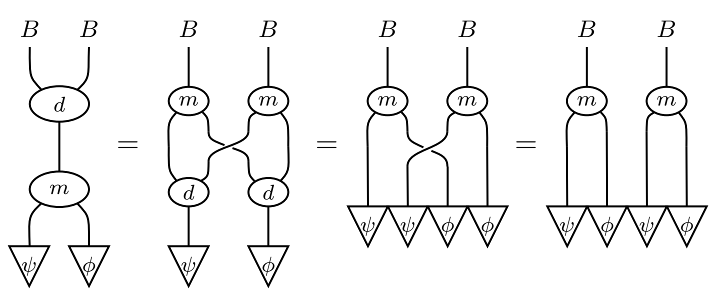
To state this precisely, (B, d, e) is considered as an object of $\mathbf{Com}^{\mathrm{l}}(\mathcal{M})$ and m and u are morphisms of this category. So $((B, d, e), m, u)$ is an internal monoid in $\mathbf{Com}^{\mathrm{l}}(\mathcal{M})$. What is interesting is that, given the above diagrams, we could have equivalently defined a bimonoid as an internal comonoid in $\mathbf{Mon}^{r}(\mathcal{M})$.

When using $\mathbf{Mon}^{l}(\mathcal{M})$ and $\mathbf{Com}^{\mathrm{r}}(\mathcal{M})$ there occurs a non equivalent definition of a bimonoid, which we will not use. In any case, though, bimonoids are canonically defined in symmetric monoidal categories, since the monoidal structures of internal monoids and comonoids are unambiguously defined.

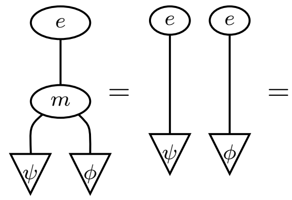
We have already seen in proposition 2.1.74 that the states of a monoid (M, m, u) internal to a monoidal category $\mathcal{M}$ form a monoid (in **Set**). In braided monoidal categories, the states of bialgebras have interesting submonoids, meaning a subset of a monoid closed under the operation of the monoid and containing the unit. We denote by $\mathbf{CS}(M)$ the set of copyable states of M, as in definition 3.1.30, by $\mathbf{DS}(M)$ the set of deletable states of $\mathcal{M}$, as in 2.1.53 and by $\mathbf{CDS}(M)$ the set $\mathbf{CS}(M) \cap \mathbf{DS}(M)$.

Lemma 3.2.8. *Let (B, m, u, d, e) be a bimonoid in a braided monoidal category $\mathcal{M}$. Then* $\mathbf{CS}(B)$, $\mathbf{DS}(B)$ *and* $\mathbf{CDS}(B)$ *are submonoids of $\mathcal{M}(I, B)$.*

Proof. We need to show that u is copyable and deletable, that the multiplication of two copyable states is copyable and the multiplication of two deletable states is deletable. For the first one observe that the fact that u is a comonoid homomorphism implies that u is both copyable (by d) and deletable (by e). Secondly, note that m being a comonoid homomorphism implies that given copyable $\psi, \phi \in \mathcal{M}(I, B)$:



so $\mu(\psi, \phi)$ is copyable. Similarly, given deletable states $\psi, \phi \in \mathcal{M}(I, B)$, we have:



which shows that $\mu(\psi, \phi)$ is deletable. Thus, $\mathbf{CS}(B)$ and $\mathbf{DS}(B)$ are submonoids of $\mathcal{M}(I, B)$ and so is their intersection $\mathbf{CDS}(B)$. □

Example 3.2.9. Given a symmetric monoidal category $\mathcal{M}$, we know that its category of internal comonoids is cartesian monoidal, due to 3.2.2. Therefore, any internal monoid in $\mathbf{Com}(\mathcal{M})$ is automatically a bimonoid. So $\mathbf{Mon}(\mathbf{Com}(\mathcal{M}))$ is the category of internal bimonoids in $\mathcal{M}$. A special case of this is when $\mathcal{M}$ is the cartesian category **Set**. Thus, every monoid in **Set** is actually a bimonoid.

Example 3.2.10. A semi-additive category is both cartesian and cocartesian. Therefore, due to 3.2.2 and its dual, every object is canonically a monoid and a comonoid, and every morphism is simultaneously both a monoid and a comonoid homomorphism. Therefore, every object of a semi-additive category is a bimonoid. A special case of this is the category **Vect** with the direct product as a monoidal product and the zero dimensional vector space as a unit. Specifically, every vector space is an abelian group and therefore a monoid under addition, but also a comonoid under the comultiplication given by the diagonal map sending a vector $v \in V$ to $(v, v) \in V \oplus V$. This way every vector space can be equipped with the structure of a bimonoid.

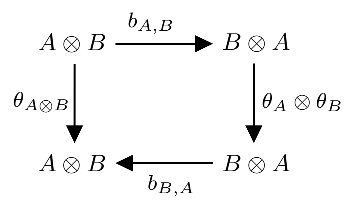
3.3 Balanced Monoidal Categories

The string diagram calculus of braided and symmetric monoidal categories depicts objects as strings. One might wonder whether a 360° twist of a string can be captured by the categorical algebra of braided/symmetric monoidal categories. This is indeed the case as we see in the following definition.

Definition 3.3.1. *Let $\mathcal{M}$ be a braided monoidal category. A natural isomorphism $\theta : \mathbb{1}_{\mathcal{M}} \Rightarrow \mathbb{1}_{\mathcal{M}}$ is called a **twist** or a **balance** if*

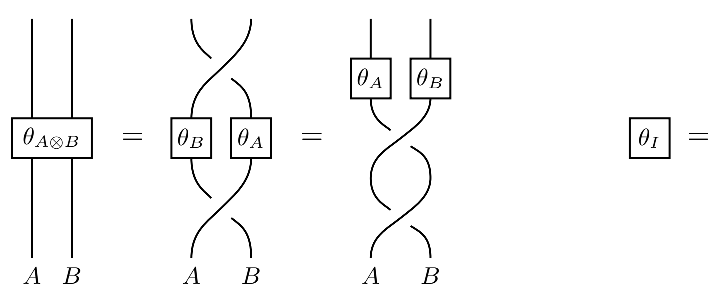
$$\theta_I = \mathrm{id}_I$$

and for every $A, B \in \mathcal{M}_0$ the following diagram commutes:

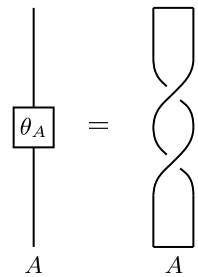


*A braided monoidal category equipped with a twist is called **balanced** and the above condition is called a **twist condition**.*

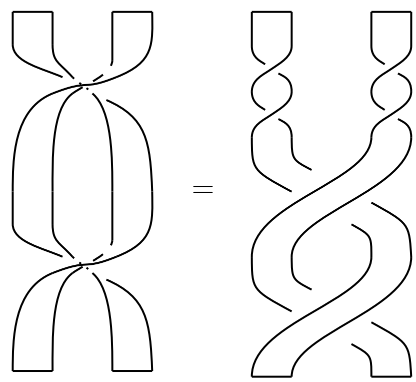
In the string diagram calculus, the above two conditions for a balance are depicted as follows:



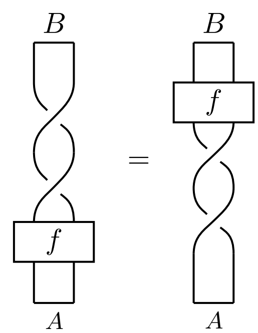
where we have used naturality of the braiding to give an equivalent form to the *twist condition*. The idea behind a twist though unfolds when introducing a variant of the string diagrams already used. In this variant, objects are not depicted as wires but as ribbons and the twist morphisms as a twist of a ribbon.



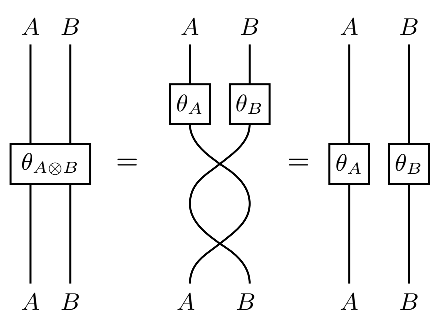
So the defining conditions for a balance are as follows:



Naturality of θ is given as:



Remark 3.3.2. Observe that every symmetric monoidal category $\mathcal{M}$ is balanced when equipped with the trivial twist, the identity $\mathrm{Id}_{\mathbb{1}_\mathcal{M}}$. But there might be other twists that make $\mathcal{M}$ a balanced monoidal category. Actually, since the identity functor $\mathbb{1}_\mathcal{M}$ is strict monoidal and the following equation holds



we see that every monoidal natural isomorphism $\mathbb{1}_\mathcal{M} \Rightarrow \mathbb{1}_\mathcal{M}$ is a twist and every twist is such a monoidal natural isomorphism.

Generally, in a braided monoidal category no monoidal natural transformation of the identity endofunctor is a twist, unless it is symmetric.

3.4 Monoidal closed categories

As it has already been discussed, monoidal categories in a sense generalise cartesian categories, in a non-trivial way. Symmetric monoidal categories have even more in common with cartesian categories, since the cartesian product is symmetric. In this section, we study how one can define a closed structure in a monoidal category and we compare it to the case of closed categories (see 1.5) and cartesian closed categories (see 1.5).

Definitions and Properties

We begin by defining what closure is for a general monoidal category.

Definition 3.4.1. *We call a monoidal category $\mathcal{M}$* ***left closed*** *if the functor $X \otimes - : \mathcal{M} \to \mathcal{M}$ has a right adjoint, denoted by $X \multimap -$, for every $X \in \mathcal{M}$. We call $\mathcal{M}$* ***right closed*** *if the functor $- \otimes Y$ has a right adjoint, denoted by $- \circ\!\!- Y$, for every $Y \in \mathcal{M}$. We call $\mathcal{M}$* ***bi-closed*** *if it is both right and left closed.*

In the case of a monoidal closed category M, the symbol $[X, Y]$ is sometimes used instead of $X \multimap Y$, for every $X, Y \in \mathcal{M}_0$. Furthermore, since $[-,-] : \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{M}$ is easily seen to be a functor, called **internal hom**, as in the case of cartesian closed categories, the diagonal components of the unit and counit isomorphisms of the adjunction with parameters $\cdot \otimes - \dashv [\cdot, -]$ are dinatural.

Remark 3.4.2. If a monoidal category, $\mathcal{M}$, is braided, then being left (equivalently right) closed, implies $\mathcal{M}$ is closed. This follows from the fact that for every $Y \in \mathcal{M}_0$, $b_{Y,-} : Y \otimes - \Rightarrow - \otimes Y$ is a natural isomorphism, so left closure is equivalent to right closure and by uniqueness of adjoints, $Y \multimap - \cong - \circ\!\!- Y$.

An interesting fact about monoidal closed categories is the interplay between the functors $\otimes - \multimap -$ and limits, which is a direct consequence of proposition 1.5.40 and the fact that $A \otimes - \dashv A \multimap -$, for every $A \in \mathcal{M}_0$.

Proposition 3.4.3. *Let $\mathcal{M}$ be a monoidal closed category and let $D : \mathcal{I} \to \mathcal{M}$ be a small diagram. For every $X \in \mathcal{M}_0$, if the limit of D exists, then*

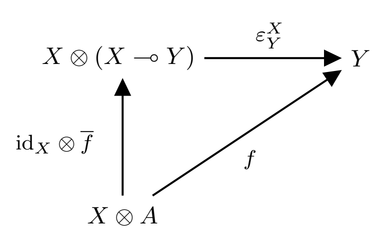
$$X \multimap \lim D \cong \lim(X \multimap D-),$$

if the colimit of D exists, then

$$X \otimes \mathrm{colimD} \cong \mathrm{colim(X \otimes D)}.$$

There are some central notions that come with a left or right monoidal closed category and consequently with a bi-closed one. We will focus on left monoidal closed catgeories and let duality take care of the rest. The first notion is the natural isomorphism between the hom-sets $c_{X,Y,Z} : \mathcal{M}(X \otimes Y, Z) \to \mathcal{M}(Y, X \multimap Z)$ called **currying**, as was the case in cartesian closed categories, and its inverse called **uncurrying**. Obviously, this is the bar ismorphism of the adjunction, but naturality in the first variable makes it an adjunction with a parameter.

The unit and the counit of this adjunction also get special names. The counit $\varepsilon^X_Y : X \otimes (X \multimap Y) \to Y$, natural in Y and dinatural in X, is called **evaluation**. That is because it satisfies the following universal property, due to 1.5.38, i.e. for every $f : X \otimes A \to Y$ there exists a unique $\bar{f} = c_{X,A,Y}(f)$ such that the following diagram commutes:



which is similar to that of an exponential object in a cartesian closed category. For the same reasons, the unit $\eta^X_Y : Y \to X \multimap (X \otimes Y)$, again natural in X and dinatural in Y, is called **coevaluation**.

In a left monoidal closed category, there is a way to internalise the currying isomorphism. Observe that given any $A, X, Y, Z \in \mathcal{M}_0$ we have natural isomorphisms:

$$\mathcal{M}(A, (Y \otimes X) \multimap Z) \cong \mathcal{M}((X \otimes Y) \otimes A, Z) \cong \mathcal{M}(Y \otimes (X \otimes A), Z) \cong \mathcal{M}(X \otimes A, Y \multimap Z) \cong \mathcal{M}(A, X \multimap (Y \multimap Z)).$$

So by full faithfulness of the Yoneda embedding we get that:

$$(Y \otimes X) \multimap Z \cong X \multimap (Y \multimap Z).$$

as objects in $\mathcal{M}$. In a right monoidal closed category this takes the form:

$$(X \otimes Y) \multimap Z \cong X \multimap (Y \multimap Z).$$

In a bi-monoidal closed category we have both of the above, implying

$$(X \otimes Y) \multimap Z \cong X \multimap (Y \multimap Z) \cong (Y \otimes X) \multimap Z,$$

which is also true in any monoidal closed category.

Remark 3.4.4. A very useful way to look at $X \multimap Y$ is through its I-shaped elements. Precisely, given any $f \in \mathcal{M}(X, Y)$, we have that $f \circ r_X \in \mathcal{M}(X \otimes I, Y) \cong \mathcal{M}(I, X \multimap Y)$. So for every $f \in \mathcal{M}(X, Y)$, there exists a unique morphism $\ulcorner f \urcorner : I \to X \multimap Y$, called the **name** of f and defined as

$$\ulcorner f \urcorner := c_{X,I,Y}(f \circ r_X).$$

Thus, the states of $X \multimap Y$ correspond bijectively to morphisms in $\mathcal{M}(X, Y)$ for every $X, Y \in \mathcal{M}_0$. This has an interesting implication if $\mathcal{M}$ is semi-cocartesian.

Proposition 3.4.5. *Let $\mathcal{M}$ be a monoidal closed category. If $\mathcal{M}$ is semi-cocartesian, then it is a preorder category.*

Proof. If $\mathcal{M}$ is semi-cocartesian, then the unit I is initial. Furthermore, since $\mathcal{M}$ is closed, we have the following natural isomorphisms for every $X, Y \in \mathcal{M}_0$

$$\mathcal{M}(X, Y) \cong \mathcal{M}(I \otimes X, Y) \cong \mathcal{M}(I, X \multimap Y).$$

But, initiality of I, implies that $\mathcal{M}(I, X \multimap Y) \cong \{*\}$, thus $\mathcal{M}(X, Y)$ is a one-element set. □

Remark 3.4.6. Note that the above proposition implies that every homset of a semi-cocartesian category is inhabited. In such cases we may speak of a **total preorder**.

Remark 3.4.7. The above proposition also shows that **Set**, considered as a monoidal category with the disjoint union as a tensor, cannot be closed. The same also holds for **Ab**, considered as a monoidal category under the direct sum.

There is also a way to "compose" names, which we may refer to as **internal composition**, denoted by $\bullet$ and defined as the currying of the following morphism:

$$X \otimes ((X \multimap Y) \otimes (Y \multimap Z)) \xrightarrow{a^{-1}} (X \otimes (X \multimap Y)) \otimes (Y \multimap Z) \xrightarrow{\varepsilon^X_Y \otimes (Y \multimap Z)} Y \otimes (Y \multimap Z) \xrightarrow{\varepsilon^Y_Z} Z$$

Proposition 3.4.8. *Internal composition $\bullet^Y_{X,Z} : (X \multimap Y) \otimes (Y \multimap Z) \to X \multimap Z$ is natural in X, Z and dinatural in Y.*

Proof. We will prove this claim for the three variables seperately. Naturality in Z amounts to the commutativity of the following diagram,

$$\begin{array}{ccc} (X \multimap Y)\otimes(Y\multimap Z) & \xrightarrow{\bullet^Y_{X,Z}} & X\multimap Z \\ \Big\downarrow{\scriptstyle (X\multimap Y)\otimes(Y\multimap h)} & & \Big\downarrow{\scriptstyle X\multimap h} \\ (X \multimap Y)\otimes(Y\multimap Z') & \xrightarrow[\bullet^Y_{X,Z'}]{} & X\multimap Z' \end{array}$$

for $h : Z \to Z'$, where we adopt a "whiskering-like" notation for identities. Observe that

$$\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ a^{-1}_{X,X\multimap Y,Y\multimap Z} = c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z})$$

is natural in Z as a vertical composition of natural transformations. In addition naturality of the currying implies,

$$\begin{array}{ccc} \mathcal{M}(X\otimes((X\multimap Y)\otimes(Y\multimap Z')),Z') & \xrightarrow{c_{X,(X\multimap Y)\otimes(Y\multimap Z'),Z'}} & \mathcal{M}((X\multimap Y)\otimes(Y\multimap Z'),X\multimap Z') \\ \Big\downarrow{\scriptstyle -\circ X\otimes((X\multimap Y)\otimes(Y\multimap h))} & & \Big\downarrow{\scriptstyle -\circ((X\multimap Y)\otimes(Y\multimap h))} \\ \mathcal{M}(X\otimes((X\multimap Y)\otimes(Y\multimap Z)),Z') & \xrightarrow[c_{X,(X\multimap Y)\otimes(Y\multimap Z),Z'}]{} & \mathcal{M}((X\multimap Y)\otimes(Y\multimap Z),X\multimap Z') \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}(X\otimes((X\multimap Y)\otimes(Y\multimap Z)),Z) & \xrightarrow{c_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}} & \mathcal{M}((X\multimap Y)\otimes(Y\multimap Z),X\multimap Z) \\ \Big\downarrow{\scriptstyle h\circ -} & & \Big\downarrow{\scriptstyle (X\multimap h)\circ -} \\ \mathcal{M}(X\otimes((X\multimap Y)\otimes(Y\multimap Z)),Z') & \xrightarrow[c_{X,(X\multimap Y)\otimes(Y\multimap Z),Z'}]{} & \mathcal{M}((X\multimap Y)\otimes(Y\multimap Z),X\multimap Z') \end{array}$$

so,

$$\begin{aligned} \bullet^Y_{X,Z'} \circ ((X\multimap Y)\otimes(Y\multimap h)) =& c_{X,(X\multimap Y)\otimes(Y\multimap Z),Z'}(c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z'),Z'}(\bullet^Y_{X,Z'}) \circ X\otimes((X\multimap Y)\otimes(Y\multimap h))) \\ =& c_{X,(X\multimap Y)\otimes(Y\multimap Z),Z'}(h\circ c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z})) \\ =& (X\multimap h)\circ \bullet^Y_{X,Z} \end{aligned}$$

Naturality in X amounts to

$$\begin{array}{ccc} (X' \multimap Y)\otimes(Y\multimap Z) & \xrightarrow{\bullet^Y_{X',Z}} & X'\multimap Z \\ \Big\downarrow{\scriptstyle (f\multimap Y)\otimes(Y\multimap Z)} & & \Big\downarrow{\scriptstyle f\multimap Z} \\ (X \multimap Y)\otimes(Y\multimap Z) & \xrightarrow[\bullet^Y_{X,Z}]{} & X\multimap Z \end{array}$$

for $f : X \to X'$. Observe that

$$c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) = \varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ a^{-1}_{X,X\multimap Y,Y\multimap Z},$$

by 1.3.20, is dinatural in X as a composition of a dinatural transformation and two natural transformations, so

$$\begin{gathered} c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) \circ (X\otimes((f\multimap Y)\otimes(Y\multimap Z))) = \\ c^{-1}_{X',(X'\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X',Z}) \circ (f\otimes((X'\multimap Y)\otimes(Y\multimap Z))). \end{gathered}$$

In addition, naturality of the currying in the first variable is equivalent to

$$\begin{array}{ccc}
\mathcal{M}(X' \otimes [(X' \multimap Y) \otimes (Y \multimap Z)], Z) & \xrightarrow{c_{X,(X'\multimap Y)\otimes(Y\multimap Z),Z}} & \mathcal{M}((X' \multimap Y) \otimes (Y \multimap Z), X' \multimap Z) \\
\Big\downarrow {\scriptstyle - \circ f \otimes [(X' \multimap Y) \otimes (Y \multimap Z)]} & & \Big\downarrow {\scriptstyle (f \multimap Z) \circ -} \\
\mathcal{M}(X \otimes [(X' \multimap Y) \otimes (Y \multimap Z)], Z) & \xrightarrow[c_{X,(X'\multimap Y)\otimes(Y\multimap Z),Z}]{} & \mathcal{M}((X' \multimap Y) \otimes (Y \multimap Z), X \multimap Z)
\end{array}$$

thus, putting things together we get,

$$\begin{aligned}
\bullet^Y_{X,Z} \circ [(f \multimap Z) \otimes (Y \multimap Z)] =& c_{X,(X'\multimap Y)\otimes(Y\multimap Z),Z}(c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) \circ (X \otimes [(f \multimap Y) \otimes (Y \multimap Z)])) \\
=& c_{X,(X'\multimap Y)\otimes(Y\multimap Z),Z}(c^{-1}_{X',(X'\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X',Z}) \circ (f \otimes [(X' \multimap Y) \otimes (Y \multimap Z)])) \\
=& (f \multimap Z) \circ \bullet^Y_{X',Z},
\end{aligned}$$

which is the naturality condition for X.

Finally, dinaturality in Y amounts to

$$\begin{array}{ccccc}
 & & X \multimap Z & & \\
 & \nearrow^{\bullet^Y_{X,Z}} & & \nwarrow^{\bullet^{Y'}_{X,Z}} & \\
(X \multimap Y) \otimes (Y \multimap Z) & & & & (X \multimap Y') \otimes (Y' \multimap Z) \\
 & \nwarrow_{(X \multimap Y) \otimes (g \multimap Z)} & & \nearrow_{(X \multimap g) \otimes (Y' \multimap Z)} & \\
 & & (X \multimap Y) \otimes (Y' \multimap Z) & &
\end{array}$$

for $g : Y \to Y'$. Observe again that by 1.3.20,

$$c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) = \varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ a^{-1}_{X,X\multimap Y,Y\multimap Z},$$

is dinatural in Y as a composition of a dinatural transformation and two natural transformations, so

$$\begin{aligned}
& c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) \circ (X \otimes ((X \multimap Y) \otimes (g \multimap Z))) = \\
& c^{-1}_{X,(X\multimap Y')\otimes(Y'\multimap Z),Z}(\bullet^{Y'}_{X,Z}) \circ (X \otimes ((X \multimap g) \otimes (Y' \multimap Z))).
\end{aligned}$$

Thus,

$$\begin{aligned}
\bullet^Y_{X,Z} \circ [(X \multimap Y) \otimes (g \multimap Z)] =& c_{X,(X\multimap Y)\otimes(Y'\multimap Z),Z}(c^{-1}_{X,(X\multimap Y)\otimes(Y\multimap Z),Z}(\bullet^Y_{X,Z}) \circ (X \otimes [(X \multimap Y) \otimes (g \multimap Z)])) \\
=& c_{X,(X\multimap Y)\otimes(Y'\multimap Z),Z}(c^{-1}_{X,(X\multimap Y')\otimes(Y'\multimap Z),Z}(\bullet^{Y'}_{X,Z}) \circ (X \otimes [(X \multimap g) \otimes (Y' \multimap Z)])) \\
=& \bullet^{Y'}_{X,Z} \circ [(X \multimap g) \otimes (Y' \multimap Z)]
\end{aligned}$$

which conclludes the proof. □

Remark 3.4.9. The name "*internal composition*" is justified for three reasons. The first one is that it is of shape $\bullet^Y_{X,Z} : (X \multimap Y) \otimes (Y \multimap Z) \to X \multimap Y$. The second one follows from the commutativity of the following square

$$\begin{array}{ccc}
\mathcal{M}(X \otimes X \multimap Z, Z) & \xrightarrow{c_{X,X\multimap Z,Z}} & \mathcal{M}(X \multimap Z, X \multimap Z) \\
\Big\downarrow {\scriptstyle - \circ (X \otimes \bullet^Y_{X,Z})} & & \Big\downarrow {\scriptstyle - \circ \bullet^Y_{X,Z}} \\
\mathcal{M}(X \otimes (X \multimap Y \otimes Y \multimap Z), Z) & \xrightarrow{c_{X,X\multimap Y\otimes Y\multimap Z,Z}} & \mathcal{M}(X \multimap Y \otimes Y \multimap Z, X \multimap Z)
\end{array}$$

so for $\varepsilon^X_Z \in \mathcal{M}(X \otimes X \multimap Z, Z)$ we get:

$$c_{X,X\multimap Y\otimes Y\multimap Z,Z}(\varepsilon^X_Z \circ (X \otimes \bullet^Y_{X,Z})) = \bullet^Y_{X,Z}.$$

Uncurrying the above we arrive at

$$\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes Y \multimap Z) \circ a_{X,X\multimap Y,Y\multimap Z} = \varepsilon^X_Z \circ (X \otimes \bullet^Y_{X,Z}).$$

This shows that evaluating a composite is repeated evaluation.

Another reason justifying the term "internal composition" is the following.

