NATIONAL TECHNICAL
UNIVERSITY OF ATHENS

SCHOOL OF APPLIED
MATHEMATICAL AND
PHYSICAL SCIENCES

SCHOOL OF MECHANICAL
ENGINEERING

NATIONAL CENTRE FOR
SCIENTIFIC RESEARCH
"DEMOKRITOS"

INSTITUTE OF NANOSCIENCE
AND NANOTECHNOLOGY

INSTITUTE OF NUCLEAR AND
PARTICLE PHYSICS

Postgraduate Thesis of the M.Sc. program
"Physics and Technological Applications"

Supersymmetry, Supergravity and Inflationary Cosmology

Υπερσυμμετρία, Υπερβαρύτητα και Πληθωριστική Κοσμολογία

MASTER THESIS
of Foivos-Thomas Dimitriou

Supervisor: Fotis Farakos

Athens, July, 2025

Acknowledgements

I would like to express my deepest gratitude to my supervisor, Fotis Farakos, for his patience and his guidance throughout this year. His support meant a lot to me, who gave me the opportunity to study aspects of Theoretical Physics and Cosmology.

I would also like to express my heartfelt gratitude to my family. They helped me a lot all this year. And, last but not least, i would like to express my gratitude to my close friends, who help me grow through their friendship and are always by my side.

Abstract

This master thesis was prepared within the framework of the interdepartmental postgraduate program: "Physics and Technological Applications" of the National Technical University of Athens, in collaboration with the National Centre for Scientific Research "Demokritos". The current thesis was written during the 2024-2025 academic year.

In the present thesis elements of supersymmetry, supergravity and inflationary cosmology are investigated. In particular, the one-loop corrections to the Kahler potential are studied for a generic 4D N=1 globally supersymmetric Lagrangian with only chiral superfields. The implications of these corrections are then investigated for inflationary models embedded in supergravity. As a working example, chaotic inflation is analyzed. Interestingly enough, the inflationary trajectory remains stabilized, and a single-field scenario is successfully realized.

Περίληψη

Η παρούσα μεταπτυχιακή εργασία εκπονήθηκε στο πλαίσιο του διατμηματικού μεταπτυχιακού προγράμματος: «Φυσική και Τεχνολογικές Εφαρμογές» του Εθνικού Μετσόβιου Πολυτεχνείου, σε συνεργασία με το Εθνικό Κέντρο Έρευνας Φυσικών Επιστημών «Δημόκριτος». Εκπονήθηκε κατά το ακαδημαϊκό έτος 2024-2025.

Στην παρούσα διπλωματική εργασία μελετώνται στοιχεία της Υπερσυμμετρίας, της Υπερβαρύτητας και της πληθωριστικής κοσμολογίας. Ειδικότερα, εξετάζονται οι διορθώσεις ενός βρόχου στο δυναμικό Kähler για μια γενική τετραδιάστατη, με N=1, ολική Υπερσυμμετρική Λαγκρανζιανή, η οποία περιέχει μόνο χειραλικά υπερπεδία. Στη συνέχεια, διερευνώνται οι επιπτώσεις αυτών των διορθώσεων σε πληθωριστικά μοντέλα ενσωματωμένα στην Υπερβαρύτητα. Ως παράδειγμα εφαρμογής, αναλύεται το μοντέλο του χαοτικού πληθωρισμού. Ενδιαφέρον παρουσιάζει το γεγονός ότι η πληθωριστική τροχιά παραμένει σταθεροποιημένη και υλοποιείται επιτυχώς ένα σενάριο με ένα μοναδικό πεδίο.

Contents

Part I

Supersymmetry

Chapter 1

Elements of Supersymmetry

1.1 What is Supersymmetry?

Supersymmetry (SUSY) is a space-time symmetry mapping particles and fields of integer spin (bosons) into particles and fields of half integer spin (fermions), and viceversa. The generators Q act as

$$Q|\text{Fermion}\rangle = |\text{Boson}\rangle \quad \text{and viceversa} \tag{1.1}$$

From its very definition, this operator has two obvious but far-reaching properties that can be summarized as follows:

- It changes the spin of a particle (meaning that Q transforms as a spin-1/2 particle) and hence its space-time properties. This is why supersymmetry is not an internal symmetry but a space-time symmetry.
- In a theory where supersymmetry is realized, each one-particle state has at least a superpartner. Therefore, in a SUSY world, instead of single particle states, one has to deal with (super)multiplets of particle states.

Supersymmetry generators have specific commutation properties with other generators. In particular: Q commutes with translations and internal quantum numbers (e.g. gauge and global symmetries), but it does not commute with Lorentz generators

$$[Q, P_\mu] = 0 \quad , \quad [Q, G] = 0 \quad , \quad [Q, M_{\mu\nu}] \neq 0. \tag{1.2}$$

This implies that particles belonging to the same supermultiplet have different spin but same mass and same quantum numbers.

A supersymmetric field theory is a set of fields and a Lagrangian which exhibit such a symmetry. As ordinary field theories, supersymmetric theories describe particles and interactions between them: SUSY manifests itself in the specific particle spectrum a theory enjoys, and in the way particles interact between themselves.

A supersymmetric model which is covariant under general coordinate transformations is called supergravity (SUGRA) model. In this respect, a non-trivial fact, which again comes from the algebra, in particular from the (anti)commutation relation

$$\{Q, \bar{Q}\} \sim P_\mu, \tag{1.3}$$

is that having general coordinate transformations is equivalent to have local SUSY, the gauge mediator being a spin 3/2 particle, the gravitino. Hence local supersymmetry and General Relativity are intimately tied together.

One can have theories with different number of SUSY generators $Q : Q^I, I = 1, \ldots, N$. The number of supersymmetry generators, however, cannot be arbitrarily large. The reason is that any supermultiplet contains particles with spin at least as large as $\frac{N}{4}$. Therefore, N can be at most as large as 4 for theories with maximal spin 1 (gauge theories) and as large as 8 for theories with maximal spin 2 (gravity). Thus stated, this statement is true in 4 space-time dimensions. Equivalent statements can be made in higher/lower dimensions, where the dimension of the spinor representation of the Lorentz group is larger/smaller (for instance, in 10 dimensions, which is the natural dimension where superstring theory lives, the maximum allowed N is 2). What really matters is the number of single state supersymmetry generators, which is a dimension-independent statement.

Finally, notice that since supersymmetric theories automatically accommodate both bosons and fermions, SUSY looks like the most natural framework where to formulate a theory able to describe matter and interactions in a unified way.

1.2 Superspace & Superfields

We now move on to discuss the space in which the 4D, N = 1 supersymmetric theories live, and also how to study the most generic Lagrangians.

When one works in Lagrangian formulation, it is generally better to write the Lagrangian in a way where its symmetries are manifest. Therefore, the usual space-time Lagrangian formulation is not the most suitable option for studying supersymmetric theories, since in the standard spacetime, supersymmetry is not manifest. This motivates one to construct mathematically an extension of spacetime, called *superspace*.

The four-dimensional N=1 supersymmetric theories can be studied if one enlarges the space-time labelled with coordinates x^μ, associated to the generators $\mathcal{P}_\mu$, by introducing 2+2 anticommuting Grassman coordinates θ_α, and their complex conjugates $\bar{\theta}_{\dot{\alpha}}$, associated to the supersymmetry generators $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}$. This construct is an eight coordinate superspace labelled by $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The set of constant Grassmann numbers that is introduced, anti-commute with everything "fermionic" and commute with everything "bosonic" in the following manner:

$$\left\{\theta^\alpha, \theta^\beta\right\} = 0 \quad , \quad \left\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\right\} = 0 \quad , \quad \left\{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\right\} = 0. \tag{1.4}$$

Grassmann parameters allow us to express the supersymmetry algebra completely in terms of commutators:

$$[\theta\mathcal{Q}, \bar{\theta}\bar{\mathcal{Q}}] = 2\theta\sigma^\mu\bar{\theta}\mathcal{P}_\mu \quad , \quad [\theta\mathcal{Q}, \theta\mathcal{Q}] = [\bar{\theta}\bar{\mathcal{Q}}, \bar{\theta}\bar{\mathcal{Q}}] = 0, \tag{1.5}$$

where the summation notation is $\theta\mathcal{Q} \equiv \theta^\alpha\mathcal{Q}_\alpha$ and $\bar{\theta}\bar{\mathcal{Q}} \equiv \bar{\theta}_{\dot{\alpha}}\bar{\mathcal{Q}}^{\dot{\alpha}}$. Discussing it from the point of view of a group theory, exponentiating this Lie algebra one gets the superPoincaré Group. A group element can then be written as:

$$G(x, \theta, \bar{\theta}, \omega) = \exp\left(ix\mathcal{P} + i\theta\mathcal{Q} + i\bar{\theta}\bar{\mathcal{Q}} + \frac{1}{2}i\omega\mathcal{M}\right). \tag{1.6}$$

A "point" in superspace (strictly speaking, the notion of a point is generalized in this context because of the anticommuting properties of the Grassmann variables $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$) gets identified with a so-called super-translation through the one-to-one map:

$$(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \longleftrightarrow e^{(x^\mu \mathcal{P}_\mu)}\, e^{(\theta\mathcal{Q}+\bar{\theta}\bar{\mathcal{Q}})}. \tag{1.7}$$

The $2+2$ anti-commuting Grassmann numbers $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ can then be thought of as coordinates in superspace. All standard spinor identities remain valid for these Grassmann variables.

Superfields are... fields in superspace: functions of the superspace coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. From the anticommutative nature of Grassman variables, any product of more than two θ's or two $\bar{\theta}$'s vanishes: given that $\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha$, we have that $\theta_\alpha^2 = 0$ and therefore $\theta_\alpha\theta_\beta\theta_\gamma = 0$, since at least two indices in this product are the same. The implication of the latter observation, is that the most general superfield $Y = Y(x, \theta, \bar{\theta})$ can be written as a Taylor-like expansion:

$$\begin{aligned} Y(x,\theta,\bar{\theta}) = y(x) + \theta\phi(x) + \bar{\theta}\,\bar{\chi}(x) + \theta\theta\, m(x) + \bar{\theta}\bar{\theta} n(x) + \\ + \theta\sigma^\mu\bar{\theta}\, v_\mu(x) + \theta\theta\bar{\theta}\,\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\,\rho(x) + \theta\theta\bar{\theta}\bar{\theta}\, d(x). \end{aligned} \tag{1.8}$$

This non-reducible representation of a superfield tells us that it is a a finite collection (a *multiplet*) of ordinary fields.

1.2.1 Integration & Derivation in Superspace

One also needs to know how integration and derivation as operations act on Grassmann numbers, since these operations will eventually occur when one writes down supersymmetric Lagrangians explicitly in terms of superfields.

Derivatives in superspace are denoted as:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} \quad \text{and} \quad \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta \quad , \quad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \text{and} \quad \bar{\partial}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\beta}}, \tag{1.9}$$

where the derivative acts as:

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \quad , \quad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \quad , \quad \partial_\alpha\bar{\theta}_{\dot{\beta}} = 0 \quad , \quad \bar{\partial}^{\dot{\alpha}}\theta^\beta = 0. \tag{1.10}$$

Now, let us discuss how integration acts on a Grassmann variable θ , or its complex conjugate $\bar{\theta}$. Integration is defined as follows:

$$\int d\theta = 0\ , \qquad \int d\theta\,\theta = 1\,. \tag{1.11}$$

The latter definition essentially means that Grassmann variables, when integrated over, behave like a delta function $\theta = \delta(\theta)$. To illustrate this, if one considers a generic function $f(\theta) = f_0 + \theta f_1$, one will obtain the following results:

$$\int d\theta\, f(\theta) = f_1\,, \qquad \int d\theta\,\delta(\theta) f(\theta) = f_0, \tag{1.12}$$

which what essentially tells us is that integration is equivalent to derivation $\int = \partial$.

The latter can be generalized for $N = 1$ superspace, for the integration measure:

$$d^2\theta \equiv \frac{1}{2} d\theta^1 d\theta^2\,, \qquad d^2\bar{\theta} \equiv \frac{1}{2} d\bar{\theta}^{\dot{2}} d\bar{\theta}^{\dot{1}}\,. \tag{1.13}$$

Some useful identities are presented as well:

$$\int d^2\theta\, \theta\theta = \int d^2\bar{\theta}\, \bar{\theta}\bar{\theta} = 1\,, \qquad \int d^2\theta\, d^2\bar{\theta}\, \theta\theta\, \bar{\theta}\bar{\theta} = 1 \tag{1.14}$$

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta\,, \qquad \int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}\,. \tag{1.15}$$

1.3 Supersymmetry Transformations

Now one arises the question: how superfields transform under SUSY transformations? And what does it mean for a function to be supersymmetric invariant i.e. to respect supersymmetry? In order to move forward, one is motivated from the ordinary spacetime translations, representing the generators of transformations in Poincaré group $\mathcal{P}_\mu$ as differential operators. Let us consider the following example of space-time translations on a field $\phi(x)$, with infinitesimal parameter α^μ, generated by $\mathcal{P}_\mu$. The latter can be written down as:

$$\phi(x+a) = e^{-ia^\mu \mathcal{P}_\mu}\phi(x)e^{ia^\mu \mathcal{P}_\mu} = \phi(x) - ia^\mu[\mathcal{P}_\mu, \phi(x)] + \cdots \tag{1.16}$$

To identify how the generator is realized as differential operator, one can Taylor expand the LHS:

$$\phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x) + \cdots \tag{1.17}$$

Equating the right hand sides of the two equations above allow us to realize:

$$[\phi(x), \mathcal{P}_\mu] = -i\partial_\mu\phi(x) = P_\mu^{'}\phi(x), \tag{1.18}$$

where $\mathcal{P}_\mu$ is the generator of translations and $P_\mu^{'}$ is its representation as a hermitian differential operator. Thus, a translation of a field give rise to a change on the field itself:

$$\delta\phi = \phi(x+a) - \phi(x) = ia^\mu P_\mu^{'}\phi. \tag{1.19}$$

From the latter example, one is in position to obtain similarly a representation of the supersymmetry generators $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}$ as differential operators. We are interested in realizing a translation in superspace (a supersymmetry transformation) of a generic superfield $Y(x, \theta, \bar{\theta})$. Instead of an infinitesimal vector-parameter, we now have the spinorial parameters $\epsilon_\alpha, \bar{\epsilon}_{\dot{\alpha}}$. The translation in superspace can be written in analogous way as before:

$$Y(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) = e^{-i(\epsilon\mathcal{Q}+\bar{\epsilon}\bar{\mathcal{Q}})}\, Y(x,\theta,\bar{\theta})\, e^{i(\epsilon\mathcal{Q}+\bar{\epsilon}\bar{\mathcal{Q}})}, \tag{1.20}$$

and the variation of the superfield is denoted as:

$$\delta_{\epsilon,\bar{\epsilon}}Y(x,\theta,\bar{\theta}) \equiv Y(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) - Y(x,\theta,\bar{\theta}). \tag{1.21}$$

As a matter of fact, from the latter equations, one is provided with a more definition of a superfield: a superfield is a field in superspace which transforms under a super-translation according to (1.20). To begin with, one has to derive the explicit expressions for the translations $\delta x, \delta\theta, \delta\bar{\theta}$. Interestingly enough, we have to assume $\delta x \neq 0$, despite the fact that we do not apply spacetime translations i.e.

acting with $\mathcal{P}_\mu$. The variation reads:

$$Y(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) = e^{-i(\epsilon\mathcal{Q}+\bar{\epsilon}\overline{\mathcal{Q}})}e^{-i(x\mathcal{P}+\theta\mathcal{Q}+\bar{\theta}\overline{\mathcal{Q}})}\,Y_0\,e^{i(x\mathcal{P}+\theta\mathcal{Q}+\bar{\theta}\overline{\mathcal{Q}})}e^{i(\epsilon\mathcal{Q}+\bar{\epsilon}\overline{\mathcal{Q}})}. \tag{1.22}$$

Therefore, one needs to compute the expression $\exp\big[i(x\mathcal{P}+\theta\mathcal{Q}+\overline{\theta\mathcal{Q}})\big]\,\exp\big[i(\epsilon\mathcal{Q}+\bar{\epsilon}\overline{\mathcal{Q}})\big]$. The Baker-Campbell-Hausdorff formula [1], allow us to compute the product of exponentials for non-commuting mathematical objects. One calculates:

$$\begin{aligned}
&\exp\{i(x\mathcal{P}+\theta\mathcal{Q}+\overline{\theta\mathcal{Q}})\}\exp\{i(\epsilon\mathcal{Q}+\bar{\epsilon}\overline{\mathcal{Q}})\} = \\
&= \exp\left\{ix^\mu\mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar{\epsilon}+\bar{\theta})\overline{\mathcal{Q}} - \tfrac{1}{2}[\overline{\theta\mathcal{Q}}, \epsilon\mathcal{Q}] - \tfrac{1}{2}[\theta\mathcal{Q}, \bar{\epsilon}\overline{\mathcal{Q}}]\right\} \\
&\overset{(1.5)}{=} \exp\left\{ix^\mu\mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar{\epsilon}+\bar{\theta})\overline{\mathcal{Q}} + \epsilon\sigma^\mu\bar{\theta}\mathcal{P}_\mu - \theta\sigma^\mu\bar{\epsilon}\mathcal{P}_\mu\right\} \\
&= \exp\left\{i\left(x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}\right)\mathcal{P}_\mu + i(\epsilon+\theta)\mathcal{Q} + i(\bar{\epsilon}+\bar{\theta})\overline{\mathcal{Q}}\right\}
\end{aligned} \tag{1.25}$$

From the latter, one identifies the expressions for the translations as:

$$\delta x^\mu = i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}\,, \qquad \delta\theta^a = \epsilon^a\,, \qquad \delta\bar{\theta}^{\dot{a}} = \bar{\epsilon}^{\dot{a}} \tag{1.26}$$

Notice that one reaches to a non-trivial answer: the spinorial parameters determine a non-vanishing spacetime translation. This is in line with the construction of supersymmetry algebra: $\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} \sim P_\mu$. Thus, the result above, comes from the fact that two subsequent supersymmetry transformations generate a space-time translation.

We can now find the representation of the supersymmetry generators $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ as differential operators. Let us Taylor expand the right hand side of (1.21)which becomes

$$\begin{aligned}
\delta_{\epsilon,\bar{\epsilon}}Y(x,\theta,\bar{\theta}) &= Y(x,\theta,\bar{\theta}) + i\left(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}\right)\partial_\mu Y(x,\theta,\bar{\theta}) + \\
&\quad + \epsilon^\alpha\partial_\alpha Y(x,\theta,\bar{\theta}) + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}}Y(x,\theta,\bar{\theta}) + \cdots - Y(x,\theta,\bar{\theta}) \\
&= \left[\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i\left(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}\right)\partial_\mu + \ldots\right]Y(x,\theta,\bar{\theta})
\end{aligned} \tag{1.27}$$

We can now Taylor expand the exponentials of (1.20) and obtain:

$$\begin{aligned}
\delta_{\epsilon,\bar{\epsilon}}Y(x,\theta,\bar{\theta}) &= (1 - i\epsilon\mathcal{Q} - i\bar{\epsilon}\bar{\mathcal{Q}} + \cdots)\,Y(x,\theta,\bar{\theta})\,(1 + i\epsilon\mathcal{Q} + i\bar{\epsilon}\bar{\mathcal{Q}} + \cdots) - Y(x,\theta,\bar{\theta}) \\
&= -i\epsilon^\alpha\left[\mathcal{Q}_\alpha, Y(x,\theta,\bar{\theta})\right] + i\bar{\epsilon}^{\dot{\alpha}}\left[\bar{\mathcal{Q}}_{\dot{\alpha}}, Y(x,\theta,\bar{\theta})\right] + \cdots\,,
\end{aligned} \tag{1.28}$$

where one takes into account that $i\epsilon\mathcal{Q} \equiv i\epsilon_\alpha\mathcal{Q}^\alpha = -i\bar{\epsilon}^{\dot{\alpha}}\mathcal{Q}_{\dot{\alpha}}$. Defining the commutators in terms of the representation of SUSY generators:

$$[Y, \mathcal{Q}_\alpha] \equiv Q_\alpha Y, \quad \left[Y, \bar{\mathcal{Q}}_{\dot{\alpha}}\right] \equiv \bar{Q}_{\dot{\alpha}}Y, \tag{1.29}$$

1

$$e^A e^B = e^C \quad \text{where} \quad C = \sum_{n=1}^{\infty}\frac{1}{n!}C_n(A,B), \tag{1.23}$$

with quantities C_n being:

$$C_1 = A+B \quad , \quad C_2 = [A,B] \quad , \quad C_3 = \frac{1}{2}[A,[A,B]] - \frac{1}{2}[B,[B,A]] \quad \cdots\,. \tag{1.24}$$

one derives the supersymmetry variation of a superfield by parameters $\epsilon, \bar{\epsilon}$ as:

$$\delta_{\epsilon,\bar{\epsilon}} Y = (i\epsilon Q + i\bar{\epsilon}\bar{Q})Y. \tag{1.30}$$

As for the case of spacetime translations, equating the RHS of (1.30) with (1.27), we get the following expression for the differential operators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$:

$$\begin{cases} Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} = +i\bar{\partial}_{\dot{\alpha}} - \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{cases} \tag{1.31}$$

Notice that, consistently, $Q^\dagger_\alpha = \bar{Q}_{\dot{\alpha}}$ (recall that $(\sigma^\mu_{\alpha\dot{\beta}})^\dagger = \sigma^\mu_{\beta\dot{\alpha}}$).

The two differential operators close the supersymmetry algebra, and therefore are valid representations of SUSY generators, since the following equations hold:

$$\{Q_\alpha, Q_\beta\} = \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 0, \quad \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu. \tag{1.32}$$

1.4 Supersymmetric Actions

One is in position to construct manifestly supersymmetric invariant actions. It easy to see that the integral in superspace of any arbitrary superfield is a supersymmetric invariant quantity. This can be expressed through the following integral:

$$\mathcal{I} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; Y(x, \theta, \bar{\theta}), \tag{1.33}$$

is manifestly supersymmetric invariant, if Y is a superfield. Why? Let us break down this statement. The integration measure is invariant under supertranslations:

$$\int d\theta\theta = \int d(\theta + \xi)\,(\theta + \xi) = 1 \tag{1.34}$$

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{I} = \delta_{\epsilon,\bar{\epsilon}} \int d^4x d^2\theta d^2\bar{\theta}\, Y(x, \theta, \bar{\theta}) = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \delta_{\epsilon,\bar{\epsilon}} Y(x, \theta, \bar{\theta}) \tag{1.35}$$

We can write the SUSY transformation of the general superfield Y as:

$$\delta_{\epsilon,\bar{\epsilon}} Y = \epsilon^\alpha\partial_\alpha Y + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} Y + \partial_\mu\left[-i\left(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}\right)Y\right] \tag{1.36}$$

The integral evaluates to:

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{I} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\epsilon^\alpha\partial_\alpha Y + \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} Y + \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \partial_\mu[\cdots]. \tag{1.37}$$

The first two terms vanish, since the integration measure contains more θ's than the expressions of $\partial_\alpha Y, \bar{\partial}_{\dot{\alpha}} Y$. Thus, we find that the integral $\mathcal{I}$ is SUSY invariant up to a total space-time derivative (which applying the Gauss's theorem, gives a surface term which also vanishes). Thus, under supersymmetry transformation, the full integral is left invariant:

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x\, d^2\theta\, d^2\bar{\theta}\, Y(x, \theta, \bar{\theta}) = 0. \tag{1.38}$$

From the latter observation, one constructs supersymmetric invariant actions by integrating in superspace an appropriate superfield. The idea is that a suitable superfield , call it $\mathcal{Y}$, is being integrated over the Grassmann variables in order to give rise to a real Lagrangian density, transforming as a scalar density under Poincaré transformations. This would lead to a SUSY invariant action $\mathcal{S}$:

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \mathcal{Y}(x,\theta,\bar{\theta}) = \int d^4x \mathcal{L}(\phi(x), \psi(x), A_\mu(x), \dots). \tag{1.39}$$

If the origin of the action $\mathcal{S}$ comes from an integral of a superfield in superspace, implies that automatically is SUSY invariant by construction, and the Lagrangian $\mathcal{L}$, a function of just ordinary fields, is guaranteed to be supersymmetric invariant, up to total space-time derivatives.

