ΑΛΕΞΑΝΔΡΟΣ ΣΑΠΛΑΟΥΡΑΣ

Lévy διαδικασίες και G-Lévy διαδικασίες

Διπλωματική εργασία για το Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
Μαθηματική Προτυποποίηση στις Σύγχρονες Τεχνολογίες και την Οικονομία

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝ῞Ν ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΑΘΗΝΑ, 2012

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Lévy processes and G-Lévy processes

Thesis for the Interdepartmental Postgraduate Course Programme
Mathematical Modelling in Modern Technologies and Financial Engineering

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Περίληψη

Στην λήψη κάθε απόφασης υπάρχει αβεβαιότητα ως προς την έκβαση του τελικού αποτελέσματος. Ένας πιθανός τρόπος για την το κατά δυνατόν πιο ‘ασφαλή’ λήψη απόφασης είναι να λάβουμε υπ΄ όψιν περισσότερα του ενός μέτρα πιθανότητας. Μέχρι στιγμής στην βιβλιογραφία, το σύνολο των μέτρων πιθανότητας που λαμβάναμε υπ΄ όψιν αποτελούνταν από απόλυτα συνεχή μέτρα ως προς ένα μέτρο αναφοράς. Αυτό σημαίνει ότι έχουμε πλήρη γνώση των ενδεχομένων που *αποκλείεται* να συμβούν, το οποίο σαν υπόθεση έρχεται κατά κάποιο τρόπο σε σύγκρουση με την ύπαρξη της αβεβαιότητας.

Σε μοντέλα διάχυσης, από το θεώρημα Girsanov στο πλαίσιο της προηγούμενης παραγράφου, μπορούμε να αντιμετωπίσουμε την αβεβαιότητα ως προς τον μέσο και όχι ως προς την τυπική απόκλιση (βλ. [5, 6]). Επομένως, ενδιαφέροντα προβλήματα δεν μπορούν να αντιμετωπισθούν με την συγκεκριμένη θεωρία. Ένα χαρακτηριστικό παράδειγμα δίνεται από την περιοχή των χρηματοοικονομικών. Η τιμή ενός παραγώγου σε μια μετοχή εξαρτάται σε μεγάλο βαθμό από την μεταβλητότητα που παρουσιάζει η μετοχή. Στο πιο απλό μοντέλο, αυτό των Black-Scholes η τιμή της παραμέτρου σ θεωρείται σταθερή. Εάν επιθυμούσαμε να αλλάξουμε μια νέα ΣΔΕ με διαφορετική παράμετρο σ τότε τα μέτρα υπό τα οποία θα αναζητούμε την λύση θα είναι κάθετα.

Το κατάλληλο πλαίσιο για να αντιμετωπίσουμε προβλήματα αυτού του τύπου αποτελεούν οι χώροι υπογραμμικής μέσης τιμής. Σε αυτούς μπορεί να γενικευθεί η κίνηση Brown έτσι ώστε η τετραγωνική κύμανση να είναι πλέον μια στοχαστική διαδικασία η οποία περιορίζεται εντός του επιθυμητού (ανάλογα με την επιλογή του τελεστή G) διαστήματος.

Σκοπός της εργασίας αυτής είναι να μελετηθούν οι διαδικασίες Lévy στους χώρους υπογραμμικής μέσης τιμής και να συγκριθούν τα αποτελέσματα με αυτά της κλασσικής περίπτωσης.

Abstract

Almost all choice situations exhibit ambiguity. One possible way to model decisions under ambiquity is to take into account more than one probability measures. Until now, however, most literature essentially concentrates on the modelling of multiple priors with respect to some reference measure. The reference measure plays the crucial role of fixing the sets of measure zero. That means that the decision maker can be certain about events that are regarded impossible to happen. In other words, an endogenous characteristic of the model, certainty regarding "impossible" events, conflicts with the acceptance of ambiguity presence in reality.