Proposition 3.4.10. *Let $\mathcal{M}$ be a left monoidal closed category, $X, Y, Z \in \mathcal{M}_0$ and let $f \in \mathcal{M}(X, Y)$ and $g \in \mathcal{M}(Y, Z)$ be composable morphisms. Then the following diagram commutes:*

$$\begin{array}{ccc}
X \multimap Y \otimes Y \multimap Z & \xrightarrow{\bullet^Y_{X,Z}} & X \multimap Z \\
\uparrow{\scriptstyle \ulcorner f \urcorner \otimes \ulcorner g \urcorner} & & \uparrow{\scriptstyle \ulcorner g \circ f \urcorner} \\
I \otimes I & \xrightarrow[\ell_I]{} & I
\end{array}$$

Proof. Naturality of currying implies that the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{M}(X \otimes (X \multimap Y \otimes Y \multimap Z), Z) & \xrightarrow{c_{X,X\multimap Y,Y\multimap\ Z}} & \mathcal{M}((X \multimap Y \otimes Y \multimap Z), X \multimap Z) \\
\downarrow{\scriptstyle - \circ (\mathrm{id}_X \otimes (\ulcorner f \urcorner \otimes \ulcorner g \urcorner))} & & \downarrow{\scriptstyle - \circ (\ulcorner f \urcorner \otimes \ulcorner g \urcorner)} \\
\mathcal{M}(X \otimes (I \otimes I), Z) & \xrightarrow{c_{X,I\otimes I,\ Z}} & \mathcal{M}(I \otimes I, X \multimap Z) \\
\downarrow{\scriptstyle - \circ \mathrm{id}_X \otimes \ell_I^{-1}} & & \downarrow{\scriptstyle - \circ \ell_I^{-1}} \\
\mathcal{M}(X \otimes I, Z) & \xrightarrow{c_{X,I,\ Z}} & \mathcal{M}(I, X \multimap Z)
\end{array}$$

So picking $\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ a^{-1}_{X,X\multimap Y,Y\multimap Z} \in \mathcal{M}(X \otimes (X \multimap Y \otimes Y \multimap Z), Z)$ we have the following:

$$\begin{aligned}
\bullet^Y_{X,Z} \circ (\ulcorner f \urcorner \otimes \ulcorner g \urcorner) \circ \ell_I^{-1} &= c_{X,I,Z}\{\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ a^{-1}_{X,X\multimap Y,Y\multimap Z} \circ (\mathrm{id}_X \otimes (\ulcorner f \urcorner \otimes \ulcorner g \urcorner)) \circ (\mathrm{id}_X \otimes \ell_I^{-1})\} \\
&= c_{X,I,Z}\{\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes \mathrm{id}_{Y\multimap Z}) \circ ((\mathrm{id}_X \otimes \ulcorner f \urcorner) \otimes \ulcorner g \urcorner) \circ a^{-1}_{X,I,I} \circ (\mathrm{id}_X \otimes \ell_I^{-1})\} \\
&= c_{X,I,Z}\{\varepsilon^Y_Z \circ [(\varepsilon^X_Y \circ (\mathrm{id}_X \otimes \ulcorner f \urcorner)) \otimes \ulcorner g \urcorner] \otimes (r_X^{-1} \otimes \mathrm{id}_I)\} \\
&= c_{X,I,Z}\{\varepsilon^Y_Z \circ (\mathrm{id}_Y \otimes \ulcorner g \urcorner) \circ ((f \circ r_X) \otimes \mathrm{id}_I) \circ (r_X^{-1} \otimes \mathrm{id}_I)\} \\
&= c_{X,I,Z}\{g \circ r_Y \circ (f \otimes \mathrm{id}_I)\} \\
&= c_{X,I,Z}\{g \circ f \circ r_X\} \\
&= \ulcorner g \circ f \urcorner,
\end{aligned}$$

where the second equality follows from naturality and invertibility of the associator, the third one follows from the triangle law and functoriality of the tensor, the fourth one from the interchange law and the definition of name, the fifth one from the definition evaluation and functoriality of the tensor, the sixth one from naturality of the right unitor and the last one by the definition of name. Thus we have proven the result. □

Remark 3.4.11. Names and their internal composition can also be defined in any right monoidal closed category. In this case, the order in which the names appear in the tensor product would be reversed. All this is also true in any (bi-)monoidal closed category.

Remark 3.4.12. The above proposition shows that internal composition is compatible with external composition.

We can use the notion of names to shed some light to the closed structure (definition 1.5.48) of left, right or bi-closed monoidal categories. To begin with, **the internal identities** are given as the names of the corresponding identity morphisms. So for any $X \in \mathcal{M}_0$ we have a morphism $j_X : I \to X \multimap X$, defined as

$$j_X := \ulcorner \mathrm{id}_X \urcorner.$$

By definition, this means that $j_X = c_{X,I,X}(r_X)$. Furthermore, $j_x : I \to X \multimap X$ is dinatural in X. To prove this, firstly observe that

$$(\mathrm{id}_X \multimap f) \circ j_X = (\mathrm{id}_X \multimap f) \circ c_{X,I,X}(r_X) = c_{X,I,Y}(f \circ r_X) = \ulcorner f \urcorner,$$

since $X \otimes - \dashv X \multimap -$. Secondly, naturality of the currying in the first variable implies:

$$(f \multimap Y) \circ j_Y = c_{X,I,Y}(r_Y \circ (f \otimes \mathrm{id}_I))$$

and naturality of the right unitor implies $f \otimes \mathrm{id}_I = f \circ r_X$, so

$$(f \multimap Y) \circ j_Y = c_{X,I,Y}(f \circ r_X) = \ulcorner f \urcorner.$$

Thus, $(\mathrm{id}_X \multimap f) \circ j_X = (f \multimap Y) \circ j_Y$ holds, which is the dinaturality condition for $j : I \dot{\to} (- \multimap -)$.

This morphism, j, is one of the three (di)natural transformations needed to define a closed category (definition 1.5.48). The second such morphism is the currying of the left unitor, i.e. for every $X \in \mathcal{M}_0$,

$$i_X := c_{I,X,X}(\ell_X) : X \to I \multimap X.$$

This is natural in X, since $X \otimes - \dashv X \multimap -$ and naturality of the left unitor imply:

$$\begin{aligned}(\mathrm{id}_X \multimap f) \circ i_X &= c_{I,X,Y}(f \circ \ell_X) \\ &= c_{I,X,Y}(\ell_Y \circ (\mathrm{id}_Y \otimes f)) \\ &= i_Y \circ f,\end{aligned}$$

which is equivalent to the naturality of $i : \mathbb{1}_{\mathcal{M}} \Rightarrow I \multimap -$.

The final piece of a closed structure is the morphism $L : Y \multimap Z \to (X \multimap Y) \multimap (X \multimap Z)$, which by the following calculations will be seen to correspond to a version of post-composition. So for $X, Y, Z \in \mathcal{M}_0$ we define:

$$L^X_{Y,Z} := c_{X \multimap Y, Y \multimap Z, X \multimap Z}(\bullet^Y_{X,Z}).$$

This morphism is natural in Y, Z and dinatural in X. To prove these (di)naturality conditions let $f : X \to X', g : Y \to Y'$ and $h : Z \to Z'$. For the naturality in Z observe that:

$$\begin{aligned}L^X_{Y,Z'} \circ (Y \multimap h) =& c_{X \multimap Y, Y \multimap Z', X \multimap Z}(\bullet^Y_{X,Z'} \circ ((X \multimap Y) \otimes (Y \multimap h))) \\ =& c_{X \multimap Y, Y \multimap Z', X \multimap Z}((X \multimap h) \circ \bullet^Y_{X,Z}) \\ =& [(X \multimap Y) \multimap (Y \multimap Z)] \circ L_{Y,Z},\end{aligned}$$

by naturality of the currying and internal composition. Similarly, for Y observe that:

$$\begin{aligned}[(X \multimap g) \multimap (X \multimap Z)] \circ L^Y_{Y',Z} =& c_{X \multimap Y, Y' \multimap Z, X \multimap Z}(\bullet^{Y'}_{X,Z} \circ [(X \multimap g) \otimes (Y \multimap Z)]) \\ =& c_{X \multimap Y, Y' \multimap Z, X \multimap Z}(\bullet^{Y}_{X,Z} \circ [(X \multimap Y) \otimes (g \multimap Z)]) \\ =& L^X_{Y,Z} \circ (g \multimap Z),\end{aligned}$$

by naturality of the currying in the second variable and dinaturality of internal composition in Y. Finally, for dinaturality of L in X observe that:

$$\begin{aligned}[(f \multimap Y) \multimap (X \multimap Z)] \circ L^X_{Y,Z} &= c_{X' \multimap Y, Y \multimap Z, X \multimap Z}(\bullet^Y_{X,Z} \circ [(f \multimap Y) \otimes (Y \multimap Z)]) \\ &= c_{X' \multimap Y, Y \multimap Z, X \multimap Z}((f \multimap Z) \circ \bullet^Y_{X',Z}) \\ &= [(X' \multimap Y) \multimap (f \multimap Z)] \circ L^{X'}_{Y,Z},\end{aligned}$$

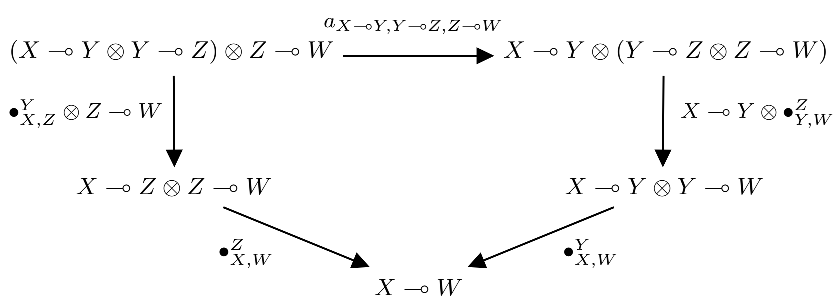
by naturality of the currying and naturality of internal composition in X.

It is not a trivial task to prove that these three morphisms actually satisfy the closed category axioms. We will not prove these directly, since they are beyond the scope of this thesis, but we will prove some of their curried versions, in what follows, since they provide a way to get some special internal monoids.

We now move on to some further results for monoidal closed categories, connected to internal monoids and modules.

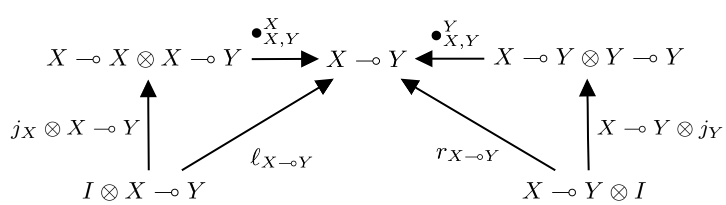
Proposition 3.4.13. *In a monoidal closed category, $\mathcal{M}$, the following hold.*

1. *For every $X, Y, Z, W \in \mathcal{M}$*



commutes and is called ***associativity.***

2. *For every $X, Y \in \mathcal{M}$*

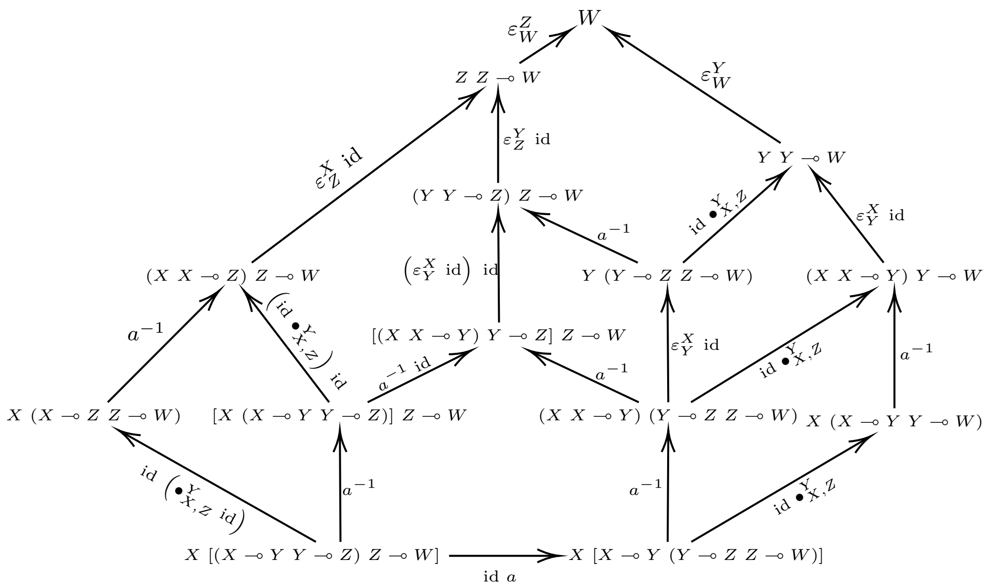


commute and are called ***left*** *and* ***right unitality****.*

Proof. We will give a proof for the uncurryed versions (namely the transpose maps under the tensor-hom adjunction) of the above morphisms and we will use the following relation

$$\varepsilon^Y_Z \circ (\varepsilon^X_Y \otimes Y \multimap Z) \circ a^{-1}_{X, X \multimap Y, Y \multimap Z} = \varepsilon^X_Z \circ (X \otimes \bullet^Y_{X,Z})$$

from 3.4.9, the pentagon and the triangle identities of monoidal categories. So for 1. observe that the following commutes:



where the components of the associators and the tensor symbols have been suppressed. The outer morphisms are indeed the uncurryed versions of the associativity morphisms, therefore the associativity diagram commutes.

To prove 2. note that by the definition of identities

$$\varepsilon_A^A \circ (A \otimes j_A) = r_A,$$

for every $A \in \mathcal{M}_0$. So, for left unitality we have

$$\begin{aligned}
\varepsilon_Y^X \circ (\varepsilon_X^X \otimes X \multimap Y) \circ a^{-1} \circ [X \otimes (j_X \otimes X \multimap Y)] &= \varepsilon_Y^X \circ (\varepsilon_X^X \otimes X \multimap Y) \circ [(X \otimes j_X) \otimes X \multimap Y] \circ a^{-1} \\
&= \varepsilon_Y^X \circ [(\varepsilon_X^X \circ (X \otimes j_X)) \otimes X \multimap Y] \circ a^{-1} \\
&= \varepsilon_Y^X \circ (r_X \otimes X \multimap Y) \circ a^{-1} \\
&= \varepsilon_Y^X \circ (X \otimes \ell_{X \multimap Y}) \\
&= c^{-1}_{X, I \otimes X \multimap Y, Y}(\ell_{X \multimap Y})
\end{aligned}$$

and for right unitality:

$$\begin{aligned}
\varepsilon_Y^Y \circ (\varepsilon_Y^X \otimes Y \multimap Y) \circ a^{-1} \circ (X \otimes (X \multimap Y \otimes j_Y)) &= \varepsilon_Y^Y \circ (\varepsilon_Y^X \otimes Y \multimap Y) \circ ((X \otimes X \multimap Y) \times j_Y) \circ a^{-1} \\
&= \varepsilon_Y^Y \circ (Y \otimes j_Y) \circ (\varepsilon_Y^X \otimes I) \circ a^{-1} \\
&= r_Y \circ (\varepsilon_Y^X \otimes I) \circ a^{-1} \\
&= \varepsilon_Y^X \circ r_{X \otimes X \multimap Y} \circ a^{-1} \\
&= \varepsilon_Y^X \circ (X \otimes r_{X \multimap Y}) \\
&= c^{-1}_{X, X \multimap Y \otimes Y \multimap Y, Y}(r_{X \multimap Y}).
\end{aligned}$$

□

Corollary 3.4.14. *Let $\mathcal{M}$ be a monoidal closed category. For every object $A \in \mathcal{M}_0$, $(A \multimap A, \bullet^A_{A,A}, j_A)$ is an internal monoid.*

Remark 3.4.15. The above monoid should be considered as an object of internal endomorphisms of the object A. In a right monoidal closed category every such monoid, say $(A \multimap A, \bullet, j_A)$, can act (one the right) action on A via the evaluation morphism $\varepsilon_A^A : A \otimes (A \multimap A) \to A$. The right module axioms hold by 3.4.9 and the definition of j_A. Similarly, in a left monoidal closed category such an internal monoid can act on A, on the *left*.

The existence of such "endo-monoids" in a monoidal closed category provides another version of monoid actions.

Proposition 3.4.16. *Let $\mathcal{M}$ be a monoidal (right) closed category, (M, m, u) be an internal monoid and A an object. The (hom)set of monoid homomorphisms of type $M \to A \multimap A$ is in bijection with the set of right M-actions on A.*

Proof. We will prove one direction of the above since the other one is similar. We will denote both currying and uncurrying by a bar. So let $r : A \otimes M \to A$ be an action. Observe that $c_{A,M,A}(r) : M \to A \multimap A$. This morphisms satisfies:

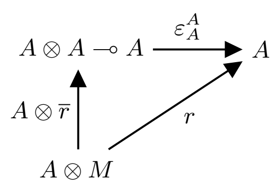
$$\overline{r} \circ u = \overline{r \circ (A \otimes u)} = \overline{r_A} = j_A,$$

which gives the unitality condition for $\overline{r}$ to be an internal monoid homomorphism. Furthermore, note that the associativity axiom for actions, the interchange law and 3.4.9 imply

$$\begin{aligned}
\bullet^A_{A,A} \circ \overline{r} \otimes \overline{r} &= \overline{\varepsilon^A_A \circ (\varepsilon^A_A \otimes \mathrm{id}_{A \multimap A}) \circ a^{-1} \circ (\mathrm{id}_A \otimes (\overline{r} \otimes \overline{r}))} \\
&= \overline{\varepsilon^A_A \circ (\varepsilon^A_A \otimes \mathrm{id}_{A \multimap A}) \circ ((\mathrm{id}_A \otimes \overline{r}) \otimes \overline{r}) \circ a^{-1}} \\
&= \overline{\varepsilon^A_A \circ (A \otimes \overline{r}) \circ [(\varepsilon^A_A \circ (A \otimes \overline{r})) \otimes M] \circ a^{-1}} \\
&= \overline{r \circ (r \otimes \mathrm{id}_M) \circ a^{-1}} \\
&= \overline{r \circ (\mathrm{id}_A \otimes m)} \\
&= \overline{r} \circ m
\end{aligned}$$

which shows that $\overline{r}$ is a monoid homomorphism. Similarly, given a monoid homomorphism $f : M \to A \multimap A$ one can show that $\overline{f} : A \otimes M \to A$ is a right action. □

Remark 3.4.17. According to the above proposition, every M-action on an object A of a monoidal closed category corresponds uniquely to a monoid homomorphism $M \to A \multimap A$. The correspondence is given by (un)currying. Thus, the following diagram commutes:



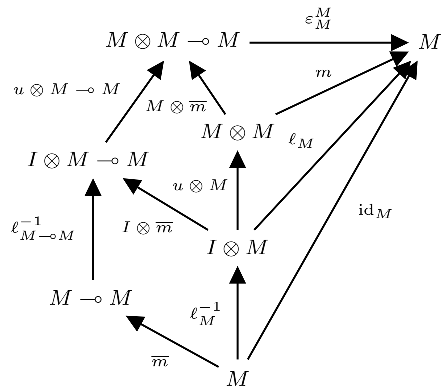
Since (M, M, m) is a right M-module for every monoid we can apply the above to get that $\overline{m} : M \to M \multimap M$ is a monoid homomorphism. This actually leads to an internal monoid version of the Cayley theorem, as we will prove in the next proposition.

Proposition 3.4.18. *Let (M, m, u) be a monoid in a monoidal closed category $\mathcal{M}$. The monoid homomorphism $\overline{m} : M \to M \multimap M$ has a left inverse, thus it is a monomorphism.*

Proof. To prove this we construct a candidate left inverse for $\overline{m}$ as the following composite:

$$M \multimap M \xrightarrow{\ell^{-1}_{M \multimap M}} I \otimes M \multimap M \xrightarrow{u \otimes M \multimap M} M \otimes M \multimap M \xrightarrow{\varepsilon^M_M} M.$$

This is indeed a left inverse for $\overline{m}$ since naturality of the left unitor, the interchange law, unitality for M and the universal property of the exponential object $M \multimap M$, imply that the following diagram commutes:



so $\overline{m}$ is a monomorphism. □

The above proposition is indeed an internal version of the Cayley embedding theorem for monoids. This will be apparent in the examples of monoidal closed categories that follow. The final example will be the ordinary Cayley embedding theorem for monoids.

Example 3.4.19. The first example of a monoidal closed category is **Set**, where the existence of exponentials provides the closed structure. **Set** is a cartesian closed category, which constitutes a general class of examples. Similarly, **Cat** is a monoidal closed category, being cartesian closed.

Example 3.4.20. There are also monoidal closed categories that are not cartesian. The category **Vect** is such an example. The internal hom functor is given by taking the set of linear maps between vector spaces and equipping it with the vector space operations pointwise. This construcion provides a right adjoint to the tensor product functor for vector spaces. Actually, this construction can be carried over in **Ab**, of which **Vect** is a subcategory.

Example 3.4.21. The category of Banach spaces with bounded linear maps as morphisms is a symmetric monoidal category with the underlying field as a unit. Furthermore, the set of bounded linear maps between two Banach spaces is a vector space with the pointwise vector space operations. This hom-vector space can be equipped with a norm making it complete. So the category of Banach spaces is monoidal closed. As a non-example, there is a subcategory of the category of Banach spaces, **Hilb**, whose homsets are not Hilbert spaces. But if one restricts to **FdHilb** then this category is again monoidal closed.

Example 3.4.22. A discrete monoidal category $\mathcal{M}$ is necessarily a monoid, since all coherent isomorphisms would have to reduce to identity morphisms. If in addition such a monoid is equipped with evaluation morphisms, i.e. counits of the tensor-hom adjunction, then for every $X \in \mathcal{M}_0$, the morphism

$$X \otimes (X \multimap I) \to I$$

would have to be an identity. Thus every object of the monoid $\mathcal{M}_0$ would have a left inverse, which is equivalent to having a right inverse, therefore being a group. Here a group can be viewed as an example of a monoidal closed category.

Example 3.4.23. If the category $\mathcal{M}$ of the above example was a poset, then it would again be a monoid under the tensor product, but the existence of exponentials would not make it a group. Instead it would be a structure called a **monoidal closed poset**. Such a category is quite general, with different very interesting subcases.

On the one hand, there is the case where the tensor product is cartesian, thus providing a meet operation and rendering the monoidal unit a top element $\top$. In this case, the image of the internal hom functor is called a **relative pseudocomplement**. If in addition this cartesian closed poset is also cocartesian, i.e. equipped with a join operation, then we may speak of a **Heyting Algebra** and the internal hom is called **implication**. These kinds of algebras serve both as models of intuitionistic propositional calculus and, in the case where not only finite but all joins exist, of point-free topologies. So when such categories are cocomplete (all small joins exist) they are called **frames**.

On the other hand, if the tensor product is not cartesian, but all joins exist[3], we have a structure called a **quantal**. In both cases the tensor-hom adjunction and the fact that arbitrary joins are small colimits force the tensor product to preserve the join operation, by proposition 1.5.40.

Example 3.4.24. A monoid M inside **Set** is an ordinary monoid. Since **Set** is cartesian closed, there is a notion of internal composition, which applied to $M^M \times M^M$ results in the ordinary composition of functions. This immediately shows that M^M is an internal monoid, but also that the action of M on itself, defined by multiplication, translates to an injective homomorphism $M \to M^M$. This is used to show that any abstract monoid can be realised as a monoid of endomorphisms of some set and it is called the Cayley embedding theorem.

This can be extended to **Cat**, being cartesian closed, to show that any internal monoid (strict monoidal category) can be embedded in the category of endofunctors of some category. This is actually a weaker form of the strictification theorem we already saw in 2.2

Moving to the context of semi-additive categories (see definition 1.6.21), there is an interesting interplay between the tensor product, the internal hom and the biproduct therein. This is again a consequence of 1.5.40, 3.4.3 and the fact that a biproduct and a zero object are simultaneously both limits and colimits. The following proposition makes this interplay rigorous.

Proposition 3.4.25. *In a monoidal closed category which is also semi-additive, the following hold naturally for every* $A, B, C \in \mathcal{M}_0$.

1. $A \otimes 0 \cong 0 \cong 0 \otimes A$
2. $A \multimap 0 \cong 0 \cong 0 \multimap A$
3. $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ *and* $(B \oplus C) \otimes A \cong (B \otimes A) \oplus (C \otimes A)$
4. $A \multimap (B \oplus C) \cong (A \multimap B) \oplus (A \multimap C)$
5. $(A \oplus B) \multimap C \cong (A \multimap C) \oplus (B \multimap C)$.

Remark 3.4.26. The isomorphisms in the above proposition are both natural and unique. Naturality follows from functoriality of $\otimes$, $\multimap$ and $\oplus$. Uniqueness is a little more delicate to prove.

We show that this is the case for 3. and 5. since the rest are similar in nature or depend on 3.4.3 and 1.6.19.

Proof. For 3., observe that $A \otimes - : \mathcal{M} \to \mathcal{M}$ preserves coproducts, therefore $(A \otimes (B \oplus C), \mathrm{id}_A \otimes \iota)$ is a coproduct whenever $(B \oplus C, \iota)$ is. According to 1.6.19, there is always a unique choice of $p_{A\otimes B} : A \otimes (B \oplus C) \to A \otimes B$ and $p_{A\otimes C} : A \otimes (B \oplus C) \to A \otimes C$ such that $(A \otimes (B \oplus C), p, \mathrm{id}_A \otimes \iota)$ is a bi-product. The defining properties of $p_{A\otimes B}$ are the following:

$$p_{A\otimes B} \circ (\mathrm{id}_A \otimes \iota_B) = \mathrm{id}_{A\otimes B} \text{ and } p_{A\otimes B} \circ (\mathrm{id}_A \otimes \iota_C = 0_{A\otimes B, A\otimes C}),$$

but functoriality of the tensor product implies that $\mathrm{id}_A \otimes \pi_B$ also satisfies these properties, where π gives the projections of $B \oplus C$. So uniqueness forces $p_{A\otimes B} = \mathrm{id}_A \otimes \pi_B$ and similarly $p_{A\otimes C} = \mathrm{id}_A \otimes \pi_C$. From this we see that $(A \otimes (B \oplus C), \mathrm{id}_A \otimes \pi, \mathrm{id}_A \otimes \iota)$ is a biproduct and therefore unique up to unique isomorphism.

For 5. note that for every $X \in \mathcal{M}_0$

$$\begin{aligned}
\mathcal{M}(X, (A \oplus B) \multimap C) &\cong \mathcal{M}(X \otimes (A \oplus B), C) \\
&\cong \mathcal{M}((X \otimes A) \oplus (X \otimes B), C) && \text{(by 3.)} \\
&\cong \mathcal{M}(X \otimes A, C) \times \mathcal{M}(X \otimes B, C) && \text{(since } \oplus \text{ is a categorical coproduct)} \\
&\cong \mathcal{M}(X, A \multimap C) \times \mathcal{M}(X, B \multimap C) \\
&\cong \mathcal{M}(X, (A \multimap C) \oplus (B \multimap C)) && \text{(since } \oplus \text{ is a categorical product and by 1.6.19)}
\end{aligned}$$

[3]Such a poset is called a sup-lattice.