1.5 Chiral Superfields

A *chiral superfield* Φ is defined as:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{1.40}$$

and an anti-chiral one $\bar{\Phi}$ is:

$$D_\alpha \bar{\Phi} = 0. \tag{1.41}$$

This is solved by observing that

$$D_\alpha \bar{\theta}^{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}\theta^\alpha = D_\alpha y^\mu = \bar{D}_{\dot{\alpha}} y^\mu = 0, \tag{1.42}$$

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \qquad \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}.$$

Thus Φ depends only on θ and y^μ (in essence all $\bar{\theta}$ dependence is through y^μ), and $\bar{\Phi}$ only on $\bar{\theta}$ and $\bar{y}^\mu$. For the chiral Φ, we have the following expansion:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y), \tag{1.43}$$

or Taylor expanding in terms of x, θ and $\bar{\theta}$:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \theta\theta F(x) \\ &\quad - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x). \end{aligned} \tag{1.44}$$

Physically, such a chiral superfield describes one complex scalar ϕ and one Weyl fermion ψ. The field F turns out to be an auxiliary field (a non-propagating degree of freedom). For $\bar{\Phi}$ we similarly have

$$\begin{aligned} \bar{\Phi}(\bar{y},\bar{\theta}) &= \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) - \bar{\theta}\bar{\theta}\bar{F}(\bar{y}) \\ &= \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) - \bar{\theta}\bar{\theta}\bar{F}(x) \\ &\quad + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{\phi}(x). \end{aligned} \tag{1.45}$$

We now write the susy variations of the component fields: First, for chiral superfields it is useful to change variables from $x^\mu, \theta, \bar{\theta}$ to y^μ, θ. Then

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha}, \qquad \bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} \tag{1.46}$$

so that the variation reads:

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta) &= \left(\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\cdot\bar{\epsilon}^{\dot{\beta}}\frac{\partial}{\partial y^\mu}\right)\Phi(y,\theta)\\ &= \sqrt{2}\epsilon\psi - 2\epsilon\theta F + 2i\theta\sigma^\mu\bar{\epsilon}\left(\frac{\partial}{\partial y^\mu}\phi + \sqrt{2}\theta\frac{\partial}{\partial y^\mu}\psi\right)\\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta\left(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi\right) - \theta\theta\left(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\frac{\partial}{\partial y^\mu}\psi\right).\end{aligned} \tag{1.47}$$

Therefore, the final expression for the supersymmetry variation of the components of the chiral superfield is written as:

$$\begin{cases}\delta\phi = \sqrt{2}\epsilon\psi\\ \delta\psi_\alpha = \sqrt{2}i\left(\sigma^\mu\bar{\epsilon}\right)_\alpha\partial_\mu\phi - \sqrt{2}\epsilon_\alpha F\\ \delta F = i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon}\end{cases} \tag{1.48}$$

1.6 Chiral Models

We discussed above how one can construct susy invariant actions, from an integral in superspace.We should now move on to discuss the most general SUSY invariant action one can write, for the chiral sector. Firstly, one can convince himself that products of superfields are of course superfields. Also, products of (anti) chiral superfields are (anti) chiral superfields. Typically, one will have a *superpotential* $W(\Phi)$ which is again chiral. What is W? This W may depend on several different Φ_i. Using the y and θ variables one is able to Taylor expand:

$$W(\Phi) = W(\phi(y)) + \sqrt{2}\frac{\partial W}{\partial\phi^i}\theta\psi_i(y) - \theta\theta\left(\frac{\partial W}{\partial\phi^i}F_i(y) + \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi_i(y)\psi_j(y)\right) \tag{1.49}$$

where it is understood that $\partial W/\partial\phi$ and $\partial^2 W/\partial\phi\partial\phi$ are evaluated at $\phi(y)$. Why are we discussing about this quantity? One can see that any integral of a superfield in full superspace can be rewritten as an integral in half superspace, but the converse is not true. Why? Because if one considers a general superfield Y:

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\; Y = \frac{1}{4}\int d^4x\, d^2\theta\, \bar{D}^2 Y \tag{1.50}$$

On the other hand, the converse is not true in general. Consider a term like:

$$\int d^4x\, d^2\theta\, \Phi^n, \tag{1.51}$$

where Φ is a chiral superfield. This integral cannot be converted into an integral in full superspace, essentially because there are no covariant derivatives to do the trick (they contain $\theta\theta\, F$ terms). Integrals which cannot be converted into integral in full superspace, are called F-terms. All others, are called D-terms. Thus, one observes that any generic Lagrangian of the form:

$$\int d^2\theta d^2\overline{\theta}\, K(x,\theta,\overline{\theta}) + \int d^2\theta\, W(\Phi) + \int d^2\overline{\theta}\, \bar{W}(\bar{\Phi}) \tag{1.52}$$

is automatically SUSY invariant, i.e. it transforms at most by a total derivative in space-time. Let us obtain this. The SUSY variation of any superfield is given by (1.30) and, since the ϵ and $\overline{\epsilon}$ are constant spinors and the Q and $\overline{Q}$ are differential operators in superspace, it is again a total derivative in all of superspace:

$$\delta_{\epsilon,\bar{\epsilon}}K = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha K) + \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}(-\overline{\epsilon}^{\dot{\alpha}} K) + \frac{\partial}{\partial x^\mu}\left[-i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\epsilon})K\right]. \tag{1.53}$$

Integration $\int d^2\theta d^2\overline{\theta}$ leaves the last term which is a total space-time derivative as claimed. If now one studies the W i.e. a chiral superfield one changes variables to θ and y and one has:

$$\delta_{\epsilon,\bar{\epsilon}}W = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha W(y,\theta)) + \frac{\partial}{\partial y^\mu}\left[-i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\epsilon})W(y,\theta)\right]. \tag{1.54}$$

Integrating $\int d^2\theta$ only leaves the last term which becomes $\frac{\partial}{\partial x^\mu}[\cdots]$ and is a total derivative in space-time. The analogous result holds for an anti chiral superfield $\overline{W}(\overline{\phi})$ and integration $\int d^2\overline{\theta}$. This proves the supersymmetry of the action resulting from the space-time integral of the Lagrangian discussed above (1.52).

Now, let us discuss the integrands: K and W. Also, what is K? It is the so called *Kähler potential.* With Kähler potential, one is in position to describe the kinetic terms of fields. It is a superfield. It is also real and scalar function. It is a generic function of $\Phi, \bar{\Phi}$, and it should not contain covariant derivatives. The most general expression is:

$$K(\Phi,\bar{\Phi}) = \sum_{m,n=1}^{\infty} c_{mn}\bar{\Phi}^m\Phi^n \ , c_{mn} = c^*_{nm} \tag{1.55}$$

Now, let us discuss the superpotential. It is a holomorphic function of Φ, and its hermitian conjugate has been introduced to the lagrangian, in order to make $\mathcal{L}$ real. As we said earlier, W must be a chiral superfield. It should not contain covariant derivatives, and the general expression is written as:

$$W(\Phi) = \sum_{n=1}^{\infty} a_n\Phi^n \tag{1.56}$$

The scalar potential of scalar fields, for the model of interest, can be found by:

$$V(\phi,\bar{\phi}) = \left(K^{-1}\right)^i_j W_i W^j, \tag{1.57}$$

where

$$K^j_i = \frac{\partial^2}{\partial\phi^i\partial\bar{\phi}_j}K(\phi,\bar{\phi}), \tag{1.58}$$

is the so called Kähler metric, and one can speak about a manifold $\mathcal{M}$ called the Kähler manifold.

Chapter 2

One-loop Kähler potential

2.1 d=4, N=1 Globally Supersymmetric Lagrangian

In the following analysis, from the work of [8], one considers a general d=4 N=1 globally supersymmetric lagrangian involving chiral (and vector) superfields, with arbitrary superpotential and Kähler potential. One is in position to obtain perturbative quantum corrections from adopting a component field approach that respects supersymmetry. We comment that for the full lagrangian, one needs to take into account if the field approach respects also the background gauge invariance, and one works in supersymmetric Landau gauge. In the component field approach, one receives quantum corrections in terms of component Feynman diagrams.

One starts this discussion by studying the most general 4D, N=1 globally supersymmetric theory defined by a tree-level Lagrangian:

$$\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_V, \tag{2.1}$$

where we are interested only in the Lagrangian obtained from chiral superfields ($\mathcal{L}_V$ is constructed from vector superfields that describe radiation):

$$\mathcal{L}_\Phi = \int d^2\theta\, d^2\bar{\theta}\; K\left(\Phi^i, \bar{\Phi}^{\bar{j}}\right) + \left[\int d^2\theta\, w\left(\Phi^i\right) + \text{ h.c. }\right] \tag{2.2}$$

The superpotential w and the Kähler potential K are arbitrary up to constraints by gauge invariance. The scalar field derivatives of superpotential $w_{ij} = \partial^2 w/\partial\phi^i\phi^j$ are symmetric. The Kähler potential is real, and the Kähler metric in terms of the scalar component field $\hat{K}_{\bar{i}j} \equiv \partial^2 K/\partial\bar{\varphi}^{\bar{i}}\partial\varphi^j$ is hermitian. We should mention that in the special case of a renormalizable theory, w is at most cubic in the chiral superfields, and K is quadratic (i.e. canonical). Here one proceeds to full generality, and do not impose renormalizability. The latter lagrangian is the most general one that contains no more than two space-time derivatives on component fields. It could come from a more fundamental theory at low energies. In fact, since we take w, K to be arbitrary, the total lagrangian $\mathcal{L}$ in (2.1) necessarily describes an effective theory, applicable below a cutoff scale Λ. One takes this $\mathcal{L}$ as a general classical bare lagrangian and study the contributions from one-loop corrections.

Quantum corrections to a classical supersymmetric lagrangian can be computed by various techniques, as pointed in [8]. One might consider using the background field method, where the superfields Φ^i are split into background ($\hat{\Phi}^i$) and quantum parts (Φ^i) (adopting notation from [8]):

$$\Phi^i \to \hat{\Phi}^i + \Phi^i \tag{2.3}$$

We are not going into any details of background field method. However, plugging (2.3) (and also the split of vector superfields, with their gauge fixing) into the original lagrangian (2.1), the resulting lagrangian can be expanded in powers of the quantum superfields Φ^i. The zero-th order part is just the original lagrangian for the classical (background) superfields, $\mathcal{L}(\hat{\Phi}, \hat{V})$. The linear terms to Φ^i do not contribute to the effective action and can be dropped. The part bilinear in quantum superfields, $\mathcal{L}_{\text{bil}}(\phi, V, \dots; \hat{\phi}, \hat{V})$, is the relevant one for the computation of one-loop quantum corrections. One has to integrate out the quantum superfields in the theory defined by $\mathcal{L} + \mathcal{L}_{\text{bil}}$. The final product of this operation gives the one-loop corrections to the effective lagrangian in terms of background superfields only, and will obtain the form:

$$\Delta\mathcal{L} = \left[\int d^2\theta\, \Delta w(\hat{\Phi}) + \text{h.c.}\right] + \int d^4\theta \left[\Delta K(\hat{\bar{\Phi}}, \hat{\Phi})\right] + \dots \tag{2.4}$$

where the dots are terms associated with so called supercovariant derivatives and the correction to the Fayet-Iliopoulos terms, which are not the purpose of this analysis.

Now, one maintains the whole off-shell structure of quantum supermultiplets and chooses to work with component fields instead of superfields. The component expansion of the chiral quantum superfields reads [1]:

$$\Phi^i = \varphi^i + \sqrt{2}\theta\psi^i + \theta\theta F^i - i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^i - \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi^i - \frac{1}{4}\theta\theta\overline{\theta\theta}\Box\varphi^i, \tag{2.5}$$

where, as usual, φ^i are complex scalar fields, F^i the auxiliary scalars, and ψ^i are complex Weyl fields. Notice again that these expressions are valid for the expanded quantum superfields. What about the background ones? Since supersymmetry induces one-loop corrections to have the form (2.4), a useful choice of the background superfields can simplify the task of evaluating the corrections $\Delta w, \Delta K$. In [8], the choice made for the background superfield is as follows:

$$\hat{\Phi}^i = \hat{\varphi}^i + \theta\theta\hat{F}^i, \tag{2.6}$$

where both the scalars $\hat{\varphi}^i, \hat{F}^i$ are taken as constants, i.e., space-time independent. This will give the advantage of simplifying the evaluation of one-loop contributions. As a result of the latter choice, it is easy to check that the effective Lagrangian is expressed as:

$$\Delta\mathcal{L} = \left[\Delta w_i(\hat{\varphi})\hat{F}^i + \text{ h.c. }\right] + \Delta K_{\bar{\imath}j}(\hat{\bar{\varphi}}, \hat{\varphi})\hat{\bar{F}}^{\bar{\imath}}\hat{F}^j + \dots \tag{2.7}$$

The dots are higher order terms in $\hat{F}$ which are omitted. The convenience of using a background with constant $\hat{\varphi}, \hat{F}$ is that one-loop diagrams have vanishing external momenta. Once the results of the one-loop computation have been cast in the form (2.7), the functions Δw and ΔK can be obtained from their derivatives $\Delta w_i, \Delta K_j, \Delta K_{\bar{\imath}j}$. One is in position to proceed as follows. First, one expands the lagrangian in a background with constant $\hat{\varphi}, \hat{F}$. Then, the one-loop corrections are evaluated to the terms linear and quadratic in $\hat{F}$, from which Δw and ΔK can be derived.

So far, we established the background chiral superfields in the form (2.6), with constant $\phi, \hat{F}$ fields, and plug the background-quantum splitting (2.3) in (2.1). The lagrangian is then expanded and one keeps only the bilinear part in the quantum fields ϕ, V. One obtains the form $\mathcal{L}_{\text{bil}} = \mathcal{L}_{\phi\phi} + \mathcal{L}_{VV} + \mathcal{L}_{\phi V}$. The terms that are contained in $\mathcal{L}_{\text{bil}}$ have at most quadratic dependence on $\hat{F}$. The lagrangian part of our interest $\mathcal{L}_{\phi\phi}$ can be decomposed as $\mathcal{L}_{\phi\phi} = \mathcal{L}^{(0)}_{\phi\phi} + \mathcal{L}^{(\hat{F})}_{\phi\phi} + \mathcal{L}^{(\hat{F}\hat{F})}_{\phi\phi}$. The reason that $\mathcal{L}_{\phi V}$ is not needed for the one-loop corrections, is that mixed (chiral-vector) terms are annihilated as a result of the use of supersymmetric Landau gauge. Thus, $\mathcal{L}_{\phi\phi}$ is what we need. The lagrangian parts of interest are

[1] Our conventions in this Part of thesis are: for the metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, for the Pauli matrices $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, and for the supersymmetric derivatives $D_\alpha = \partial/\partial\theta^\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$, $\bar{D}_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$.

presented below:

$$\mathcal{L}^{(0)}_{\phi\phi} = \hat{K}_{\bar{i}j}\left(-\bar{\varphi}^{\bar{i}}\Box\varphi^j + \bar{F}^{\bar{i}}F^j + i\bar{\psi}^{\bar{i}}\bar{\sigma}^\mu\partial_\mu\psi^j\right) + \left[\hat{w}_{ij}\left(F^i\varphi^j - \frac{1}{2}\psi^i\psi^j\right) + \text{ h.c. }\right] \tag{2.8}$$

$$\mathcal{L}^{(\hat{F})}_{\phi\phi} = \hat{F}^j\left[\frac{1}{2}\hat{w}_{ikj}\varphi^i\varphi^k + \hat{K}_{\bar{i}kj}\bar{F}^{\bar{i}}\varphi^k + \hat{K}_{\bar{i}\bar{k}j}\left(\bar{F}^{\bar{i}}\bar{\varphi}^{\bar{k}} - \frac{1}{2}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{k}}\right)\right] + \text{ h.c.} \tag{2.9}$$

$$\mathcal{L}^{(\hat{F}\hat{F})}_{\phi\phi} = \hat{\bar{F}}^{\bar{i}}\hat{F}^j\left[\hat{K}_{\bar{i}j\bar{k}\ell}\bar{\varphi}^{\bar{k}}\varphi^\ell + \frac{1}{2}\left(\hat{K}_{\bar{i}jk\ell}\varphi^k\varphi^\ell + \hat{K}_{\bar{i}j\bar{k}\bar{\ell}}\bar{\varphi}^{\bar{k}}\bar{\varphi}^{\bar{\ell}}\right)\right] \tag{2.10}$$

We remind that matrices $\hat{K}_{\bar{i}j}, \hat{w}_{ij}$ are metrics in terms of background $\hat{\varphi}$:

$$\hat{K}_{\bar{i}j} \equiv K_{\bar{i}j}(\hat{\bar{\varphi}}, \hat{\varphi}), \quad \hat{w}_{ij} \equiv w_{ij}(\hat{\varphi}).$$

2.2 Propagators

In the following analysis, we examine the quantum bilinears within a constant $\hat{\varphi}$ background and compute the corresponding $\hat{\varphi}$-dressed propagators for the quantum fields. One is in position to compute the $\hat{\varphi}$ - dressed one (tadpoles) and two-point (loops) functions of $\hat{F}$, which will allow us to determine the quantum corrections $\Delta K, \Delta w$.

We start from the Lagrangian of our interest (which has no dependence on $\hat{F}$):

$$\mathcal{L}^{(0)}_{\phi\phi} = \hat{K}_{\bar{i}j}\left(-\overline{\varphi}^{\bar{i}}\Box\varphi^j + \overline{F}^{\bar{i}}F^j + i\overline{\psi}^{\bar{i}}\bar{\sigma}^\mu\partial_\mu\psi^j\right) + \left[\hat{w}_{ij}\left(F^i\varphi^j - \frac{1}{2}\psi^i\psi^j\right) + \text{h.c.}\right] \tag{2.11}$$

We split the latter to its scalar and its fermionic part:

$$\mathcal{L}^{(0)}_{\phi\phi,S} = \hat{K}_{\bar{i}j}\left(-\overline{\varphi}^{\bar{i}}\Box\varphi^j + \overline{F}^{\bar{i}}F^j\right) + \hat{w}_{ij}F^i\varphi^j + \hat{\bar{w}}_{\bar{i}\bar{j}}\overline{F}^{\bar{i}}\overline{\varphi}^{\bar{i}} \tag{2.12}$$

$$\mathcal{L}^{(0)}_{\phi\phi,F} = \hat{K}_{\bar{i}j}\left(i\overline{\psi}^{\bar{i}}\bar{\sigma}^\mu\partial_\mu\psi^j\right) - \frac{1}{2}\hat{w}_{ij}\psi^i\psi^j - \frac{1}{2}\hat{\bar{w}}_{\bar{i}\bar{j}}\overline{\psi}^{\bar{i}}\overline{\psi}^{\bar{j}} \tag{2.13}$$

2.2.1 Scalar Propagators

We begin from scalar propagators. By recasting the scalar part of Lagrangian using matrices, we obtain the form:

$$\mathcal{L}^{(0)}_{\phi\phi,S} = \begin{pmatrix}\overline{\varphi}^{\bar{k}} & F^l\end{pmatrix}\begin{pmatrix}-\hat{K}_{\bar{k}k}\Box & \hat{\bar{w}}_{\bar{l}\bar{k}} \\ \hat{w}_{lk} & \hat{K}_{\bar{l}l}\end{pmatrix}\begin{pmatrix}\varphi^k \\ \overline{F}^{\bar{l}}\end{pmatrix}, \tag{2.14}$$

where we denote as K_B the matrix:

$$K_B = \begin{pmatrix}-\hat{K}_{\bar{k}k}\Box & \hat{\bar{w}}_{\bar{l}\bar{k}} \\ \hat{w}_{lk} & \hat{K}_{\bar{l}l}\end{pmatrix} \tag{2.15}$$

The procedure one does schematically, is to invert the matrix that includes kinetic terms. We discuss first why the inverse K_B^{-1} gives us the scalar propagators.