The assumption of existing a reference probability measure mathematically can be interpreted as all the family measures be at least (locally) absolutely continuous with respect to the reference measure. In diffusion models, by Girsanov's theorem that setting can only lead in mean uncertainty of the considered stochastic process (see [5, 6]). In other words, many interesting applications with uncertainty in the standard deviation cannot be treated in an effective theoretical way in this framework. A characteristic example comes from Finance. The price of an option written on a risky stock heavily depends on the underlying volatility. Also the value of a portfolio consisting of risky positions is closely connected with the volatility levels of the corresponding assets. Considering the simplest case for the stock price, Black-Scholes model

$$dS_t = rS_t dt + \sigma s S_t dB_t$$

where $B.$ is the classical Brownian motion, we are forced to work with constant volatility. Otherwise, if we wanted volatility to take values between a given set, let $[\underline{\sigma}^2, \overline{\sigma}^2]$, our choice would be to model S using the family of processes $S_t^\sigma, \sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]$ which leads to a family of singular measures.

The suitable framework to handle such kind of problems is that of Sublinear Expectation Spaces. Roughly speaking, a nonlinear expectation $\hat{\mathbb{E}}$ is a monotone constant preserving functional defined on a linear space of random variables. To become sublinear we need $\hat{\mathbb{E}}$ to satisfy positive homogeneity and subadditivity. In these spaces, basic notions of classical probability theory, such us distribution, independence etc, can be defined. Going one step further, Peng ([19, 20, 21, 22]), the generalisation of Normal distribution can be obtained -which presents uncertainty in variance. Using the new G-Normal distribution (where the letter G arises because of a given operator G which plays crucial role in construction) a new type of "G-Brownian motion" can be introduced. The interesting fact is that quadratic variation of G-Brownian motion is not a deterministic function, but a stochastic process. Using suitable operator G, ($G(y) = \frac{1}{2}\sup_{\theta\in[\underline{\sigma},\overline{\sigma}]}\theta y$), it can be proved that

$$\underline{\sigma}^2 \le \frac{d\langle B\rangle_t}{dt} \le \overline{\sigma}^2,$$

so the aforemntioned problem could be treated under a generalised Black-Scholes model

$$dS_t = rS_t dt + S_t dB_t$$

where the canonical process $B.$ is a G-Brownian motion.

The main target of the current thesis is to study the Lévy Stochastic Processes [**?**]in Sublinear Expectation Spaces. The thesis is organised as follows. In the first part Lévy processes in the classical case and the basic results are presented. In the second part, Chapter 2, Sublinear Expectation Spaces are introduced. In Chapter 3 the existence and

characterisations of maximal and Normally distributed r.v. are given. Chapter 4 is devoted to the basic properties of G-Brownian motion and Chapter 5 to G-Lévy processes.

Acknowledgements

In the next few lines I would like to thank the people who devoted part of their time so that this thesis to be accomplished. First of all I would like to thank Prof. Dr. Antonis Papapantoleon. His serene comments and explanations made clear any obscure arised. Moreover, the freedom he provided me in exploring my own ideas enhanced my confidence and increased the pleasure of engaging the whole procedure. I would like also to thank Prof. Dr. I. Spiliotis and Prof. Dr. M. Loulakis since they have been real teachers. Prof. Dr. I. Spiliotis was the teacher who introduced me in Stochastic Calculus and it was him that transformed something incomprehensible and unsettling into clear and charming and Prof. Dr. Loulakis was the teacher by whose intuitive examples cast light in the remaining dark points.

I would like to thank the group of Financial Mathematics in TU Berlin for their warm hospitality and the interested discussions we had.

I would like to express my deepest gratitude to my family, and especially my parents whose constant support, sucrifices and patience enabled me to achieve my targets.

This work was partially supported by the IKYDA 2012-2013 program of the German Academic Exchange Service (DAAD) and the Greek State Scholarships Foundation (I.K.Y.). In that point I would like to thank once again Prof. Dr. A. Papapantoleon and Prof. Dr. M. Loulakis for the trust they put on me.