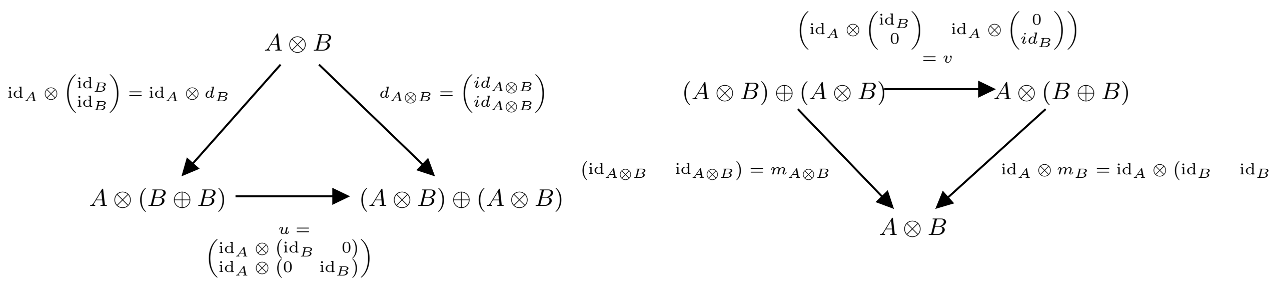
naturally, so by the Yoneda embedding being fully faithful,

$$(A \oplus B) \multimap C \cong (A \multimap C) \oplus (B \multimap C)$$

$\square$

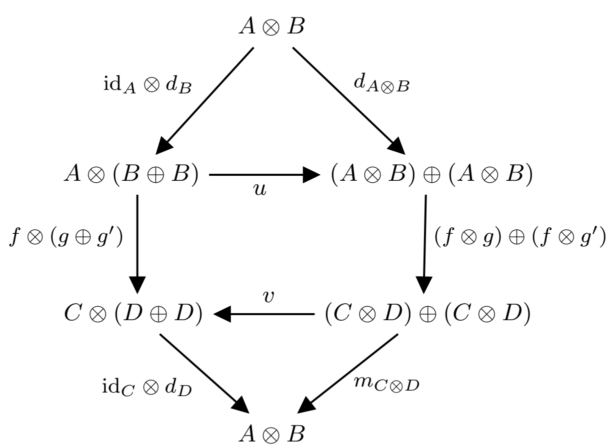
Somewhat surprisingly the above proposition and uniqueness implies that $\otimes$ and $\multimap$ are strong monoidal functors when considering biproducts as monoidal products.

Since a biproduct is preserved by tensoring and biproducts are unique up to unique isomorphism, we get the following two commutative triangles involving diagonals and codiagonals with respect to the bi-product,

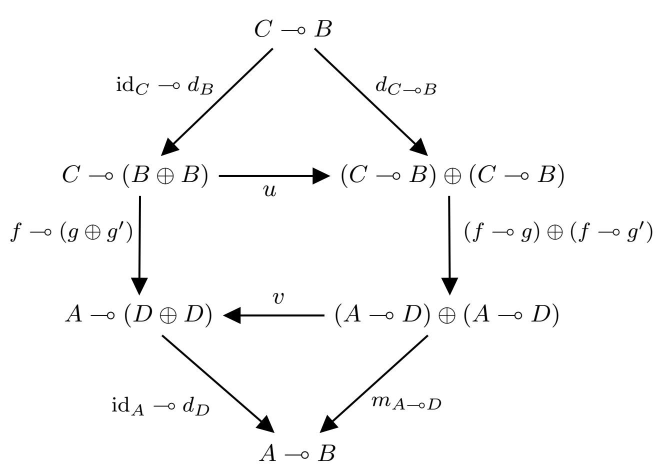


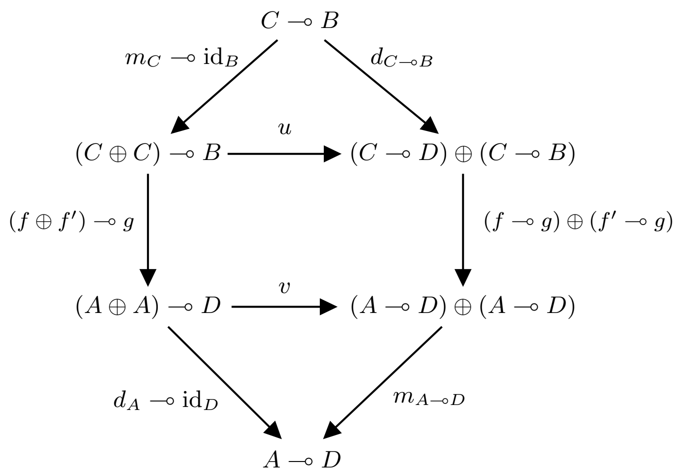
where u and v are the unique morphisms given by the above proposition and are mututal inverses. Observe that the only part where the closed structure is needed is when uniqueness and invertibility of u and v is asserted. The above triangles commute in any monoidal category with biproducts.

According to the above we get that tensoring with a zero morphism and/or currying the resulting morphism with a zero morphism yields again a zero morphism, since any morphism of this form factors through a zero object. Furthermore, we get that the following diagram commutes:



by naturality of u or v, for every $f, f' : A \to C$ and $g, g' : B \to D$. Similarly, one can show that the following two diagrams also commute:





where the u's and v's are defined in each case similarly. A synopsis of the above is given in the following proposition.

Proposition 3.4.27. *Let $\mathcal{M}$ be a monoidal closed semi-additive category, $A, B, C, D \in \mathcal{M}_0$ are objects and $f, f' \in \mathcal{M}(A, B), g, g' \in \mathcal{M}(C, D)$ are morphisms. Then the following hold:*

1. $f \otimes 0_{C,D} = 0_{A \otimes C, B \otimes D} = 0_{A,B} \otimes g$
2. $f \multimap 0_{C,D} = 0_{B \multimap C, A \multimap D} = 0_{B,A} \multimap g$
3. $f \otimes (g + g') = (f \otimes g) + (f \otimes g')$ *and* $(f + f') \otimes g = (f \otimes g) + (f' \otimes g)$
4. $f \multimap (g + g') = (f \multimap g) + (f \multimap g')$ *and* $(f + f') \multimap g = (f \multimap g) + (f' \multimap g)$

Finally, there is a notion of preservation of the internal hom by monoidal functors, but there is a caveat. Colax monoidal functors cannot play such a role, but lax ones, and consequently strong and strict ones, can. This kind of preservation is given by a morphism of type $\gamma_{X,Y} : F(X \multimap Y) \to FX \multimap FY$, natural in X and Y.

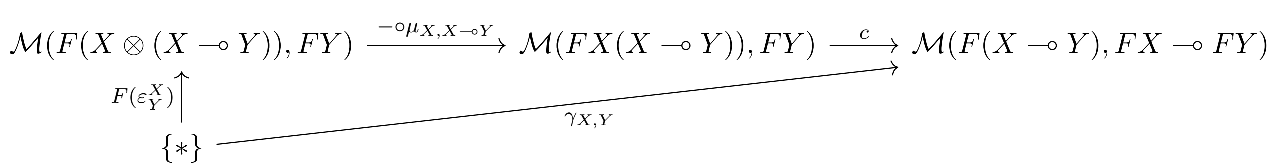
Given monoidal closed categories $\mathcal{M}$ and $\mathcal{N}$ and a lax monoidal functor $(F, \mu, \phi) : \mathcal{N} \to \mathcal{M}$ we have

$$\mathcal{M}(F(X \otimes (X \multimap Y)), FY) \xrightarrow{-\circ \mu_{X, X \multimap Y}} \mathcal{M}(FX \otimes F(X \multimap Y), FY) \cong \mathcal{M}(F(X \multimap Y), FX \multimap FY).$$

So picking $F\varepsilon^X_Y \in \mathcal{M}(F(X \otimes (X \multimap Y)), FY)$, which is guaranteed to exist since $\mathcal{M}$ is monoidal closed, we define

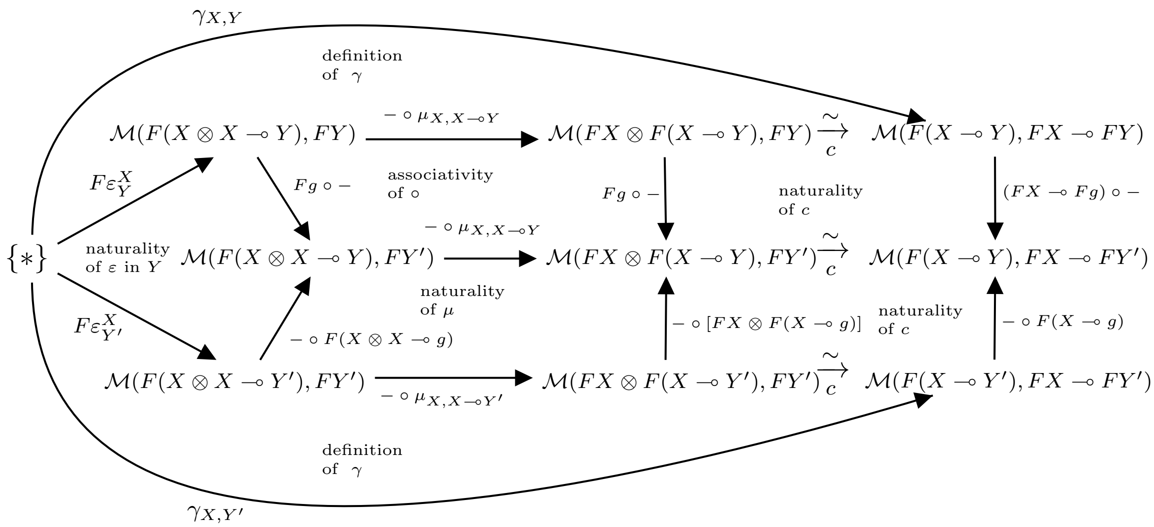
$$\gamma_{X,Y} := c_{FX, F(X \multimap Y), FY}(F\varepsilon^X_Y \circ \mu_{X, X \multimap Y}),$$

that is:

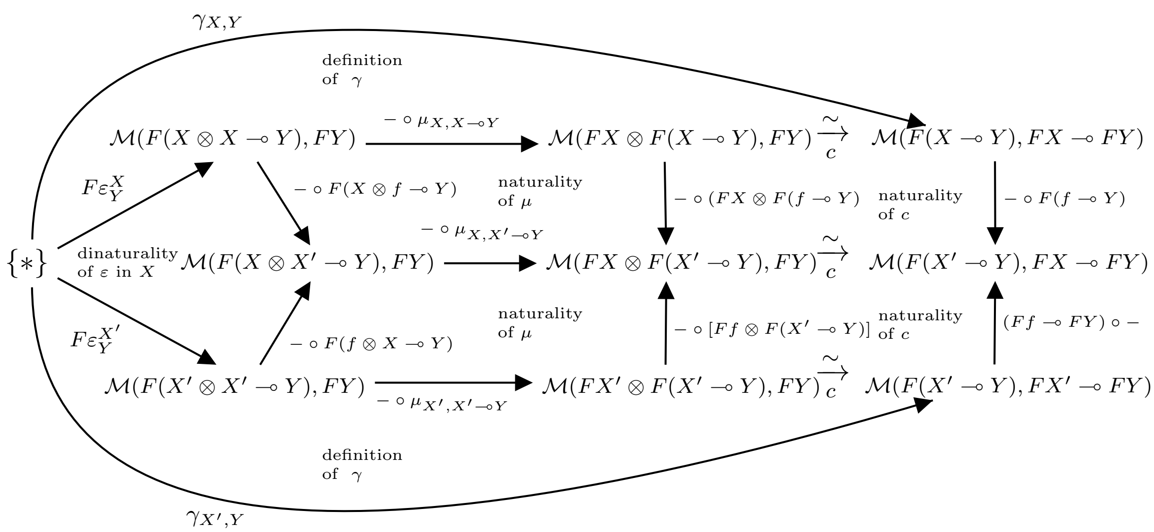


and call γ the **implicator**[4] of F. This is natural in X and Y. To prove this let $f : X \to X'$ and $g : Y \to Y'$ and observe that the outer diagram below commutes

[4]This name is inspired by the use of closed categories in formal logic.



which shows naturality in Y, while the outer diagram below also commutes



which amounts to naturality in X.

Remark 3.4.28. The definition of γ depends on μ, the multiplicator. In the case of strong or strict monoidal functors μ might be an isomorphism or an identity. This though is not sufficient to make γ an isomorphism. So in general lax, strong and strict monoidal functors give rise to **lax closed functors**. On the other hand, there are cases of **strong or strict closed functors**.

Example 3.4.29. If $\mathcal{M}, \mathcal{N}$ are monoidal closed sup-lattices, i.e. quantales, then a lax monoidal functor $F : \mathcal{M} \to \mathcal{N}$ between them is a **lax morphism of quantales**. If F is strong monoidal, then we may speak of a **strong morphism of quantales**. Note that in both these cases the internal hom is, in general, laxly preserved. So when it is strongly preserved we have an even "stronger" version of a strong morphism of quantales. Under lax morphisms we may form the category of quantales, with evident subcategories formed by restricting to stronger versions of morphisms.

If the tensor product of such quantals is cartesian, then these quantales are actually frames. In this case only strong monoidal cocontinuous functors are considered. Since isomorphic objects are equal in posets, strict and strong morphisms are the same. Strong monoidal cocontinuous functors and frames form the category of frames. If frames were to be considered as generalised topological spaces, then their morphisms should be considered as generalised continuous functions. But only inverse images of continuous functions preserve arbitrary joins (i.e. unions of open sets) and finite meets. So such morphisms are considered as "inverse images" of continuous maps and in order to form a category resembling **Top**, we take the opposite of the category of frames. This category is called the category of **Locales**.

Finally, restricting to monoidal closed categories equipped with finite joins, i.e. Heyting algebras, the only version of morphism between them would be the strong one, i.e. strongly preserving the implication.

We now focus on natural transformations. Let $\mathcal{M}, \mathcal{N}$ be monoidal categories and $(F, \mu, \phi), (G, \nu, \psi) : \mathcal{N} \to \mathcal{M}$ be lax monoidal functors. Then, given a natural transformation $\eta : F \Rightarrow G$ there are two evident ways to get from $F(X \multimap Y) \to FX \multimap GY$, namely:

$$(\eta_X \multimap GY) \circ \beta_{X,Y} \circ \eta_{X \multimap Y} \text{ and } (FX \multimap \eta_Y) \circ \gamma_{X,Y},$$

where γ and β are the implicators of F and G, respectively. When these composites are equal we may say that η is a **closed natural transformation**. What is interesting is that if η is monoidal, it is automatically closed.

Proposition 3.4.30. *A monoidal natural transformation of lax monoidal functors between monoidal closed categories is a closed natural transformation.*

Proof. In the following calculation we use naturality of the currying and the multiplicators of F and G, but also the fact that η is monoidal and natural. Observe that:

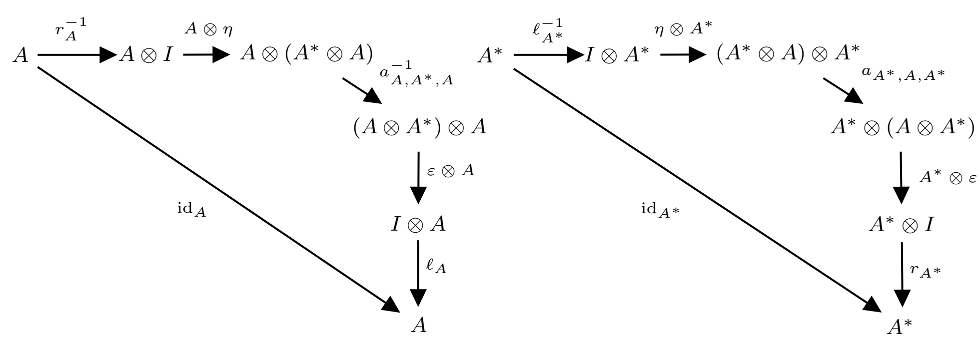
$$\begin{aligned}
(\eta_X \multimap GY) \circ \beta_{X,Y} \circ \eta_{X \multimap Y} &= (\eta_X \multimap GY) \circ c_{GX, GX \multimap Y, GY}(G(\varepsilon_Y^X) \circ \nu_{X, X \multimap Y}) \circ \eta_{X \multimap Y} \\
&= c_{FX, FX \multimap Y, GY}(G(\varepsilon_Y^X) \circ \nu_{X, X \multimap Y} \circ (\eta_X \otimes \eta_{X \multimap Y})) \\
&= c_{FX, FX \multimap Y, GY}(G(\varepsilon_Y^X) \circ (\eta_{X \otimes (X \multimap Y)}) \circ \mu_{X, X \multimap Y}) \\
&= c_{FX, FX \multimap Y, GY}((\eta_Y \circ F(\varepsilon_Y^X) \circ \mu_{X, X \multimap Y}) \\
&= (FX \multimap \eta_Y) \circ \gamma_{X,Y}
\end{aligned}$$

□

3.5 Rigid, pivotal, ribbon and compact closed monoidal categories

Monoidal closed categories can be manipulated in a way to give some sort of inverse or complement to an object as witnessed in examples 3.4.22 and 3.4.23. The notion of a dual for an object, which nevertheless still resembles $A \multimap I$ in **Vect**, can be expressed in a different way.

Definition 3.5.1. *Let $\mathcal{M}$ be a monoidal category and let $A \in \mathcal{M}_0$. An object $A^r \in \mathcal{M}_0$ is called a* ***right dual*** *of A, denoted by $A \dashv A^r$, if it is equipped with morphisms $\eta : I \to A^r \otimes A$, called the* ***counit****, and $\varepsilon : A \otimes A^r \to I$, called the* ***unit****, such that the following diagrams commute:*



called ***snake equations****. The object* A *is called a* ***left dual*** *of* A^r *and* $(A, A^r, \eta, \varepsilon)$ *is called a* ***duality****. If* A *has a left dual,* A^l*, and a right dual* A^r*, then* A *is called* ***rigid****. If* $A^l \cong A^r$*, then* A *and* A^r *are called* ***duals*** *of each other. If* $A \dashv A$*, then* A *is called* ***self-dual****.*

If every object of $\mathcal{M}$ *has a right (resp. left) dual then* $\mathcal{M}$ *is called a* ***right (left) rigid monoidal category****. If every object of* $\mathcal{M}$ *is rigid, then it is called a* ***rigid monoidal category.***

Remark 3.5.2. We might call an object A in a monoidal category $\mathcal{M}$ **dualisable**, if there exists and object $A^r \in \mathcal{M}_0$ such that $A \dashv A^r$.

Remark 3.5.3. In a strict monoidal category $\mathcal{M}$, the above diagrams take the following easier form for a duality $A \dashv A^r$:

$$(\varepsilon \otimes A) \circ (A \otimes \eta) = \mathrm{id}_A \text{ and } (A^r \otimes \varepsilon) \circ (\eta \otimes A^r) = \mathrm{id}_{A^r},$$

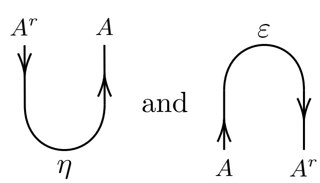
which can also be used in a general monoidal category due to the coherence theorme. These should look more than just familiar, since they are very similar to the triangle laws of an adjunction. To shed some light to this similarity we give the following example.

Example 3.5.4. Given a category $\mathcal{C}$, we already know that the functor category $End(\mathcal{C})$ is a strict monoidal category with composition of functors as the tensor product functor and the identity endofunctor as the unit. A duality in $End(\mathcal{C})$ amounts to the following data: two endofunctors $F, G \in End(\mathcal{C})$ and two natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{C}}$, such that:

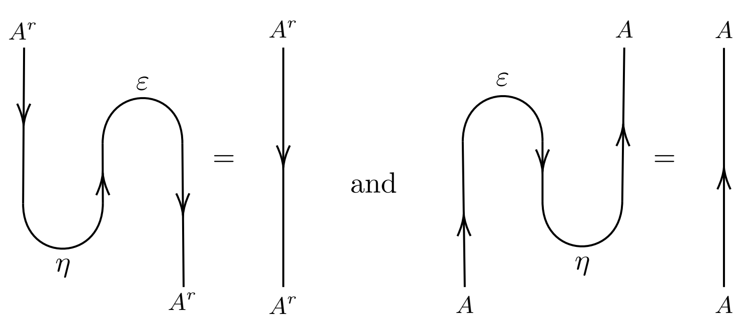
$$(\varepsilon \circ F) \cdot (F \circ \eta) = \mathrm{Id}_F \text{ and } (G \circ \varepsilon) \cdot (\eta \circ G) = \mathrm{Id}_G.$$

These of course amount to the triangle laws of an adjuntion $F \dashv G$. A unification of the two concepts of duality and adjunction is achieved in terms of internal adjunctions at the level of bicategories.

In terms of the graphical language for monoidal categories, the unit and the counit of a duality would be drawn simply as a state and an effect, respectively, of a tensor product of two objects (two wires), which are claimed to be dual. To allow for the duality of two objects to be present in the graphical calculus, the wires representing objects are considered explicitly as directed, but also reversed arrows are introduced to denote the duals. This allows for a representation of the unit and the counit as a **cup** and a **cap**, respectively, as follows:

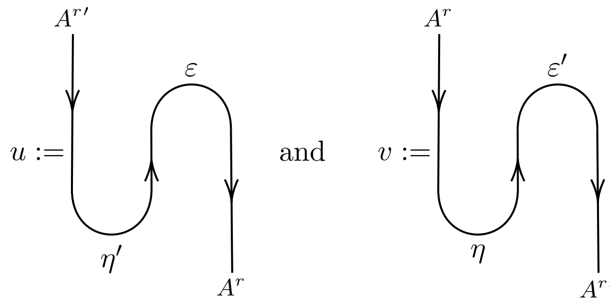


Under these added rules, the snake equations, η and ε must satisfy, are depicted as follows:

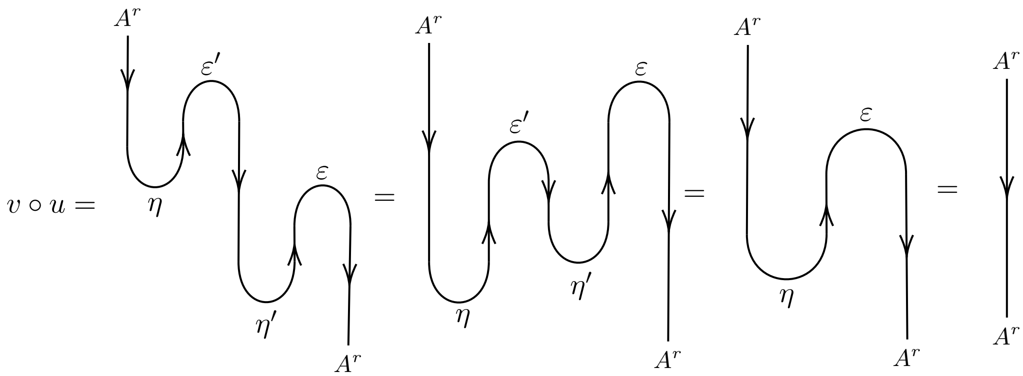


Proposition 3.5.5. *If* $A, A^r, A^{r\prime}$ *are objects in a monoidal category* $\mathcal{M}$ *where* $A \dashv A^r$*, then* $A \dashv A^{r\prime}$ *if and only if* $A^r \cong A^{r\prime}$*.*

Proof. Let $(A, A^r, \eta, \varepsilon)$ and $(A, A^{r\prime}, \eta', \varepsilon')$ be dualities. Define the following morphisms:

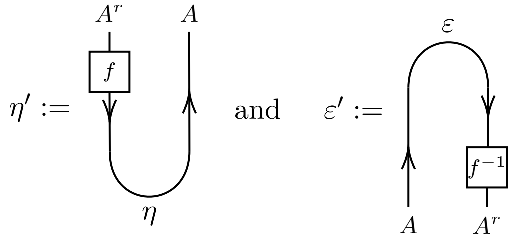


Then the snake equations imply:

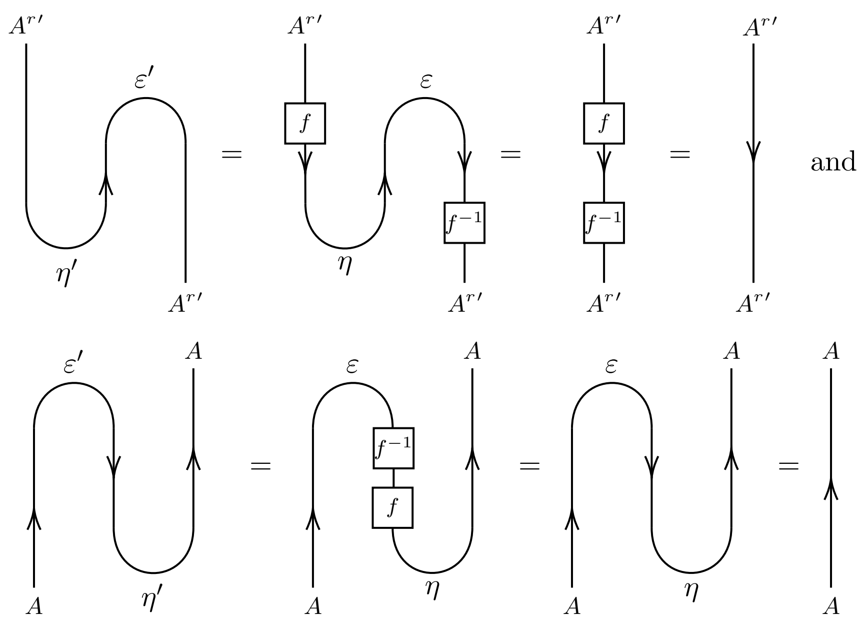


and similarly that $u \circ v = \mathrm{id}_{A^{r\prime}}$.

On the other hand, if $(A, A^r, \eta, \varepsilon)$ is a duality and $f : A^r \to A^{r\prime}$ is an isomorphism, then define:



and observe that:

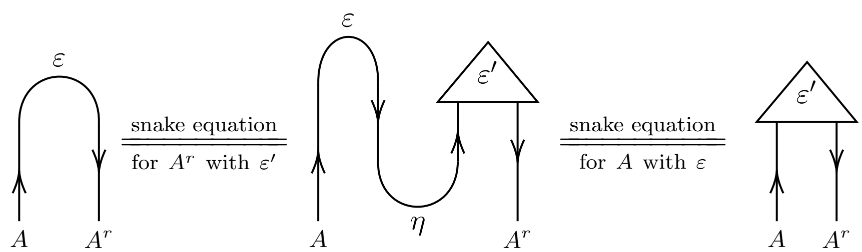


make $(A, A^{r\prime}, \eta', \varepsilon')$ a duality. □

Remark 3.5.6. Similarly, one can prove that left duals are unique up to isomorphism and that every object isomorphic to a dual is again a dual. Furthermore, in the above proof we saw that given an isomorphism between a right dual to A and another object $A^{r\prime} \in \mathcal{M}_0$, we create a unit and a counit η' and ε' so that $(A, A^{r\prime}, \eta', \varepsilon')$ is a duality. A subtle thing to note at this point is that the assignement of units and counits is dependent on the chosen isomorphism $f : A^r \to A^{r\prime}$ in a bijective fashion. This follows very easily from the snake equations. In this context, every isomorphism between right duals is uniquely associated to a unique duality. Concluding, a right dual and similarly a left dual are unique up to (not necessarily unique) isomorphism, but a duality *is unique up to unique isomorphism.* Another way to look at this is given by the following proposition.