We consider the generating functional for complex scalars and a general K differential operator:

$$Z[J^\dagger, J] = \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \, \exp\big[i \int d^4x \; \phi^\dagger K \phi + J^\dagger \phi + \phi^\dagger J\big] \tag{2.16}$$

We now do the standard trick to derive the propagator, by completing the square as $\pm \int J^\dagger K^{-1} J$. This allow us to write the following:

$$Z[J^\dagger, J] = \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \, \exp\big[i \int \; \big(\phi^\dagger + J^\dagger K^{-1}\big) K \big(\phi + K^{-1} J\big)\big] \times \exp\big[-i \int J^\dagger K^{-1} J\big] \tag{2.17}$$

We can now shift as:

$$\phi \to \phi + K^{-1} J \;, \;\; \phi^\dagger \to \phi^\dagger + J^\dagger K^{-1}$$

Thus, we are obtaining:

$$Z[J^\dagger, J] = \int \mathcal{D}\phi^\dagger \mathcal{D}\phi \, \exp\big[i \int \; \phi^\dagger K \phi\big] \times \exp\big[-i \int J^\dagger K^{-1} J\big] = Z(0) \exp\big[-i \int J^\dagger K^{-1} J\big] \tag{2.18}$$

Finally, we write:

$$Z[J^\dagger, J] = \mathcal{N} \times \exp\big[- \int J^\dagger (iK^{-1}) J\big], \tag{2.19}$$

where $\mathcal{N}$ is a normalization constant which will be proportional to a determinant. Thus, the propagator will be of the form:

$$\langle 0 |\, \phi(x)\, \phi^\dagger(x') \; |0\rangle = \frac{1}{i^2} \frac{\delta}{\delta J^\dagger(x)} \frac{\delta}{\delta J(x')} Z[J^\dagger, J]\Big|_{J^\dagger = J = 0} = iK^{-1} \delta(x - x'), \tag{2.20}$$

and that is the justification why the matrix elements of iK^{-1} determine the scalar (or fermionic) propagators. We are now ready to derive the inverse kinetic matrix (2.15). This is allowed from the fact that K_B^{-1} satisfies:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} -\hat{K}_{\bar{k}k} \Box & \hat{\bar{w}}_{\bar{l}\bar{k}} \\ \hat{w}_{lk} & \hat{K}_{\bar{l}l} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.21}$$

We get the following 2 systems of equations:

$$-X_1 \, \hat{K}_{\bar{k}k} \Box + X_2 \, \hat{w}_{lk} = 1 \tag{2.22a}$$

$$X_1 \, \hat{\bar{w}}_{\bar{l}\bar{k}} \; + X_2 \, \hat{K}_{\bar{l}l} = 0, \tag{2.22b}$$

and also:

$$-X_3 \, \hat{K}_{\bar{k}k} \Box + X_4 \, \hat{w}_{lk} = 0 \tag{2.23a}$$

$$X_3 \, \hat{\bar{w}}_{\bar{l}\bar{k}} \; + X_4 \, \hat{K}_{\bar{l}l} = 1 \tag{2.23b}$$

We begin from the system (2.22), which will give us the X_1, X_2 matrices. Taking the second expression, we write:

$$X_1 \, \hat{\bar{w}}_{\bar{l}\bar{k}} \; + X_2 \, \hat{K}_{\bar{l}l} = 0 \;\Rightarrow\; X_2 = -X_1 \, \hat{K}^{-1}_{\bar{l}l} \, \hat{\bar{w}}_{\bar{l}\bar{k}} \;\;\Rightarrow\;\; X_2 = -X_1 \, \hat{\bar{w}}_{\bar{k}\bar{l}} \, \hat{K}^{-1T}_{\bar{l}l}, \tag{2.24}$$

where we remind ourselves that $\hat{w}_{ij}$ is symmetric. The latter expression implies:

$$\begin{aligned} &- X_1 \, \hat{K}_{\bar{k}k} \Box + X_2 \, \hat{w}_{lk} = 1 \;\Rightarrow\; -X_1 \, \hat{K}_{\bar{k}k} \Box - X_1 \; \hat{\bar{w}}_{\bar{k}\bar{l}} \, \hat{K}^{-1T}_{\bar{l}l} \, \hat{w}_{lk} = 1 \;\Rightarrow \\ &X_1 \big[- \hat{K} \Box - \hat{\bar{w}} \, \hat{K}^{-1T} \, \hat{w} \big]_{\bar{k}k} = 1 \;\Rightarrow\; X_1 = \big[(-\hat{K} \Box - \hat{\bar{w}} \, \hat{K}^{-1T} \, \hat{w})^{-1} \big]_{k\,\bar{k}} \end{aligned} \tag{2.25}$$

Now, for the X_2 we obtain:

$$X_2 = -X_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\,\hat{K}^{-1T}_{\bar{l}l} = -\big[(-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]_{k\bar{k}}\,\hat{\bar{w}}_{\bar{k}\bar{l}}\,\hat{K}^{-1T}_{\bar{l}l} \tag{2.26}$$

The system (2.23) reads as follows:

$$-X_3\,\hat{K}_{\bar{k}k}\,\Box + X_4\,\hat{w}_{lk} = 0 \;\Rightarrow\; X_3\,\hat{K}_{\bar{k}k}\,\Box = X_4\,\hat{w}_{lk} \;\Rightarrow\; X_3 \;=\; X_4\,\hat{w}_{lk}\,\hat{K}^{-1}_{k\bar{k}}\,\frac{1}{\Box} \tag{2.27}$$

The last relation allow us to determine X_4 as:

$$\begin{aligned}&X_3\,\hat{\bar{w}}_{\bar{l}\bar{k}} \;+\; X_4\,\hat{K}_{\bar{l}l} = 1 \;\Rightarrow\; X_4\,\hat{w}_{lk}\,\hat{K}^{-1}_{k\bar{k}}\,\hat{\bar{w}}_{\bar{k}\bar{l}}\,\frac{1}{\Box} + X_4\,\hat{K}_{\bar{l}l} = 1 \;\Rightarrow\; X_4\hat{\bar{w}}_{\bar{l}\bar{k}}\,\hat{K}^{-1T}_{\bar{k}k}\,\hat{w}_{kl} \;+\; X_4\,\hat{K}_{\bar{l}l}\,\Box = \Box\\ &\Rightarrow X_4\big[-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w}\big]_{\bar{l}l} = -\Box \;\Rightarrow\; X_4 = -\Box\big[(-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]_{l\bar{l}}\end{aligned} \tag{2.28}$$

Finally, we find X_3:

$$X_3 = -\big[(-\hat{K}^T\,\Box - \hat{w}\,\hat{K}^{-1}\,\hat{\bar{w}})^{-1}\big]_{\bar{l}l}\,\hat{w}_{lk}\,\hat{K}^{-1}_{k\bar{k}} \tag{2.29}$$

We are now able to write the scalar propagators of our interest, in momentum space. One essentially does the following replacement $-\Box \to p^2$, and then one writes:

$$\langle\hat{\varphi}|\,\varphi^k\,\overline{\varphi}^{\bar{k}}\,|\hat{\varphi}\rangle = i\tilde{X}_1 \;\Rightarrow\; \langle\hat{\varphi}|\,\varphi^k\,\overline{\varphi}^{\bar{k}}\,|\hat{\varphi}\rangle = i\big[(\hat{K}\,p^2 - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]^{k\,\bar{k}} \tag{2.30a}$$

$$\langle\hat{\varphi}|\,\varphi^k\,F^l\,|\hat{\varphi}\rangle = i\tilde{X}_2 \;\Rightarrow\; \langle\hat{\varphi}|\,\varphi^k\,F^l\,|\hat{\varphi}\rangle = -i\big[(\hat{K}\,p^2 - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\;\hat{\bar{w}}\,\hat{K}^{-1T}\big]^{kl} \tag{2.30b}$$

$$\langle\hat{\varphi}|\,\overline{F}^{\bar{l}}\,\overline{\varphi}^{\bar{k}}\,|\hat{\varphi}\rangle = i\tilde{X}_3 \;\Rightarrow\; \langle\hat{\varphi}|\,\overline{F}^{\bar{l}}\,\overline{\varphi}^{\bar{k}}\,|\hat{\varphi}\rangle = -i\big[(\hat{K}^T\,p^2 - \hat{w}\,\hat{K}^{-1}\,\hat{\bar{w}})^{-1}\,\hat{w}\,\hat{K}^{-1}\big]^{\bar{l}\,\bar{k}} \tag{2.30c}$$

$$\langle\hat{\varphi}|\,F^l\,\overline{F}^{\bar{l}}\,|\hat{\varphi}\rangle = i\tilde{X}_4 \;\Rightarrow\; \langle\hat{\varphi}|\,F^l\,\overline{F}^{\bar{l}}\,|\hat{\varphi}\rangle = ip^2\big[(\hat{K}\,p^2 - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]^{l\,\bar{l}} \tag{2.30d}$$

2.2.2 Fermionic Propagators

Now, we do the same procedure for the fermionic propagators. We write in matrix form the fermionic part of the Lagrangian as:

$$\begin{aligned}\mathcal{L}^{(0)}_{\phi\phi,F} &= \hat{K}_{\bar{i}j}\left(i\overline{\psi}^{\bar{i}}\,\bar{\sigma}^\mu\partial_\mu\psi^j\right) - \frac{1}{2}\hat{w}_{ij}\psi^i\psi^j - \frac{1}{2}\hat{\bar{w}}_{\bar{i}\bar{j}}\overline{\psi}^{\bar{i}}\overline{\psi}^{\bar{j}} =\\ &= \frac{1}{2}\begin{pmatrix}\overline{\psi}^{\bar{k}} & \psi^l\end{pmatrix}\begin{pmatrix} i\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu & -\hat{\bar{w}}_{\bar{k}\bar{l}}\\ -\hat{w}_{lk} & i\hat{K}^T_{l\bar{l}}\,\sigma^\mu\partial_\mu\end{pmatrix}\begin{pmatrix}\psi^k\\ \overline{\psi}^{\bar{l}}\end{pmatrix}\end{aligned} \tag{2.31}$$

The matrix that contains the kinetic terms, denoted as K_F is:

$$K_F = \begin{pmatrix} i\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu & -\hat{\bar{w}}_{\bar{k}\bar{l}}\\ -\hat{w}_{lk} & i\hat{K}^T_{l\bar{l}}\,\sigma^\mu\partial_\mu\end{pmatrix} \tag{2.32}$$

Again we have to invert the matrix that includes kinetic terms. The fermionic propagators will be determined by the entries of iK_F^{-1}. The justification is analogous to that of the complex scalar fields: we have the generating functional:

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\exp\Big[i\int d^4x\;\bar{\psi}\,K\,\psi + \bar{\eta}\psi + \bar{\psi}\eta\Big] \tag{2.33}$$

By completing the square:

$$\bar{\psi}K\psi + \bar{\eta}\psi + \bar{\psi}\eta = \left(\bar{\psi} + \bar{\eta}K^{-1}\right) K \left(\psi + K^{-1}\eta\right) - \bar{\eta}K^{-1}\eta, \tag{2.34}$$

we obtain that:

$$Z[\bar{\eta},\eta] = \mathcal{C} \times \exp\big[- \int \bar{\eta}(iK^{-1})\eta\big]. \tag{2.35}$$

The 2-point functions read as:

$$\langle 0|\psi_\alpha(x_1)\bar{\psi}_b(y_1)|0\rangle = \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_a(x)} i \frac{\delta}{\delta\eta_b(x)} Z[\bar{\eta},\eta]\Big|_{\eta=\bar{\eta}=0} = iK^{-1}\,\delta(x_1 - y_1) \tag{2.36}$$

We now focus on deriving the inverse kinetic matrix for fermions. Again, we proceed as follows:

$$\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} i\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu & -\hat{\bar{w}}_{\bar{k}\bar{l}} \\ -\hat{w}_{lk} & i\hat{K}^T_{l\bar{l}}\,\sigma^\mu\partial_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.37}$$

This leads to the system of our interest:

$$iS_1\,\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu - S_2\,\hat{w}_{lk} = 1 \tag{2.38a}$$
$$-S_1\,\hat{\bar{w}}_{\bar{k}\bar{l}} + iS_2\,\hat{K}^T_{l\bar{l}}\,\sigma^\mu\partial_\mu = 0, \tag{2.38b}$$

From (2.38b) we are able to write:

$$-S_1\,\hat{\bar{w}}_{\bar{k}\bar{l}} + iS_2\,\hat{K}^T_{l\bar{l}}\,\sigma^\mu\partial_\mu = 0 \;\Rightarrow\; S_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\,\hat{K}^{-1T}_{\bar{l}l}\,\bar{\sigma}^\nu\partial_\nu = iS_2(\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu), \tag{2.39}$$

and by using the $(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu) = 2\eta^{\mu\nu}$ we have that $\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu = \Box$ (similarly for $\bar{\sigma}\sigma$). Finally:

$$S_2 = -iS_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\hat{K}^{-1T}_{\bar{l}l}\bar{\sigma}^\nu\partial_\nu\frac{1}{\Box} \tag{2.40}$$

Now we can go to (2.38a), and write:

$$iS_1\,\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu - S_2\,\hat{w}_{lk} = 1 \;\Rightarrow\; iS_1\,\hat{K}_{\bar{k}k}\,\bar{\sigma}^\mu\partial_\mu + iS_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\hat{K}^{-1T}_{\bar{l}l}\hat{w}_{lk}\,\bar{\sigma}^\nu\partial_\nu\frac{1}{\Box} = 1,$$

where we can act with $\sigma^q\partial_q$ and we continue as:

$$-S_1\,\hat{K}_{\bar{k}k}\,\Box - S_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\hat{K}^{-1T}_{\bar{l}l}\hat{w}_{lk} = i\sigma^q\partial_q \;\Rightarrow S_1 = i\sigma^q\partial_q\big[(-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]_{k\bar{k}} \tag{2.41}$$

Thus, we find also S_2:

$$\begin{aligned} S_2 &= -iS_1\,\hat{\bar{w}}_{\bar{k}\bar{l}}\,\hat{K}^{-1T}_{\bar{l}l}\bar{\sigma}^\nu\partial_\nu\frac{1}{\Box} = \big[(-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]_{k\bar{k}}\hat{\bar{w}}_{\bar{k}\bar{l}}\hat{K}^{-1T}_{\bar{l}l} \\ &= \Big[\big[(-\hat{K}\,\Box - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]\,\hat{\bar{w}}\hat{K}^{-1T}\Big]_{kl} \end{aligned} \tag{2.42}$$

We are now able to write the fermionic propagators of interest, in momentum space. Again, we do the correspondence $-\Box \to p^2$ and also $i\partial_\mu \to p_\mu$, and then one writes:

$$\langle\hat{\varphi}|\;\psi^k_\alpha\,\overline{\psi}^{\bar{k}}_{\dot{\alpha}}\,|\hat{\varphi}\rangle = i\tilde{S}_1 \;\Rightarrow\; \langle\hat{\varphi}|\;\psi^k_\alpha\,\overline{\psi}^{\bar{k}}_{\dot{\alpha}}\,|\hat{\varphi}\rangle = ip_\mu\sigma^\mu_{\alpha\dot{\alpha}}\big[(\hat{K}\,p^2 - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\big]^{k\,\bar{k}} \tag{2.43a}$$
$$\langle\hat{\varphi}|\;\psi^k_\alpha\,\psi^{\beta l}\;|\hat{\varphi}\rangle = i\tilde{S}_2 \;\Rightarrow\; \langle\hat{\varphi}|\;\psi^k_\alpha\,\psi^{\beta l}\;|\hat{\varphi}\rangle = i\delta^\beta_\alpha\big[(\hat{K}\,p^2 - \hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w})^{-1}\;\hat{\bar{w}}\,\hat{K}^{-1T}\big]^{kl} \tag{2.43b}$$

2.3 Quantum Corrections of Superpotential

We now proceed with the computation of the $\hat{\varphi}$-dressed one-point function of $\hat{F}$ (its tadpole). This gives us the term $\Delta w_i(\hat{\varphi})\hat{F}^i$, which allows us to verify the corrections that the superpotential receives. Now, the lagrangian of our interest is (2.9), which we wish to contract with propagators, in order to close the $\hat{F}$ tadpole. The only possible contributions to the $\hat{F}$ tadpole [8] come from the third and fourth terms in (2.9), which can be closed with propagators. The one-loop level contributions are:

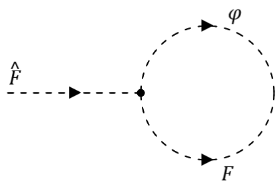

Figure 2.1: Scalar contribution.

We can work with the hermitian conjugate of the third term: $\hat{\bar{F}}^{\bar{j}}\hat{K}_{ik\bar{j}}F^i\varphi^k$, and contracting with the propagator (2.30b), we find that the scalar contribution can be evaluated as:

$$\begin{aligned}\mathcal{T}_s &= \hat{\bar{F}}^{\bar{j}}\int\frac{d^4p}{(2\pi)^4}\hat{K}_{ik\bar{j}}\ \langle\hat{\varphi}|\ F^i\varphi^k\ |\hat{\varphi}\rangle \\ &= -\hat{\bar{F}}^{\bar{j}}\int\frac{d^4p}{(2\pi)^4}i\,\mathrm{Tr}\left[\hat{K}_{\bar{j}}\left(\hat{K}\,p^2-\hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w}\right)^{-1}\,\hat{\bar{w}}\,\hat{K}^{-1T}\right]\end{aligned} \tag{2.44}$$

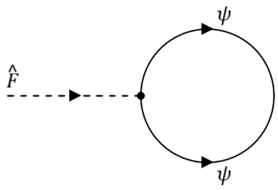

Figure 2.2: Fermion contribution.

We do the same procedure for with the hermitian conjugate of the fourth term: $-\frac{1}{2}\hat{\bar{F}}^{\bar{j}}\hat{K}_{ik\bar{j}}\psi^i\psi^k$, and one needs to take into account that contracting with propagator can be done in terms of (2.43b), by considering:

$$\psi^i_a\psi^{\beta k} = \epsilon_{\alpha\gamma}\psi^{\gamma i}\psi^{\beta k} = -\frac{1}{2}\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta}\psi^i\psi^k = -\frac{1}{2}\delta^{\beta}_{\alpha}\psi^i\psi^k, \tag{2.45}$$

and thus one-loop level is computed as:

$$\begin{aligned}\mathcal{T}_f &= -\frac{1}{2}\hat{\bar{F}}^{\bar{j}}\int\frac{d^4p}{(2\pi)^4}\hat{K}_{ik\bar{j}}\ \langle\hat{\varphi}|\ \psi^i\,\psi^k\ |\hat{\varphi}\rangle \\ &= +\hat{\bar{F}}^{\bar{j}}\int\frac{d^4p}{(2\pi)^4}i\,\mathrm{Tr}\left[\hat{K}_{\bar{j}}\left(\hat{K}\,p^2-\hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w}\right)^{-1}\,\hat{\bar{w}}\,\hat{K}^{-1T}\right]\end{aligned} \tag{2.46}$$

Since the scalar and fermion contributions are equal in magnitude but opposite in sign, we find that $\mathcal{T}_s + \mathcal{T}_f = 0$. This cancellation indicates that the superpotential remains uncorrected $\Delta w = 0$. The lack of one-loop corrections to the superpotential provides a clear validation of the well-established non-renormalization theorem.

2.4 One Loop Level Contributions to the Kähler potential

We move on to compute the two-point function of $\langle\hat{\varphi}|\,\hat{\bar{F}}^i\hat{F}^j\,|\hat{\varphi}\rangle$, which is at one-loop level and vanishing external momenta. The one-loop diagrams will lead us to the corrected Kähler potential, which is the main goal. Why? We remind that in the effective Lagrangian there is this term : $\Delta K_{\bar{i}j}(\hat{\bar{\varphi}},\hat{\varphi})\hat{\bar{F}}^{\bar{i}}\hat{F}^j$, which can be obtained from the one-loop diagrams and thus find ΔK. We remind that here we target to find the contributions from the chiral sector only, thus we are interested in pure ϕ loops, and ignore the contribution from the V sector.

To obtain the two-point function $\langle\hat{\varphi}|\,\hat{\bar{F}}^i\hat{F}^j\,|\hat{\varphi}\rangle$, one uses the first interaction term in $\mathcal{L}^{(\hat{F}\hat{F})}_{\phi\phi}$ and also considers the product of terms in $\mathcal{L}^{(\hat{F})}_{\phi\phi}$, in order to combine the first and second terms (along with their hermitian conjugates), since the terms involving $\hat{\bar{F}}^i\hat{F}^j$ are of particular interest. The combined terms will determine the vertices, and then as before, one contracts with the propagators. In the following, the different one-loop contributions are presented (as in [8]), along with the values of the corresponding diagrams (denoted as $\mathcal{I}_{1,\cdots,n}$):

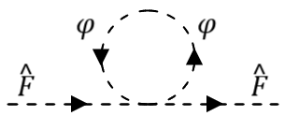

Figure 2.3: Loop contribution "1", from the interaction term in $\mathcal{L}^{(\hat{F}\hat{F})}_{\phi\phi}$

$$\mathcal{I}_1 = \hat{\bar{F}}^i\hat{F}^j\int\frac{d^4p}{(2\pi)^4}i\,\mathrm{Tr}\left[\hat{K}_{ij}\left(\hat{K}p^2-\hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\right] \tag{2.47}$$

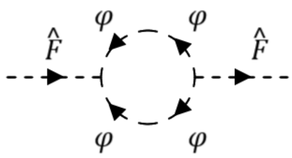

Figure 2.4: Loop contribution "2"

$$\mathcal{I}_2 = -\frac{1}{2}\hat{\bar{F}}^i\hat{F}^j\int\frac{d^4p}{(2\pi)^4}i\,\mathrm{Tr}\left[\left(\hat{K}p^2-\hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\hat{\bar{w}}_{\bar{\imath}}\left(\hat{K}^Tp^2-\hat{w}\hat{K}^{-1}\hat{\bar{w}}\right)^{-1}\hat{w}_j\right] \tag{2.48}$$

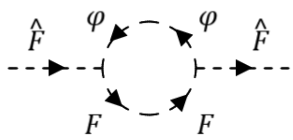

Figure 2.5: Loop Contribution "3"

$$\mathcal{I}_3 = -\hat{\bar{F}}^{\bar{\imath}}\hat{F}^j \int \frac{d^4p}{(2\pi)^4} ip^2 \,\mathrm{Tr}\left[\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\hat{K}_{\bar{\imath}}\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\hat{K}_j\right] \tag{2.49}$$

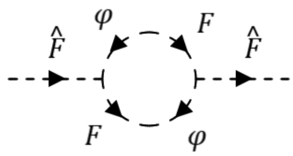

Figure 2.6: Loop contribution "4"

$$\mathcal{I}_4 = -\hat{\bar{F}}^{\bar{\imath}}\hat{F}^j \int \frac{d^4p}{(2\pi)^4} i \,\mathrm{Tr}\left[\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\hat{\bar{w}}\hat{K}^{-1T}\hat{K}^T_{\bar{\imath}}\left(\hat{K}^Tp^2 - \hat{w}\hat{K}^{-1}\hat{\bar{w}}\right)^{-1}\hat{w}\hat{K}^{-1}\hat{K}_j\right] \tag{2.50}$$

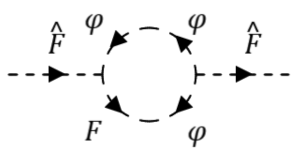

Figure 2.7: Loop Contribution "5"

$$\mathcal{I}_5 = \hat{\bar{F}}^i\hat{F}^j \int i\frac{d^4p}{(2\pi)^4} \,\mathrm{Tr}\left[\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1}\hat{\bar{w}}_{\bar{\imath}}\left(\hat{K}^Tp^2 - \hat{w}\hat{K}^{-1}\hat{\bar{w}}\right)^{-1}\hat{w}\hat{K}^{-1}\hat{K}_j\right] \tag{2.51}$$

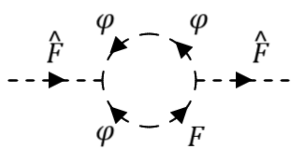

Figure 2.8: Loop Contribution "6"

$$\mathcal{I}_6 = \hat{\bar{F}}^i \hat{F}^j \int \frac{d^4p}{(2\pi)^4} i \,\mathrm{Tr}\left[\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1} \hat{K}_{\bar{\imath}} \left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)^{-1} \hat{\bar{w}}\hat{K}^{-1T}\hat{w}_j\right] \tag{2.52}$$

In the latter notation, we should note that $\hat{w}_i, \hat{K}_j, \hat{K}_{\bar{\imath}j}, \cdots$ denote third, or fourth derivatives of w or K. We should also point out a "beautiful cancellation" of two extra diagrams that exist, using the third and fourth terms (+ h.c.) in $\mathcal{L}^{(\hat{F})}_{\phi\phi}$. The contribution of these two diagrams is of equal magnitude but opposite sign, and thus they cancel each other, since are scalar + fermionic loops:

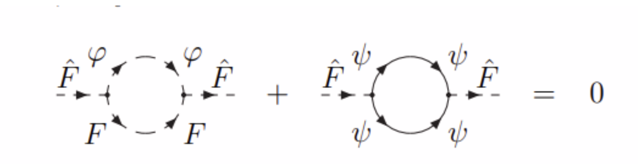

Figure 2.9: Scalar + fermionic loop of equal magnitude and opposite sign.