Proposition 3.5.7. *Let $\mathcal{M}$ be a monoidal category and let $(A, A^r, \eta, \varepsilon)$ and $(A, A^r, \eta, \varepsilon')$ be dualities. Then $\varepsilon = \varepsilon'$.*

Proof. Since both $(A, A^r, \eta, \varepsilon)$ and $(A, A^r, \eta, \varepsilon')$ are dualities, they both satisfy the snake equations. Thus we get:



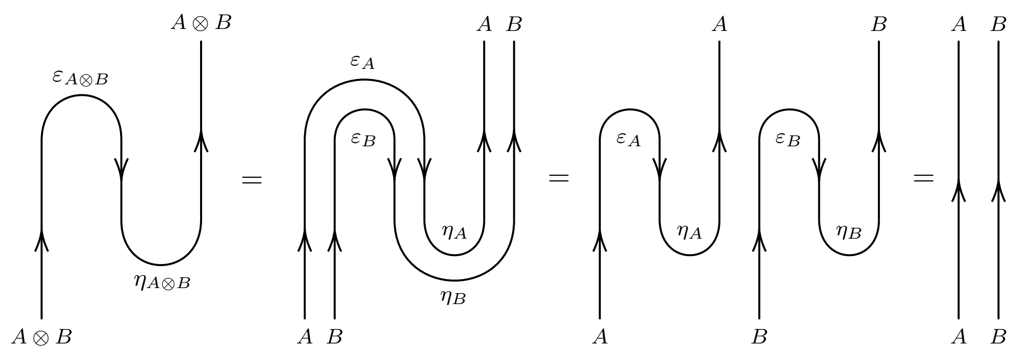
□

Remark 3.5.8. Similarly, one can prove that there is a unique η such that $(A, A^r, \eta, \varepsilon)$ is a duality.

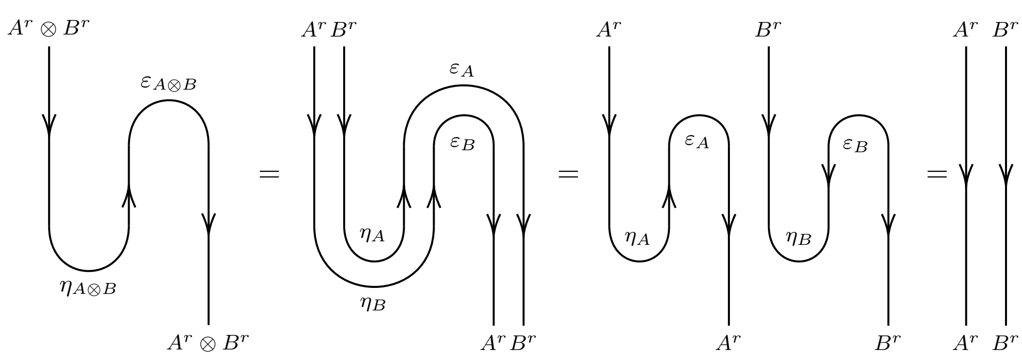
Proposition 3.5.9. *In a monoidal category $\mathcal{M}$ with dualities $(A, A^r, \eta_A, \varepsilon_A)$ and $(B, B^r, \eta_B, \varepsilon_B$ we have*

$$A \otimes B \dashv B^r \otimes A^r.$$

Proof. Let $(A, A^r, \eta_A, \varepsilon_A)$ and $(B, B^r, \eta_B, \varepsilon_B)$. We will define $\eta_{A\otimes B}$ and $\varepsilon_{A\otimes B}$ and prove that $(A \otimes B, B^r \otimes A^r, \eta_{A\otimes B}, \varepsilon_{A\otimes B})$ is a duality, using only the graphical calculus. So define:



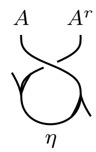
and

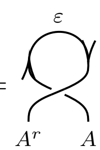


which are the snake equations for $(A \otimes B, B^r \otimes A^r, \eta_{A\otimes B}, \varepsilon_{A\otimes B})$. $\square$

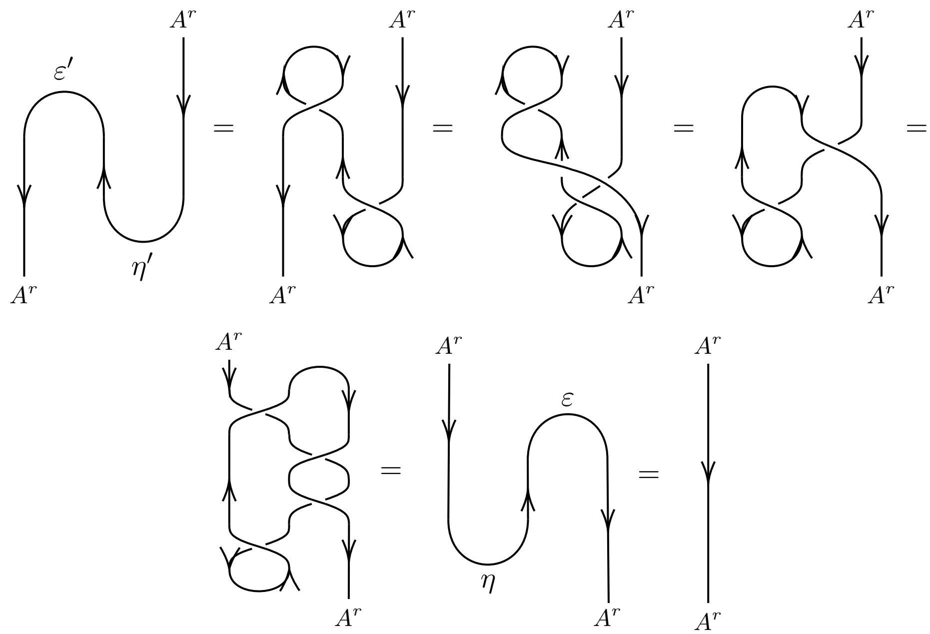
Proposition 3.5.10. *Let $\mathcal{M}$ be a braided monoidal category and $A \in \mathcal{M}_0$. Then A has a left dual A^l if and only if it has a right dual A^r. Furthermore, $A^l \cong A^r$.*

Proof. To prove this we will show diagrammatically that a right dual A^r is equivalently a left dual of A, since by proposition 3.5.5 the isomorphism $A^l \cong A^r$ comes for free. So let $(A, A^r, \eta, \varepsilon)$ be a duality and define:

$\eta' :=$ 

and
$\varepsilon' :=$ 

Since η, η' are states and $\varepsilon, \varepsilon'$ are effects, and by the snake equations for $A \dashv A^r$ we have the following:



and similarly the other snake equation holds. So $A^r \dashv A$. Using the same trick when beginning with a left dual A^l yields $A \dashv A^l$. So in a braided monoidal category $A^l \cong A^r$. $\square$

Proposition 3.5.11. *In any monoidal category $\mathcal{M}$, $I \dashv I$ i.e. the unit is self dual.*

Proof. In a monoidal category the right and the left unitors coincide at the identity. So if we were to pick a unit and a counit such that $(I, I, \eta, \varepsilon)$ is a duality, then the two snake equations would amount to the following composites:

$$I \otimes I \xrightarrow{\eta \otimes I} (I \otimes I) \otimes I \xrightarrow{a_{I,I,I}} I \otimes (I \otimes I) \xrightarrow{I \otimes \varepsilon} I \otimes I \text{ and}$$

$$I \otimes I \xrightarrow{I \otimes \eta} I \otimes (I \otimes I) \xrightarrow{a^{-1}_{I,I,I}} (I \otimes I) \otimes I \xrightarrow{\varepsilon \otimes I} I \otimes I$$

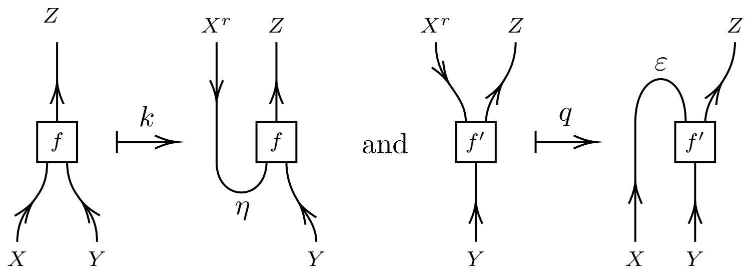
being the identity on $I \otimes I$. But picking $\eta := \ell_I^{-1}$ and $\varepsilon := \ell_I$, both of the above composites are mutual inverses and do equal $\mathrm{id}_{I \otimes I}$ by proposition 2.1.11. $\square$

We have already seen that in a monoidal closed category there is a notion of dual for an object A given by $A \multimap I$. The following proposition sheds light to the connection between a dual for A and $A \multimap I$.

Proposition 3.5.12. *Let $\mathcal{M}$ be a monoidal closed category and suppose $X \in \mathcal{M}_0$ has a right dual A^r with unit η and counit ε. Then for every $Z \in \mathcal{M}_0$,*

$$X \multimap Z \cong X^r \otimes Z.$$

Proof. We will prove that $\mathcal{M}(Y, X \multimap Z) \cong \mathcal{M}(Y, X^r \otimes Z)$ for every $Y \in \mathcal{M}_0$, naturally both in Y and Z. Since closure of $\mathcal{M}$ implies that $\mathcal{M}(Y, X \multimap Z) \cong \mathcal{M}(X \otimes Y, Z)$, we only need to show that $\mathcal{M}(X \otimes Y, Z) \cong \mathcal{M}(Y, X^r \otimes Z)$ naturally. To this end we construct two functions $k : \mathcal{M}(X \otimes Y, Z) \to \mathcal{M}(Y, X^r \otimes Z)$ and $q : \mathcal{M}(Y, X^r \otimes Z) \to \mathcal{M}(X \otimes Y, Z)$ as follows:

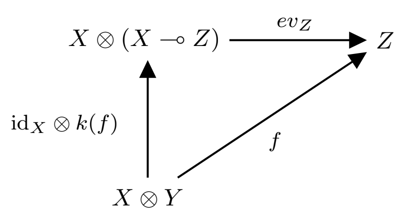


for $f : X \otimes Y \to Z$ and $f' : Y \to X^r \otimes Z$. Using the snake equations it easy to check that k and q are mutual inverses, thus $q = k^{-1}$. Furthermore, k and its inverse are evidently natural in both Y and Z, since in diagrammatic terms pre-composing with any $g : Y' \to Y$ or post-composing with any $h : Z \to Z'$ commutes with k and k^{-1}. So $X \otimes - \dashv X^r \otimes -$ and by uniqueness of adjoints we get that $X \multimap - \cong X^r \otimes -$. $\square$

Remark 3.5.13. Similarly, one can prove that if $X \in \mathcal{M}_0$ has a left dual, X^l, then $- \otimes X \dashv - \otimes X^l$. In addition, the natural transformation k above, actually, provides the currying of the morphism f. Defining the evaluation morphism as:

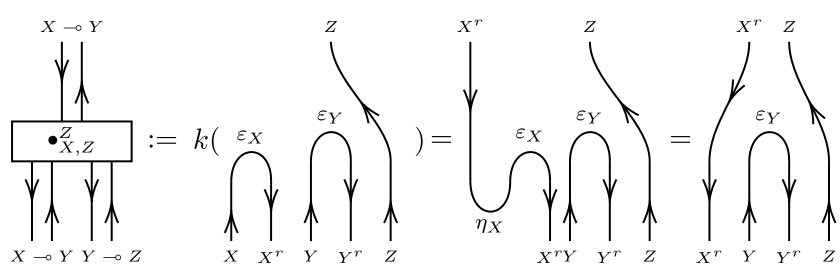
$$ev_Z := \ell_z \circ (\varepsilon \otimes Z) \circ a^{-1}_{X,X^r,Z}$$

we can see that we get the universal property for exponentials, i.e. the following diagram commutes:



Note also that the above proof gives an exponential object, independently of $\mathcal{M}$ being monoidal closed, so the existence of duals implies the existence of exponentials.

Going a little further, we may depict the internal composition morphism, assuming X and Y have right duals, as the following:

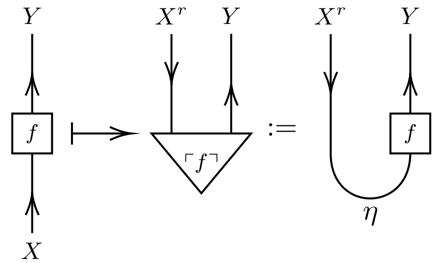


which is of a much simpler form.

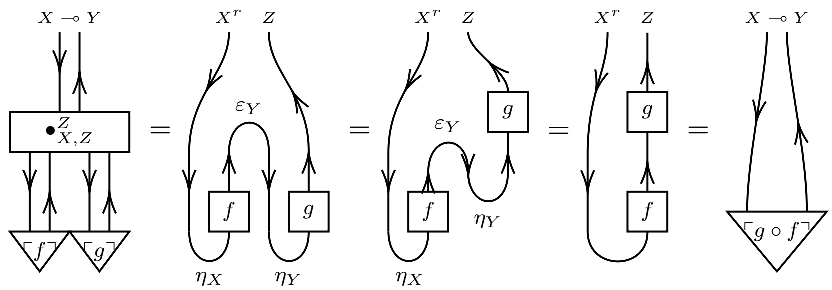
Corollary 3.5.14. *In a monoidal closed category $\mathcal{M}$, if an object X has a right dual, then $X^r \cong X \multimap I$.*

Corollary 3.5.15. *A (right/left) rigid monoidal category is a (right/left) monoidal closed category, whose internal hom is given by tensoring with the dual.*

Every hom-set in a monoidal closed category is in bijection with the set of states of a tensor product object via the name construction. Given a duality $X \dashv X^r$ and a morphism $f : X \to Y$, the name (see remark 3.4.4) of f takes a simple form using string diagrams. That is:



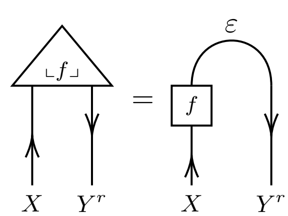
The simple diagrammatic form of names is in exact accordance with the definition of a name in a monoidal closed category. This can easily be deduced by the fact that $\ulcorner f \urcorner = k(f \circ r_X)$ and remark 3.5.13. In the graphical language we have introduced, given $f \in \mathcal{M}(X, Y)$ and $g \in \mathcal{M}(Y, Z)$, where X and Y are dualisable, we can compose names as follows:



which is again in exact accordance with the corresponding property in monoidal closed categories, as in 3.4.10. On the other hand, if the codomain of a morphism has a right dual, we can define a dual notion to tha of names, namely conames.

Furthermore, the isomorphism $\ulcorner - \urcorner : \mathcal{M}(X, Y) \to \mathcal{M}(I, X^r \otimes Y)$ restates the classic result that any linear map between finite dimensional vector spaces can be written as the tensor product of a linear functional and a vector in the target vector space.

Definition 3.5.16. *Let $\mathcal{M}$ be a monoidal category, $X, Y \in \mathcal{M}_0$ be objects such that Y has a right dual and let $f : X \to Y$ be a morphism. The state $\llcorner f \lrcorner : X \otimes Y^r \to I$ is called the **coname** of f if:*



Remark 3.5.17. Any morphism can be assigned to and retrieved from its coname, by applying the snake equations, similarly to how they were applied in the proof of 3.5.12. Plainly, there is a bijection

$$\llcorner \lrcorner : \mathcal{M}(X, Y) \cong \mathcal{M}(X \otimes Y^r, I).$$

Finally, we could also have defined names and conames using left duals instead of right duals. Although we will not carefully distinguish between "left" or "right" names and conames, this distinction will be helpful in discussing the transpose of a morphism subsequently.

The above propositions and the definition of conames are crucial in making the following examples and non-examples clearer.

Example 3.5.18. Every object $V \in$ **Vect** has a dual vector space, $V^* = V \multimap I$, but only finite dimensional vector spaces are dualisable[5]. To illustrate this, let $V \in$ **FdVect**, V^* be its dual vector space, $\{v_i\}_{i=1}^{\dim V}$ be

[5] This is due to the fact that a unit is ill-defined in a infinite dimensional vector space, due to the fact that it should be an infinite sum

a basis for V and $\{f_j\}_{j=1}^{\dim V}$ be the dual basis of V^*. Define $\varepsilon : V \otimes V^* \to I$ as the linear function[6] for which

$$V \otimes V^* \ni v \otimes f \mapsto f(v)$$

and η as the linear extension of

$$I \ni 1 \mapsto \sum_{i=1}^{\dim V} f_i \otimes v_i.$$

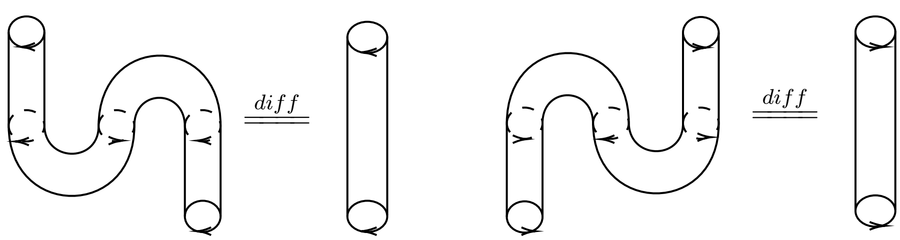
Rather informally,

$$v_j \mapsto v_j \otimes \sum_{i=1}^{\dim V} f_i \otimes v_i = \sum_{i=1}^{\dim V} v_j \otimes f_i \otimes v_i \mapsto \sum_{i=1}^{\dim V} f_i(v_j) \otimes v_i = \sum_{i=1}^{\dim V} \delta_{i,j} v_i = v_j$$

is the identity on V, which gives the one snake equation, while the other snake equation holds similarly. Therefore, every object of **FdVect** is dualisable and since this is a symmetric monoidal category left and right duals give the dual vector space making **FdVect** a rigid monoidal category. It is worth noting that the definition of η looks basis dependent, but remark 3.5.8 assures that it isn't.

Example 3.5.19. A very similar example to **FdVect** is **FdHilb**. Again, only finite dimensional Hilbert spaces have dual objects and the units and counits are constructed in the same way, where the pairing is redefined in terms of the inner product. Thus, **FdHilb** is rigid.

Example 3.5.20. The category n**Cob** has duals for every object. The dual of a closed manifold is the same manifold with its orientation reversed. The unit and counit of the duality are given by bending the identity cobordism of the original manifold so that the input becomes a component of the output and vice versa. We illustrate this for the $n = 2$ case.



Example 3.5.21. As we have already seen a rigid monoidal category is also a monoidal closed category. Therefore a discrete rigid monoidal category is a group, as was the case for discrete monoidal closed categories (see example 3.4.22).

Example 3.5.22. A monoidal preorder which is left and right rigid is a preordered monoid that additionally satisfies:

$$I \leq a^r \otimes a, \quad a \otimes a^r \leq I \text{ and}$$
$$I \leq a \otimes a^l, \quad a^l \otimes a \leq I,$$

for every $a \in \mathcal{M}_0$, where a^l, a^r denote the left and right dual of a, respectively. Such a category is called a **pregroup** if it is skeletal.

[6] This is actually the unique linear map associated to the bilinear map $< -, - >: V \times V^* \to I$ called *pairing*.

Example 3.5.23. In the category of sets with its cartesian structure, the only objects with right duals are the one element sets. This is a consequence of terminality and the isomorphism $B^A \cong \{*\}^{(A\times B^r)} \cong \{*\}$ for every $A, B \in \mathbf{Set}$, where B has a right dual. Therefore, neither **Set** nor **FinSet** are rigid.

The argument in the above example hints at something more general, which has a nice interpretation in the context of reconstructing quantum theory in terms of monoidal categories.

Theorem 3.5.24 (No Deleting)**.** *A semi-cartesian category which is rigid, is necessarily a (total) preorder.*

Proof. Let $\mathcal{M}$ be semi-cartesian and rigid. Then the coname construction implies that for every $A, B \in \mathcal{M}_0$

$$\mathcal{M}(A, B) \cong \mathcal{M}(A \otimes B^r, I).$$

But I being terminal reveals that $\mathcal{M}(A, B) \cong \{*\}$, thus $\mathcal{M}$ is a monoidal preorder. □

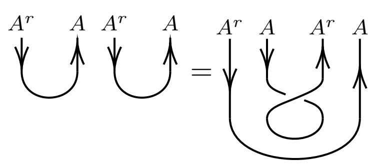
Remark 3.5.25. This theorem should be considered as a dual of proposition 3.4.5, which can only be formulated in rigid categories instead of general monoidal closed categories.

Remark 3.5.26. An immediate conclusion drawn by the above theorem is that rigid cartesian categories are total preorders. This shows that nontrivial rigid structures can only exist in monoidal categories whose tensor product is not cartesian and that monoidal closed categories are a nontrivial generalisation of cartesian closed categories.

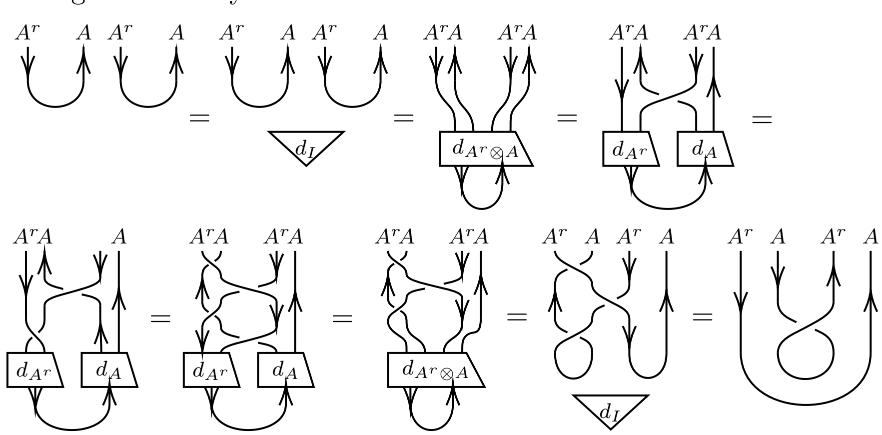
Remark 3.5.27. The above theorem reveals that the existence of uniform deleting in a rigid monoidal category trivializes the whole category, in the sense that it makes it a total preorder. Thus every reformulation of even a fragment of quantum theory in terms of rigid monoidal categories is either doomed to not be able to express a multitude of morphisms between quantum systems or it has to not allow for uniform deleting, a fact that is true for quantum systems. Hence, the above theorem is called the "No Deleting" theorem.

Another way in which cartesian closed categories differ from braided monoidal closed categories is given by the "no cloning" theorem for braided rigid monoidal categories. This theorem is also very important to categorical quantum mechanics. To show this we firstly prove two lemmas.

Lemma 3.5.28. *If a braided rigid monoidal category $\mathcal{M}$ has uniform copying, then the following equality holds for every object $A \in \mathcal{M}$:*



Proof. We show this diagrammaticaly as follows:

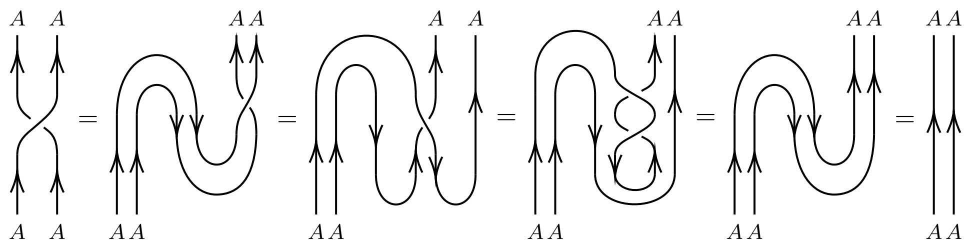


where the first, the third and the sixth equality follow from the fact that d is a monoidal natural transformation, the second and the seventh equality follow from naturality of d, the fourth follows from "cocommutativity" of d and the rest follow by isotopy. □

Using the above lemma we may prove the following about the diagonal components of the braiding.

Lemma 3.5.29. *Let $\mathcal{M}$ be a braided rigid monoidal category. If $\mathcal{M}$ has uniform copying, then the diagonal components of the braiding are identity morphisms.*

Proof. Let $A \in \mathcal{M}_0$. Then:

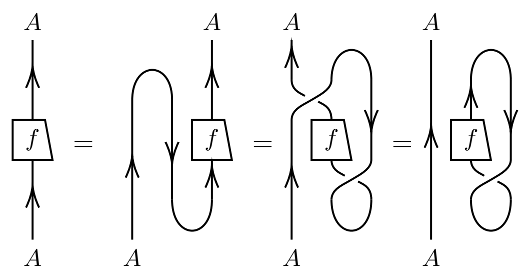


where the third equality follows from lemma 3.5.28 and the rest follow by isotopy. □

We can now prove the "No Cloning" theorem for braided rigid monoidal categories.

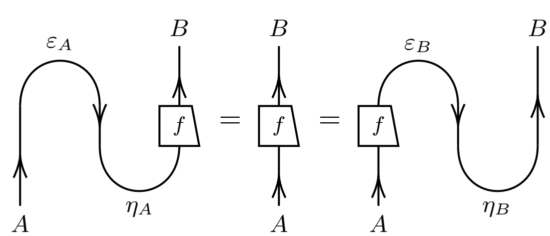
Theorem 3.5.30 (No Cloning)**.** *Let $\mathcal{M}$ be a braided rigid monoidal category. If $\mathcal{M}$ has uniform copying, then every endomorphism is a scalar multiple of the identity endomorphism.*

Proof. Let $A \in \mathcal{M}_0$ and $f : A \to A$. Then:



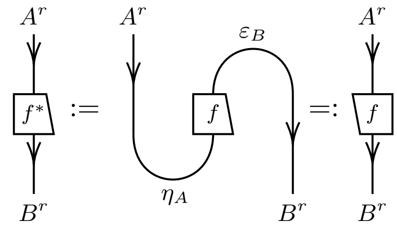
where the third equality follows from lemma 3.5.29 and the others by isotopy. □

Given a morphism between (right-)dualisable objects $f : A \to B$, we may create its name and its coname. The original morphism can be reconstructed by its name and its coname by post-composing the A^r-output object with the counit ε_A or by pre-composing the B^r-input object with the unit η_B.



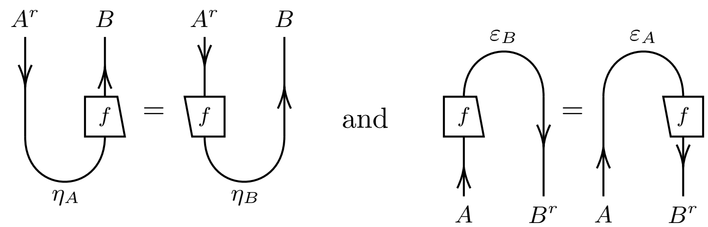
Observe that the morphism f is not depicted as a box but as a trapezoid whose largest edge is down. We will adopt this notation to distinguish from the case where instead of using units and counits to reconstruct the original morphism we perform the following trick. We will also use quadrilaterals for states for the same reason.