Now we have to sum the diagrams above, in order to derive the total contribution. For the ϕ sector, one finds:

$$\sum_{i=1}^{6} \mathcal{I}_i = \hat{\bar{F}}^{\bar{\imath}} \hat{F}^j \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \left[\frac{1}{2}\mathrm{Tr}\log\hat{K} + \frac{1}{2}\mathrm{Tr}\log\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)\right]_{\bar{\imath}j} \tag{2.53}$$

By comparing the latter expression with the term of interest in (2.7), one finally identifies the one-loop correction to the Kähler potential as:

$$\Delta K(\hat{\bar{\varphi}}, \hat{\varphi}) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \left[\frac{1}{2}\mathrm{Tr}\log\hat{K} + \frac{1}{2}\mathrm{Tr}\log\left(\hat{K}p^2 - \hat{\bar{w}}\hat{K}^{-1T}\hat{w}\right)\right] \tag{2.54}$$

For the final task of this section, we perform the momentum integration of the latter expression. If we do a Wick rotation and regulate the momentum integral with a simple ultraviolet cutoff Λ, we can find the one-loop correction of Kähler potential in closed form, for the general theory of study. Finally, the result reads:

$$\Delta K[\Lambda] = \frac{\Lambda^2}{16\pi^2}\log\det\hat{K} - \frac{1}{32\pi^2}\left[\mathrm{Tr}\left(\mathcal{M}^2_\phi\left(\log\frac{\mathcal{M}^2_\phi}{\Lambda^2} - 1\right)\right)\right], \tag{2.55}$$

where the field dependent mass matrix in the chiral sector is:

$$\mathcal{M}^2_\phi \equiv \hat{K}^{-1/2}\hat{\bar{w}}\hat{K}^{-1T}\hat{w}\hat{K}^{-1/2} \tag{2.56}$$

The dependence on $\hat{\bar{\varphi}}, \hat{\varphi}$ is carried in the matrices $\hat{w}, \hat{K}$. Thus, the latter expression gives the full one-loop correction to the Kähler potential in a closed form, for the general theory under study. The first term contains quadratically divergent contributions, whereas the second term contains logarithmically divergent and finite contributions. Last note for this chapter, is that nn the special case of renormalizable theories, the superpotential is at most cubic and the metric is canonical in chiral sector, so $\hat{K}_{\bar{\imath}j} = \delta_{\bar{\imath}j}$.

Here, we close the chapter of the study of one-loop contributions.

Part II

Inflationary Cosmology

Chapter 3

Inflation Theory

3.1 FRW Spacetime & Kinematics

The spacetime of the Universe on the largest scales is described by the Friedmann-Robertson-Walker (FRW) metric, which assumes homogeneity (spatial translation invariance) and isotropy (spatial rotation invariance) for spacelike hypersurfaces. These hypersurfaces are maximally symmetric three-dimensional spaces. However, the relative size of spatial slices evolves with time, characterized by the *scale factor* $\alpha(t)$. The FRW spacetime, for comoving observers (the frame in which the Universe is homogeneous and isotropic) is written as:

$$\begin{aligned} & ds^2 = -dt^2 + \alpha^2(t)\,\gamma_{ij}\,dx^i dx^j \\ \textit{or} \quad & ds^2 = -dt^2 + \alpha^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right], \end{aligned} \tag{3.1}$$

with $K = -1$ (negatively curved) or $K = 0$ (flat) or $K = +1$ (positively curved) being the curvature parameter of the spatial slices. The γ_{ij} is the metric of a maximally symmetric 3-space. In the case of flat Universe ($K = 0$) the metric induces to $\gamma_{ij} = \delta_{ij}$.

In order for one to study the causal structure, it is useful to define the so called *conformal time* as:

$$d\eta = \frac{dt}{\alpha(t)} \tag{3.2}$$

Conformal time intuitively can be thought of as a 'clock' that decelerates as the universe expands. The integral defining conformal time can be re-written in a way that is going to be really useful:

$$\eta = \int \frac{dt}{\alpha} = \int \frac{1}{\dot{\alpha}}\frac{da}{\alpha} = \int \frac{1}{\alpha H}\, dln\alpha \tag{3.3}$$

In this form, a particularly important quantity appears, which is the *comoving Hubble radius* $r_H = 1/\alpha H$. Conformal time depends on the evolution of the comoving Hubble radius.

3.2 Evolution of the Scale Factor

The time evolution of the scale factor, and thus the dynamics of the Universe, is governed by the Einstein Field Equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{3.4}$$

The Einstein tensor is defined as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{3.5}$$

and the Ricci tensor is:

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{3.6}$$

Finally, the Ricci scalar is simply:

$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{3.7}$$

Recall that Christoffel symbols read as:

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left[\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}\right] \tag{3.8}$$

For the RHS of equation (3.4), one needs to consider the energy-momentum tensor (the "matter content") of the Universe. The requirements of isotropy and homogeneity constrain the tensor to be that of a perfect fluid. For the case of a perfect fluid the stress-energy tensor reads as:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{3.9}$$

where ρ is the proper energy density and P the pressure in the fluid rest frame. The timelike velocity 4-vector of the fluid is denoted as $u^\mu \equiv dx^\mu/d\tau$, where τ is the proper time. For observers comoving with the fluid, it is a convenient choice to consider $u^\mu = (1, 0, 0, 0)$. The energy momentum tensor then reads:

$$T^\mu_\nu = diag[-\rho, P, P, P]$$

From the latter analysis, the Einstein equations reduce to the form of the *Friedmann Equations*, which describe the time evolution of the scale factor. They read as:

$$H^2 = \frac{1}{3M_{pl}^2}\rho - \frac{K}{\alpha^2} \tag{3.10a}$$

$$\dot{H} + H^2 = -\frac{1}{6M_{pl}^2}(\rho + 3P), \tag{3.10b}$$

where $H \equiv \dot{\alpha}/\alpha$ is the Hubble parameter, and $M_{pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass. A first observation is that the second Friedmann equation shows that accelerated expansion of the universe occurs when $\rho + 3P < 0$ or $w \equiv P/\rho < -1/3$. Thus, inflation requires negative pressure or a violation of the strong energy condition. A second observation is that the two Friedmann equations (3.10) imply the continuity equation:

$$\dot{\rho} = -3H(\rho + P) \tag{3.11}$$

3.3 Horizon Problem

The particle horizon is the maximal distance that a signal can travel between the time corresponding to the initial singularity, $t_i = 0$, and a later time t. In physical coordinates, this distance is given by

$$D(t) = a(t)\int_{t_i}^{t} \frac{dt'}{a(t')} = a(t)\int_0^t \frac{dt'}{a(t')}. \tag{3.12}$$

If the early universe was filled by ordinary matter, then $a(t) \propto t^\alpha$ with $\alpha < 1$. In that case, the integral is dominated by late times and converges to a finite value:

$$a(t) \propto \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \end{cases} \implies D(t) = \begin{cases} 3t & \text{matter} \\ 2t & \text{radiation} \end{cases}. \tag{3.13}$$

This leads to a puzzle: because the age of the universe (t_0) is much larger than the time of recombination (t_*), the CMB naively consists of many causally disconnected patches.

The following questions immediately arise: Why is the CMB homogeneous? And, more importantly, why are the observed CMB fluctuations correlated on large scales and not just random noise?

The horizon problem is solved if the early universe experienced a sustained period of accelerated expansion, i.e., inflation, $\ddot{a} > 0$. In that case, the integral is dominated by early times and the particle horizon diverges in the past.

Signals were therefore able to travel a much larger distance than suggested by the naive extrapolation of the standard FRW expansion.

3.4 Flatness Problem

The curvature parameter $\Omega_k \equiv 1 - \frac{\rho}{\rho_{\text{crit}}}$, where $\rho_{\text{crit}} \equiv \frac{3H^2}{\kappa}$, may be interpreted as the difference between the average potential energy and the average kinetic energy of a region of space.

Spacetime in General Relativity is dynamical, curving in response to matter in the universe. We can then use the Friedmann Equation to quantify the problem in the Euclidean space. To quantify the problem, consider

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}. \tag{3.14}$$

Recall that the comoving wavelength of fluctuations is time-independent, while the comoving Hubble radius is time-dependent, written as

$$1 - \Omega_k(a) = \frac{\rho(a)}{\rho_{\text{crit}}(a)} = 3H(a)^2, \tag{3.15}$$

where

$$\Omega_k(a) = \frac{\rho(a)}{\rho_{\text{crit}}(a)}, \quad \rho_{\text{crit}}(a) = 3H(a)^2. \tag{3.16}$$

Note that $\Omega_k(a)$ is defined to be time-dependent. The critical density ρ_{crit} is the density at the critical point (i.e., flat universe). Therefore, in standard Big Bang cosmology without inflation, the "almost flatness" today, i.e., $\Omega_k(a_0) \sim 1$, requires an extreme fine-tuning of $\Omega_k \sim 1$ close to 1 in the early universe. More specifically, one finds that the deviation from flatness at Big Bang Nucleosynthesis (BBN), during the GUT era and at the Planck scale, respectively, has to satisfy the following conditions:

$$|\Omega_k(\text{BBN}) - 1| \leq O(10^{-16}), \tag{3.17}$$
$$|\Omega_k(\text{GUT}) - 1| \leq O(10^{-54}), \tag{3.18}$$
$$|\Omega_k(\text{Planck}) - 1| \leq O(10^{-60}). \tag{3.19}$$

Another way of understanding this problem is by deriving the differential equation.

$$\frac{d\Omega_k}{dt} = (1+3w)\Omega_k(\Omega_k - 1). \tag{3.20}$$

The latter equation is derived by differentiating and using the continuity equation. This makes it apparent that $\Omega_k = 1$ is an unstable fixed point of the strong energy condition satisfied

$$\frac{d\Omega_k}{dt} = 0 \quad \text{if} \quad 1+3w \geq 0. \tag{3.21}$$

Again, why is $\Omega_k(a) \sim O(1)$ and not much smaller or much larger?

The solution might be inflation: If the comoving Hubble radius decreases this drives the universe toward flatness (rather than away from it). This solves the flatness problem! The solution $\Omega = 1$ is an attractor during inflation.

3.5 Slow-Roll Inflation

The typical model when one deals with Inflation, is the slow-roll inflation. Consider a single scalar field, $\phi(t, \mathbf{x})$, called the *inflaton*. The scalar field is minimally coupled to standard Einstein gravity. Note that the scalar field plays the role of parametrization of the time-evolution of the energy density. The dynamics of the inflaton is described by the action:

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi\, \nabla_\nu \phi - V(\phi) \right] = S_{GR} + S_\phi, \tag{3.22}$$

where R is the Ricci scalar determined by the metric $g_{\mu\nu}$ from the standard Einstein-Hilbert action, and S_ϕ is the scalar action with canonical kinetic term. The potential term $V(\phi)$, which describes the self-interactions of the inflaton, is an arbitrary function that typically has the following profile:

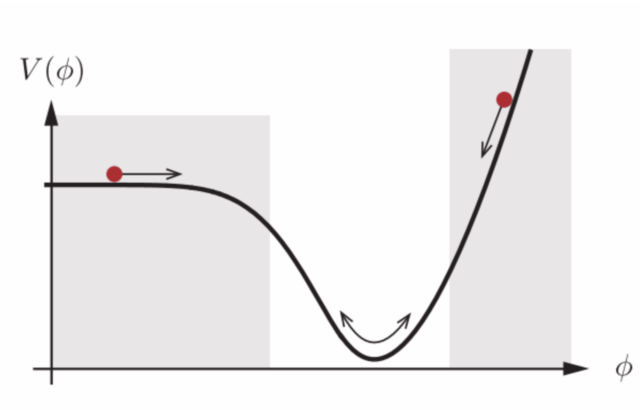

Figure 3.1: Profile of a slow-roll potential [9]. Inflation period lasts in the shaded part of the potential.The inflaton slowly rolls down the potential toward the reheating phase.

We proceed with the energy-momentum tensor of the scalar field. Using that $\nabla_\mu \phi = \partial_\mu \phi$, the variation of the Klein-Gordon action, with respect to the inverse metric, $g^{\mu\nu}$, reads:

$$\delta S_\phi = \int \mathrm{d}^4x \left\{ \delta\sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - V(\phi) \right] + \sqrt{-g} \left[-\frac{1}{2} \delta g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi \right] \right\}, \tag{3.23}$$

and by applying the relation

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu},$$

we obtain:

$$\delta S_\phi = \int \mathrm{d}^4x\ \sqrt{-g}\ \delta g^{\mu\nu}\left[-\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\ + \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + V(\phi)\right)\right], \tag{3.24}$$

Thus, the energy momentum tensor is found to be:

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + V(\phi)\right). \tag{3.25}$$

We now wish to extract the field equation of motion for the inflaton. Performing the variation of scalar action with respect to ϕ, we derive:

$$\begin{aligned}
\delta S_\phi &= -\int \mathrm{d}^4x\ \left[\frac{1}{2}\sqrt{-g}g^{\mu\nu}\nabla_\mu(\delta\phi)\nabla_\nu\phi\ + \frac{1}{2}\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu(\delta\phi)\ +\ \sqrt{-g}\frac{dV(\phi)}{d\phi}\delta\phi\right] \\
&= -\int \mathrm{d}^4x\ \left[\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu(\delta\phi)\ +\ \sqrt{-g}\frac{dV(\phi)}{d\phi}\delta\phi\right] \\
&= -\int \mathrm{d}^4x\ \left[\sqrt{-g}\ \nabla_\nu\left(g^{\mu\nu}\nabla_\mu\phi\,\delta\phi\right)\right] + \int \mathrm{d}^4x\,\sqrt{-g}\ \left[\nabla^\mu\nabla_\mu\phi - \frac{dV(\phi)}{d\phi}\right]\delta\phi
\end{aligned} \tag{3.26}$$

Now, using the generalized Stoke's theorem for non-trivial spacetimes:

$$\int_M d^4x\sqrt{-g}\,\nabla_\mu(A^\mu) = \int_{\partial M} d^3x\sqrt{|h|}\,n_\mu\,A^\mu, \tag{3.27}$$

the first term on the RHS of (3.26) vanishes. Finally, we obtain:

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) - V'(\phi) = 0 \tag{3.28}$$

Now, we are able to derive the specific properties of the model. Considering the FRW metric with $K = 0$ for $g_{\mu\nu}$, implies that the background value of the scalar field is homogeneous; thus it depends only on time $\phi = \phi(t)$.

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) - V'(\phi) = 0\ \Rightarrow\ \frac{1}{\sqrt{-g}}\,\partial_0\left(\sqrt{-g}\,g^{00}\,\partial_0\phi\right) - V'(\phi) = 0 \\
&\frac{1}{\alpha^3}\partial_0\left(-\alpha^3\dot{\phi}\right) - V'(\phi) = 0\ \Rightarrow\ \frac{1}{\alpha^3}\left(3\alpha^2\dot{\alpha}\dot{\phi}\ +\ \alpha^3\ddot{\phi}\right) + V'(\phi) = 0 \Rightarrow \boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0}
\end{aligned} \tag{3.29}$$

We now want to derive an expression for the energy density and the isometric pressure of the scalar field. Thus, we do the following:

$$\begin{aligned}
T^\mu_\nu = g^{\mu\alpha}T_{\alpha\nu} &= g^{\mu\alpha}\partial_\alpha\phi\partial_\nu\phi - g^{\mu\alpha}g_{\alpha\nu}\left(\frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + V(\phi)\right) \\
&= \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_\nu\left(\frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + V(\phi)\right)
\end{aligned} \tag{3.30}$$

The 00-th component will extract the energy density of the inflaton. It reads:

$$T^0_0 = -\dot{\phi}^2 - \left(-\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = -\rho_\phi\ \Rightarrow \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.31}$$

Now, for the spatial indices, the ij-th component can be written as:

$$T_i^j = \partial^j \phi \, \partial_i \phi - \delta_i^j \left(-\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) = P_\phi \, \delta_i^j \;\Rightarrow\; P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.32}$$

We comment that the energy density is the summation of the kinetic term $\frac{1}{2}\dot{\phi}^2$ and the potential $V(\phi)$. The equation of state, is formed as:

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.33}$$

Let us summarize the equations that govern the dynamics of the (homogeneous) scalar field and the FRW geometry:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \;,\; 3M_{pl}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.34}$$

Notice that the gradient of the potential plays the role of a force, while the expansion of the universe acts like a friction term, $H\dot{\phi}$. The value of this *Hubble drag* term, is determined by the Friedmann equation.

3.6 Conditions for Slow-Roll

We mark that the inflation condition can be satisfied i.e. $w_\phi < -1/3$ (negative pressure) if the potential energy dominates over the kinetic energy. Thus, we need the scalar field to "slow-roll". This is the first condition for inflation and is called *slow-roll approximation*:

$$\dot{\phi}^2 \ll V(\phi) \tag{3.35}$$

Now, taking the derivative of the Friedmann equation in (3.34) gives us:

$$6M_{pl}^2 H\dot{H} = \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \;\Rightarrow\; 6M_{pl}^2 H\dot{H} = -3H\dot{\phi}^2 \;\Rightarrow\; \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} \tag{3.36}$$

where we introduced the slow-roll parameter ϵ, that will quantify the approximation. We see that:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} \propto \frac{\rho_{\text{kin}}}{\rho_{\text{tot}}} \ll 1, \tag{3.37}$$

the ϵ - parameter is related to the ratio of the inflaton's kinetic energy density $\rho_{kin} \approx \frac{1}{2}\dot{\phi}^2$ to the total energy density $\rho_{tot} = 3M_{pl}^2 H^2$. The slow-roll approximation tells us that this ratio should be very small, meaning the potential energy dominates the dynamics of the Universe, during the inflation era.

There is also a second condition. In order for inflation to last sufficiently long, the acceleration of the scalar field should be small enough. That means:

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, \; |V'_\phi|. \tag{3.38}$$

A useful parameter to evaluate this, is to introduce the dimensionless acceleration per Hubble time:

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{3.39}$$

Now, taking the time derivative of (3.37) we obtain:

$$\dot{\epsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{\rm pl}^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_{\rm pl}^2 H^3} \Rightarrow \eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\epsilon - \delta), \tag{3.40}$$

where we introduced another parameter. For $|\eta| < 1$, the fractional change of ϵ per Hubble time is small, so the persistence of inflation is secured.

The two conditions discussed above, can be expressed in terms of the shape of the potential function. Firstly, the Klein-Gordon and the Friedmann equation can be approximated as:

$$3H\dot{\phi} \approx -V'(\phi) \;\;,\;\; 3M_{pl}^2 H^2 \approx V(\phi) \tag{3.41}$$

Thus, the ϵ-parameter (3.37) can be approached as:

$$\epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} \approx \frac{V'^2}{6H^2}\frac{1}{V} \Rightarrow \boxed{\epsilon_V \equiv \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2} \tag{3.42}$$

Taking the derivative of the simplified Klein-Gordon leads to:

$$3H\dot{\phi} = -V'(\phi) \;\Rightarrow\; 3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi} \;\Rightarrow \frac{\dot{H}}{H^2} + \frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{V''}{3H^2} \quad \Rightarrow \boxed{\eta_V \equiv \epsilon + \delta \equiv M_{pl}^2 \frac{V''}{V}} \tag{3.43}$$

From the two conditions we conclude that successful slow-roll inflation takes place when the *potential slow- roll parameters* are small-scale: $\{\epsilon_V, |\eta_V|\} \ll 1$.

Inflation ends when the first condition is violated, i.e. $\epsilon_V \approx 1$. One is able to derive the number of *e-folds*, which measures the amount of expansion, during the inflation:

$$N_e(\phi) = \int_a^{a_f} \mathrm{d}\ln\alpha = \int_t^{t_f} H\mathrm{d}t = \int_\phi^{\phi_f} \frac{H}{\dot{\phi}}\mathrm{d}\phi \approx \int_\phi^{\phi_f} \frac{1}{M_{pl}^2}\frac{V}{V'}|\mathrm{d}\phi| = \int_\phi^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}}\frac{|\mathrm{d}\phi|}{M_{pl}} \tag{3.44}$$

The number of e-folds required for inflation to solve the horizon problem is at least between 40 and 60 e-folds (with the precise value depending on the reheating procedure).

3.6.1 The $m^2\phi^2$ Inflation

Let us now study slow-roll inflation for the case of the simplest model: single-field inflation driven by a Klein-Gordon mass term:

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{3.45}$$

The slow-roll parameters are

$$\varepsilon_V(\phi) = \eta_V(\phi) = 2\left(\frac{M_{pl}}{\phi}\right)^2 . \tag{3.46}$$

In order for slow-roll conditions to hold $\{\varepsilon_V, |\eta_V|\} < 1$, we need to consider super-Planckian values for the inflaton:

$$\phi > \sqrt{2}M_{pl} \equiv \phi_f . \tag{3.47}$$

The number of e-folds as a function of the scalar field, before the end of inflation is:

$$N(\phi) = \int_{\phi}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}} \frac{|\mathrm{d}\phi|}{M_{pl}} = \frac{1}{2M_{pl}^2} \int_{\phi}^{\phi_f} \phi\, |\mathrm{d}\phi| = \frac{\phi^2}{4M_{pl}^2} - \frac{1}{2}. \tag{3.48}$$

Solving the horizon problem then requires that the initial value of the field, ϕ_i, satisfies

$$\phi_i > \phi_{60} \equiv 2\sqrt{60}\, M_{pl} \sim 15\, M_{pl}. \tag{3.49}$$

We note that the total field excursion is super-Planckian, $\Delta\phi = \phi_i - \phi_f \gg M_{pl}$.

3.7 Reheating Phenomenon

During inflation, the energy density of the universe is dominated by the inflaton potential $V(\phi)$. Inflation ends when the inflaton has gained enough kinetic energy. The energy stored in the inflaton field must then be transferred to Standard Model particles—a process known as *reheating*—which marks the beginning of the Hot Big Bang.

At the end of the period of inflation, the inflaton field has rolled at the bottom of the potential, and we are having scalar field oscillations. Assuming that in this region the potential behaves as $V(\phi) = \frac{1}{2}m^2\phi^2$, the Klein Gordon equation reads:

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi \tag{3.50}$$

As the universe expands, the expansion timescale becomes much longer than the oscillation period, $H^{-1} \gg m^{-1}$. This allows us to neglect the Hubble friction term, and the field oscillates $\ddot{\phi} \approx -m^2\phi$ with frequency m. The energy continuity equation can then be written as:

$$\dot{\rho}_\phi + 3H\rho_\phi = -3HP_\phi = -\frac{3}{2}H\left(m^2\phi^2 - \dot{\phi}^2\right) \tag{3.51}$$

Over one oscillation period, the right-hand side averages to zero. As a result, the oscillating field behaves as dust-like matter (pressure-free matter), with $\rho_\phi \propto a^{-3}$.

To prevent the universe from remaining empty, the inflaton must couple to Standard Model fields. This coupling allows the energy stored in the inflaton field to be transferred to ordinary particles. If the decay is slow, the inflaton energy density evolves according to:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi\rho_\phi,$$

where Γ_ϕ denotes the decay rate of the inflaton.