Definition 3.5.31. *Let $\mathcal{M}$ be a monoidal category, $A, B \in \mathcal{M}_0$ be right dualisable objects and $f \in \mathcal{M}(A, B)$. The **transpose** of f is a morphism $f^* : B^r \to A^r$ defined as:*



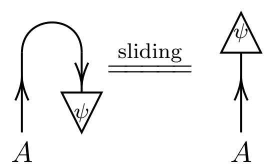
depicted by a trapezoid whose largest edge is up.

Remark 3.5.32. Observe that given a morphism f between right dualisable objects we can form its transpose f^*, which is then a morphism between left dualisable objects, by 3.5.6. So we can form its left name and coname, according to 3.5.17. These are easily seen to be related to the right name and coname of f, using the triangle equations and the definitions of name and coname. Thus, we have the following property:



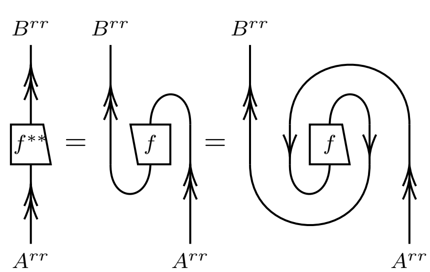
which we will call **sliding**.

Remark 3.5.33. Interestingly, the counit of a duality provides an abstract form of the pairing between a vector space and its dual. conames and sliding provide a way to associate a state of an object A^* to an effect of A as follows:



Note that the coname of the state ψ is also its transpose. So given states $\phi : I \to A$ and $\psi : I \to A^*$ we see that the scalar $\varepsilon \circ (\phi \otimes \psi)$ is the scalar $\psi^* \circ \phi$. Also note that this generally is not the inner product of states.

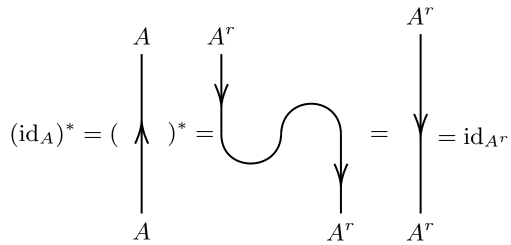
Remark 3.5.34. In a rigid monoidal category, right duals have right duals which are generally called **double duals**. So in the case where two objects of a monoidal category are doubly dualisable, we can use the graphical calculus to express the *double transpose* of a morphism between them. Note that a double dual will be depicted as a wire with a double arrow.



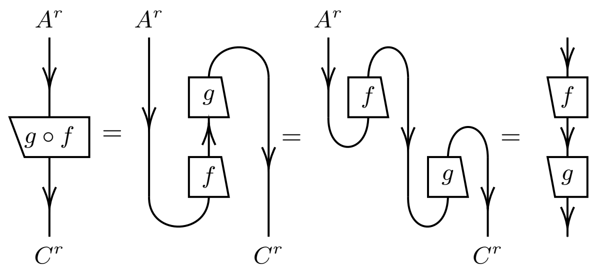
We now focus on functors between rigid monoidal categories and their transformations.

Proposition 3.5.35. *In a (right-)rigid monoidal category, the assignments $(-)^r : \mathcal{M}_0 \to \mathcal{M}_0$ and $(-)^* : \mathcal{M}(A, B) \to \mathcal{M}(B, A)$ constitute a contravariant functor, called the **right duals functor**.*

Proof. We will prove this graphically. The functoriality condition concerning identities holds since for every $A \in \mathcal{M}_0$ we have

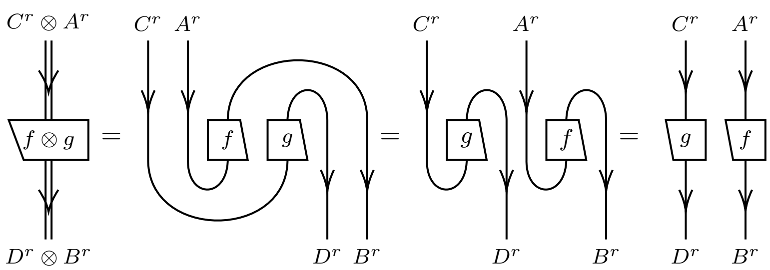


Now let $f \in \mathcal{M}(A, B)$ and $g \in \mathcal{M}(B, C)$. Then:



thus $(g \circ f)^* = f^* \circ g^*$. $\square$

Remark 3.5.36. The right duals functor is a contravariant functor of type $(-)^* : \mathcal{M} \to \mathcal{M}$. From the following graphical manipulation for $(f, g) \in \mathcal{M}(A, B) \times \mathcal{M}(C, D)$, we can see that it is also a monoidal functor, but it is not an endofunctor of $\mathcal{M}$.



Instead, given the above and 3.5.9, it is a contravariant monoidal functor of type $\mathcal{M}^{\text{rev}} \to \mathcal{M}$, or $\mathcal{M} \to \mathcal{M}^{\text{rev}}$ depending on preference.

The monoidal functor axioms follow from the following facts. When we have a choice of a right dual for every object of a rigid monoidal category, we get a right duals functor. Since dualities are unique up to unique isomorphism, according to 3.5.5, there is a unique isomorphism $(X \otimes Y)^r \to Y^r \otimes X^r$ for every $X, Y \in \mathcal{M}_0$, which constitutes the multiplicator of the right duals functor. The unit of a monoidal category is self dual, so the unique isomorphism $I^r \cong I$ provides the unit constraint. Thus combining all the above with the coherence theorem (or the trinagle law) for monoidal categories we get that the coherence conditions for a strong monoidal functor are satisfied.

Generally, in a right rigid monoidal category every object has a right dual. So right duals have right duals, which are in the image of the right duals functor composed with itself. But the functor $(-)^{**} : \mathcal{M} \to \mathcal{M}^{\text{rev}} \to \mathcal{M}$ is a covariant endofunctor in $\mathcal{M}$ which is also strong monoidal, as a composite of strong monoidal functors.

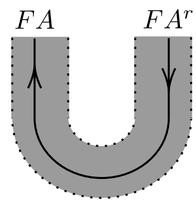
Definition 3.5.37. *Let $\mathcal{M}$ be a rigid monoidal category. The functor $(-)^{**} : \mathcal{M} \to \mathcal{M}$ is called the* ***double duals functor****.*

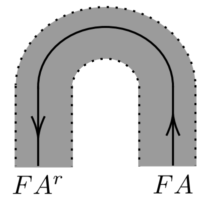
We denote the uniquely defined multiplicator of $(-)^{**}$ by $\phi : (-)^{**} \otimes (-)^{**} \Rightarrow (- \otimes -)^{**}$ and the uniquely defined unitor by ϕ_I.

The right duals functor preserves duals. That is, in a rigid monoidal category, given $X \in \mathcal{M}_0$ such that $X \dashv X^r$, then $X^r \dashv (X^r)^r$. Preservation of duals is a general property all strong monoidal functors possess.

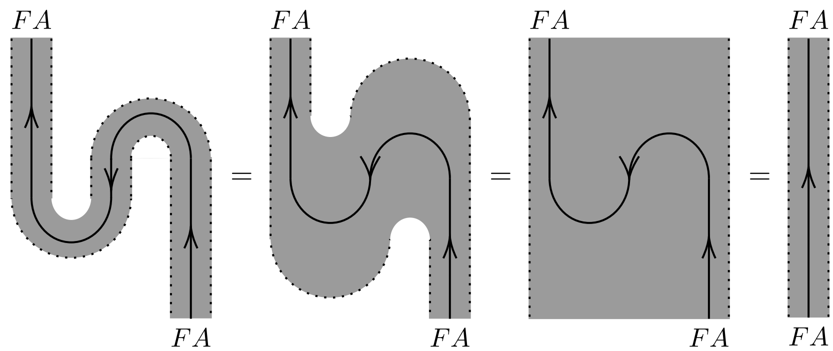
Proposition 3.5.38. *Let $\mathcal{M},\mathcal{N}$ be monoidal categories, $(F,\mu,\phi):\mathcal{M}\to\mathcal{N}$ a strong monoidal functor and let $(A,A^r,\eta_A,\varepsilon_A)$ be a duality in $\mathcal{M}$. Then $(FA,FA^r,\mu^{-1}_{A^r,A}\circ F\eta_A\circ\phi,\phi^{-1}\circ F\varepsilon_A\circ\mu_{A,A^r})$ is a duality in $\mathcal{N}$.*

Proof. We will prove this graphically. Observe that for the new unit and counit in $\mathcal{N}$ we have:

$\mu^{-1}_{A,A^r}\circ F\eta_X\circ\phi =$ 

$\phi^{-1}\circ F\varepsilon_X\circ\mu_{A^r,A} =$ 

Now the first snake equation is satisfied because:

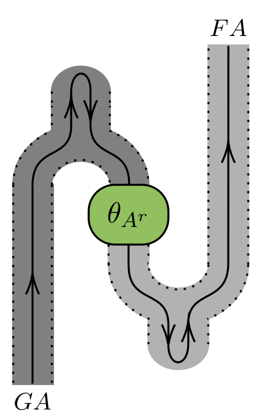


and similarly one can prove the second one. □

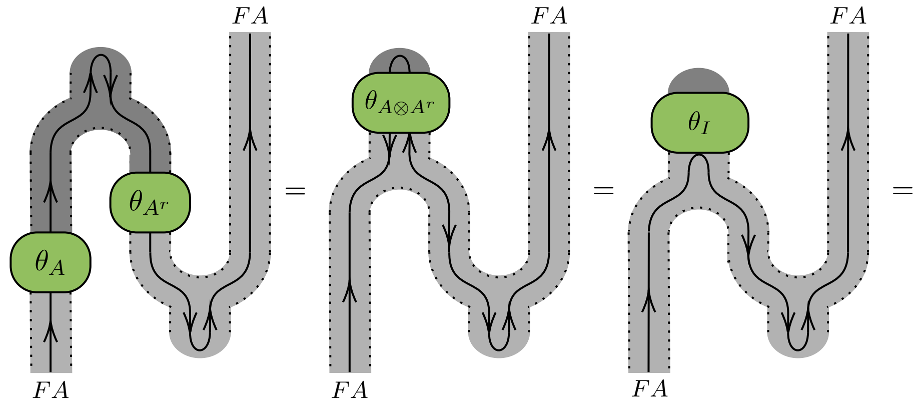
In the cases of braided and symmetric monoidal categories there was a distinct notion of braided monoidal functor, but monoidal natural transformations between braided/symmetric monoidal functors were in a sense already braided/symmetric. The case is different for closed and rigid monoidal categories when considering monoidal functors. In closed categories only lax monoidal functors were closed. As we saw above any kind of monoidal functor preserves duals, so monoidal functors between rigid monoidal categories are already "rigid". Interestingly monoidal natural transformations between strong monoidal functors between rigid monoidal categories are natural isomorphisms. We prove a slightly more general result in what follows.

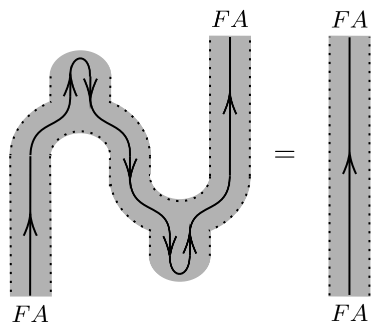
Proposition 3.5.39. *Let $\mathcal{M},\mathcal{N}$ be monoidal categories, $(F,\mu,\phi),(G,\nu,\chi):\mathcal{M}\to\mathcal{N}$ be strong monoidal functors, $(A,A^r,\eta_A,\varepsilon_A)$ be a duality in $\mathcal{M}$ and let $\theta:F\Rightarrow G$ be a monoidal natural transformation. Then θ_A and θ_{A^r} are invertible.*

Proof. We will prove this using the graphical calculus for monoidal natural transformations. Firstly define the following morphism



which is clearly $(\theta_{A^r})^*$, the transpose of θ_{A^r}. Now observe that precomposing with θ_A gives:





and similarly post-composing with θ_A results in the identity on FA^r. Using the other snake equation one might prove that θ_{A^r} is also invertible. □

Corollary 3.5.40. *If $\mathcal{M}$ and $\mathcal{N}$ are rigid monoidal categories, then* $\mathbf{MonCat}(\mathcal{M},\mathcal{N})$ *is a groupoid.*

We saw in proposition 3.5.12, that rigid monoidal categories are monoidal closed categories, where the tensor-hom adjunction is given as $X \otimes - \dashv X^r \otimes -$, for every object X. This allows us to restate the results in 3.4.25 and 3.4.27 about the coexistence of a monoidal and a semi-additive structure in the context of rigid categories.

Proposition 3.5.41. *Let $\mathcal{M}$ be a rigid monoidal category with a semi-additive structure. Then for every $A, B, C, D \in \mathcal{M}_0$ and every $f, f' \in \mathcal{M}(A, B)$ and $g, g' \in \mathcal{M}(C, D)$:*

1. $A \otimes 0 \cong 0 \cong 0 \otimes A$
2. $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$ *and* $(B \oplus C) \otimes A \cong (B \otimes A) \oplus (C \otimes A)$
3. $(A \oplus B)^r \cong A^r \oplus B^r$
4. $f \otimes 0_{C,D} = 0_{A\otimes C, B\otimes D} = 0_{A,B} \otimes g$
5. $f \otimes (g + g') = f \otimes g + f \otimes g'$ *and* $(f + f') \otimes g = f \otimes g + f' \otimes g$

Remark 3.5.42. Note that the first two properties are a consequence of proposition 3.4.3 and proposition 1.6.19. The third property follows from the fact that

$$(A \oplus B)^r \cong (A \oplus B) \multimap I \cong (A \multimap I) \oplus (B \multimap I) \cong A^r \oplus B^r.$$

Furthermore, all of the above hold under weaker conditions. What is only needed is that the domains or the codomains of the morphisms involved have duals.

Proposition 3.5.43. *In a monoidal category with a zero object,* $0 \dashv 0$.

Proof. Terminality and initiality of the zero object imply that every morphism involved in the snake diagrams is uniquely defined, and by this uniqueness the triangle laws hold. □

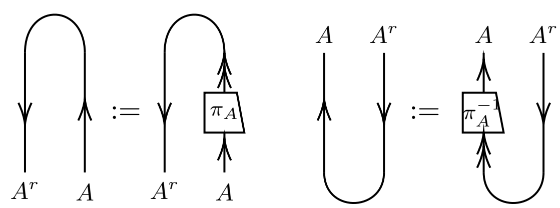
In some rigid monoidal categories such as **FdVect** or **FdHilb**, every object is isomorphic to its double dual. This does not follow from the axioms of a rigid monoidal category, but it is a special case.

Definition 3.5.44. *Let $\mathcal{M}$ be a (right-)rigid monoidal category. We call $\mathcal{M}$* ***pivotal*** *if it is equipped with a monoidal natural transformation of type $\pi : \mathbb{1}_{\mathcal{M}} \Rightarrow (-)^{**}$. This monoidal natural transformation is called a* ***pivot****.*

Remark 3.5.45. A pivot is a monoidal natural transformation of strong monoidal functors between rigid monoidal categories. Thus, according to 3.5.39, π is a natural isomorphism.

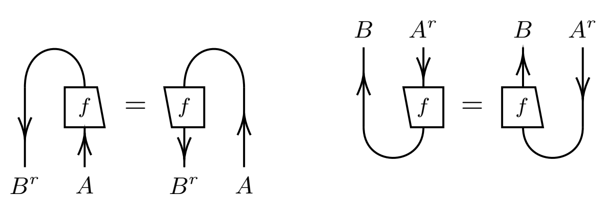
Remark 3.5.46. Every functor preserves isomorphisms, by proposition 1.2.2. Thus the image of the components of a pivot under an arbitrary functor between rigid monoidal categories are isomorphisms. According to proposition 3.5.38, though, strong monoidal functors are needed to preserve duals. Furthermore, strong monoidal functors have enough structure so that a pivot is transported through them. Note that the pivot provided this way might not coincide with a pre-existing pivot on the target category.

A double dual is a right dual of a right dual. Pre-composing the counit of such a duality with a pivot and post-composing the unit with the inverse of the pivot we get morphisms of type $A^r \otimes A \to I$ and $I \to A \otimes A^r$, for every object A of a pivotal monoidal category. We may interpret this as a way to bend wires in the opposite direction than that allowed by the string diagram calculus for rigid monoidal categories. This is achieved through the following definitions/identifications:

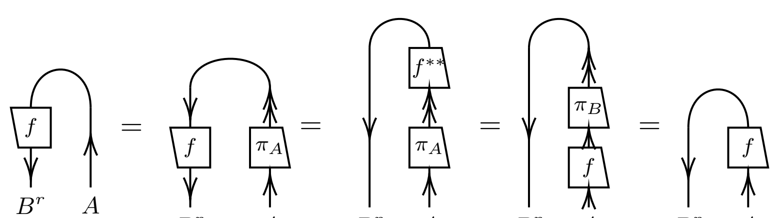


This allows us to generalise the sliding property for names and conames. It also allows us to explicitly construct the duality $A^r \dashv A$ which is guaranteed to exist by the pivot.

Lemma 3.5.47. *Let $\mathcal{M}$ be a pivotal monoidal category. Then for every $f : A \to B$*



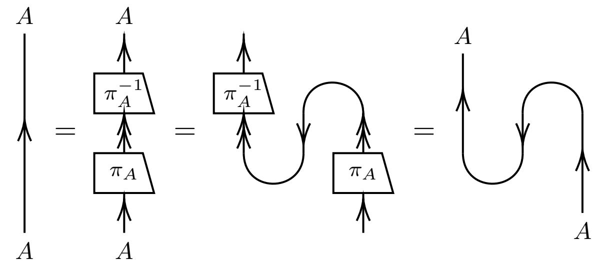
Proof. For the proof of this we use the sliding for names and conames and naturality of the pivot. So the first one goes as follows.



The second one follows from similar arguements. □

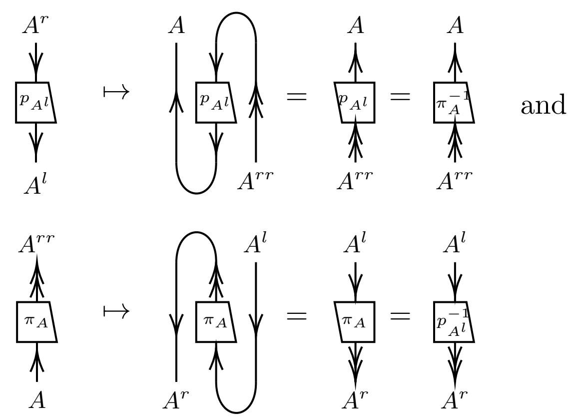
Lemma 3.5.48. *In a pivotal monoidal category every right dual is a left dual.*

Proof. We will prove this using the graphical calculus and the opposite bending introduced above. The first snake equation is shown as follows:

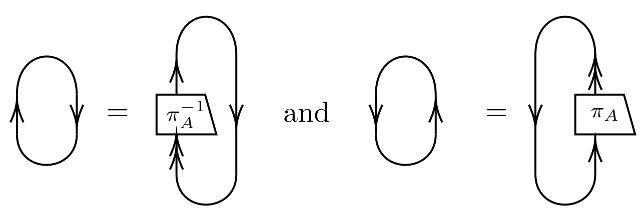


and the second one follows similarly. □

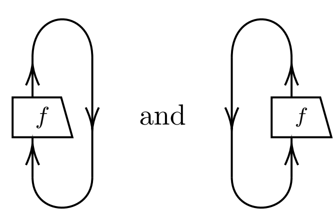
Remark 3.5.49. According to the above lemma, a pivot and uniqueness of duals up to isomorphism, provides isomorphisms between left and right duals. In case, $\mathcal{M}$ is rigid, i.e. has right and left duals for every object, a pivot may also be defined as a monoidal natural transformation relating left duals to right duals. That is, we may view a pivot as being of shape $p : (-)^l \Rightarrow (-)^r$. This equivalence can be achieved explicitly by taking transposes as follows.



The graphical calculus for pivotal categories allows us to form loops for every object such as:

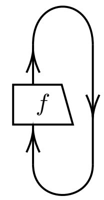


Obviously these loops are states. Furthermore, for every morphism $f : A \to B$ we may form two such loops.



We give the first one a special name.

Definition 3.5.50. *Let $\mathcal{M}$ be a pivotal monoidal category and $f : A \to A$ an endomorphism. The **trace** of f is the scalar:*

$Tr(f) :=$ 

Remark 3.5.51. Non-graphically the trace of an endomorphism $f : A \to A$ is given as the following composite:

$$Tr(f) := \varepsilon_A \circ [(f \circ \pi_A^{-1}) \otimes \mathrm{id}_{A^r}] \circ \eta_{A^r} : II$$

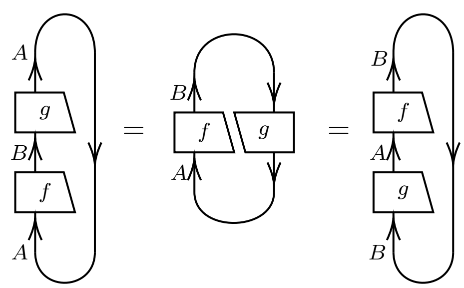
This shows the pivot can be a means to define traces. Also note that generally the trace so defined depends on the chosen dualities.

This kind of trace shares some common properties with the notion of trace in **FdHilb** and **FdVect**.

Lemma 3.5.52. *Let $\mathcal{M}$ be a pivotal monoidal category. For every $f : A \to B$, $g : B \to A$, $h, k : A \to A$, $h' : B \to B$ and $s \in \mathcal{M}(I, I)$ the following hold:*

- $Tr(g \circ f) = Tr(f \circ g)$, *for every $f : A \to B$ and $g : B \to A$ (cyclicity)*
- $Tr(s) = s$ *for every $s \in \mathcal{M}(I, I)$*
- *If there is a zero object, then $Tr(0_{A,A}) = 0_{I,I}$.*
- *If there is a superposition rule, then $Tr(h + k) = Tr(h) + Tr(k)$ (linearity).*
- $Tr(s \cdot h) = s \cdot Tr(h)$ *for every $h : A \to A$ and $s \in \mathcal{M}(I, I)$.*
- *If $\mathcal{M}$ is braided, then $Tr(f \otimes g) = Tr(f) \cdot Tr(g) = Tr(f) \circ Tr(g)$.*

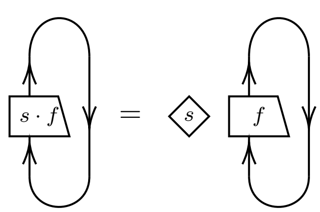
Proof. We will prove the above diagrammatically. For the first one observe that:

$Tr(g \circ f) :=$  $= Tr(f \circ g)$

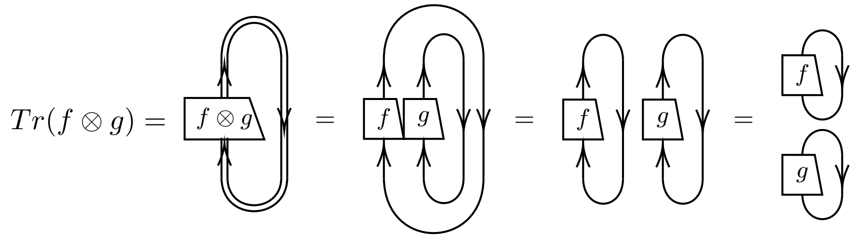
due to sliding. The second property follows trivially by $I \dashv I$ and the third one is a consequence of $0_{A,A} \otimes \mathrm{id}_{A^r} = 0_{A \otimes A^r, A \otimes A^r}$. The fourth follows from distributivity of composition and the tensor over the superposition rule and from the following manipulation:

$$\begin{aligned} Tr(h + k) &= \varepsilon_A \circ [((f + g) \circ \pi_A^{-1}) \otimes A^r] \circ \eta_{A^r} \\ &= \varepsilon_A \circ [(f \circ \pi_A^{-1}) \otimes A^r] \circ \eta_{A^r} + \varepsilon_A \circ [(g \circ \pi_A^{-1}) \otimes A^r] \circ \eta_{A^r} \\ &= Tr(f) + Tr(g). \end{aligned}$$

For the fifth one observe that:

$Tr(s \cdot f) =$  $= s \otimes Tr(f)$

For the last one observe that:



∎

Corollary 3.5.53. *In a semi-additive pivotal category $\mathcal{M}$, for every $f : A \oplus B \to A \oplus B$ and $g : A \to A,\ h : B \to B$,*

$$Tr(f) = Tr(f_{1,1}) + Tr(f_{2,2}) \text{ and } Tr(g \oplus h) = Tr(g) + Tr(h).$$

Proof. The first one follows from cyclicity and linearity of a trace in the presence of a superposition rule, but also from $Tr(0_{A,A}) = 0_{I,I}$. Explicitly we have

$$\begin{aligned} Tr(f) &= Tr(\sum_{i,j} \pi_i \circ f \circ \iota_j) \\ &= \sum_{i,j} Tr(\pi_i \circ f \circ \iota_j) \\ &= Tr(f_{1,1}) + Tr(f_{2,2}) + \sum_{i \neq j} Tr(f \circ \iota_j \circ \pi_i) \\ &= Tr(f_{1,1}) + Tr(f_{2,2}) + 0_{I,I} + 0_{I,I} \\ &= Tr(f_{1,1}) + Tr(f_{2,2}). \end{aligned}$$

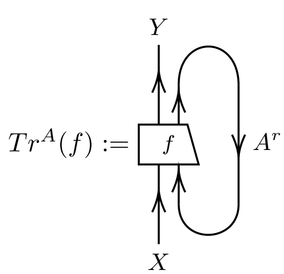
The second equation is an immediate consequence of the first one and the fact that

$$g \oplus h = \begin{pmatrix} g & 0_{B,A} \\ 0_{A,B} & h \end{pmatrix}.$$

∎

The last two properties in the above lemma hint at a more general notion of trace.