The particles produced from inflaton decay interact and generate additional particles through various reactions, forming a dense particle soup. Over time, this system reaches thermal equilibrium at a temperature T_{rh}. This reheating temperature is set by the energy density ρ_{rh} at the end of the reheating period, which must satisfy $\rho_{\mathrm{rh}} < \rho_{\phi,E}$, where $\rho_{\phi,E}$ is the inflaton energy density at the end of inflation. If reheating is slow, it's possible that $\rho_{\mathrm{rh}} \ll \rho_{\phi,E}$.

The thermalization process is typically assumed to occur due to particle interactions, but some particles—such as gravitinos—may never thermalise because their interactions are extremely weak. Still, as long as the particles' momenta are much larger than their masses, the universe's energy density behaves like that of radiation, regardless of the precise momentum distribution. Once at least the baryons, photons, and neutrinos have thermalised, the standard Hot Big Bang era begins.

3.8 Inflation as an Effective Field Theory

In the absence of a complete microscopic theory of inflation, we describe inflation in the context of an effective field theory. We are then obliged to include in the inflationary action all operators consistent with the assumed symmetries of the inflaton,

$$\mathcal{L}_{\text{eff}}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) + \sum_n c_n \frac{V(\phi)}{\Lambda^n}\left(\frac{\partial_\mu \phi}{\Lambda}\right)^{2n} + \sum_d d_d \frac{(\partial_\mu \phi)^{2n}}{\Lambda^{d-4}} + \cdots . \tag{3.52}$$

One of the remarkable features of inflation is that it is sensitive even to Planck-suppressed operators.

Now, let us discuss the so called *Eta problem.* Successful inflation requires that the inflaton mass m_ϕ is parametrically smaller than the Hubble scale H:

$$\eta_V = \frac{m_\phi^2}{3H^2} \ll 1. \tag{3.53}$$

It is difficult to protect this hierarchy against high-energy corrections. We know that some new degrees of freedom must appear at a $\Lambda \lesssim M_{\text{Pl}}$ to give a UV-completion of gravity. In string theory, the scale is often found to be significantly below the Planck scale, $\Lambda \lesssim M_{\text{Pl}}$. If ϕ has order-one couplings to any massive fields ψ (with $m_\psi \sim \Lambda$), then integrating out the fields ψ yields the effective action (3.52) for ϕ with order-one couplings c_n and d_d. The above argument makes us worry that integrating out the massive fields ψ yields corrections to the potential of the form

$$\Delta V = c_1 V'(\phi)\frac{\phi^2}{\Lambda^2}, \tag{3.54}$$

where $c_1 \sim \mathcal{O}(1)$. If this term arises, then the eta parameter receives the following correction

$$\Delta\eta_V = \frac{M_{\text{Pl}}^2}{V}(\Delta V)' \approx 2c_1\left(\frac{M_{\text{Pl}}}{\Lambda}\right)^2 \gg 1, \tag{3.55}$$

where the final inequality follows from $\Lambda \lesssim M_{\text{Pl}}$. Notice that this problem is independent of the energy scale of inflation. All inflationary models have to address the eta problem.

Large-field inflation The Planck-scale sensitivity of inflation is dramatically enhanced in models with observable gravitational waves, $r \geq 0.01$. In this case, the inflaton field moves over a super-Planckian range during the last 60 e-folds of inflation, $\Delta\phi \gg M_{\text{Pl}}$, and an infinite number of operators contribute equally to the effective action.

Chapter 4

Fluctuations during Inflation

Inflation equips one with a natural mechanism of initial conditions for the hot Big Bang. Since the inflaton field $\phi(t)$ gonverns the energy density of the Universe, it controls the end of inflation. Quantum mechanical treatment of inflation requires some variance, so the inflaton will obtain spatial fluctuations $\delta\phi(t, \mathbf{x})$ around the classical evolution. Thus, different regions of space inflate by different amounts. These variances in the local expansion histories implies differences $\delta\rho(t, \mathbf{x})$ in the local densities, after the end of inflation, and to curvature perturbations $\zeta(\mathbf{x})$ (gauge dependently).

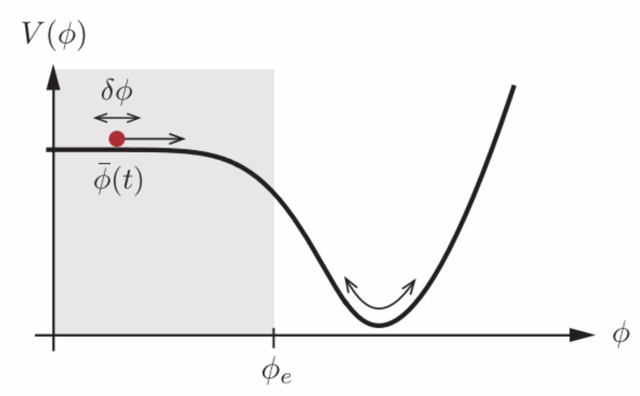

Figure 4.1: Quantum fluctuations $\delta\phi(t, \mathbf{x})$ around the classical background evolution discussed above [9]. Different regions of the universe undergo different evolutions. After inflation, space is left with density fluctuations $\delta\rho(t, \mathbf{x})$. Notice that the theory was not enginnered to produce these fluctuations: their emergence a natural consequence of Quantum Mechanics.

4.1 Test scalar field in de Sitter Space

We first study the toy model of a free, massless scalar field in de Sitter background. One basic assumption that simplifies the problem, is that the field sustains a small amount of the total energy density, and thus, it exerts no influence on the underlying de Sitter structure. The action for this model reads:

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,\alpha^3\left[\dot{\phi}^2 - \frac{1}{\alpha^2}\left(\nabla\phi\right)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\,\alpha^2\left[\phi'^2 - \left(\nabla\phi\right)^2\right], \end{aligned} \tag{4.1}$$

where we changed to conformal time, η, and the derivative with respect to η is denoted as $\phi' \equiv d\phi/d\eta$. In de Sitter geometry, we have:

$$\eta(t) = \int_{-\infty}^{t} \frac{\mathrm{d}t'}{\alpha(t')} = -\frac{1}{H}e^{-Ht} = -\frac{1}{H\alpha} \Rightarrow \alpha(\eta) = -\frac{1}{H\eta}, \ \ \eta \in (-\infty, 0). \tag{4.2}$$

Now, one is able to introduce a canonically normalized field, namely a *Mukhanov* variable $v \equiv \alpha\phi$. Now, integration by parts

$$\int \mathrm{d}\eta \, \alpha^2 \phi'^2 = \int \mathrm{d}\eta \left[v'^2 + \frac{\alpha''}{\alpha} v^2 \right],$$

allow us to rewrite the action as:

$$S = \frac{1}{2} \int \mathrm{d}\eta \, \mathrm{d}^3x \left[v'^2 - (\nabla v)^2 + \frac{\alpha''}{\alpha} v^2 \right], \tag{4.3}$$

and thus one recovers the standard kinetic term. Now the next step of the analysis is for one to proceed to field equation of motion. In terms of Euler-Lagrange equation, we obtain:

$$\frac{\partial}{\partial \eta}\left(\frac{\partial \mathcal{L}}{\partial v'}\right) + \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i v)}\right) - \frac{\partial \mathcal{L}}{\partial v} = 0$$

$$\Rightarrow v'' - \nabla^2 v - \frac{\alpha''}{\alpha} v = 0 \Rightarrow v'' - \left(\nabla^2 + \frac{\alpha''}{\alpha}\right) v = 0 \tag{4.4}$$

Let us define the Fourier modes:

$$v(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{x}} \, v_{\boldsymbol{k}}(\eta). \tag{4.5}$$

Inserting the ansatz into the equation of motion, we derive the *Mukhanov-Sasaki equation* for the Fourier modes $v_{\boldsymbol{k}}$ of the field:

$$\boxed{v''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a}\right) v_{\boldsymbol{k}} = 0} \tag{4.6}$$

We have infinitely many harmonic oscillators, one for each k, with the following time-dependent frequency:

$$\omega_k^2(\eta) = k^2 - \frac{a''}{a} = k^2 - \frac{2}{\eta^2}.$$

Let us realize the exact de Sitter background. The effective mass of the normalized field $m_{\text{eff}}^2(\eta) = 2/\eta^2$ encodes the expansion of the universe. To develop our intuition, we examine specific limiting cases of (4.6).

If we go back to the early times, we can see that the contribution of mass is much smaller than the value of momentum $|\boldsymbol{k}|$ of the relevant modes. Thus, we can see that in this limit $k \gg |1/\eta|$, the Mukhanov-Sasaki equation reduces to that of a standard flat-space harmonic oscillator:

$$v''_{\boldsymbol{k}} + k^2 v_{\boldsymbol{k}} \approx 0. \tag{4.7}$$

Thus, all modes of cosmological interest were deep inside the horizon at early times, oscillating: $v_{\boldsymbol{k}} \propto e^{\pm ik\eta}$.

Now let's examine modes with wavelength much larger than the horizon $k \ll |1/\eta|$, which modes of interest correspond to later times (as $\eta \to 0$). In this limit, the equation takes the form:

$$v''_{\boldsymbol{k}} - \frac{2}{\eta^2} v_{\boldsymbol{k}} \approx 0. \tag{4.8}$$

These modes behave as: $v_{\boldsymbol{k}} \propto \alpha \propto \eta^{-1}$, and therefore $\phi_{\boldsymbol{k}} = v_{\boldsymbol{k}}/\alpha = const.$ The latter implies that a mode "freezes" upon horizon exit, which is the behavior we are interested in when we examine perturbations on *superhorizon* scales.

4.1.1 Canonical Quantization

Quantization is not much more difficult than quantizing the simple harmonic oscillator in quantum mechanics, except for a brief discussion regarding the non-uniqueness of the vacuum, which arises from the time dependence of the oscillator frequencies $\omega(\eta)$. Other than that, the canonical quantization procedure is standard, and we treat conformal time in the same way we treat time in the simple harmonic oscillator. Thus, in an analogous way, the canonical momentum can be identified as:

$$\pi(\eta, \mathbf{x}) \equiv \frac{\partial \mathcal{L}}{\partial v'} = v' \tag{4.9}$$

We now promote the fields $v(\eta, \mathbf{x}), \pi(\eta, \mathbf{x})$ to field operators: $\hat{v}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x})$. We impose the following equal-time commutation relations (for $\hbar \equiv 1$):

$$[\hat{v}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \tag{4.10a}$$

$$[\hat{v}(\eta, \mathbf{x}), \hat{v}(\eta, \mathbf{x}')] = [\hat{\pi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = 0 \tag{4.10b}$$

For the Fourier modes, we derive a similar commutator:

$$[\hat{v}_{\boldsymbol{k}}(\eta), \hat{\pi}_{\boldsymbol{k'}}(\eta)] = \int d^3x \int d^3x' \; [\hat{v}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, e^{-i\mathbf{k}'\cdot\mathbf{x}'} \tag{4.11}$$

$$= i(2\pi)^3 \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}') \tag{4.12}$$

Since we are dealing with Heisenberg operators, the operator solution of the Mukhanov equation is determined by two initial conditions $\hat{v}_{\boldsymbol{k}}(\eta_i)$, $\hat{\pi}_{\boldsymbol{k}}(\eta_i)$. But instead, since the equation is linear, we can write the operator solution in terms of two linearly independent classical solutions, with coefficients being the two time-independent integration operators $\hat{a}_{\pm\boldsymbol{k}}$. These constant operators will define initial conditions which may depend on direction, but the evolution depends only on $|\boldsymbol{k}|$. Thus, the most general solution can be written as:

$$\hat{v}_{\boldsymbol{k}}(\eta) = f_k(\eta)\, \hat{a}_{\mathbf{k}} + f_k^*(\eta)\, \hat{a}^\dagger_{-\mathbf{k}}\,, \tag{4.13}$$

where, as mentioned above, the complex mode function $f_k(\eta)$ is the classical solution of (4.6). Thus, the field operator is expanded as follows:

$$\hat{v}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left[f_k(\eta)\, \hat{a}_{\mathbf{k}} + f_k^*(\eta)\, \hat{a}^\dagger_{-\mathbf{k}} \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.14}$$

One now has the freedom of proper normalization choice for the classical solutions (by rescaling them) as:

$$W[f_k, f_k^*] = f_k f_k^{*\prime} - f_k' f_k^* \equiv i, \tag{4.15}$$

where $W[f_k, f_k^*]$ is the Wronskian. The latter intention becomes clear if one substitutes (4.15) and (4.13) into the commutation relation for Fourier modes. Then, one will obtain:

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}'), \tag{4.16}$$

which is the proper commutation relation for the creation and annihilation operators for the simple harmonic oscillator. The vacuum state in Hilbert space is defined via:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0. \tag{4.17}$$

Quantum states of Hilbert space are being produced by repeated application of creation operators. However the choice of vacuum is not unique.

4.1.2 Bunch-Davies Vacuum

The most general solution of the Mukhanov-Sasaki equation (4.6) is:

$$f_k(\eta) = c_1 \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + c_2 \left(1 + \frac{i}{k\eta}\right) e^{ik\eta}, \tag{4.18}$$

where the constraint for the coefficients of linearly independent solutions, comes from normalisation of the mode functions, $W[f_k, f_k^*] \equiv i$, which implies

$$|c_1|^2 - |c_2|^2 = \frac{1}{2k}. \tag{4.19}$$

Why is vacuum not unique at this point? The classical solutions $f_k(\eta)$ are not uniquely defined, since c_1, c_2 are running parameters (in such a way that the constraint is satisfied). A change in the classical solution $f_k(\eta)$ can be followed by a change in $\hat{a}_{\mathbf{k}}$, so that the operator $\hat{v}_{\boldsymbol{k}}(\eta)$ remains fixed. A change in $\hat{a}_{\mathbf{k}}$ however implies a change in the vacuum state (4.17). However, one is in position to require the vacuum state $|0\rangle$ to be the ground state of the Hamiltonian, and this is how we will get rid of one-parameter family of solutions, and work with a special classical solution.

We are writing down the Hamiltonian operator:

$$\hat{H} = \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{v})^2 + \frac{1}{2}\frac{a''}{a}\hat{v}^2\right]. \tag{4.20}$$

Introducing the field operator expansion (4.14) into this, we find:

$$\hat{H} = \frac{1}{4}\int d^3k \left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\mathcal{F}_k^* + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\mathcal{F}_k + \left(2\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \delta(\mathbf{0})\right)\mathcal{E}_k\right], \tag{4.21}$$

where the appearing expressions $\mathcal{E}, \mathcal{F}$ are:

$$\mathcal{E}_k(\eta) = |f_k'|^2 + \omega_k^2 |f_k|^2 \tag{4.22}$$

$$\mathcal{F}_k(\eta) = {f_k'}^2 + \omega_k^2 {f_k'}^2 \tag{4.23}$$

From the latter result, the vacuum expectation value of the Hamiltonian is found to be:

$$\langle 0|\hat{H}|0\rangle = \frac{1}{4}\delta(\mathbf{0})\int d^3k\ \mathcal{E}_k(\eta) \tag{4.24}$$

Ignoring the usual, non-physical divergent factor, we can define the energy density of the vacuum as:

$$\varepsilon_k = \frac{1}{4}\int d^3k\ \mathcal{E}_k(\eta). \tag{4.25}$$

This quantity is minimized for each-k mode separately, which translates into minimizing the $\mathcal{E}_k(\eta)$ function. Taking the behavior of Fourier modes at early times:

$$\lim_{\eta\to-\infty} f_k(\eta) = c_1 e^{-ik\eta} + c_2 e^{ik\eta}, \tag{4.26}$$

we get:

$$\lim_{\eta\to-\infty} \mathcal{E}_k(\eta) = 2(|c_1|^2 + |c_2|^2)\, k^2 \tag{4.27}$$

Taking into account the constraint we mentioned before, the latter function is minimized for [1]:

$$|c_1| = \frac{1}{\sqrt{2k}}, \quad c_2 = 0 \tag{4.28}$$

And at this point, we found the special mode function we wished to determine. The *Bunch-Davies* solution reads:

$$f_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{4.29}$$

To complete the discussion about the Hamiltonian and the vacuum state, notice that the quantities are found to be:

$$\lim_{\eta\to-\infty} \mathcal{E}_k(\eta) = k \tag{4.30}$$

$$\lim_{\eta\to-\infty} \mathcal{F}_k(\eta) = 0, \tag{4.31}$$

which implies that the limit of the Hamiltonian is simply:

$$\lim_{\eta\to-\infty} \hat{H} = \int d^3k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\delta(\mathbf{0})\right]\omega_k. \tag{4.32}$$

Now, it is clear that the vacuum state $|0\rangle$ is the ground state of harmonic oscillator, with energy $\omega_k/2$.

4.1.3 Power Spectrum

Quantum fluctuations give rise to the useful observable of the power spectrum. We write the field expansion, as before:

$$\hat{v}(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}_{\mathbf{k}} f_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} f^*_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\right].$$

The vacuum expectation value (VEV) of the field operator $\hat{v}$ does not contribute: $\langle\hat{v}\rangle \equiv \langle 0|\hat{v}|0\rangle = 0$. However, one is able to calculate the 2-point correlation function, which measures the variance of

[1] As $\eta \to 0$, we find that $\mathcal{E}_k = 2(2|c_2|^2 + \frac{1}{2k})\, k^2 = 4|c_2|^2k^2 + k \geq 0$, is minimized for $c_2 = 0$ and thus $|c_1| = 1/\sqrt{2k}$.

inflaton fluctuations:

$$\begin{aligned}
\langle|\hat{v}|^2\rangle &\equiv \langle 0|\hat{v}(\eta,\mathbf{0})\hat{v}(\eta,\mathbf{0})|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\langle 0|\left(\hat{a}_{\mathbf{k}}f_k(\eta) + \hat{a}^\dagger_{\mathbf{k}}f^*_k(\eta)\right)\left(\hat{a}_{\mathbf{k}'}f_{k'}(\eta)+\hat{a}^\dagger_{\mathbf{k}'}f^*_{k'}(\eta)\right)|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} f_k(\eta)f^*_{k'}(\eta)\langle 0|[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}]|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}|f_k(\eta)|^2 \\
&= \int d\ln k\,\frac{k^3}{2\pi^2}|f_k(\eta)|^2 .
\end{aligned} \tag{4.33}$$

We define the (dimensionless) *power spectrum* as

$$\boxed{\mathcal{P}_v(k,\eta) \equiv \frac{k^3}{2\pi^2}|f_k(\eta)|^2}. \tag{4.34}$$

An interesting feature is that the power spectrum is determined by the square measure of the classical solutions. Substituting the Bunch-Davies solutions, we obtain:

$$\mathcal{P}_\varphi(k,\eta) = \frac{\mathcal{P}_v(k,\eta)}{a^2(\eta)} = \left(\frac{H}{2\pi}\right)^2\left[1+(k\eta)^2\right] \tag{4.35}$$

For the superhorizon limit where $k\eta \to 0$, for all k-modes, we get:

$$\lim_{k\eta\to 0}\mathcal{P}_\varphi(k,\eta) = \left(\frac{H}{2\pi}\right)^2, \tag{4.36}$$

which is the famous *scale invariant* power spectrum of the massless field, since it approaches the same constant for all modes.

Another thing to keep in mind is that one generally works in *quasi-de Sitter* spacetime. In pure de Sitter space we have $\dot{H} = 0$ and thus the slow-roll parameter $\epsilon = 0$. This translates as that for perfect de Sitter inflation never ends. Of course, we know that inflation has to end and that the spacetime during inflation has to deviate from the de Sitter idealization. But these formulas also hold for slow-roll approximation of quasi-de Sitter spacetime.

4.1.4 Massive Fields

We are also interested to generalize the above discussion for any massive fields during the period of inflation, and study their power spectrum.

The equation of motion for a massive spectator field is:

$$f''_k + \left(k^2 + m^2a^2 - \frac{a''}{a}\right) f_k = 0, \tag{4.37}$$

which, in pure de Sitter background, it can be written as:

$$f''_k + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right) f_k = 0, \quad \text{where} \quad \nu^2 \equiv \frac{9}{4} - \frac{m^2}{H^2}. \tag{4.38}$$

The most general classical solution is

$$f_k(\eta) = \sqrt{|\eta|}\left[c_1 H_\nu^{(1)}(|k\eta|) + c_2 H_\nu^{(2)}(|k\eta|)\right], \tag{4.39}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of the first and second kind. Imposing the Bunch-Davies initial condition, it is found:

$$c_1 = \sqrt{\frac{\pi}{2}} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}, \quad c_2 = 0. \tag{4.40}$$

We now investigate the 2 cases. For the first case, let us assume that ν is real, and thus: $m < \frac{3}{2}H$, which is the case of light fields . For the limit of our interest (the superhorizon limit) we get:

$$\mathcal{P}_\varphi(k,\eta) = \frac{\mathcal{P}_v(k,\eta)}{a^2(\eta)} \Rightarrow \lim_{k\eta\to 0} \mathcal{P}_\varphi(k,\eta) = \left(\frac{H}{2\pi}\right)^2 (k\eta)^{3/2-\nu}. \tag{4.41}$$

We observe that the superhorizon limit of the spectrum is not scale-invariant, and also undergoes time evolution. A standard way to quantify the deviation from scale invariance is through the scalar spectral index:

$$n_\varphi \equiv \frac{\mathrm{d}\ln \mathcal{P}_\varphi}{\mathrm{d}\ln k} = \frac{3}{2} - \nu = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \xrightarrow{m\ll H} n_\varphi \simeq \frac{m^2}{3H^2}. \tag{4.42}$$

As expected, the limit $m \to 0$, brings us back to the scale-invariant spectrum.

Now, for the case of heavy fields i.e. for $m > \frac{3}{2}H$, the degree of the Hankel function becomes imaginary, $\nu = i\mu$, where

$$\mu \equiv \sqrt{\frac{m^2}{H^2} - \frac{9}{4}} \xrightarrow{m\gg H} \mu \simeq \frac{m}{H}. \tag{4.43}$$

There is an exponential suppression of the amplitude of the classical solution:

$$|f_k(\eta)| = \sqrt{\frac{\pi}{2}} e^{-\pi\mu/2} \sqrt{|\eta|}\, |H_{i\mu}^{(1)}(|k\eta|)|. \tag{4.44}$$

Therefore, an important distinction between light and heavy particles during inflation, is that the power spectrum of very massive fields is highly suppressed, $\mathcal{P}_\varphi \propto e^{-\pi m/H}$, for $m \gg \frac{3}{2}H$.

4.2 Primordial Fluctuations

We now proceed with the fluctuations in the inflaton field during inflation. These fluctuations cannot be treated as decoupled from fluctuations in the metric, since the two are coupled by the Einstein equations. This leads to somewhat more complex equations, but conceptually the quantization of the coupled inflaton-metric fluctuations is essentially the same.

4.2.1 Metric fluctuations

We first study the fluctuations of the metric in the *ADM formalism.* In ADM, spacetime is sliced into three-dimensional hypersurfaces and one temporal component (foliation). The perturbed line element reads:

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \tag{4.45}$$

Here, $h_{ij} \equiv h_{ij}(t,\mathbf{x})$ is the the three-dimensional metric on spatial slices of constant time t. In order to seperate the spacetime metric into its spatial and temporal parts, one introduces the lapse function $N(t,\mathbf{x})$, that measures proper time separation between slices, and the shift vector $N^i(t,\mathbf{x})$ that encodes the spatial coordinate shift from one slice to the next. The geometry of the spatial slices is described by the 3-Ricci tensor $R^{(3)}_{ij}$ of the emergent metric h_{ij} and also from the extrinsic curvature:

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right) = \frac{1}{N}E_{ij} \tag{4.46}$$

Now, the 4D Ricci scalar, R, which will be introduced to the action of interest, can be written in terms of the three-dimensional one, $R^{(3)}$, and the extrinsic curvature as:

$$R = R^{(3)} + \frac{1}{N^2}\left(E^{ij}E_{ij} - E^2\right) \tag{4.47}$$

We should note that indices are raised or lowered with h^{ij} (not the full metric), and we also denoted $E \equiv h^{ij}E_{ij}$.

The action of our interest, which will give rise to fluctuations of the metric, is (setting $M_{pl} = 1$):

$$\begin{aligned} S &= \frac{1}{2}\int d^4x\sqrt{-g}\left[R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V\right] \\ &= \frac{1}{2}\int d^4x\sqrt{h}N\left[R^{(3)} - 2V + \frac{1}{N^2}\left(E^{ij}E_{ij} - E^2\right) + \frac{1}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi\right] \end{aligned} \tag{4.48}$$

Before we proceed, let us discuss how one reaches in (4.48). The full metric, as a block matrix, can be written as:

$$g^{(4)}_{\mu\nu} = \begin{pmatrix} -N^2 + N_iN^i & N_j \\ N_i & h_{ij} \end{pmatrix} \tag{4.49}$$

From properties of determinants, we are able to write:

$$det(g^{(4)}_{\mu\nu}) = det(h_{ij}) \cdot det[-N^2 + N_iN^i - N_jh^{ij}N_i] = -N^2h \;\Rightarrow \sqrt{-g} = \sqrt{h}N. \tag{4.50}$$

The inverse metric is found to be:

$$g^{(4)\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2h^{ij} - N^iN^j \end{pmatrix} \tag{4.51}$$

Also the expansion of the canonical term for the inflaton reads:

$$\begin{aligned} g^{(4)\mu\nu}\,\partial_\mu\phi\partial_\nu\phi &= g^{00}\dot{\phi}^2 + 2g^{0i}\dot{\phi}\,\partial_i\phi + g^{ij}\partial_i\phi\partial_j\phi \\ &= -\frac{1}{N^2}\dot{\phi}^2 + 2\frac{N^i}{N^2}\dot{\phi}\,\partial_i\phi - \frac{\left(N^i\partial_i\phi\right)^2}{N^2} + h^{ij}\partial_i\phi\partial_j\phi \\ &= -\frac{1}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 + h^{ij}\partial_i\phi\partial_j\phi. \end{aligned} \tag{4.52}$$

Substituting the relevant relations for R, the canonical term for the inflaton, and for $\sqrt{-g}$, we obtain the action (4.48). Now, if one notices this action, it becomes clear that N and N_i are not dynamical variables, since they do not appear with time dependence. This means that their values will be fixed by constraint equations. So, we begin by varying the action with respect to N. The variation will give us:

$$\frac{\delta S}{\delta N} = \frac{1}{2}\int d^4x\sqrt{h}\left[R^{(3)} - 2V - \frac{1}{N^2}\left(E^{ij}E_{ij} - E^2\right) - \frac{1}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi\right] = 0$$
$$\Rightarrow R^{(3)} - 2V - h^{ij}\partial_i\phi\partial_j\phi - \frac{1}{N^2}\left[E^{ij}E_{ij} - E^2 + \left(\dot{\phi} - N^i\partial_i\phi\right)^2\right] = 0 \tag{4.53}$$

Now, varying the action with respect to N^i gives the following constraint:

$$\frac{\delta S}{\delta N^i} = -\frac{1}{2}\int d^4x\sqrt{h}N^{-1}\frac{\delta}{\delta N^i}\left[\left(E^{kj}E_{kj} - E^2\right) + \left(\dot{\phi} - N^k\partial_k\phi\right)^2\right] = 0$$
$$\Rightarrow \nabla_i\left[N^{-1}\left(E^i_j - E\delta^i_j\right)\right] - N^{-1}\left(\dot{\phi} - N^k\partial_k\phi\right)\partial_i\phi = 0 \tag{4.54}$$

Substituting the solutions for N and N^i into the action leaves ϕ and h_{ij} to determine the dynamics of coupled inflaton-metric fluctuations. This is what one does essentially when one perturbs the action (4.48). Now, before one proceeds into perturbation theory, to determine scalar perturbations, one has to commit to choose a gauge for the inflaton ϕ and the induced metric h_{ij}, in order to fix space and time parameterizations. In this thesis, we will discuss the results from the spatially flat gauge and the comoving gauge.

4.2.2 Spatially Flat Gauge

In the spatially flat gauge, the induced metric is taken to be unperturbed, thus there is no deviation from the flat metric:

$$h_{ij} = a^2\delta_{ij}. \tag{4.55}$$

The latter implies the perturbations are transferred to the inflaton. The perturbed quantities of inflaton, the lapse and the shift are written as:

$$\phi = \bar{\phi}(t) + \phi_p(t,\mathbf{x}), \quad N = 1 + \alpha_p(t,\mathbf{x}), \quad N_i = \partial_i\beta_p(t,\mathbf{x}). \tag{4.56}$$

Plugging the perturbed quantities and the flat 3D metric into the constraint equations discussed before, one derives certain equations to *linear order*. The reason is that the second order terms in N and N_i will be coupled to zeroth order constraints, which vanish when the background equations of motions will be implemented. For the $\bar{\phi}(t)$, the background equation of motion that satisfies is simply the Klein Gordon equation:

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \frac{\mathrm{d}V}{\mathrm{d}\bar{\phi}} = 0 \tag{4.57}$$

The solution to the perturbations are found to be:

$$\alpha_p = \frac{\dot{\bar{\phi}}}{2H}\,\phi_p\,, \quad \partial^2\beta_p = \frac{(\dot{\bar{\phi}})^2}{2H^2}\frac{d}{dt}\left(-\frac{H}{\dot{\bar{\phi}}}\,\phi_p\right). \tag{4.58}$$

Inserting perturbations into the action (4.48), expanding to quadratic action, and working some integrations by parts, one reaches to:

$$S_{(2)} = \frac{1}{2}\int dt\, d^3x\; a^3\left[(\dot{\phi}_p)^2 - \frac{1}{a^2}(\partial\phi_p)^2 - \left[\ddot{V} - 2(3\epsilon - \epsilon^2 + \epsilon\eta)H^2\right]\phi_p^2\right]. \tag{4.59}$$

We now proceed with familiar steps we already done: we switch to conformal time, and derive the equation of motion for the Fourier modes of the canonically normalized field $v = a\phi$, which is:

$$v_k'' + \left(k^2 + a^2\left[\ddot{V} - 6\epsilon H^2\right] - \frac{a''}{a}\right) v_k = 0, \tag{4.60}$$

where, along the road to derivation , one drops slow-roll suppressed terms of higher order in the effective mass.

Notice that $a^2\left[\ddot{V} - 6\epsilon H^2\right] \sim \mathcal{O}(\epsilon,\eta)(aH)^2 \ll a''/a = 2(aH)^2$. However, one wishes to work with non-evolving quantities on superhorizon scales, and thus it is useful to map the solution (taken at $k = aH$) to the *comoving curvature perturbation*, which is defined as:

$$\zeta = -\frac{H}{\dot{\phi}}\,\phi_p \tag{4.61}$$

Thus, the power spectrum is computed:

$$\mathcal{P}_\zeta(k) = \left(\frac{H}{\dot{\phi}}\,\phi_p\right)^2 \mathcal{P}_{\phi_p}(aH,\eta) \tag{4.62}$$

By dropping $\mathcal{O}(\epsilon,\eta)$, one gets the Mukhanov equation for the massless case, which leads to the scale-invariant power spectrum. Finally, one finds:

$$\mathcal{P}_\zeta(k) = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH} \tag{4.63}$$

4.2.3 Comoving Gauge

In comoving gauge, there is a more explicit study discussed in Appendix A. Now we will just display the results from perturbation theory. In this gauge, the perturbations are transferred to the induced metric. This implies that the inflaton field is taken to its background value:

$$\phi = \bar{\phi}(t). \tag{4.64}$$

The perturbations in this gauge are:

$$h_{ij} = a^2\left(1 + 2\zeta(t,\mathbf{x})\right)\delta_{ij}\,, \quad N = 1 + \alpha_p(t,\mathbf{x}), \quad N_i = \partial_i\beta_p(t,\mathbf{x}), \tag{4.65}$$

where ζ is the comoving curvature perturbation. As before, plugging the perturbations into the constraint equations, we get:

$$\alpha_p = \frac{\dot{\zeta}}{H}\,, \quad \partial^2\beta_p = -\frac{\partial^2\zeta}{H} + a^2\frac{(\dot{\bar{\phi}})^2}{2H^2}\dot{\zeta} \tag{4.66}$$

These solutions of perturbations, applied to the action, expanding to second order and a few integration by parts, one finally obtains:

$$S_{(2)} = \int dt\, d^3x\; a^3\,\epsilon\left[(\dot{\zeta})^2 - \frac{1}{a^2}(\partial\zeta)^2\right]. \tag{4.67}$$

Since there is no "real mass" term, the curvature perturbation field is conserved outside the horizon, as mentioned in the flat gauge. That is the advantage of the curvature perturbation, as it was discussed earlier. To check this in comoving gauge the equation of motion derived from varying the latter action, for the Fourier components, reads:

$$\ddot{\zeta}_k + (3+\eta)H\dot{\zeta}_k + \frac{k^2}{a^2}\zeta_k = 0 \tag{4.68}$$

On super-Hubble scales, $k \ll aH$, this reduces to:

$$\ddot{\zeta}_k + (3+\eta)H\dot{\zeta}_k \approx 0,$$

which implies that the solution is $\zeta_k = const\,, \forall k$.

One now switches as before to conformal time and proceeds with working on the equation of motion for the canonically normalized field, $v \equiv a\sqrt{2\epsilon}\,\zeta \equiv z\,\zeta$:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \tag{4.69}$$

Now, the effective mass in a de Sitter background reads:

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\eta^2}, \quad \text{with} \quad \nu \approx \frac{3}{2} + \epsilon + \frac{\eta}{2}. \tag{4.70}$$

This equation looks familiar...it is the equation of motion for a *massive* scalar field in de Sitter space. Applying the same logic as before, the Bunch–Davies mode function, for the superhorizon limit, is:

$$|v_k(\eta)| = \frac{\sqrt{\pi}}{2}\sqrt{|\eta|}\left|H_\nu^{(1)}(-k\eta)\right| \xrightarrow{k\eta\to 0} \lim_{k\eta\to 0}|v_k(\eta)| = \frac{2^\nu\,\Gamma(\nu)}{2\sqrt{\pi}}\frac{1}{\sqrt{k}}(-k\eta)^{-\nu+1/2}. \tag{4.71}$$

Knowing the superhorizon limit of the classical solution, the power spectrum of ζ reads as:

$$\begin{aligned}\mathcal{P}_\zeta(k) &= \lim_{k\eta\to 0}\frac{k^3}{2\pi^2}|\zeta_k(\eta)|^2 = \frac{1}{z^2(\eta)}\lim_{k\eta\to 0}\frac{k^3}{2\pi^2}|v_k(\eta)|^2 \\ &= \frac{1}{2a^2\epsilon}\frac{k^2}{4\pi^2}(-k\eta)^{-2\nu+1} = \frac{1}{16\pi^2}\frac{H^2(\eta)}{\epsilon(\eta)}(-k\eta)^{-2\nu+3}.\end{aligned} \tag{4.72}$$

An important feature is that the time dependence of $H(\eta)$ and $\epsilon(\eta)$ precisely cancels the time dependence of the final factor in (4.72), so that the power spectrum is *time independent*, as expected. Now, one is able to choose a k_* to be a reference scale that exits the horizon at time $\eta_* = -1/k_*$. Equation (4.72) can then be recasted as:

$$\mathcal{P}_\zeta(k) = \frac{1}{8\pi^2\epsilon_*}\frac{H_*^2}{M_{pl}^2}\left(\frac{k}{k_*}\right)^{-2\epsilon_*-\eta_*}. \tag{4.73}$$

4.3 Scalar Perturbations

Let us concentrate what we know about curvature perturbations, and make a connection with observations. The power spectrum of the curvature perturbation ζ, as obtained from spatially flat gauge

(4.63) and comoving gauge (4.73), takes a power law form:

$$\mathcal{P}_\zeta(k) = A_s \left(\frac{k}{k_*}\right)^{n_s-1} , \tag{4.74}$$

where the amplitude of perturbations, and also the spectral index are defined in the following manner:

$$A_s \equiv \frac{1}{8\pi^2}\frac{1}{\epsilon_*}\frac{H_*^2}{M_{\rm pl}^2}, \tag{4.75}$$

$$n_s \equiv 1 - 2\epsilon_* - \eta_* \; . \tag{4.76}$$

From CMB measurements, the observational constraint on the scalar spectral index is $n_s = 0.9603 \pm 0.0073$. The observed percent-level deviation from the scale-invariant value, $n_s = 1$, are the first direct measurement of time dependence in the inflationary dynamics. The measured amplitude of the scalar spectrum is: $A_s = (2.196 \pm 0.060) \times 10^{-9}$, for the reference scale $k_* = 0.05\ Mpc^{-1}$.

For a following analysis, we need to write the observed quantities in terms of ϵ_V, η_V parameters computed from the shape of a candidate inflaton potential. The latter quantities can be rewritten as:

$$A_s = \frac{1}{24\pi^2}\frac{1}{\epsilon_V}\frac{V}{M_{\rm pl}^4}, \tag{4.77}$$

$$n_s \equiv 1 - 6\epsilon_V + 2\eta_V \; . \tag{4.78}$$

4.4 Gravitational Waves

One of the most important and model-independent prediction of inflation is a spectrum of primordial gravitational waves, with an amplitude determined directly from the Hubble scale H during inflation. A potential measurement of primordial gravitational waves would give pure information about the energy scale of inflation. An interesting feature is that inflationary gravitational waves give rise to a unique signature in the polarization of the CMB.

So, we are now interested in studying the quantum generation of tensor perturbations to the spatial metric. In this case, the problem is more manageable than scalar perturbations, since first-order tensor perturbation are *gauge-invariant* and do not backreact on the inflationary background. Expansion of the Einstein-Hilbert action will give us the second-order action for tensor fluctuations.

The spatial metric, reads:

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + (\delta_{ij} + 2\gamma_{ij})\mathrm{d}x^i\mathrm{d}x^j\right], \tag{4.79}$$

where γ_{ij} is transverse and traceless $\gamma_{ii} = 0\,,\ \partial^i\gamma_{ij} = 0$. Substituting the latter into the Einstein–Hilbert action and expanding to second order gives:

$$S = \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R \quad\Rightarrow\quad S_{(2)} = \frac{M_{pl}^2}{8}\int \mathrm{d}\eta\,\mathrm{d}^3x\,a^2\left[\gamma_{ij}'^2 - (\partial\gamma_{ij})^2\right]. \tag{4.80}$$

It is now useful to use rotational symmetry to align the z-axis of the coordinate system with the momentum of the mode, i.e., $\vec{k} \equiv (0,0,k)$, and write

$$\frac{M_{pl}}{2} a\gamma_{ij} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} v_+ & v_\times & 0 \\ v_\times & -v_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.81}$$

The quadratic action (4.80) can be recasted as:

$$S_{(2)} = \frac{1}{2} \sum_{\lambda=+,\times} \int \mathrm{d}\eta\, \mathrm{d}^3x \left[v_\lambda'^2 - (\partial v_\lambda)^2 + \frac{a''}{a} v_\lambda^2 \right], \tag{4.82}$$

which is just two duplicates of the action of a massless scalar field, one for each available polarization mode of the gravitational wave, $v_{+,\times}$. The equation of motion for each polarization mode is:

$$v_k'' + \left(k^2 - \frac{a''}{a} \right) v_k = 0, \tag{4.83}$$

where the effective mass can be written as

$$\frac{a''}{a} = \frac{\nu^2 - 1/4}{\eta^2}, \quad \text{with} \quad \nu \approx \frac{3}{2} + \epsilon. \tag{4.84}$$

The Bunch–Davies mode function is then given by (7.59). The superhorizon limit of the power spectrum of the tensor fluctuations then is

$$\begin{aligned} \mathcal{P}_\gamma(k) &= 2 \times \lim_{k\eta \to 0} \frac{k^3}{2\pi^2} |\gamma_k(\eta)|^2 = 2 \times \left(\frac{2}{aM_{pl}} \right)^2 \lim_{k\eta \to 0} \frac{k^3}{2\pi^2} |v_k(\eta)|^2 \\ &= \frac{2}{\pi^2} \frac{H^2(\eta)}{M_{pl}^2} (-k\eta)^{-2\nu+3}. \end{aligned} \tag{4.85}$$

Introducing the reference scale k_*, this can be written as

$$P_\gamma(k) = \frac{2}{\pi^2} \frac{H^2}{M_{pl}^2} (k/k_*)^{n_t - 1}. \tag{4.86}$$

This result is arguably the most robust and model-independent prediction of inflation. We see that the form of the tensor power spectrum is again a power law,

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k_*} \right)^{n_t}, \tag{4.87}$$

where the amplitude and the spectral index are

$$A_t = \frac{2}{\pi^2} \frac{H^2}{M_{pl}^2}, \tag{4.88}$$

$$n_t = -2\epsilon_*. \tag{4.89}$$

Notice that the tensor amplitude is a direct measure of the expansion rate H during inflation. This is in contrast to the scalar amplitude which depends on both H and ϵ. The tensor tilt is a direct measure of ϵ, whereas the scalar tilt depends both on ϵ and η. Observationally, a small value for n_t is hard to distinguish from zero. The tensor amplitude is often normalized with respect to the measured scalar amplitude, $A_s = (2.196 \pm 0.060) \times 10^{-9}$ (at $k_* = 0.05\,\mathrm{Mpc}^{-1}$):

$$r = \frac{A_t}{A_s} = 16\epsilon_*, \tag{4.90}$$

where r is the *tensor-to-scalar ratio.*

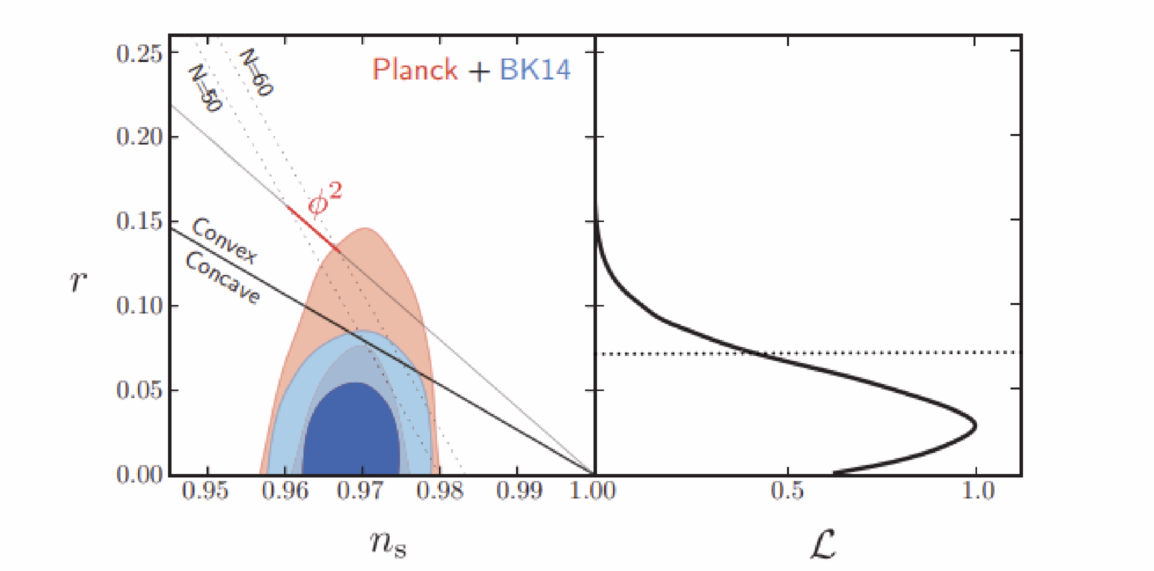

Figure 4.2: Current constraints on n_s and r from CMB measurements of Planck and BICEP [9]. For the ϕ^2 case and for $N_* = 60$ e-folds, we find $r \approx 0.13\,$, $n_s \approx 0.97$. These predictions are excluded by the latest CMB data.

Part III

Supergravity & Inflationary Potentials

Chapter 5

Chaotic Inflation

5.1 Inflaton Potentials in Supergravity

The discussion about Supergravity is long, thus in this section we are only going to present the Lagrangian and the potential of our interest, which are the objects one essentially finds In $N = 1$ Supergravity. However, we note that one starts from the free chiral model and only ask to make the global supersymmetry local. As a result of the procedure one finds that Einstein gravity will emerge and that one is unavoidably led to supergravity. In $N = 1$ Supergravity, the bosonic scalar part reads (setting $M_{pl} = 1$):

$$e^{-1}\mathcal{L}_{bos} = -\frac{1}{2}R - K_{i\bar{j}}\, g^{\mu\nu}\partial_\mu\Phi^i\partial_\nu\bar{\Phi}^{\bar{j}} - V(\Phi,\bar{\Phi}), \tag{5.1}$$

where the potential reads:

$$V(\Phi,\bar{\Phi}) = e^{\mathcal{K}}\left(K^{i\bar{j}}D_iWD_{\bar{j}}\bar{W} - 3|W|^2\right) \tag{5.2}$$

Finding a proper inflaton potential in supergravity is not an easy task to achieve. The main reason, is that in minimal $\mathcal{N} = 1$ supergravity, the canonical term for Kähler potential is proportional to $\Phi\bar{\Phi}$. The F-term part of the scalar potential is proportional to $e^{\mathcal{K}}$, and therefore the potential scales like $e^{|\Phi|^2}$. This is much too steep for chaotic inflation at $\Phi \gg 1$. Also, the presence of terms such as $e^{|\Phi|^2}$, suggests that the typical scalar masses are of order $\mathcal{O}(H)$, which leave no room to support inflation even at values of $|\Phi| < 1$.

One discussion about the problem, is one to find flat directions of the inflaton potential in supergravity. One idea (which will be adopted in this thesis) is that instead of considering a minimal Kähler potential containing $\Phi\bar{\Phi}$, one may instead consider the dependence $\mathcal{K}[(\Phi - \bar{\Phi})^2/2]$. This potential now has shift symmetry: it does not depend on the field combination $\Phi + \bar{\Phi}$, and therefore the dangerous exponential term $e^{\mathcal{K}}$ is also independent of $\Phi + \bar{\Phi}$, which makes the potential flat and appropriate for chaotic inflation, with the structure $\Phi + \bar{\Phi}$ essentially playing the role of the inflaton.