Definition 3.5.54. *Let $\mathcal{M}$ be a pivotal monoidal category and $f : X \otimes A \to Y \otimes A$. The morphism:*



is called the ***(partial) trace*** *of f* ***with respect to*** *A.*

Remark 3.5.55. It is immediate that the above definition of trace, subsumes the previous one. This follows from the fact that

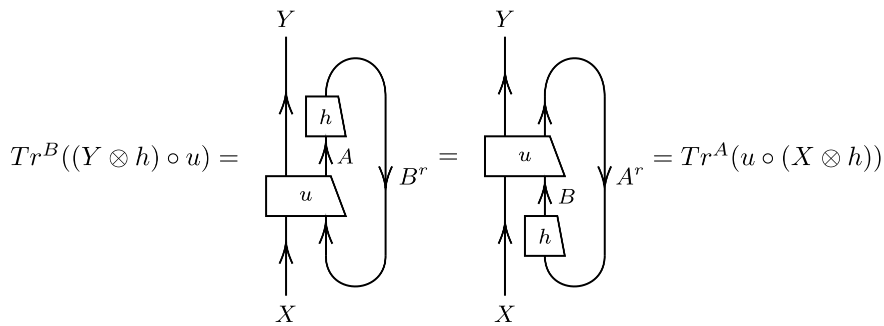
$$Tr(f) = Tr^A(\ell_A \circ (\mathrm{id}_I \otimes f) \circ \ell_A^{-1})$$

for every $f : A \to A$ and that for every $g : X \to Y$ we have

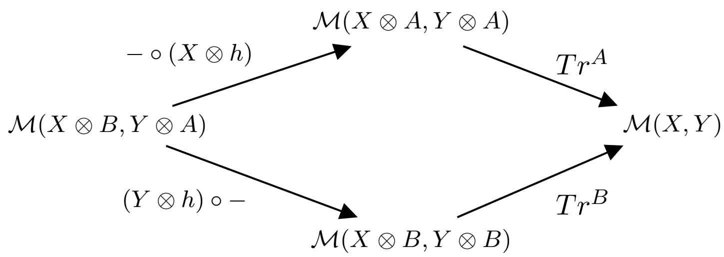
$$Tr^A(g \otimes f) = g \otimes Tr(f).$$

Also note that this trace also depends on the chosen dualities.

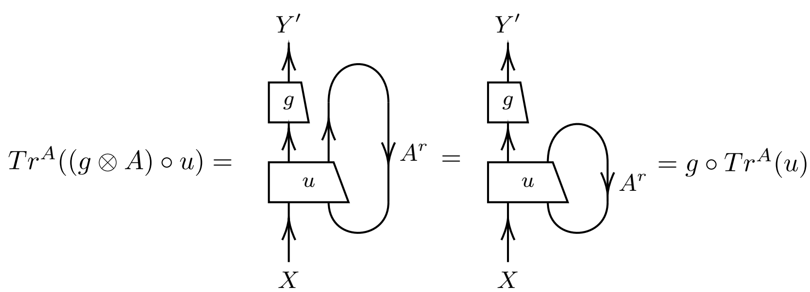
Remark 3.5.56. Note that the trace may be regarded as an operation of type $Tr^A : \mathcal{M}(X \otimes A, Y \otimes A) \to \mathcal{M}(X, Y)$, which is natural in X, Y and dinatural in A. Given $u : X \otimes B \to Y \otimes A$ we have:



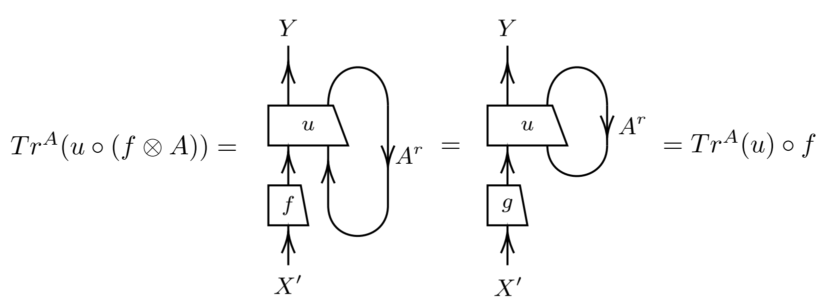
so we the following dinaturality sqaure commutes:



Furthermore, for every $f : X \to X'$, $g : Y \to Y'$ and $u : A \to A$ we have:



which shows naturality in Y, while naturality in X is shown as follows.

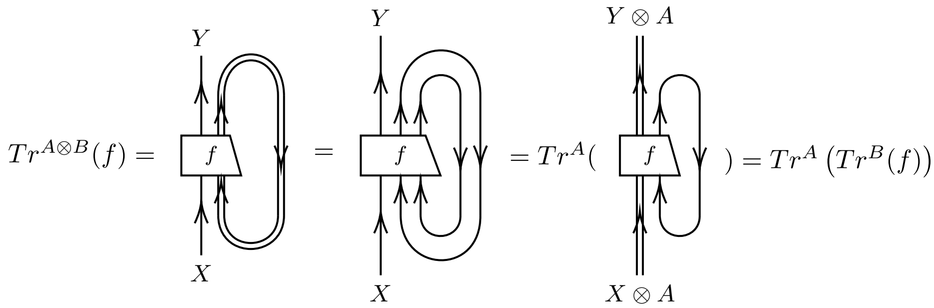


While the above remark already gives some properties of the trace, we give another one.

Lemma 3.5.57. *In a pivotal category $\mathcal{M}$ for every $f : X \otimes (A \otimes B) \to Y \otimes (A \otimes B)$ we have*

$$Tr^{A \otimes B}(f) = Tr^A(Tr^B(f)).$$

Proof. This follows from the following graphical manipulation:



□

Traces allow to give an abstract definition of the dimension of an object in a pivotal monoidal category.

Definition 3.5.58. *Let M be a pivotal monoidal category. The* ***dimension*** *of an object $A \in \mathcal{M}_0$ is the scalar given by the trace of the identity morphism:*

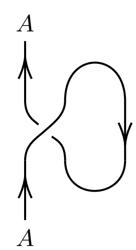
$$\dim(A) := Tr(\mathrm{id}_A).$$

The properties of the dimension will be listed in the following proposition without a proof, since they are direct consequences of the properties of the trace.

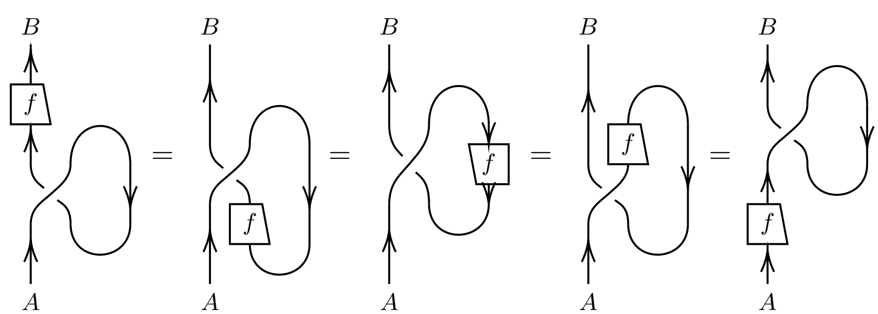
Proposition 3.5.59. *In a pivotal monoidal category, $\mathcal{M}$, the following hold.*

- $\dim(I) := \mathrm{id}_I$
- *Isomorphic objects have the same dimension.*
- *If there is a zero object, then* $\dim(0) = 0_{I,I}$.
- *If $\mathcal{M}$ is semi-additive, then* $\dim(A \oplus B) = \dim(A) + \dim(B)$*, for every $A, B \in \mathcal{M}_0$.*
- *If $\mathcal{M}$ is braided, then* $\dim(A \otimes B) = \dim(A) \cdot \dim(B)$.

Observe that there are braided monoidal categories that are rigid. What is interesting is that pivotality forces such categories to be balanced. Conversely, a twist on a braided rigid monoidal category induces a pivot. The string diagram calculus in such cases is exceptionally intuitive since the pivot actually becomes a pivot of a string. Before the proof of the sketched equivalence, note that in a braided monoidal category $\mathcal{M}$ which is pivotal, there is a family of morphisms indexed by objects of $\mathcal{M}$.



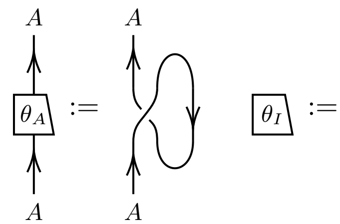
This morphism is natural in A, since the sliding relations for pivotal categories and naturality of the braiding give:



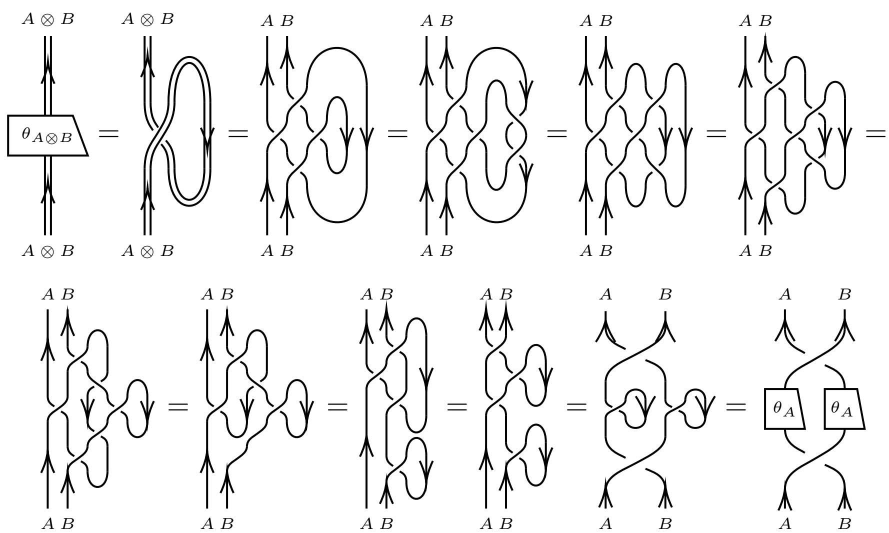
for every $f : A \to B$. Actually this morphism is the trace of the braiding $Tr^A(b_{A,A})$.

Theorem 3.5.60. *Let $\mathcal{M}$ be a braided monoidal category which is also rigid. Every twist on $\mathcal{M}$ defines a pivot and vice versa.*

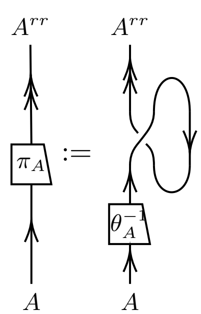
Proof. Let $\mathcal{M}$ be braided pivotal. We will define a twist as follows.



We have already seen that θ as defined above, is natural. So we only need to show that it satisfies the twist condition. Let $A, B \in \mathcal{M}_0$ and observe that:



For the other direction, define the pivot as:



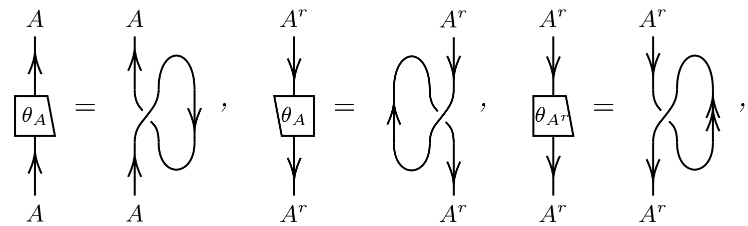
and work similarly. □

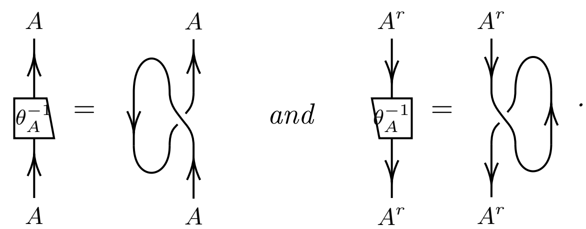
According to the above result, some braided rigid monoidal categories are pivotal or equivalently are balanced. What is interesting is that according to 3.5.49, 3.5.10 and 3.5.6, in a braided rigid monoidal category we already have a pivot, given as the unique isomorphism between left and right duals. Uniqueness up to unique isomorphism implies monoidality and naturality. Thus **every braided rigid monoidal category is also balanced and pivotal.**

Obviously, a different choice of left or right duals results in different twists or pivots. So braided rigid categories are balanced and pivotal with the pivot and the balance given by the fact that a right dual is also a left dual due to the braiding.

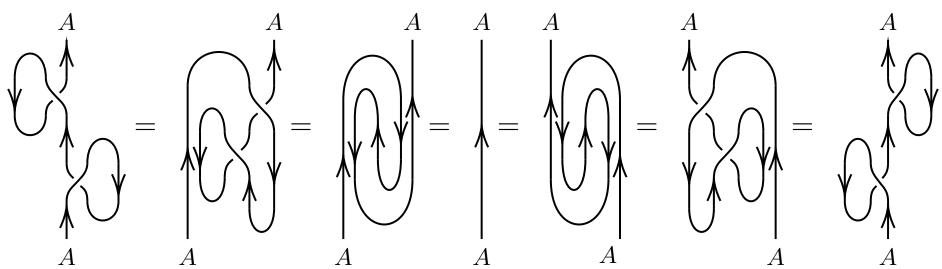
A balance, a braiding and a rigid structure are not enough to define ribbon categories. Ribbon categories make the ribbon diagram notation of balanced monoidal categories sublime. To see why the need for an additional condition, we give the following proposition for a braided pivotal monoidal category, where the pivot need not be the one already provided.

Proposition 3.5.61. *In a braided pivotal monoidal category $\mathcal{M}$, the following hold.*

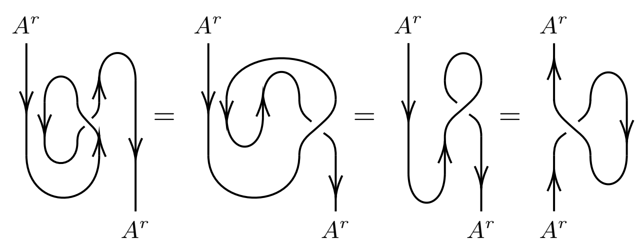




Proof. We have already shown the first one and all the rest are similar, so we will only prove the last two. So observe that for every $A \in \mathcal{M}_0$:

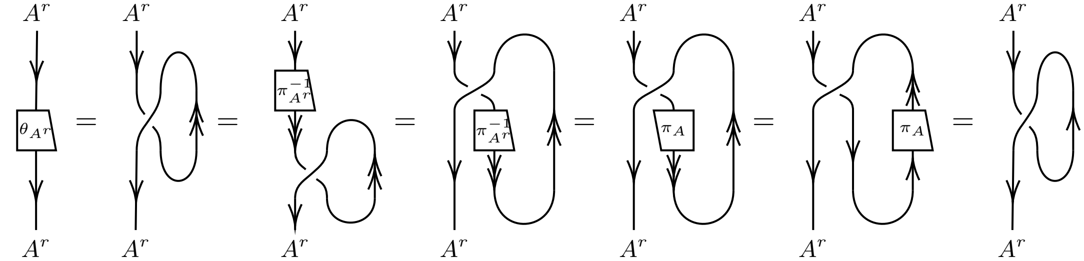


and



□

We can massage the graphical representation of θ_{A^r}, given 3.5.49, as follows.

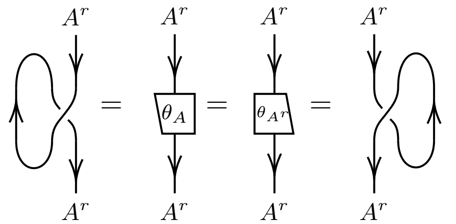


It is clear that this need not equal θ^r_A, but when it does we have:

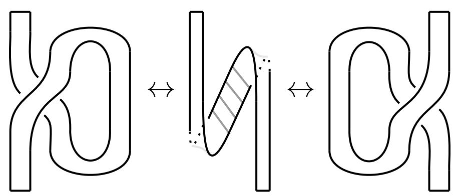
Definition 3.5.62. *A braided pivotal monoidal category is a* ***ribbon category****, if for every* $A \in \mathcal{M}_0$

$$\theta_{A^r} = (\theta_A)^r.$$

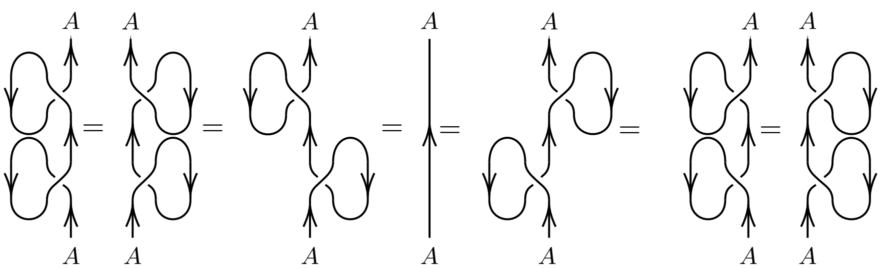
Corollary 3.5.63. *In a ribbon category* $\mathcal{M}$*, for every* $A \in \mathcal{M}_0$*:*



Remark 3.5.64. The above corollary is the reason why ribbon categories are called this way. Ribbons really satisfy the above relation, in a topological sense, as we can see below.



Furthermore, the equalities above are a restatement of the definition of a ribbon category, so they also serve as a characterization. Finally, the above condition holds for every right dual, but since in a pivotal category every object is a right dual, this holds for any object. This has immediate consequences when combined with 3.5.61, as seen below:

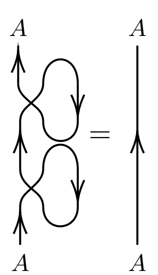


So far we have used the string diagram calculus without any mention to the topological aspect of it. That is, we have used it as a formal tool. So to be able to use it "topologically", we can only perform:

- planar oriented isotopies, for pivotal categories,
- framed isotopies of $\mathbb{R}^3$, for ribbon categories.

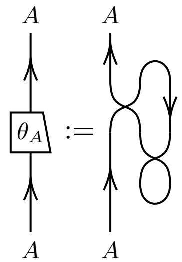
We can lower the dimension of the graphical calculus for ribbon categories if they are also symmetric. In this case the twist is a "square root" of the identity morphism. This is a corrolary of the above remark.

Corollary 3.5.65. *In a symmetric ribbon category,* $\mathcal{M}$*, for every object* $A \in \mathcal{M}_0$ *we have:*



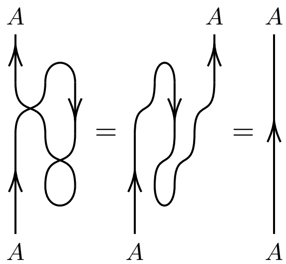
So in a symmetric ribbon category there are serious constraints for the twist. A symmetry forces the twist to be a monoidal natural transformation of the identity endofunctor.

Definition 3.5.66. *A rigid symmetric monoidal category, $\mathcal{M}$, is a* ***compact closed*** *category if it is equipped with the twist, given canonically by:*



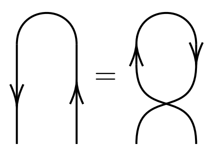
for every $A \in \mathcal{M}_0$.

Remark 3.5.67. Notice that in compact closed categories the twist is actually the identity, since:

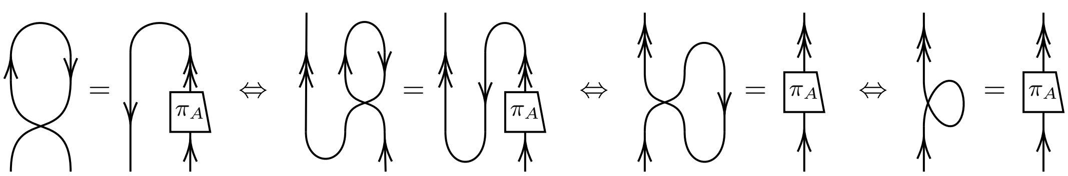


Thus, $\theta_{A^r} = (\theta_A)^r$ holds trivially. So a compact closed category can equivalently be defined as a symmetric ribbon category with the identity natural transformation as a twist.

In addition, we have that the pivot is the one provided by the following equality:



That is, for every $A \in \mathcal{M}_0$ we have:

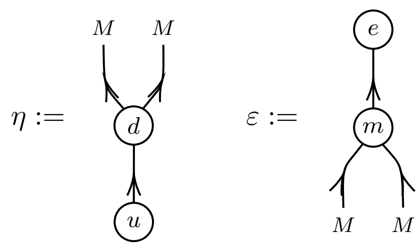


Example 3.5.68. One can easily check that **FdHilb** and **FdVect** are symmetric monoidal and rigid. Therefore, they are compact. Similarly, n**Cob** is symmetric and rigid and thus canonically compact.

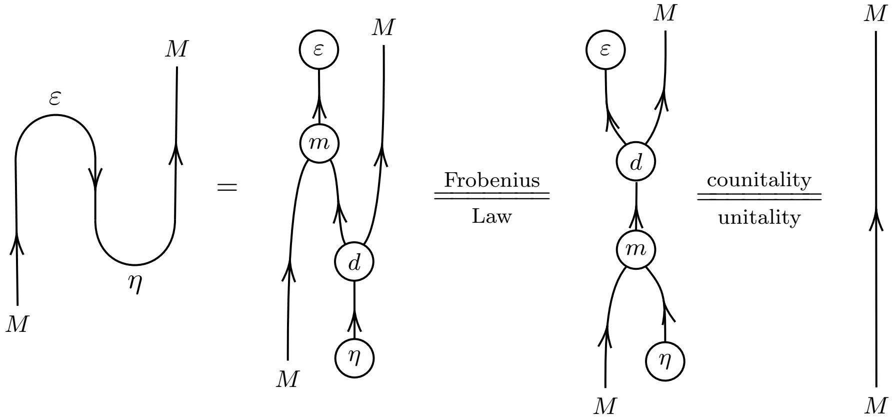
We now revisit Frobenius objects (definition 2.1.75), from the point of view of duality.

Proposition 3.5.69. *Let (M, m, u, d, e) be a Frobenius object internal to a monoidal catgeory $\mathcal{M}$. Then M is self-dual.*

Proof. We will prove this using string diagrams. Define the unit and the counit as follows:



Then we get the second snake equation:



and similarly we derive the first one. So $(M, M, d \circ u, e \circ m)$ is indeed a duality. $\square$

We already saw that a Frobenius object internal to $\mathbf{Vect}_k$ is a Frobenius algebra M. But by the above proposition, a Frobenius object in any monoidal category is self-dual, so we have that $M \cong M \multimap k$.

Corollary 3.5.70. *Any Frobenius algebra is finite dimensional.*

Frobenius objects are self-dual, so special Frobenius objects (definition 2.1.75) are also self-dual. The speciality condition hints at a property which is related to another notion, similar to duality.

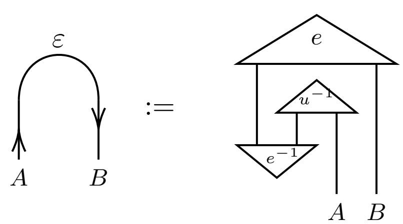
Definition 3.5.71. *Let $\mathcal{M}$ be a monoidal category and $A, B \in \mathcal{M}_o$ be objects. We call the objects A and B* ***weak inverses*** *of each other, if there exist isomorphisms $e : A \otimes B \to I$ and $u : B \otimes A \to I$.*

Example 3.5.72. The best example of weak inverses is given by endofunctors of a category $\mathcal{C}$, which are equivalences. Indeed, given two functors $F, G \in End(\mathcal{C})$, the existence of natural isomorphisms $e : A \otimes B \to I$ and $u : B \otimes A \to I$ makes (F, G, u, e) an equivalence of $\mathcal{C}$ with itself.

Remark 3.5.73. We have already discussed how a monoidal category is a categorified version of a monoid. Thus, a monoidal category $\mathcal{M}$ in which every object of has a weak inverse, is a categorified version of a group. This is actually a (bad[7]) vertical categorification of the notion of group, the horizontal categorification being a groupoid.

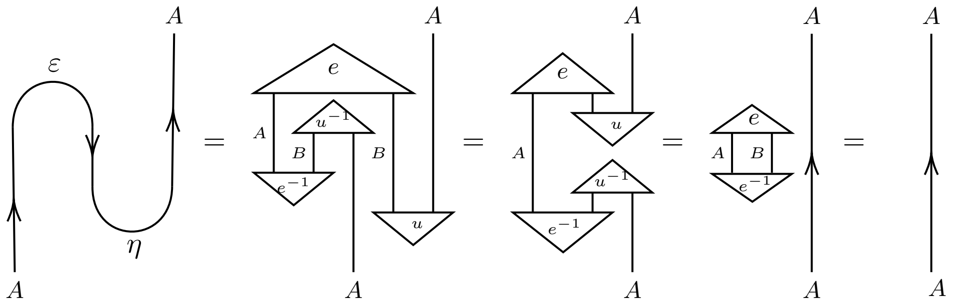
Proposition 3.5.74. *Let $\mathcal{M}$ be a monoidal category and assume that (A, B, u, e) form an internal equivalence in the one object bicategory corresponding to $\mathcal{M}$. Then $A \dashv B$ and $B \dashv A$.*

Proof. Define $\eta := u$ and ε as follows:

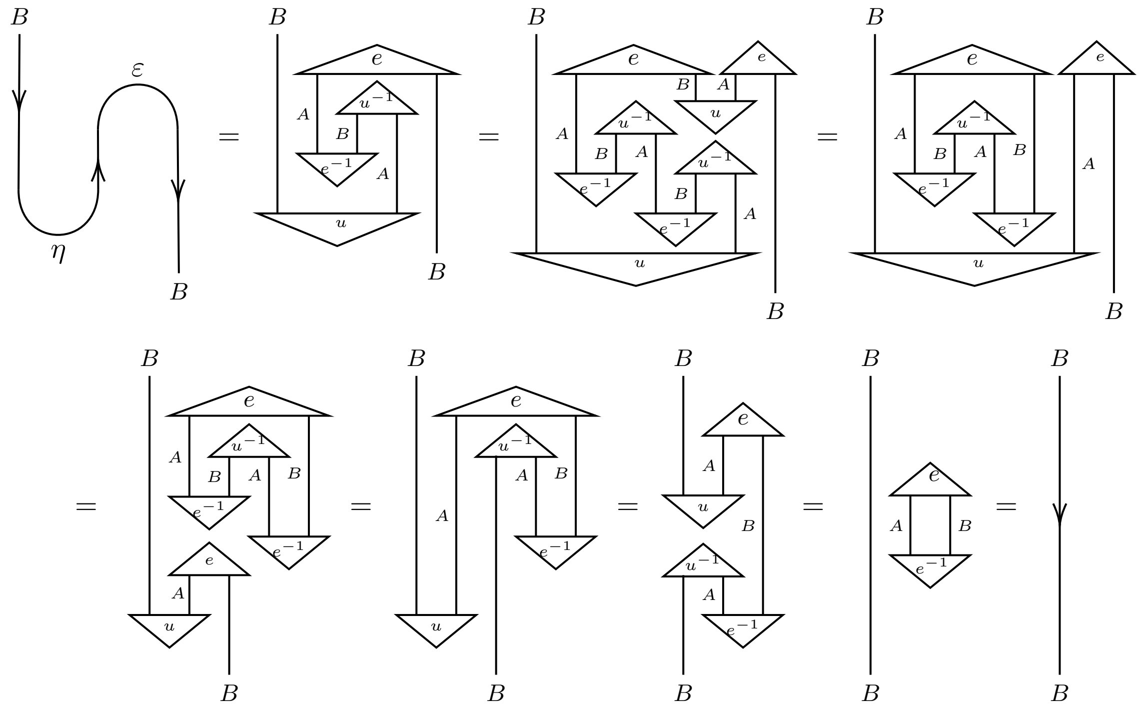


Observe that invertibility of e and u implies:

[7]The "good" categorifications of groups include that $\mathcal{M}$ is also a groupoid. Such objects are generally called **2-groups**.



which gives the first snake equation, while this first equation and invertibility of e and u give:



which proves the second one. $\square$

3.6 Dagger categories and dagger monoidal categories

Dagger categories are categories that allow a notion of "reversal" of their morphisms. In a sense, they constitute an abstract approach to such reversals. In this section, we firstly present dagger categories and then continue by discussing the ways a dagger can interact with monoidal categories and their variants that have already been introduced.