In Kallosh-Linde-Rube models [15, 16, 17] , the philosophy is that one considers two complex scalar fields $\mathcal{T}$ and $\mathcal{S}$, and also makes a proper choice for the form of the Kähler potential (as discussed above) such that the scalar potential is sufficiently flat along the Re$\mathcal{T}$ direction, and thus, suitable for inflation. The way one obtains a single-field inflation scenario, is by realizing Re$\mathcal{T}$ as the inflaton field, and by acknowledging the stabilizing role of the Im$\mathcal{T}$ and $\mathcal{S}$ fields. As we will see, the inflationary trajectory can be stabilized at $\mathcal{S} = Im\mathcal{T} = 0$, as the inflaton field slow-rolls.

The flatness of the potential is broken only by the superpotential $\mathcal{W}$, which is taken to be of the form:

$$\mathcal{W} = \mathcal{S}f(\mathcal{T}), \tag{5.3}$$

where $f(\mathcal{T})$ is a real holomorphic function. This superpotential has a number of good properties. First, both $\mathcal{W}$ and $D_{\mathcal{T}}\mathcal{W}$ vanish at $\mathcal{S} = 0$. As such, the only non-vanishing contribution to the scalar potential comes from $F_{\mathcal{S}} = D_{\mathcal{S}}\mathcal{W} = f(\mathcal{T})$. Along the inflaton's trajectory where $\mathrm{Im}\,\mathcal{T} = 0$, we obtain the amazingly simple potential:

$$V = |f(\mathcal{T})|^2. \tag{5.4}$$

Secondly, the reality condition implies that both $|f(\mathcal{T})|^2$, and $K((\mathcal{T}-\bar{\mathcal{T}})^2, \mathcal{S}\bar{\mathcal{S}})$ are invariant under $\mathcal{T} \to \bar{\mathcal{T}}$, making $\mathrm{Im}\,\mathcal{T} = 0$ an extremum.

For the stability of the model, we need to take into account two conditions. Firstly, the inflationary trajectory has to be resistant from tachyonic instabilities

$$m_b^2, m_s^2, m_c^2 \geq 0.$$

An other important condition for single field inflation, is that the stabilizer $\mathcal{S}$ and $\mathrm{Im}\mathcal{T}$ will not generate any perturbations of long-wavelength, and that means that they are not light fields. They need to be heavier than the Hubble parameter, so that their power spectrum is highly suppressed and they will remain fixed at their vacuum expectation values—namely, zero. The second condition is

$$m_b^2, m_s^2, m_c^2 \gtrsim H^2.$$

The inflaton is realized to be: $t = \sqrt{2}\mathrm{Re}\mathcal{T}$. From the latter, one can write the bosonic sector of the effective theory during inflation as (setting $M_{pl} = 1$):

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{2}\partial\phi\partial\phi - f^2\left(\phi/\sqrt{2}\right) \tag{5.5}$$

Now, we proceed with a specific model of chaotic inflation, which is the usual, simple $m^2\phi^2$ potential.

5.2 Chaotic Inflation in Quadratic Potential

We now move on to the part of calculation, which is the main task of the current thesis: our goal is to understand the KLR models and how the one-loop corrected Kähler potential affects the chaotic inflation models in supergravity. We begin from the standard supergravity model, and we consider the following [15, 16], with Kähler potential:

$$\mathcal{K} = -\frac{1}{2}(\mathcal{T}-\bar{\mathcal{T}})^2 + \mathcal{S}\bar{\mathcal{S}} - \zeta(\mathcal{S}\bar{\mathcal{S}})^2, \tag{5.6}$$

and the superpotential is taken to be of the simple form (to ensure that we will derive the $m^2\phi^2/2$ potential):

$$\mathcal{W} = m\mathcal{T}\mathcal{S}. \tag{5.7}$$

In order to ensure stability, the $\zeta(\mathcal{S}\bar{\mathcal{S}})^2$ term has been introduced to the Kähler potential. We will derive a condition for the ζ parameter that secures a single field-inflation model free from tachyonic instabilities. The Kähler matrix for the model [5.6] is found to be:

$$\hat{K} = \begin{pmatrix} \mathcal{K}_{\mathcal{T}\bar{\mathcal{T}}} & \mathcal{K}_{\mathcal{T}\bar{\mathcal{S}}} \\ \mathcal{K}_{\mathcal{S}\bar{\mathcal{T}}} & \mathcal{K}_{\mathcal{S}\bar{\mathcal{S}}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-4\zeta\mathcal{S}\bar{\mathcal{S}} \end{pmatrix} \tag{5.8}$$

The scalar potential as a function of the fields (without the one-loop correction) reads (setting $M_{pl} = 1$):

$$\begin{aligned} V(\mathcal{T},\bar{\mathcal{T}},\mathcal{S},\bar{\mathcal{S}}) &= e^{\mathcal{K}}\left(\mathcal{K}^{\mathcal{T}\bar{\mathcal{T}}}D_{\mathcal{T}}\mathcal{W}\ D_{\bar{\mathcal{T}}}\bar{\mathcal{W}} + \mathcal{K}^{\mathcal{S}\bar{\mathcal{S}}}D_{\mathcal{S}}\mathcal{W}\ D_{\bar{\mathcal{S}}}\bar{\mathcal{W}} - 3|\mathcal{W}|^2\right) \\ &= \exp\left[-\frac{1}{2}(\mathcal{T}-\bar{\mathcal{T}})^2 + \mathcal{S}\bar{\mathcal{S}} - \zeta(\mathcal{S}\bar{\mathcal{S}})^2\right]\times \\ m^2&\left[-3\mathcal{S}\bar{\mathcal{S}}\mathcal{T}\bar{\mathcal{T}} + \mathcal{S}\bar{\mathcal{S}}\left(1+(\mathcal{T}-\bar{\mathcal{T}})\bar{\mathcal{T}}\right)\left(1+\mathcal{T}(-\mathcal{T}+\bar{\mathcal{T}})\right) - \frac{\mathcal{T}\bar{\mathcal{T}}\left(1+\mathcal{S}\bar{\mathcal{S}}-2\zeta\mathcal{S}^2\bar{\mathcal{S}}^2\right)^2}{-1+4\zeta\mathcal{S}\bar{\mathcal{S}}}\right] \end{aligned} \tag{5.9}$$

The complex fields are splitted as:

$$\mathcal{T} = \frac{1}{\sqrt{2}}(t+ib)\ ,\quad \mathcal{S} = \frac{1}{\sqrt{2}}(s+ic) \tag{5.10}$$

Firstly, one confirms that we indeed are in the case of the quadratic potential, if one write down (5.9) in terms of t and set $\mathcal{S} = \mathrm{Im}\mathcal{T} = 0$. Then, we get:

$$V_{inf}(t) = \frac{m^2}{2}t^2. \tag{5.11}$$

Along the inflationary trajectory, we will study the VEV of $\mathcal{S},\bar{\mathcal{S}}$, and see if the stabilizers are...stabilized. One can find the VEV of the complex fields as:

$$\langle\mathcal{S}\rangle = \left.\frac{\partial V}{\partial\mathcal{S}}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0}\ ,\quad \langle\bar{\mathcal{S}}\rangle = \left.\frac{\partial V}{\partial\bar{\mathcal{S}}}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} \tag{5.12}$$

From (5.9), it is not difficult to see that the stabilizer does not receive VEV:

$$\langle\mathcal{S}\rangle = \langle\bar{\mathcal{S}}\rangle = 0 \tag{5.13}$$

For the $b = \mathrm{Im}\mathcal{T}$ field, we can find that:

$$\langle b\rangle = \frac{i}{\sqrt{2}}\left.\frac{\partial V}{\partial\mathcal{T}}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} - \frac{i}{\sqrt{2}}\left.\frac{\partial V}{\partial\bar{\mathcal{T}}}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} = 0. \tag{5.14}$$

Now, near the inflationary trajectory with $\mathcal{S} = 0$, one can compute the mass (squared) of b, according to:

$$m_b^2 = -\frac{1}{2}\left.\frac{\partial^2 V}{\partial\mathcal{T}^2}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} + \left.\frac{\partial^2 V}{\partial\mathcal{T}\partial\bar{\mathcal{T}}}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} - \frac{1}{2}\left.\frac{\partial^2 V}{\partial\bar{\mathcal{T}}^2}\right|_{\mathcal{S}=\bar{\mathcal{S}}=0} \tag{5.15}$$

$$= m^2(1+t^2) = 6H^2 + m^2 > H^2. \tag{5.16}$$

Thus, we can see that the imaginary part of $\mathcal{T}$ field, is stabilized at $b = 0$, since it is heavy enough to not experience any inflationary fluctuations. We should also note that trivially one finds that $m_t^2 = m^2$.

Now, let us calculate the mass of $\mathcal{S}$ (it turns out to be the same for s and c). The formula to compute the mass of interest is:

$$m_s^2 = \frac{1}{2}\frac{\partial^2 V}{\partial \mathcal{S}^2}\bigg|_{\mathcal{S}=\bar{\mathcal{S}}=0} + \frac{\partial^2 V}{\partial \mathcal{S}\partial \bar{\mathcal{S}}}\bigg|_{\mathcal{S}=\bar{\mathcal{S}}=0} + \frac{1}{2}\frac{\partial^2 V}{\partial \bar{\mathcal{S}}^2}\bigg|_{\mathcal{S}=\bar{\mathcal{S}}=0} \tag{5.17}$$

$$= m^2(1+2t^2\zeta) \;\Rightarrow m_s^2 = 12H^2\zeta + m^2. \tag{5.18}$$

From the latter expression for the mass squared of the stabilizer $\mathcal{S}$, we see that one has to constrain the values of ζ, for the sake of model stability. For example for $\zeta = 0$ the mass of $\mathcal{S}$ is reduced to inflaton mass, which is not allowed, since it will obtain large inflationary fluctuations. And here we see the role of the ζ parameter: it helps us regulate the mass of the stabilizer to be larger than H, in order to deal with single-field inflation. The condition can be found by:

$$m_s^2 \geq H^2 \quad \Rightarrow 12H^2\zeta + m^2 \geq H^2 \quad \Rightarrow 12\frac{m^2t^2}{6}\zeta + m^2 \geq \frac{m^2t^2}{6} \quad \Rightarrow \zeta \geq \frac{1}{12} - \frac{1}{2t^2} \tag{5.19}$$

Thus, in order to secure that the mass of s is stabilized for all values of the inflaton during inflation, we find that

$$\zeta \geq \frac{1}{12}. \tag{5.20}$$

Thus, we obtain a well-stabilized inflationary trajectory, where t is the inflaton, and the other fields are fixed at $b = s = c = 0$, and $V_{inf}(t) = m^2t^2/2$. The $\epsilon = \eta = 2/t^2$ parameters are the well established, discussed before. Finally, we should point out that the value of scalar perturbations gives us a measure of the mass scale of the inflaton, since:

$$A_s = \frac{1}{24\pi^2}\frac{1}{\epsilon_V}V_{inf}(t) = \frac{1}{24\pi^2}\frac{t^2}{2}m^2\frac{t^2}{2} = \frac{m^2t^4}{96\pi^2} \tag{5.21}$$

We also know that $A_s \sim 10^{-9}$ at $t_* = 4\sqrt{15}$. This implies that $m \sim 10^{-6}M_{pl}$.

Now, we move on to employ one-loop corrections.

5.3 Chaotic Inflation & One-Loop Corrections

Now, we introduce the one-loop correction to the Kähler potential. The correction will be employed as:

$$\Delta K_{tot} = \Delta K[\Lambda] - \Delta K[q\Lambda],$$

where for the chiral sector it is found:

$$\Delta K[\Lambda] = \frac{\Lambda^2}{16\pi^2}\log\det\hat{K} - \frac{1}{32\pi^2}\left[\mathrm{Tr}\left(\mathcal{M}_\phi^2\left(\log\frac{\mathcal{M}_\phi^2}{\Lambda^2} - 1\right)\right)\right].$$

We also mention that the parameter q comes from the Wilsonian approach to Renormalization theory, and one-loop corrections are associated with the high-momentum region $q\Lambda < |k| < \Lambda$. The $\mathcal{M}_\phi^2$ is the field-dependent mass matrix with the expression:

$$\mathcal{M}_\phi^2 \equiv \hat{K}^{-1/2}\,\hat{\bar{w}}\,\hat{K}^{-1T}\,\hat{w}\,\hat{K}^{-1/2} \tag{5.22}$$

Let us compute the correction for (5.6). We are able to find the superpotential matrix for the current model as:

$$\hat{w} = \begin{pmatrix} \mathcal{W}_{\mathcal{T}\mathcal{T}} & \mathcal{W}_{\mathcal{T}\mathcal{S}} \\ \mathcal{W}_{\mathcal{S}\mathcal{T}} & \mathcal{W}_{\mathcal{S}\mathcal{S}} \end{pmatrix} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} = \hat{\bar{w}} \tag{5.23}$$

The mass matrix is then computed:

$$\mathcal{M}_\phi^2 = \frac{m^2}{1 - 4\zeta\mathcal{S}\bar{\mathcal{S}}}\,\mathbb{I}_2 \tag{5.24}$$

Finally, we derive the one-loop correction for the supergravity model of chaotic inflation:

$$\Delta K_{tot} = (1-q^2)\frac{\Lambda^2}{16\pi^2}\log[1-4\zeta\mathcal{S}\bar{\mathcal{S}}] - \frac{1}{8\pi^2}\frac{m^2}{1-4\zeta\mathcal{S}\bar{\mathcal{S}}}\log q \tag{5.25}$$

Introducing the one-loop correction to the structure of Kähler potential and metric: $\mathcal{K} \to \mathcal{K} + \Delta K_{tot}$, we derive the scalar potential of interest (denoting as $\mathcal{V}$ the corrected one, and we do not set M_{pl} to one):

$$\tilde{\mathcal{V}} = \exp\left\{-\frac{1}{2}(\mathcal{T}-\bar{\mathcal{T}})^2 + \mathcal{S}\bar{\mathcal{S}} - \zeta(\mathcal{S}\bar{\mathcal{S}})^2 + (1-q^2)\frac{\Lambda^2}{16\pi^2}\log[1-4\zeta\mathcal{S}\bar{\mathcal{S}}] - \frac{1}{8\pi^2}\frac{m^2}{1-4\zeta\mathcal{S}\bar{\mathcal{S}}}\log q\right\}\times$$
$$\Bigg\{12m^2M_{\rm pl}^2\mathcal{S}\bar{\mathcal{S}}\mathcal{T}\bar{\mathcal{T}} - 4m^2\mathcal{S}\bar{\mathcal{S}}\left(M_{\rm pl}^2 + (\mathcal{T}-\bar{\mathcal{T}})\bar{\mathcal{T}}\right)\left(M_{\rm pl}^2 + \mathcal{T}(-\mathcal{T}+\bar{\mathcal{T}})\right)$$
$$+\frac{\mathcal{T}\bar{\mathcal{T}}\left(m(-1+4\mathcal{S}\bar{\mathcal{S}}\zeta)\left(4M_{\rm pl}^2\pi^2(1-4\mathcal{S}\bar{\mathcal{S}}\zeta) + \mathcal{S}\bar{\mathcal{S}}\left(L^2(-1+q^2)\zeta + 4\pi^2(1-6\mathcal{S}\bar{\mathcal{S}}\zeta + 8\mathcal{S}^2\bar{\mathcal{S}}^2\zeta^2)\right)\right) + 2m^3\mathcal{S}\bar{\mathcal{S}}\zeta\log[q]\right)^2}{\pi^2(-1+4\mathcal{S}\bar{\mathcal{S}}\zeta)\left((-1+4\mathcal{S}\bar{\mathcal{S}}\zeta)\left(-L^2(-1+q^2)\zeta + 4\pi^2(-1+4\mathcal{S}\bar{\mathcal{S}}\zeta)^3\right) - 2m^2\zeta(1+4\mathcal{S}\bar{\mathcal{S}}\zeta)\log[q]\right)}\Bigg\}$$

Figure 5.1: Scalar potential $\mathcal{V} = -\frac{1}{4}M_{pl}^{-4}\tilde{\mathcal{V}}$ with one-loop corrections

We should also stress that since we work in Wilsonian approach, one has to rescale the quantities

$$M_{pl} \to \frac{M_{pl}}{q}, \quad m \to \frac{m}{q}, \tag{5.26}$$

and from here on, we are working with the rescaled masses. We also use some ratios:

$$X = \frac{\Lambda}{M_{pl}}, \quad Y = \frac{m}{M_{pl}}$$

, where $0 < X < 1$ and $0 < Y < 1$. As before, firstly, one confirms that we indeed are in the case of the quadratic potential, if one write down the scalar potential in terms of t and set $\mathcal{S} = \text{Im}\mathcal{T} = 0$. Then, we get the modified inflaton potential:

$$\boxed{V_{inf}(t) = \frac{2M_{pl}^2\pi^2 q^{-2-\frac{Y^2}{8\pi^2}}Y^2}{4\pi^2 + M_{pl}^2(-1+q^2)X^2\zeta - \frac{2M_{pl}^2Y^2\zeta\log[q]}{q^2}}t^2} \tag{5.27}$$

The important note here is that the potential has mixed all of the parameters, even if ΔK does not contain any $\mathcal{T}, \bar{\mathcal{T}}$ dependence. We also want to study the VEV of $\mathcal{S}, \bar{\mathcal{S}}$, and see if they are fixed at zero. One can find the VEV of the complex fields as:

$$\langle \mathcal{S} \rangle = \left. \frac{\partial \mathcal{V}}{\partial \mathcal{S}} \right|_{\mathcal{S}=\bar{\mathcal{S}}=0}, \quad \langle \bar{\mathcal{S}} \rangle = \left. \frac{\partial \mathcal{V}}{\partial \bar{\mathcal{S}}} \right|_{\mathcal{S}=\bar{\mathcal{S}}=0} \tag{5.28}$$

From Mathematica, it turns out that indeed $\langle \mathcal{S} \rangle = 0$. The same holds also for the imaginary part of $\mathcal{T}$:

$$\langle b \rangle = \frac{i}{\sqrt{2}} \left. \frac{\partial \mathcal{V}}{\partial \mathcal{T}} \right|_{\mathcal{S}=\bar{\mathcal{S}}=0} - \frac{i}{\sqrt{2}} \left. \frac{\partial \mathcal{V}}{\partial \bar{\mathcal{T}}} \right|_{\mathcal{S}=\bar{\mathcal{S}}=0} = 0. \tag{5.29}$$

Now, masses are obtained from the formulas before. In order to secure the stability of the model, we will choose a value for the ζ parameter. Moreover, we are using $q = 0.9$ as a value for the model. One chooses q close to 1, in order to investigate what's going on... if we had chosen $q = 0$, it is as someone claims that knows the underlying theory. For q=0.9 we find a condition for the denominator D of (5.27):

$$\begin{aligned} D &= 4\pi^2 + \frac{1}{q^2} M_{pl}^2 (-1 + q^2) X^2 \zeta - \frac{1}{q^2} 2 M_{pl}^2 Y^2 \zeta \log[q] \\ &= 4\pi^2 - 0.2 M_{pl}^2 X^2 \zeta + 0.26 M_{pl}^2 Y^2 \zeta \\ &= 4\pi^2 - 0.2 M_{pl}^2 \zeta (X^2 - Y^2), \text{ for } X \gg Y \\ &\approx 4\pi^2 - 0.2 \Lambda^2 \zeta \end{aligned} \tag{5.30}$$

From the latter expression, one is able to obtain a (non-rigorous) bound on ζ, in order to have positive m_t^2. Thus, we write down:

$$\Lambda^2 \zeta < 20\pi^2. \tag{5.31}$$

So in order for $\Lambda \leq M_{pl}$ we have:

$$\zeta < \frac{20\pi^2}{M_{pl}^2} \tag{5.32}$$

In the following, we fix ζ to be: $\zeta = \frac{q^2}{M_{pl}^2}$, and also $q = 0.9$. The goal is to see if one gets a stable model and single-field inflation.

We evaluate the $\epsilon - \eta$ parameters as:

$$\epsilon_V = \frac{M_{pl}^2}{2} \left(\frac{V'_{inf}(t)}{V_{inf}(t)} \right)^2 = \frac{2 M_{pl}^2}{q^2 t^2} \tag{5.33}$$

$$\eta_V = M_{pl}^2 \frac{V''_{inf}(t)}{V_{inf}(t)} = \frac{2 M_{pl}^2}{q^2 t^2}. \tag{5.34}$$

Notice that the wilsonian parameter q appears in parameters of inflation, but they essentially obtain the same value, since the inflaton t takes values of M_{pl}/q and thus, the dependence on q cancels each other out. The q parameter essentially makes the values of ϕ even more super-planckian.

Now, the value of scalar perturbations will provide us a measure of the mass scale of the inflaton, as well as the order of magnitude for the UV cutoff Λ . Thus, we will obtain a measure for the ratios X and Y . We have:

$$A_s = \frac{q^{4 - \frac{Y^2}{8\pi^2}} Y^2}{24 M_{pl}^4 \left(4\pi^2 + q^2 \left(-1 + q^2\right) X^2 - 2 Y^2 \log[q]\right)} t^4 \tag{5.35}$$

We also know that $A_s \sim 10^{-9}$ at $t_* = 4\sqrt{15}(M_{pl}/q)$. This can be done if $X \sim 10^{-2}$ and $Y \sim 10^{-5}$. Now, we can take these values and plug them into the masses of the fields, to check for stability.

Leaving this complex work to mathematica, we obtain the conditions $m_b^2, m_s^2, m_c^2 \gtrsim H^2$, for $t > 1.57\, M_{pl}$, which quite interestingly leaves us with a stable , single-field inflationary scenario, with chiral corrections.

Finally, we have finished the part of loop corrections employed into the chaotic inflation.

Chapter 6

Summary and Conclusions

In this thesis, we explored fundamental aspects of the basic global $d = 4\,,\, N = 1$ Supersymmetry, for thechiral sector. We also studied how one can obtain the full one-loop quantum corrections to the Kähler potential, when one uses the background field method, for the theory under study. We also saw that the superpotential does not receive any corrections, which is in agreement with the non-renormalization theorem in SUSY.

We then moved on to study elements of Inflationary Cosmology, and we investigated how quantum mechanical treatment of the inflaton field induces curvature perturbations (scalars) and gravitational waves (tensors). We then proceed to how to construct inflationary scenarios embedded in $N = 1$ supergravity, using the Kallosh-Linde-Rube models. We saw the simple example of chaotic inflation (ϕ^2 potential), and warmed up ourselves for the main study of interest: chaotic inflation where the one-loop corrections of the Kahler potential, for the chiral sector, were employed. Interestingly enough (since it is definitely non-trivial), we found a single-field scenario, free from tachyonic instabilities or isocurvature perturbations from potential heavy masses of the stabilizer fields.

One question one can definitely propose is: How to trust corrections from a global SUSY, employed to a local SUSY model (Supergravity)? The answer is...we do not know. Further investigation needs to be done, in order for one to see how the full contributions to Kähler potential affect the inflationary models. For example one can employ the vector's contributions to the Kähler as well. Moreover, one is in position to compute the contribution from gravity in order to realize the full matter + gravity corrections, and then check for the stability of the complete model, and if they really address the so called eta problem. However, employing only the chiral components and obtaining a non-disastrous model, one is confident this direction is worth exploring further, by including contributions from matter (vector components) and gravity to the Kähler Potential.