Definition 3.6.1. *A category $\mathcal{C}$ is called a* ***dagger category*** *if it is equipped with a contravariant functor $\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ that satisfies:*

i) $A^\dagger = A$ *and* $\mathrm{id}_A^\dagger = \mathrm{id}_A$ *for all* $A \in \mathcal{C}_0$ *and*

ii) $(f^\dagger)^\dagger = f$*, for every morphism.*

The image of a morphism under the dagger functor is called its ***adjoint****.*

Remark 3.6.2. Observe that the functor $(-)^\dagger$ is an isomorphism, its opposite being its inverse. This may easily be deduced by the property *ii*). Every functor with such a property is called an **involution**. So a dagger category is equipped with an *identity-on-objects involution*. We can immediately deduce that, for every $A, B \in \mathcal{C}_0$, $\mathcal{C}(A, B) \cong \mathcal{C}(B, A)$.

In a dagger category, there are certain kinds of morphism which play a very important role.

Definition 3.6.3. *Let $\mathcal{C}$ be a dagger category and $f : A \to B$ be a morphism. We say that f is:*

1. ***unitary** if $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$,*
2. *an **isometry** or a **dagger monomorphism** if $f^\dagger \circ f = \mathrm{id}_A$,*
3. *a **coisometry** or a **dagger epimorphism** if $f \circ f^\dagger = \mathrm{id}_B$.*

If in addition $A = B$, we call f:

1. ***normal** if $f^\dagger \circ f = f \circ f^\dagger$,*
2. ***self-adjoint** if $f = f^\dagger$,*
3. ***projection** if $f^\dagger \circ f = f$,*
4. ***positive** if there exist $B \in \mathcal{C}_0$ and $g : A \to B$ such that $f = g^\dagger \circ g$.*

Remark 3.6.4. It is worth noting that a projection can equivalently be defined as a self-adjoint morphism which is also idempotent. It is obvious that these two conditions for an endomorphism f imply that $f = f^\dagger \circ f$. On the other hand, given the first condition, we get that

$$f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f = f,$$

which shows that f is self-adjoint and consequently

$$f = f^\dagger \circ f = f \circ f,$$

thus it is also idempotent. Furhtermore, the original definition of a projection immediately implies that a projection is always positive.

Remark 3.6.5. The identity morphism of every object is unitary, self-adjoint, positive and a projection.

Lemma 3.6.6. *Let $\mathcal{C}$ be a dagger category and let $f : A \to A$ be a positive map. Then the following hold:*

1. *f is self-adjoint and*
2. *for every $h : B \to A$, $h^\dagger \circ f \circ h$ is positive.*

Proof. Note that positivity of f imlpies the existence of a $g : A \to X$, such that $f = g^\dagger \circ g$. Thus, for the first one,

$$f^\dagger = (g^\dagger \circ g)^\dagger = g^\dagger \circ g = f.$$

For the second claim, observe that

$$h^\dagger \circ g^\dagger \circ g \circ h = h^\dagger \circ f \circ h = (g \circ h)^\dagger \circ (g \circ h),$$

so $h^\dagger \circ f \circ h$ is positive. □

Definition 3.6.7. *Let $\mathcal{C}$ be a dagger category. A subcategory $\mathcal{C}' \subseteq \mathcal{C}$, is a **dagger subcategory** if it is closed under the dagger functor inherited from $\mathcal{C}$.*

Example 3.6.8. Every groupoid, seen as a category whose every morphism is an isomorphism, is a dagger category: the dagger is given by taking inverses. Consequently, every group, as a one object category, is also a dagger category, again by taking inverses as daggers.

If on the other hand we take a dagger category with one object, then we may speak of an **involutive monoid**. Involutive monoids are monoids equipped with an involution/dagger satisfying all the axioms of a one-object dagger category.

Example 3.6.9. The best example of a dagger category is **Hilb**. The operation of taking daggers amounts to forming the adjoint of a bounded linear operator. All the terminology introduced in 3.6.3 comes from the category **Hilb**, since the corresponding notions in the theories of Hilbert spaces and operator algebras coincide with the ones given above. For example, a dagger monomorphism $f : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry, since for every $a, b \in \mathcal{H}_1$,

$$< b, a >=< f^{\dagger}(f(b)), a >=< f(b), f(a) > .$$

Furthermore, **FdHilb** is evidently a dagger subcategory of **Hilb**.

The construction of adjoints depends on the inner product structure of the Hilbert spaces involved, $\mathcal{H}$ and $\mathcal{H}'$, since $f^{\dagger}$ is defined as the unique bounded linear map such that

$$< f^{\dagger}(y), x >:=< y, f(x) >,$$

for every $f : \mathcal{H} \to \mathcal{H}'$ and every $x \in \mathcal{H},\ y \in \mathcal{H}'$. Interestingly, given a dagger structure in **Vect**, we may reconstruct an inner product as follows.

Firstly, vectors in any Hilbert space, say $b \in \mathcal{H}$, are in bijection with states $|b >: \mathbb{C} \to \mathcal{H}$, such that $|b > (1) = b$. Then $|b >^{\dagger}: \mathcal{H} \to \mathbb{C}$ is an effect. So, by the definition of adjoints we have the following equalities:

$$< 1||b >^{\dagger} (a) >_{\mathbb{C}}=< (|b > (1))|a >_{\mathcal{H}}=< b|a >_{\mathcal{H}}$$

for every $a \in \mathcal{H}$. Finally, according to the above equalities, the composite $|b >^{\dagger} \circ |a >: \mathbb{C} \to \mathbb{C}$ satisfies:

$$< 1|(|b >^{\dagger} \circ|a >)(1) >_{\mathbb{C}}=< 1||b >^{\dagger} (a) >_{\mathbb{C}}=< b|a >_{\mathcal{H}}=< 1| < b|a >_{\mathcal{H}}>_{\mathbb{C}}$$

and by non-degeneracy of the inner product,

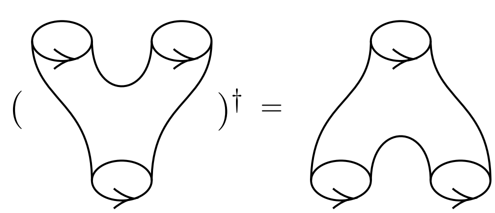
$$|b >^{\dagger} \circ|a >=< b|a >,$$

which shows that we may reproduce the inner product of two vectors using only composition and daggers. This may also be considered as a way to abstract the notion of inner product to any dagger category. A hint towards this direction is that every bounded linear operator $f : \mathcal{H} \to \mathcal{H}'$ satisfies:

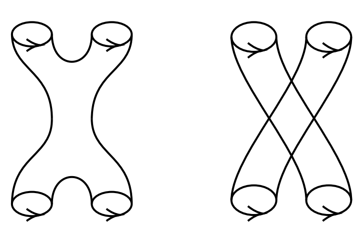
$$|b >^{\dagger} \circ f \circ |a >= (f^{\dagger} \circ |b >)^{\dagger} \circ |a >,$$

for every vector $a \in \mathcal{H}$ and $b \in \mathcal{H}'$. For this to be meaningful we need a notion of states, thus a monoidal structure will also be needed.

Example 3.6.10. The category n**Cob** is also a dagger category. The dagger operation amounts to considering the original cobordism with its input and output exchanged, which is in a sense the (time-)reversal of the cobordism. We illustrate this for the case $n = 2$.



Two examples of self-adjoint cobordismss are:



the second one being the symmetry of $2\mathbf{Cob}$ at two copies of the oriented circle, which is also unitary.

Definition 3.6.11. *Let $\mathcal{C}, \mathcal{D}$ be dagger categories. A* ***dagger functor*** *from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that for every $f : X \to Y$, in $\mathcal{C}$,*

$$F(f^\dagger) = (Ff)^\dagger.$$

In other words, a dagger functor commutes with the daggers of the categories. Interestingly natural transformations between dagger functors have an property. Their components have adjoints.

Proposition 3.6.12. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be dagger functors between dagger categories and let $\eta : F \Rightarrow G$ be a natural transformation. Then $\{\eta_A^\dagger\}_{A \in \mathcal{C}_0}$ is a natural transformation from $G \circ \dagger$ to $F \circ \dagger$.*

Proof. Let $f \in \mathcal{C}(X, Y)$. Naturality of η implies that $G(f^\dagger) \circ \eta_Y = \eta_X \circ F(f^\dagger)$, so taking adjoints we get:

$$\eta_Y^\dagger \circ G(f^\dagger)^\dagger = F(f^\dagger)^\dagger \circ \eta_X^\dagger \Leftrightarrow$$
$$F(f) \circ \eta_X^\dagger = \eta_Y^\dagger \circ G(f)$$

which is the desired naturality condition. □

Definition 3.6.13. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be dagger functors between dagger categories and let $\eta : F \Rightarrow G$ be a natural transformation. We call $\eta^\dagger := \{\eta_A^\dagger\}_{A \in \mathcal{C}_0}$ the* ***adjoint*** *natural transformation of η.*

Remark 3.6.14. It is fairly straightforward that the identity functor of a dagger category is a dagger functor, but also that the composition of dagger functors is again a dagger functor. So there is a subcategory, **DagCat**, of **Cat** whose *objects are dagger categories and whose morphisms are dagger functors.* Furthermore, given two dagger categories, $\mathcal{C}, \mathcal{D}$ we can form their cartesian product inside **CAT**. What is interesting is that we can equip $\mathcal{C} \times \mathcal{D}$ with a dagger functor defined as

$$(f, g)^\dagger := (f^\dagger, g^\dagger),$$

for every pair of morphisms $(f, g) \in \mathcal{C}(A, B) \times \mathcal{D}(C, D)$, which makes $\mathcal{C} \times \mathcal{D}$ a dagger category. Then the projection functors are trivially †-functors and the universal property of the product in **DagCat** holds. In addition, the terminal category $\mathbf{1}$ is trivially a dagger category therefore, $(\mathbf{DagCat}, \times, \mathbf{1})$ is a cartesian category.

Example 3.6.15. Any functor between groupoids preserves inverses, so it is automatically a dagger functor. The same obviously holds for groups, but need not hold for involutive monoids.

Example 3.6.16. A dagger functor $\mathbf{B}G \to \mathbf{FdHilb}$ from a group is a unitary representation of the group, since the inverses will be translated to adjoints, but the invertibility condition should still hold by functoriality. Furthermore, the functor category $\mathbf{DagCat}(\mathbf{B}\mathrm{G}, \mathbf{FdHilb})$ is essentially the category of finite dimensional unitary representations of G, since natural transformations are identified with intertwinning operators.

Some properties a category might possess get "enhanced" in the presence of a dagger structure, while some structures get refined versions. This is the content of the following, where we begin by presenting a strengthened condition for the existence of zero objects, introduce dagger kernels and dagger equalisers and continue with dagger biproducts, biproduct dagger categories, while highlighting the interplay between all these notions.

Proposition 3.6.17. *Let $\mathcal{C}$ be a dagger category and $T \in \mathcal{C}_o$. Then T is initial if and only if it is terminal if and only if it is a zero object.*

Proof. We will only prove that terminal implies initial, since the other two proofs are similar. Let T be terminal. Then, remark 3.6.2 implies that for every $X \in \mathcal{C}_0$,

$$\{*\} \cong \mathcal{C}(X, T) \cong \mathcal{C}(T, X),$$

thus T is initial. □

Remark 3.6.18. The above proposition shows that the components of the natural transformation $u : \Delta_T \Rightarrow \mathbb{1}_{\mathcal{C}}$ have as adjoints the uniform deleting morphisms $\{e_X : X \to T\}_{X \in \mathcal{C}}$. An immediate consequence of this is that the adjoints of zero morphisms are zero morphisms since for every $X, Y \in \mathcal{C}_0$,

$$0^{\dagger}_{X,Y} = (u_Y \circ e_X)^{\dagger} = e^{\dagger}_X \circ u^{\dagger}_Y = u_X \circ e_Y = 0_{Y,X}.$$

Furthermore, since the identity morphism of a zero object is the zero morphism, the uniform deleting morphisms are isometries, while $\{u_A\}_{A \in \mathcal{C}_0}$ are coisometries.

Remark 3.6.19. The zero endomorphism of every object can easily be seen to be self-adjoint, positive and idempotent, thus also a projection.

Definition 3.6.20. *In a dagger category $\mathcal{C}$, the **dagger equaliser**, also denoted by **†-equaliser**, of two parallel $f, g : A \to B$ is an equaliser (E, e) of f, g such that e is a dagger monomorphism. A dagger category in which every pair of parallel morphisms has a †−equaliser is called a **dagger category with dagger equalisers**.*

Remark 3.6.21. Given a †-equaliser (E, e) of two parallel morphisms and any other cone (X, k) there is a unique morphism $u : X \to E$, such that

$$e \circ u = k \Rightarrow u = e^{\dagger} \circ k,$$

since e is a dagger monomorphism. Thus we freely get an expression for u.

Remark 3.6.22. General equalisers are unique up to unique isomorphism. Given two dagger equalisers (E, e) and (E', e'), the isomorphism $u : E' \to E$ and its inverse have to satisfy

$$u = e^{\dagger} \circ e' \text{ and } u^{-1} = e'^{\dagger} \circ e,$$

by uniqueness. Thus,

$$u^{\dagger} = u^{-1},$$

meaning that dagger equalisers are unique up to unitary isomorphism.

With the notion of dagger equalisers, we can define dagger kernels of morphisms.

Definition 3.6.23. *In a dagger category with a zero object, the **dagger kernel**, also written as **†-kernel**, of a morphism $f : A \to B$ is the dagger equaliser of f and $0_{A,B}$.*

Remark 3.6.24. The morphisms out of a zero object are always isometries. So a kernel of a morphism, in a dagger category with a zero object, is automatically a †-kernel.

Remark 3.6.25. A dagger category with dagger kernels is not necessarily a dagger category with dagger equalisers, but the converse is true if there is also a zero object. Interestingly, both hold in **Hilb**.

Example 3.6.26. The category of Hilbert spaces and bounded linear maps is a dagger category. It has a zero object which is the 0-dimensional vector space. It also has †-kernels for all morphisms, since the kernel of every bounded linear map is a closed subspace, as the preimage of the closed set $\{0\}$, and its inclusion is an isometry. This implies that the †-equaliser of a pair $f, g \in \mathbf{Hilb}(\mathcal{H}_1, \mathcal{H}_2)$ exists, since it is the †-kernel $(\ker\{f - g\}, \iota)$, where $\iota : \ker\{f - g\} \to \mathcal{H}_1$ is an inclusion.

Proposition 3.6.27. *In a dagger category $\mathcal{C}$ with a zero object, every †-monomorphism $f : A \to B$, has a †-kernel, which in addition is a zero object. Consequently, unitaries always have the zero object as a dagger kernel.*

Proof. Consider the dagger kernel $(K, \ker(f))$. Since, f and $\ker(f)$ are monomorphisms, we have that

$$f \circ \ker(f) = 0_{K,B} \Rightarrow \mathrm{id}_K = 0_{K,K}$$

so by 1.6.1 K is a zero object and $\ker(f)$ a zero morphism. □

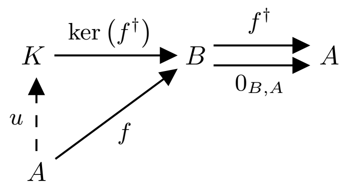
The dagger functor gives a way to generalise inner products. The candidate definition is

$$< g, f >:= g^\dagger \circ f$$

for every $f, g : A \to B$. We will gradually give some conditions under which a composite of the form $g^\dagger \circ f$ behaves like an inner product. First of all, if we substitute positive definiteness with positivity, then for every morphism f, $f^\dagger \circ f$ is positive. The second one is the following, which amounts to the non-degeneracy condition.

Proposition 3.6.28. *In a dagger category, $\mathcal{C}$, with a zero object and dagger kernels, for every morphism $f \in \mathcal{C}(A, B)$, if $f^\dagger \circ f = 0_{A,A}$ then f is a zero morphism.*

Proof. Since $\mathcal{C}$ has †-kernels and $f^\dagger \circ f = 0_{A,A}$, there exists a †-kernel of $f^\dagger$, $(K, \ker(f^\dagger))$, and a unique $u : A \to K$ such that the following diagram commutes:



On the one hand, we get that $f^\dagger \circ \ker(f^\dagger) = 0_{K,A}$ which, by taking daggers, implies

$$\ker(f^\dagger)^\dagger \circ f = 0_{A,K}.$$

On the other hand, $\ker(f^\dagger)$ being a dagger monomorphism implies

$$u = \ker(f^\dagger)^\dagger \circ f.$$

Thus, combining the above we have $u = 0_{A,K}$, so

$$f = \ker(f^\dagger) \circ u = 0_{A,B}.$$

□

Definition 3.6.29. *Let $\mathcal{C}$ be semi-additive and dagger and $A_1, A_2 \in \mathcal{C}_0$. A **dagger biproduct** also written as †-**biproduct**, of A_1 and A_2 is a biproduct $(A_1 \oplus A_2, \pi_1, \pi_2, \iota_1, \iota_2)$ such that*

$$\pi_1^\dagger = \iota_1 \text{ and } \pi_2^\dagger = \iota_2.$$

*A dagger category with a zero object is called a **biproduct dagger category**, if all binary dagger biproducts exist.*

Remark 3.6.30. A biproduct dagger category is automatically semi-additive since dagger biproducts are also biproducts. On the other hand, a semi-additive category with a dagger involution need not be a biproduct dagger category.

Remark 3.6.31. Given a dagger biproduct of two objects A_1, A_2 in a semi-additive category with a dagger involution, we can see that the projections and the coprojections satisfy the following:

$$\pi_i \circ \pi_i^\dagger = \mathrm{id}_{A_i} = \iota_i^\dagger \circ \iota_i$$

for $i = 1, 2$. That is, projections are dagger epimorphisms and coprojections are dagger monomorphisms. In addition, the property that links the product and the coproduct structure may be re-written in one of the two following ways:

$$\iota_1 \circ \iota_1^\dagger + \iota_2 \circ \iota_2^\dagger = \mathrm{id}_{A_1 \oplus A_2} = \pi_1^\dagger \circ \pi_1 + \pi_2^\dagger \circ \pi_2.$$

Remark 3.6.32. Since a biproduct is both a limit and a colimit, it is unique up to isomorphism. So, let $f : (B, \pi, \iota) \to (B', \pi', \iota')$ be the unique isomorphism between two dagger biproducts of A_1, A_2. Then one easily checks that $f^\dagger : B' \to B$ is also an isomorphism satisfying for $j = 1, 2$:

$$\begin{aligned}\pi_j \circ f =& \pi'_j \Leftrightarrow \\ f^\dagger \circ \pi_j^\dagger =& (\pi'_j)^\dagger \Leftrightarrow \\ f^\dagger \circ \iota_j =& \iota'_j\end{aligned}$$

and similarly $\pi'_j \circ f^\dagger = \pi_j$. Thus by uniqueness $f^{-1} = f^\dagger$, meaning that *a dagger biproduct is unique up to unitary isomorphism.*

We proceed by unravelling the interplay between daggers and dagger biproducts. To this end, the matrix calculus for semi-additive categories will be essential.

Proposition 3.6.33. *Let $\mathcal{C}$ be a semi-additive category and a dagger category and let $A_i, B_j \in \mathcal{C}_0$ for $1 \le i \le n \in \mathbb{N}$ and $1 \le j \le m \in \mathbb{N}$. If $(\bigoplus_{i=1}^n A_i, \pi, i)$ and $(\bigoplus_{j=1}^m B_j, p, \iota)$ are dagger biproducts, then for every $f : \bigoplus_{i=1}^n A_i \to \bigoplus_{j=1}^m B_j$ we have:*

$$\begin{pmatrix} f_{1,1} & \dots & f_{1,n} \\ \vdots & \ddots & \vdots \\ f_{m,1} & \dots & f_{m,n} \end{pmatrix}^\dagger = f^\dagger = \begin{pmatrix} f_{1,1}^\dagger & \dots & f_{m,1}^\dagger \\ \vdots & \ddots & \vdots \\ f_{1,n}^\dagger & \dots & f_{m,n}^\dagger \end{pmatrix}.$$

Proof. Let $f : \bigoplus_{i=1}^n A_i \to \bigoplus_{j=1}^m B_j$. The first equality above is only added for clarity of presentation and we will focus on the second one. So, by the definition of dagger biproducts we get that:

$$\begin{aligned}f^\dagger =& \mathrm{id}_{\bigoplus_{i=1}^n A_i} \circ f^\dagger \circ \mathrm{id}_{\bigoplus_{j=1}^m B_j} \\ =& \sum_{l=1}^{n} \iota_l \circ p_l \circ f^\dagger \circ \sum_{k=1}^{m} i_k \circ \pi_k \\ =& \sum_{l,k} \iota_l \circ \iota_l^\dagger \circ f^\dagger \circ \pi_k^\dagger \circ \pi_k \\ =& \sum_{l,k} \iota_l \circ (\pi_k \circ f \circ \iota_l)^\dagger \circ \pi_k \\ =& \sum_{l,k} \iota_l \circ (f_{k,l})^\dagger \circ \pi_k.\end{aligned}$$

Therefore, we have proved that $(f^\dagger)_{l,k} = (f_{k,l})^\dagger$. □

Corollary 3.6.34. *In a biproduct dagger category, $\mathcal{C}$, for every pair of morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(C, D)$*

$$(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger.$$

So the birpoduct bi-functor is a dagger functor.

Proof. Follows easily using the matrix representation of $f \oplus g$ as in proposition 1.6.29. □

Corollary 3.6.35. *In a biproduct dagger category, $\mathcal{C}$, for every parallel pair of morphisms $f, g \in \mathcal{C}(A, B)$,*

$$(f + g)^\dagger = f^\dagger + g^\dagger.$$

Proof. We have:

$$(f + g)^\dagger = (m_B \circ (f \oplus g) \circ d_A)^\dagger = d_A^\dagger \circ (f \oplus g)^\dagger \circ m_B^\dagger = m_A \circ (f^\dagger \oplus g^\dagger) \circ d_A = f^\dagger + g^\dagger,$$

by the previous corollary and proposition 1.6.29. □

Corollary 3.6.36. *In a dagger category with dagger biproducts, $\mathcal{C}$, the following hold for every endomorphism $f, g \in \mathcal{C}(A, A)$:*

1. *If f, g are self-adjoint, so are $f \oplus g$ and $f + g$.*
2. *If f, g are positive, so are $f \oplus g$ and $f + g$.*
3. *If f, g are unitary, so is $f \oplus g$.*
4. *If f, g are projections, so is $f \oplus g$.*

A biproduct dagger category $\mathcal{C}$ need not have †-kernels or †-equalisers. What is interesting is that some morphisms always have †-kernels. Furthermore, if $\mathcal{C}$ has †-equalisers, every homset has an extra property, discussed in proposition 3.6.39.

Proposition 3.6.37. *Let $(A_1 \oplus A_2, \pi, \iota)$ be a dagger biproduct of A_1, A_2 in a dagger category $\mathcal{C}$. Then (A_m, ι_m) is a dagger kernel of π_n, for $n \neq m$.*

Proof. We will prove this for $n = 1, m = 2$, since the other case is similar. So let $\chi : X \to A_1 \oplus A_2$ be such that $\pi_1 \circ \chi = 0_{X,A_1}$ and define $\chi_2 = \pi_2 \circ \chi$. Then

$$\chi = \iota_1 \circ \pi_1 \circ \chi = \iota_2 \circ \pi_2 \circ \chi = \iota_2 \circ \chi_2$$

and since ι_2 is a †-monomorphism, χ factors uniquely through A_2. Thus (A_2, ι_2) is a †-kernel of π_1. □

Remark 3.6.38. To interpret the above proposition in terms of the matrix calculus, notice that every generalised element of a dagger biproduct $(A \oplus B, \pi, \iota)$ that is in the kernel of π_A is of the form

$$\chi = \begin{pmatrix} 0_{X,A} \\ \chi_B \end{pmatrix},$$

where $\chi_B = \pi_B \circ \chi$.

Proposition 3.6.39. *In a biproduct dagger category with †-equalisers, every hom-monoid is cancellative.*

Proof. Let $\mathcal{C}(A, B)$ be a hom-monoid. Since $\mathcal{C}(A, B)$ is commutative we only need to show that it is left cancellative. So let $f, g, h : A \to B$ satisfy

$$h + f = h + g.$$

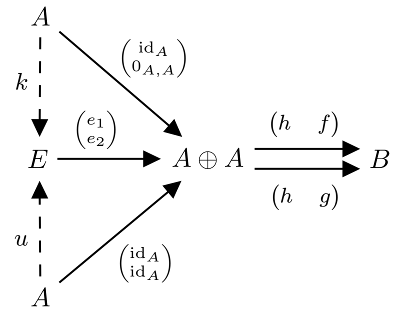
This is equivalent to

$$\begin{pmatrix} h & f \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix} = \begin{pmatrix} h & g \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix}.$$

Furthermore, by proposition 3.6.37, for the coprojection $\iota_1 : A \to A \oplus A$ we have:

$$\begin{aligned} \begin{pmatrix} h & f \end{pmatrix} \circ \iota_1 = \begin{pmatrix} h & g \end{pmatrix} \circ \iota_1 &\Leftrightarrow \\ \begin{pmatrix} h & f \end{pmatrix} \begin{pmatrix} \mathrm{id}_A \\ 0_{A,A} \end{pmatrix} = \begin{pmatrix} h & g \end{pmatrix} \begin{pmatrix} \mathrm{id}_A \\ 0_{A,A} \end{pmatrix} &\Leftrightarrow \\ h = h. \end{aligned}$$

Since, $\mathcal{C}$ has †-equalisers let (E, e) be the †-equaliser of $\begin{pmatrix} h & f \end{pmatrix}$ and $\begin{pmatrix} h & g \end{pmatrix}$ and observe that the following diagram commutes



From the fact that e is a †-monomorphism we get that

$$k = e_1^\dagger \text{ and } u = e_1^\dagger + e_2^\dagger,$$

which, together with the commutativity of the above diagram imply

$$e_1 \circ e_1^\dagger = \mathrm{id}_A,\ e_2 \circ e_1^\dagger = 0_{A,A},\ e_1 \circ e_1^\dagger + e_1 \circ e_2^\dagger = \mathrm{id}_A \text{ and } e_2 \circ e_1^\dagger + e_2 \circ e_2^\dagger = \mathrm{id}_A.$$

It easily, then, follows that

$$e_2 \circ e_2^\dagger = \mathrm{id}_A \text{ and } e_1 \circ e_2^\dagger = 0_{A,A},$$

so

$$\begin{gathered} h \circ e_1 + f \circ e_2 = h \circ e_1 + g \circ e_2 \Rightarrow \\ h \circ e_1 \circ e_2^\dagger + f \circ e_2 \circ e_2^\dagger = h \circ e_1 \circ e_2^\dagger + g \circ e_2 \circ e_2^\dagger \Rightarrow \\ f = g. \end{gathered}$$

Thus, $\mathcal{C}(A, B)$ is left cancellative. □

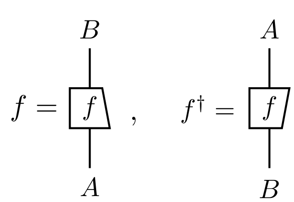
On an ordinary category, a dagger and a monoidal structure may coexist. For them to be compatible, there are some extra conditions to be satisfied. This is the content of the following definition.

Definition 3.6.40. *A dagger category $\mathcal{M}$ is called a* ***monoidal dagger category*** *if it is equipped with a monoidal structure $(\mathcal{M}, \otimes, I, a, l, r)$ such that $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a dagger functor and the coherence isomorphisms a, l, r are unitary. In addition, if $\mathcal{M}$ is equipped with a braiding/symmetry it is a* ***braided/symmetric monoidal dagger category*** *if the braiding/symmetry is unitary.*

Example 3.6.41. As the archetypal example, **Hilb** is a symmetric monoidal dagger category, since the tensor of the adjoints is the adjoint of the tensor. Also the unitors, the associator, the symmetry and their inverses are isometric isomorphisms and consequently unitary. So, in general, **Hilb** is a symmetric monoidal dagger category with dagger biproducts and dagger kernels. This is also true for **FdHilb**.

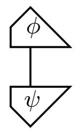
Example 3.6.42. Similarly, n**Cob** is a symmetric monoidal dagger category, since the disjoint union of the "adjoints" of two manifolds is the adjoint of the disjoint union of those two manifolds. Since the associator, the unitors and the symmetry are given by the universal property of the disjoint union, they are also unitary.

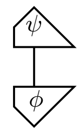
To make the graphical calculus of (braided/symmetric) monoidal categories compatible with the dagger, we do no depict morphisms as boxes, but as trapezoids. This is similar to the case of rigid monoidal categories, but there is no need for directed strings when dualities are absent. Using trapezoids allows the adjoint of a morphism to be depicted as the reflection about a horizontal axis/plane of the original trapezoid as follows:



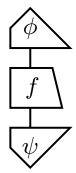
Note that this kind of reflection incorporates both contravariance of the dagger endofunctor and the fact that the tensor is a dagger functor.

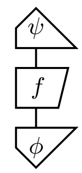
In monoidal dagger categories, there is no need to depict states as quadrilaterals, due to the absence of duals. However, we will continue using trapezoids, since no complication is introduced and, furthermore, in the presence of duals this will be helpful and necessary. This way, the inner product of two states $\phi, \psi \in_I A \in \mathcal{M}_0$ is given graphically as:

$\langle\phi|\psi\rangle = \phi^\dagger \circ \psi =$ 

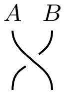
$\langle\psi|\phi\rangle = \psi^\dagger \circ \phi =$ 

and the amplitude of a morphism is given as:

$\langle\phi|f(\psi)\rangle = \phi^\dagger \circ f \circ \psi =$ 

$= \left(f^\dagger \circ \psi\right)^\dagger \circ \phi = \langle f^\dagger(\psi)|\phi\rangle = \langle\phi|f^\dagger(\psi)\rangle^\dagger =$ ()†

The graphical calculus for braided (symmetric) monoidal dagger categories incorporates unitarity of the braiding (symmetry), since:

$b^\dagger_{A,B} =$  $= b^{-1}_{A,B}$

Note that the diagonal components of a symmetry in a symmetric monoidal dagger category are also self-adjoint.

Remark 3.6.43. In a monoidal dagger category with dagger biproducts, the hom-set $\mathcal{M}(A, A)$ is an involutive rig, for every $A \in \mathcal{M}_0$. Furthermore, the set of scalars $\mathcal{M}(I, I)$ is a commutative involutive rig.

A monoidal structure on a dagger category with a zero object allows us to characterise an optimal complete set of effects. The optimization condition is independent of whether a set of effects is complete, but it is mostly useful for such collections of morphisms.

Definition 3.6.44. *Let $\Psi = \{\psi_i : B \to I\}_{i\in I}$ be a set of effects in a monoidal dagger category with a zero object. We call Ψ* ***disjoint*** *if every ψ_i is a coisometry and for ever $i, j \in I,\ i \neq j$:*

$$\psi_i \circ \psi_j^\dagger = 0_{I,I}.$$

Remark 3.6.45. Note that in the presence of dagger biproducts, if the index set is finite, $I = \{1, \ldots, n\}$, the above might be summed up into one condition, namely that the morphism

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

is a coisometry.

In a general monoidal category, we have complete sets of effects. These capture an aspect of the notion of generating set for a vector space, or rather its dual. In dagger monoidal categories, disjoint sets of effects precisely capture the notion of orthonormality. We would like to be able to abstract the notion of an orthonormal basis, at least for finite dimensional Hilbert spaces. This is achievable in the case of monoidal dagger bi-product categories, through the following notion.

Definition 3.6.46. *Let $\Psi = \{\psi_i : B \to I\}_{i=1}^n$ be a set of effects in a monoidal dagger bi-product category. We call Ψ †-**complete**, if*

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

is an isometry.

Remark 3.6.47. Note that a †-complete set of effects, Ψ, is also complete, since for every morphism $f : A \to B$

$$\psi \circ f = 0 \Leftrightarrow \psi^\dagger \circ \psi \circ f = 0 \Leftrightarrow \mathrm{id}_B \circ f = f = 0.$$

In addition, a †-complete disjoint set of effects is essentially a unitary isomorphism $\psi : A \xrightarrow{\sim} \bigoplus_{i=1}^n I$.

There is one condition that links completeness and †-completeness of disjoint sets of effects.

Proposition 3.6.48. *In a monoidal dagger biproduct category with an invertible superposition rule, a complete disjoint set of effects is †-complete.*

Proof. To prove this, we only need to show that given $\Psi = \{\psi_i : A \to I\}_{i=1}^n$ complete and disjoint, the map $\psi : A \to \bigoplus_{i=1}^n I$ is an isometry. So let $g = \psi^\dagger \circ \psi - \mathrm{id}_A$ and note that disjointness implies

$$\psi \circ g = \psi \circ \psi^\dagger \circ \psi - \psi = \psi - \psi = 0.$$

Completeness, then, implies that $g = 0$, so

$$\psi^\dagger \circ \psi = \mathrm{id}_A,$$

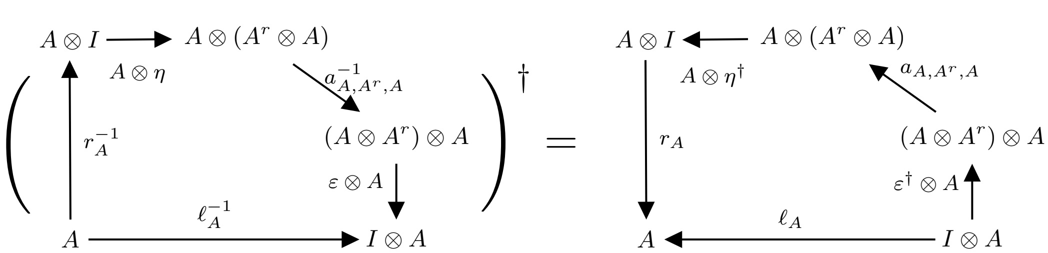
thus ψ is an isometry. □

Example 3.6.49. In **FdHilb** there is an invertible superposition rule and a dagger involution, through which the inner product can be expressed. A complete disjoint set of effects of a finite dimensional hilbert space is exactly an orthonormal basis of its dual. This allows for the identification (unitary isomorphism) of the space with the direct sum of n copies of $\mathbb{C}$, where n is the cardinality of the dual basis.

It is the case that both dualities and the dagger give ways to "reverse" a morphism. In the following, we will see how these two structures interact.

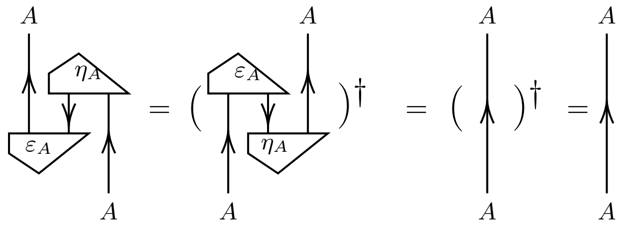
Proposition 3.6.50. *Let $\mathcal{M}$ be a monoidal †-category and $A \dashv A^r$ be a duality. Then $A^r \dashv A$ is also a duality.*

Proof. Let $(A, A^r, \eta_A, \varepsilon_A)$ be a duality. We will show that $(A^r, A, \varepsilon_A^\dagger, \eta_A^\dagger)$ is also a duality, but it is enough to show that the triangle laws hold. So the first triangle law of $A \dashv A^r$ provides the second triangle law of $A^r \dashv A$:



where †-functoriality of the tensor and unitarity of the coherent isomorphisms are used. The first triangle law of $A^r \dashv A$ follows similarly from the second triangle law of $A \dashv A^r$. □

Remark 3.6.51. The above argument can also be stated in string diagram terms as follows:



The above proposition shows that left and right rigid monoidal dagger categories are indistinguishable. We state this precisely in the following corrollary.

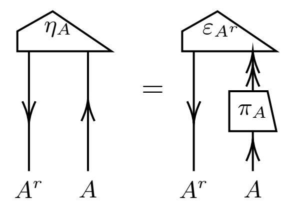
Corollary 3.6.52. *A monoidal dagger category is left rigid if and only if it is right rigid.*

In the setting of monoidal dagger categories, it is natural to study possible compatibilities between dualities and the dagger functor. To express such conditions, a pivotal structure is necessary..

Definition 3.6.53. *Let $\mathcal{M}$ be a rigid monoidal dagger category equipped with a pivot. A duality $(A, A^r, \eta, \varepsilon)$ is a* ***dagger duality*** *if*

$$\eta_A^\dagger = \varepsilon_{A^r} \circ (\mathrm{id}_{A^r} \otimes \pi_A),$$

or equivalently:

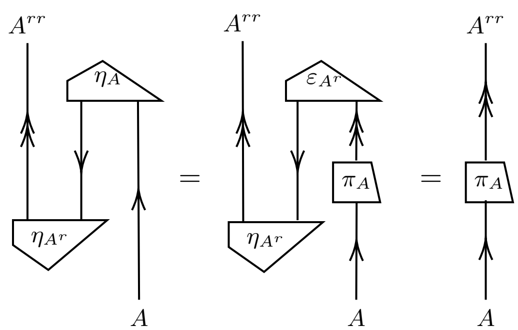


In this case, A^r is called a ***dagger dual*** *of A and vice versa.*

A rigid monoidal dagger category equipped with a pivot is called a ***pivotal dagger category****, if every dual pair is chosen as part of a dagger duality.*

Remark 3.6.54. Note that given a monoidal dagger category with a pivotal structure, it is not nessecary that every dual pair can be made part of a dagger duality. This is only possible in pivotal dagger catgegories.

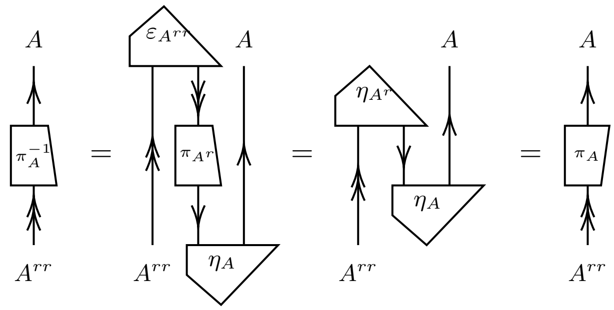
Remark 3.6.55. Note that according to the previous definition, the pivot of a pivotal monoidal category can be expressed solely through the units of the dagger dualities as follows:



This shows that the pivot is uniquely determined, once a uniform choice of dagger duals is made. Given this remark, we can prove the following proposition.

Proposition 3.6.56. *In a pivotal dagger category $\mathcal{M}$, the pivot is unitary.*

Proof. According to proposition 3.5.39, the pivot is a monoidal natural isomorphism whose inverse is given by:



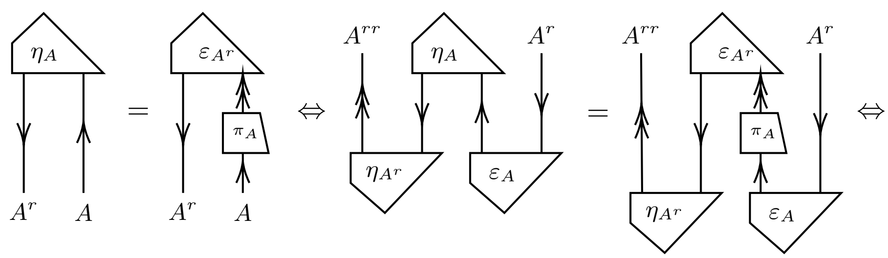
thus, $\pi_A^{-1} = \pi_A^\dagger$. □

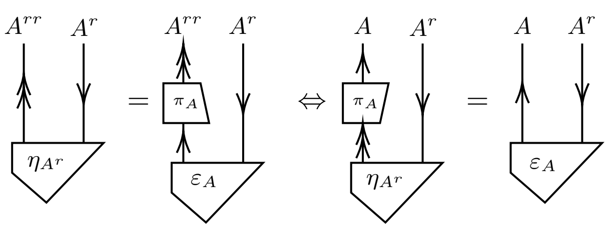
The notion of dagger duality forces the pivot to be compatible with the dagger, which in this context translates to unitarity. Furthermore, unitarity of the pivot simplifies the string diagram calculus of pivotal dagger categories. A first step towards this is the following corrolary.

Corollary 3.6.57. *For every dagger duality $(A, A^r, \eta, \varepsilon)$ in a pivotal dagger category $\mathcal{M}$, the dagger of the counit satisfies:*

$$\varepsilon_A^\dagger = (\pi_A^\dagger \otimes \mathrm{id}_{A^r}) \circ \eta_{A^r}.$$

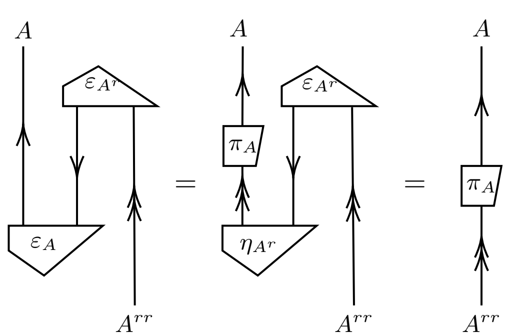
Proof. We have the following equivalences:



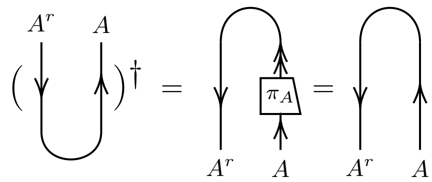


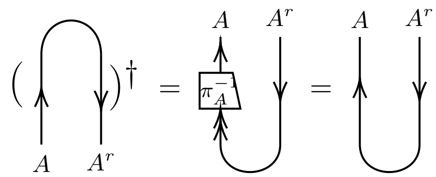
□

Remark 3.6.58. Combining the previous remark with the above corollary we immediately get that:



We can now simplify the diagrammatics of pivotal dagger categories, given the compatibility of the pivot with the dagger functor in such categories and the graphical calculus of pivotal categories, as follows:

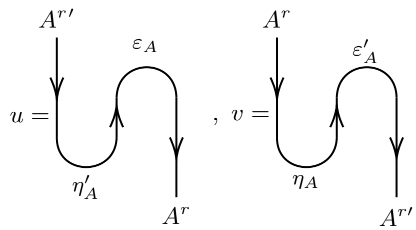




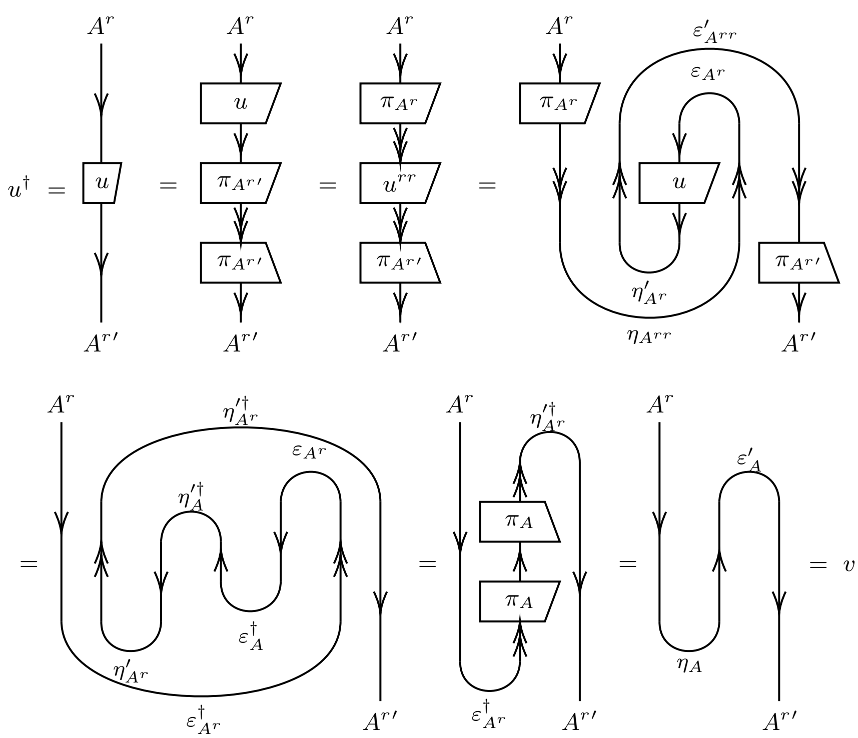
Having made these simplifications, we are able to establish the following.

Proposition 3.6.59. *In a pivotal dagger category, dagger duals are unique up to unitary isomorphism.*

Proof. Let $\mathcal{M}$ be a pivotal dagger category and let $(A, A^r, \eta, \varepsilon)$ and $(A, A^{r\prime}, \eta', \varepsilon')$ be dagger dualities. Consider the two morphisms of proposition 3.5.5:



Since dagger dualities are dualities we already know by propositions 3.5.5 and 3.5.7, that the above two morphisms are mutual inverses and that they provide the unique isomorphism between duals. So we only need to prove that they are related by daggering, in order to show that the isomorphism is unitary. To this end we have the following graphical calculation:

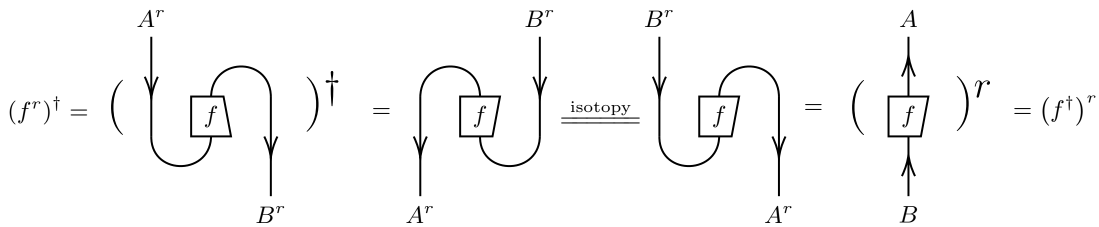


□

We continue by showing the interplay between the duals functor and the dagger functor. Note that as functors they are both contravariant, so the two ways to compose them will be covariant.

Proposition 3.6.60. *In a pivotal dagger category $\mathcal{M}$, the duals functor commutes with the dagger functor.*

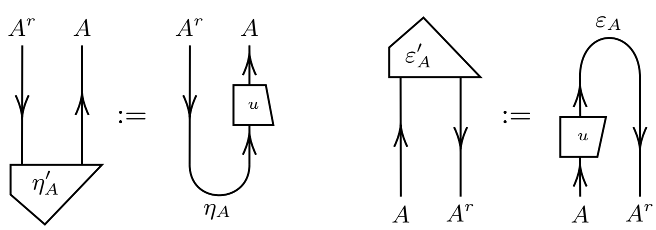
Proof. We will show this using the graphical calculus. Observe that for every morphism $f : A \to B$ in $\mathcal{M}$,

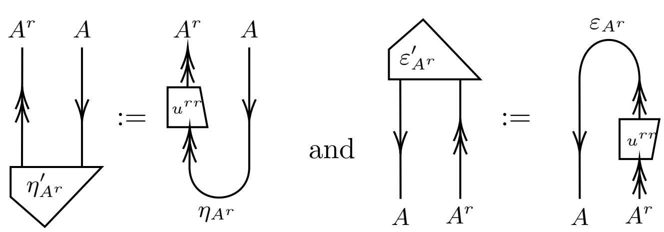


□

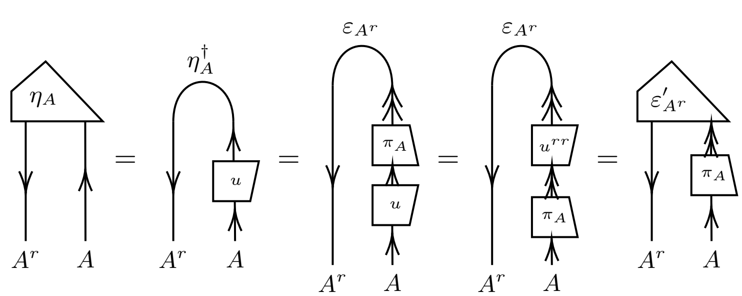
Definition 3.6.61. *Let $\mathcal{M}$ be a pivotal dagger category. The functor $(\ )^* := (\ )^{r\dagger} : \mathcal{M} \to \mathcal{M}$ will be called* ***conjugation*** *and it will be depicted as*

Remark 3.6.62. Note that dagger duals might be unique up to non-unique unitary isomorphism, but dagger dualities are unique up to unique unitary isomorphism. Furthermore, given a dagger duality $(A, A^r, \eta_A, \varepsilon_A)$ and a unitary endomorphism $u : A \to A$ in a pivotal dagger category, we can create another dagger duality $(A, A^r, \eta'_A, \varepsilon'_A)$ by defining:



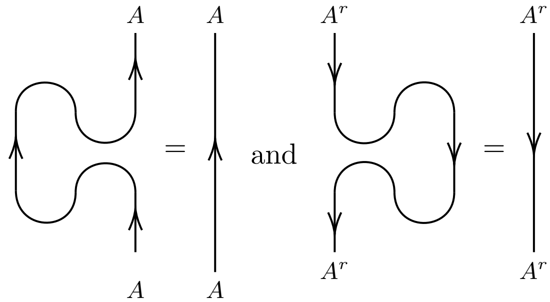


where we distinguish the chosen units and counits by depicting them as cups and caps. The triangle laws hold in this case by functoriality of taking duals and unitarity of u, so we only show that $(A, A^r, \eta'_A, \varepsilon'_A)$ is a dagger duality. To this end, observe that by the above definitions and naturality of the pivot,

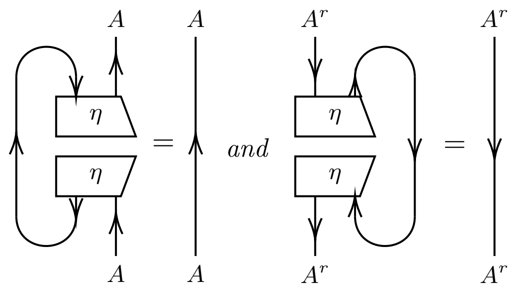


thus $(A, A^r, \eta'_A, \varepsilon'_A)$ is indeed a dagger duality.

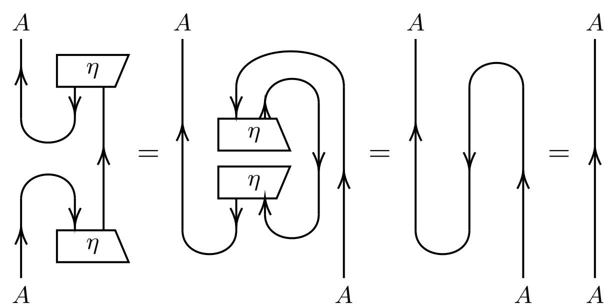
Remark 3.6.63. Note that any dagger duality $(A, A^r, \eta, \varepsilon)$ in a pivotal dagger category has the following two properties:



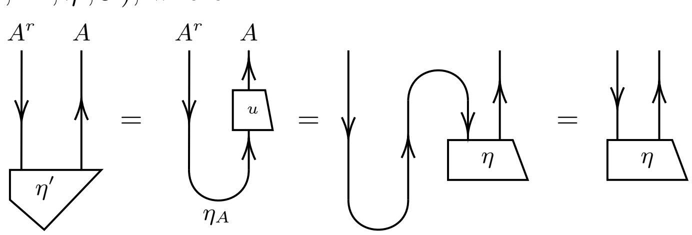
Definition 3.6.64. *Let A be an object in a pivotal dagger category $\mathcal{M}$. A state $\eta : I \to A^r \otimes A$ is called* ***maxinally entangled****, if*



Remark 3.6.65. The two conditions a maximally entangled state satisfies by definition can be used to show that the morphism $u = (\varepsilon_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \eta) : A \to A$, where we have ommited the coherent isomorphism, is unitary. Since the first equation in definition 3.6.64 is equivalent to $u \circ u^\dagger = \mathrm{id}_A$, we show that $u^\dagger \circ u = \mathrm{id}_A$. Observe that



Remark 3.6.66. By remark 3.6.63, the unit of a dagger duality is a maximally entangled state. The converse, i.e. that every maximally entangled state is the unit of a dagger duality, also holds as we shall see immediately. The endomorphism $u = (\varepsilon_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \eta) : A \to A$ of the previous remark is unitary. So by remark 3.6.62 there is a dagger $(A, A^r, \eta', \varepsilon')$, where



Note that this implies that maximally entangled states are unique up to unique unitary isomorphism.

Bringing all the structures introduced together, we give a final definition.

Definition 3.6.67. *A braided pivotal dagger category $\mathcal{M}$ is called a* ***dagger ribbon category****, if it is equipped with a unitary twist. A symmetric pivotal dagger category $\mathcal{M}$ is called a* ***dagger compact category****, if it is equipped with the identity twist.*

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