Appendix A

Perturbation Theory in Comoving Gauge

Here, we make more explicit derivations in perturbation theory in comoving gauge.

In comoving gauge, we remind that inflaton is taken to its background value $\phi = \bar{\phi}(t)$, and thus we study the following perturbations:

$$h_{ij} = a^2 \left(1 + 2\zeta(t, \mathbf{x})\right) \delta_{ij} \,, \quad N = 1 + \alpha_p(t, \mathbf{x}), \quad N_i = \partial_i \beta_p(t, \mathbf{x}). \tag{A.1}$$

The constraint equations, for comoving gauge are reduced to:

$$R^{(3)} - 2V - \frac{1}{N^2}\left[E^{ij}E_{ij} - E^2 + \dot{\bar{\phi}}^2\right] = 0 \tag{A.2a}$$

$$\nabla_i \left[N^{-1}\left(E^i_j - E\delta^i_j\right)\right] = 0 \tag{A.2b}$$

We begin by computing the three-dimensional Ricci scalar. Firstly, we need to calculate the Christoffel symbols with spatial indices:

$$\Gamma^k_{ij} = \frac{h^{kl}}{2}\left[\partial_j h_{il} + \partial_i h_{jl} - \partial_l h_{ij}\right] \tag{A.3}$$

We also need the inverse induced metric, which is found to be:

$$h^{kl} = \frac{1}{a^2\left(1 + 2\zeta\right)}\delta^{kl} \approx \frac{1}{a^2}\left(1 - 2\zeta\right)\delta^{kl} \tag{A.4}$$

To linear order in curvature perturbation ζ, Christoffel symbols read:

$$\Gamma^k_{ij} = (\partial_i\zeta)\delta^k_j + (\partial_j\zeta)\delta^k_i - (\partial^k\zeta)\delta_{ij} \tag{A.5}$$

The intrinsic curvature then is found to be:

$$\begin{aligned} R^{(3)}_{ij} &= \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \mathcal{O}(\Gamma^2) \\ &= 2\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta - 3\partial_j\partial_i\zeta - \partial_j\partial_i\zeta + \partial_j\partial_i\zeta \\ &= -\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta \end{aligned} \tag{A.6}$$

Thus, the 3D Ricci scalar is obtained from:

$$\begin{aligned} R^{(3)} &= h^{ij}R^{(3)}_{ij} = -\frac{1}{a^2}\left(1 - 2\zeta\right)\delta^{ij}\left(\partial_i\partial_j\zeta + \delta_{ij}\partial^2\zeta\right) \\ &= -\frac{1}{a^2}\partial^2\zeta - \frac{\delta^{ij}\delta_{ij}}{a^2}\partial^2\zeta \\ &= -\frac{4}{a^2}\partial^2\zeta \end{aligned} \tag{A.7}$$

The time derivative of 3D spatial metric is:

$$\dot{h}_{ij} = 2a\dot{a}(1+2\zeta)\delta_{ij} + 2a^2\dot{\zeta}\delta_{ij} = 2a^2H(1+2\zeta)\delta_{ij} + 2a^2\dot{\zeta}\delta_{ij} \tag{A.8}$$

The time evolution of the metric allows us to determine the quantities associated with extrinsic curvature, at first order:

$$E_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right) = a^2H(1+2\zeta)\delta_{ij} + a^2\dot{\zeta}\delta_{ij} - \partial_i\partial_j\beta_p \tag{A.9}$$

$$E^{ij} = h^{ik}h^{jl}E_{kl} \approx \frac{H}{a^2}(1-2\zeta)\delta^{ij} + \frac{1}{a^2}\dot{\zeta}\delta^{ij} - \frac{1}{a^4}\partial^i\partial^j\beta_p \tag{A.10}$$

$$E = h^{ij}E_{ij} = 3H + 3\dot{\zeta} - \frac{1}{a^2}\partial^2\beta_p \tag{A.11}$$

$$E_{ij}E^{ij} = 3H^2 + 6H\dot{\zeta} - \frac{2H}{a^2}\partial^2\beta_p \tag{A.12}$$

$$E^2 = 9H^2 + 18H\dot{\zeta} - \frac{6H}{a^2}\partial^2\beta_p \tag{A.13}$$

$$E^i_j = H\delta^i_j + \dot{\zeta}\delta^i_j - \frac{1}{a^2}\partial^i\partial_j\beta_p \tag{A.14}$$

The constraint equations, now obtain the following form:

$$-\frac{4}{a^2}\partial^2\zeta - 2V + 6H^2 + 12H\dot{\zeta} - 12H^2\alpha_p - \frac{4H}{a^2}\partial^2\beta_p - (1-2\alpha_p)\dot{\bar{\phi}}^2 = 0 \tag{A.15a}$$

$$2\partial_j\dot{\zeta} - 2H\partial_j\alpha_p + 6H\partial_j\zeta - 6H\partial_j\zeta = 0 \tag{A.15b}$$

From the second constraint, it is easy to check that:

$$\alpha_p = \frac{\dot{\zeta}}{H} \tag{A.16}$$

Plugging into the first constraint, we obtain:

$$-\frac{4}{a^2}\partial^2\zeta - 6H^2 + \dot{\bar{\phi}}^2 + 6H^2 - \frac{4H}{a^2}\partial^2\beta_p - (1-2\frac{\dot{\zeta}}{H})\dot{\bar{\phi}}^2 = 0$$

$$\Rightarrow \partial^2\beta_p = -\frac{\partial^2\zeta}{H} + a^2\frac{(\dot{\bar{\phi}})^2}{2H^2}\dot{\zeta} \tag{A.17}$$

Appendix B

Chaotic Inflation: Mathematica Code

Here, we list the mathematica code, for the standard chaotic inflation and for the case where one-loop corrections are employed.

B.1 Chaotic Inflation in Quadratic Potential

```
(*CHAOTIC INFLATION*)
(*Define the Khler potential K*)
K = -(1/2)*(T - Tb)^2 + S*Sb - z*(S*Sb)^2 ; Mpl = 1;
(*Define the derivatives of Kahler*)
Kt = D[K, T];
Ktb = D[K, Tb];
Ks = D[K, S];
Ksb = D[K, Sb];
Kttb = D[Kt, Tb];
Ktsb = D[Kt, Sb];
Kstb = D[Ks, Tb];
Kssb = D[Ks, Sb];
(*Kahler Matrix*)
Kijb = ( {
    {Kttb, Ktsb},
    {Kstb, Kssb}
   } );
(*Kahler Matrix Inverse*)
Kinverse = Inverse[Kijb];
(*Kahler K^(-1/2) Matrix for no TS terms*)
K12ijb = ( {
    {1/Sqrt[Kttb], 0},
    {0, 1/Sqrt[Kssb]}
   } );
(*Define the superpotential W and its conjugate barW*)
W = m*T*S;
barW = m*Tb*Sb;
(*The derivatives of W and its conjugate barW*)
Wt = D[W, T]; Wtt = D[Wt, T]; Wts = D[Wt, S];
Ws = D[W, S]; Wst = D[Ws, T]; Wss = D[Ws, S];
barWt = D[barW, Tb] ; barWtt = D[barWt, Tb] ; barWts = D[barWt, Sb];
barWs = D[barW, Sb]; barWst = D[barWs, Tb];  barWss = D[barWs, Sb];
(*Superpotential Matrix*)
Wij = ( {
    {Wtt, Wts},
    {Wst, Wss}
```

```
   } );
(*Conjugate Matrix*)
barWij = ( {
    {barWtt, barWts},
    {barWst, barWss}
   } );
(*Covariant Derivatives of W and barW*)
dTW = D[W, T] + (Kt/Mpl^2)*W;
dSW = D[W, S] + (Ks/Mpl^2)*W;
dTbarW = D[barW, Tb] + (Ktb/Mpl^2)*barW;
dSbarW = D[barW, Sb] + (Ksb/Mpl^2)*barW;
(*Compute the scalar potential V*)
V = Exp[K/
     Mpl^2]*(((Kttb)^(-1))*dTW*dTbarW + ((Kssb)^(-1))*dSW*dSbarW -
     3*W*barW/Mpl^2);
(*Simplify the expression*)
V = Simplify[V];
Print["Potential as a function of T,Tb,S,Sb (without DK): V=" V]

Potential as a function of T,Tb,S,Sb (without DK): V= e^(S Sb-1/2 (T-Tb)^2-S^2 Sb^2 z) m^2
(-3 S Sb T Tb+S Sb (1+(T-Tb) Tb) (1+T (-T+Tb))-(T Tb (1+S Sb-2 S^2 Sb^2 z)^2)/(-1+4 S Sb z))

V1[T_, Tb_, S_, Sb_] = V;
VT[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], T];
VTT[T_, Tb_, S_, Sb_] = D[VT[T, Tb, S, Sb], T];
VTTb[T_, Tb_, S_, Sb_] = D[VT[T, Tb, S, Sb], Tb];
VTb[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], Tb];
VTbTb[T_, Tb_, S_, Sb_] = D[VTb[T, Tb, S, Sb], Tb];
Vs[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], S];
Vss[T_, Tb_, S_, Sb_] = D[Vs[T, Tb, S, Sb], S];
Vssb[T_, Tb_, S_, Sb_] = D[Vs[T, Tb, S, Sb], Sb];
Vsb[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], Sb];
Vsbsb[T_, Tb_, S_, Sb_] = D[Vsb[T, Tb, S, Sb], Sb];
Vb[T_, Tb_, S_,
  Sb_] = (I/Sqrt[2])*VT[T, Tb, S, Sb] - (I/Sqrt[2])*VTb[T, Tb, S, Sb];
Vt[T_, Tb_, S_,
  Sb_] = (1/Sqrt[2])*VT[T, Tb, S, Sb] + (1/Sqrt[2])*VTb[T, Tb, S, Sb];
Print["The VEV of ImT is: <b>=", Vb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]
Print["The VEV of S is: <S>=", Vs[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]
Print["The VEV of Sb is: <Sb>=", Vsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]

The VEV of ImT is: <b>=0
The VEV of S is: <S>=0
The VEV of Sb is: <Sb>=0

mt2 = (1/2)*VTT[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   VTTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] + (1/2)*
    VTbTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
mb2 = -(1/2)*VTT[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   VTTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] - (1/2)*
    VTbTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
ms2 = (1/2)*Vss[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   Vssb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] + (1/2)*
```

```
    Vsbsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
mc2 = -(1/2)*Vss[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   Vssb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] - (1/2)*
    Vsbsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
Print["The mass of inflaton is: mt^2=", mt2]
Print["The mass of ImT is: mb^2=", mb2]
Print["The mass of ReS is: ms^2=", ms2]
Print["The mass of ImS is: mc^2=", mc2]

The mass of inflaton is: mt^2=m^2
The mass of ImT is: mb^2=m^2+m^2 t^2
The mass of ReS is: ms^2=(m^2 t^2)/2+m^2 (1-t^2/2+2 t^2 z)
The mass of ImS is: mc^2=(m^2 t^2)/2+m^2 (1-t^2/2+2 t^2 z)

Vinflaton[t_] = V1[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
Vinflatont[t_] = D[Vinflaton[t], t] ;
Vinflatontt[t_] = D[Vinflatont[t], t] ;
epsilon[t_] = (1/2)*(Vinflatont[t]/Vinflaton[t])^2 ;
eta[t_] = Vinflatontt[t]/Vinflaton[t]; eps = epsilon[t]; et = eta[t];
Print[ "The corrected phi^2 inflaton potential is V(t)=", Vinflaton[t]]
Print[ "Epsilon parameter is epsilon=", eps]
Print[ "Eta parameter is eta=", et]

The corrected phi^2 inflaton potential is V(t)=(m^2 t^2)/2
Epsilon parameter is epsilon=2/t^2
Eta parameter is eta=2/t^2

tstar = Solve[x^2/4 == 60, x, Reals] (*60 e-folds*)

{{x -> -4 Sqrt[15]}, {x -> 4 Sqrt[15]}}

x = 4* Sqrt[15]; r = 16*epsilon[x] ; ns = 1 - 6*epsilon[x] + 2*eta[x];
Print[ "Tensor to scalar ratio:", r]
Print[ "spectral tilt:", ns]

Tensor to scalar ratio:2/15
spectral tilt:29/30

(*Conditions for stability*)
Vinf = Vinflaton[t];
Reduce[ms2 > Vinf/(3*((Mpl)^2)), t, Reals]

(z >= 1/12 && (m < 0 || m > 0))

(*Scalar Perturbations*)
As[t_] = (1/(24*Pi^2))*(1/(epsilon[t]))*(Vinflaton[t]/Mpl^4);
As[4* Sqrt[15]]
(600 m^2)/ Pi^2
(*Mass parameter*)
Solve[600 m^2/Pi^2 == 2.196*10^(-9), m, Reals]
{{m -> -6.01022*10^-6}, {m -> 6.01022*10^-6}}
```

Listing B.1: Mathematica Code for chaotic inflation.

B.2 Chaotic Inflation & One-loop Corrections

```
(*CHAOTIC INFLATION & ONE-LOOP CORRECTIONS*)
(*Define the Khler potential K*)
K = -(1/2)*(T - Tb)^2 + S*Sb - z*(S*Sb)^2 ;
(*Define the derivatives of Kahler*)
Kt = D[K, T];
Ktb = D[K, Tb];
Ks = D[K, S];
Ksb = D[K, Sb];
Kttb = D[Kt, Tb];
Ktsb = D[Kt, Sb];
Kstb = D[Ks, Tb];
Kssb = D[Ks, Sb];
(*Kahler Matrix*)
Kijb = ( {
    {Kttb, Ktsb},
    {Kstb, Kssb}
   } );
(*Kahler Matrix Inverse*)
Kinverse = Inverse[Kijb];
(*Kahler K^(-1/2) Matrix for no TS terms*)
K12ijb = ( {
    {1/Sqrt[Kttb], 0},
    {0, 1/Sqrt[Kssb]}
   } );
(*Define the superpotential W and its conjugate barW*)
W = m*T*S;
barW = m*Tb*Sb;
(*The derivatives of W and its conjugate barW*)
Wt = D[W, T]; Wtt = D[Wt, T]; Wts = D[Wt, S];
Ws = D[W, S]; Wst = D[Ws, T]; Wss = D[Ws, S];
barWt = D[barW, Tb] ; barWtt = D[barWt, Tb] ; barWts = D[barWt, Sb];
barWs = D[barW, Sb]; barWst = D[barWs, Tb];  barWss = D[barWs, Sb];
(*Superpotential Matrix*)
Wij = ( {
    {Wtt, Wts},
    {Wst, Wss}
   } );
(*Conjugate Matrix*)
barWij = ( {
    {barWtt, barWts},
    {barWst, barWss}
   } );
(*Covariant Derivatives of W and barW*)
dTW = D[W, T] + (Kt/Mpl^2)*W;
dSW = D[W, S] + (Ks/Mpl^2)*W;
dTbarW = D[barW, Tb] + (Ktb/Mpl^2)*barW;
dSbarW = D[barW, Sb] + (Ksb/Mpl^2)*barW;
(*Compute the scalar potential V*)
V = Exp[K/
     Mpl^2]*(((Kttb)^(-1))*dTW*dTbarW + ((Kssb)^(-1))*dSW*dSbarW -
     3*W*barW/Mpl^2);
```

```
(*Simplify the expression*)
V = Simplify[V];
Print["Potential as a function of T,Tb,S,Sb (without DK): V=" V]
(*One loop Correction*)
Mphi2 = K12ijb . barWij . Kinverse . Wij . K12ijb;
deltaK1 = (L^2/(16*Pi^2)) *Log[Det[Kijb]] - (1/(32*Pi^2))*
    Tr[Mphi2 . (MatrixLog[Mphi2/L^2] - 1)] ;
deltaK2 = ((q*L)^2/(16*Pi^2)) *Log[Det[Kijb]]  - (1/(32*Pi^2))*
    Tr[Mphi2 . (MatrixLog[Mphi2/(q*L)^2] - 1)] ;
deltaK = deltaK1 - deltaK2;
Print["One Loop Correction is DK=", deltaK]

deltaK = (1 - q^2)*(L^2/(16*Pi^2))*
    Log[1 - 4*S*Sb*z] - (1/(8*Pi^2))*(m^2/(1 - 4*S*Sb*z))*Log[q];
deltaKt = D[deltaK, T]; deltaKttb = D[DeltaKt, Tb]; deltaKs =
 D[deltaK, S]; deltaKsb = D[deltaK, Sb]; deltaKssb =
 D[deltaKs, Sb]; deltaKtb = D[deltaK, Tb];
K = K + deltaK; Kt = Kt + deltaKt; Ks = Ks + deltaKs; Kttb =
 Kttb + deltaKttb ; Kssb = Kssb + deltaKssb;
Ktb = Ktb + deltaKtb; Ksb = Ksb + deltaKsb;
dTW = D[W, T] + (Kt/Mpl^2)*W;
dSW = D[W, S] + (Ks/Mpl^2)*W;
dTbarW = D[barW, Tb] + (Ktb/Mpl^2)*barW;
dSbarW = D[barW, Sb] + (Ksb/Mpl^2)*barW;

V = Exp[K/
     Mpl^2]*(((Kttb)^(-1))*dTW*dTbarW + ((Kssb)^(-1))*dSW*dSbarW -
     3*W*barW/Mpl^2);
V = Simplify[V];
Print["One-Loop Corrected Potential as a function of T,Tb,S,Sb: V=" V]
Mpl = mpl/q; m = m1/q; m1 = Y*mpl; L = X*mpl;
V1[T_, Tb_, S_, Sb_] = V;
VT[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], T];
VTT[T_, Tb_, S_, Sb_] = D[VT[T, Tb, S, Sb], T];
VTTb[T_, Tb_, S_, Sb_] = D[VT[T, Tb, S, Sb], Tb];
VTb[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], Tb];
VTbTb[T_, Tb_, S_, Sb_] = D[VTb[T, Tb, S, Sb], Tb];
Vs[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], S];
Vss[T_, Tb_, S_, Sb_] = D[Vs[T, Tb, S, Sb], S];
Vssb[T_, Tb_, S_, Sb_] = D[Vs[T, Tb, S, Sb], Sb];
Vsb[T_, Tb_, S_, Sb_] = D[V1[T, Tb, S, Sb], Sb];
Vsbsb[T_, Tb_, S_, Sb_] = D[Vsb[T, Tb, S, Sb], Sb];
Vb[T_, Tb_, S_,
  Sb_] = (I/Sqrt[2])*VT[T, Tb, S, Sb] - (I/Sqrt[2])*VTb[T, Tb, S, Sb];
Vt[T_, Tb_, S_,
  Sb_] = (1/Sqrt[2])*VT[T, Tb, S, Sb] + (1/Sqrt[2])*VTb[T, Tb, S, Sb];
Print["The VEV of ImT is: <b>=", Vb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]
Print["The VEV of S is: <S>=", Vs[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]
Print["The VEV of Sb is: <Sb>=", Vsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0]]

z = q^2/mpl^2;
mt2 = (1/2)*VTT[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   VTTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] + (1/2)*
```

```
    VTbTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
mb2 = -(1/2)*VTT[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   VTTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] - (1/2)*
    VTbTb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
ms2 = (1/2)*Vss[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   Vssb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] + (1/2)*
    Vsbsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
mc2 = -(1/2)*Vss[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] +
   Vssb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0] - (1/2)*
    Vsbsb[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];

Print["The mass of inflaton is: mt^2=", mt2]
Print["The mass of ImT is: mb^2=", mb2]
Print["The mass of ReS is: ms^2=", ms2]
Print["The mass of ImS is: mc^2=", mc2]

(*Cosmology*)
Vinflaton[t_] = V1[(t/Sqrt[2]), (t/Sqrt[2]), 0, 0];
Vinflatont[t_] = D[Vinflaton[t], t] ;
Vinflatontt[t_] = D[Vinflatont[t], t] ;
epsilon[t_] = ((Mpl^2)/2)*(Vinflatont[t]/Vinflaton[t])^2 ;
eta[t_] = (Mpl^2) Vinflatontt[t]/Vinflaton[t];
Print[ "The corrected phi^2 inflaton potential is V(t)=", Vinflaton[t]]
Print[ "Epsilon parameter is epsilon=", epsilon[t]]
Print[ "Eta parameter is eta=", eta[t]]

X = 10^(-2); Y = 10^(-5); q = 0.9; z = q^2/mpl^2; Vinf = Vinflaton[t]
Reduce[mt2 < Vinf/(((Mpl)^2)), t, Reals]
Reduce[mb2 > Vinf/(((Mpl)^2)), t, Reals]
Reduce[ms2 > Vinf/(((Mpl)^2)), t, Reals]
Reduce[mc2 > Vinf/(((Mpl)^2)), t, Reals]

(*Scalar Amplitude*)
As [t_]= (1/(24*Pi^2))*(1/(epsilon[t]))*(Vinflaton[t]/Mpl^4)
X = 10^(-2); Y = 10^(-5); tstar = 4* Sqrt[15]*Mpl;
Print["Value of As", As[tstar]].
```

Listing B.2: Mathematica Code for chaotic inflation with one-loop corrected Kahler Potential

Bibliography

[1] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton, USA: Univ. Pr. (1992) 259 p

[2] Michael E. Peskin and Daniel V. Schroeder. An Introduction to quantum field theory. Addison-Wesley, Reading, USA, 1995.

[3] Tom Lancaster and Stephen J Blundell. Quantum field theory for the gifted amateur. Oxford University Press, Oxford, 2014

[4] Sean M. Carroll. Spacetime and Geometry: An Introduction to General Relativity. Cam- bridge University Press, 7 2019.

[5] Introduction to Supersymmetry PHYS-F-417 - Notes - Riccardo Argurio. (Université Libre de Bruxelles).

[6] Matteo Bertolini. Lectures on supersymmetry, 2022

[7] Adel Bilal. Introduction to supersymmetry, 2001.

[8] Andrea Brignole, One-loop Kähler potential in non-renormalizable theories, Nuclear Physics B, Volume 579, Issues 1–2, 2000, Pages 101-116, ISSN 0550-3213.

[9] Daniel Baumann. Tasi lectures on primordial cosmology, 2018

[10] Daniel Baumann. Tasi lectures on inflation, 2012

[11] Daniel Baumann. Cosmological inflation: Theory and observations. Advanced Science

[12] D. Baumann , The Physics of Inflation. www.damtp.cam.ac.uk/user/db275/Inflation.pdf. Letters, 2(2):105–120, June 2009.

[13] J. M. Maldacena,Non-Gaussian features of primordial fluctuations in single field inflationary models,JHEP05(2003) 013, [astro-ph/0210603].

[14] S. Weinberg,Cosmology. Oxford University Press, 2008.

[15] R. Kallosh and A. Linde, "New models of chaotic inflation in supergravity," JCAP 1011, 011 (2010) [arXiv:1008.3375 [hep-th]].

[16] R. Kallosh, A. Linde and T. Rube, "General inflaton potentials in supergravity", Phys. Rev. D 83, 043507 (2011) [arXiv:1011.5945 [hep-th]].

[17] Renata Kallosh, Andrei Linde, Keith A. Olive, and Tomas Rube. Chaotic inflation and supersymmetry breaking. Physical Review D, 84(8), October 2011.

[18] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. Phys. Rev. D, 23:347–356, 1981.