Part I

Lévy processes

Chapter 1

Lévy Processes

1.1 The space $\mathbb{D}$

In the current section we will briefly present the space of càdlàg functions defined on $[0,1]$, denoted by $\mathbb{D}$, and we will give the basic criteria for compactness and tightness in $\mathbb{D}$. It will turn out that the space $\mathbb{D}$ is the suitable path space for Lévy processes. Another incentive to engage with the space $\mathbb{D}$ is the ingenious Skorokhod's perception of a suitable metric under which $\mathbb{D}$ becomes Polish, i.e. complete and separable metric space. We essentially follow [4, Chapter 3], which we strongly encourage to be read.

Definition 1.1. *Let $\omega : [0,1] \to \mathbb{R}$. The function ω is called* càdlàg *(French "continue à droite, limite à gauche", i.e. continuous from right, limit from left) if*

$$\omega(t+) := \lim_{s \downarrow t} \omega(s) = \omega(t) \ \text{ and } \exists\ \omega(t-) := \lim_{s \uparrow t} \omega(s)$$

In the following we will need a modulus of continuity suitable for the current space. Firstly, we need a convenient notation. For $\omega \in \mathbb{D}$ and $T \subset [0,1]$ we define

$$w_\omega(T) := \sup_{t_1, t_2 \in T} |\omega(t_1) - \omega(t_2)|$$

and

$$w_\omega(\delta) := \sup_{0 \le t \le 1-\delta} w_\omega\big([t, t+\delta]\big).$$

The following lemma generalises the notion of uniformity of a continuous function to our case:

Lemma 1.1. *Let $\omega \in \mathbb{D}$ and $\varepsilon > 0$, then there exist finite points, let $0 = t_0 < t_1 < \ldots < t_r = 1$ such that*

$$w_\omega\Big([t_{i-1}, t_i)\Big) < \varepsilon, \quad \forall i = 1, 2, \ldots, r. \tag{1.1}$$

Proof. Let

$$\tau = \sup\{t \in [0,1] \ : \ \text{exists finite partition of } [0,t) \text{ s.t. } 1.1 \text{ is satisfied } \},$$

then $\tau > 0$ from right continuity. Since $\omega(\tau-)$ exists, there is a finite partition of $[0,\tau)$ satisfying 1.1 and therefore we can extend the given partition by adding one point greater than τ. Thus, $\tau = 1$ since again $\omega(\tau) = \omega(\tau+)$. □

Remark 1.1. *We have to underline that the chosen intervals are right open. Otherwise, we would not be able to exclude jumps greater than ε in discontinuity points. Reversing the argument, the lemma fails for a left continuous with right limits function, but can be valid if we choose left open intervals.*

It is immediate that for a given positive number there can be only finitely many points with discontinuities (jumps) exceeding it. Moreover, it is immediate to obtain that ω has at most countably many discontinuities, let D the set of all that points. Indeed, let D_I the set of points that have discontinuities of height $h \in I$. Then

$$D = D_{[1,\infty)} \bigcup_{n\in\mathbb{N}} D_{[\frac{1}{n+1},\frac{1}{n})}$$

is countable.

Now we can define the required modulus of continity:

Definition 1.2. *Let $\omega \in \mathbb{D}$ and $\mathcal{P}_\delta$ a finite partition of $[0,1]$, let $\mathcal{P}_\delta = \{0 = t_0 < t_1 < \ldots < t_r = 1\}$, such that*

$$\min\{t_i - t_{i-1} : i = 1, 2, \ldots, r\} > \delta. \tag{1.2}$$

Then

$$w'_\omega(\delta) = \inf_{\mathcal{P}_\delta} \max_{0<i\leq r} w_\omega\Big([t_{i-1}, t_i)\Big).$$

Remark 1.2. *We cannot use $w_\omega(delta)$ as the modulus of continuity since*

$$\inf_{\delta>0} w_\omega(\delta) > 0$$

whenever ω has a jump. Especially, $\inf_{\delta>0} w_\omega(\delta)$ is equal to the maximum jump that ω has.

Proposition 1.1. *A function ω lies in $\mathbb{D}$ if and only if $\lim_{\delta\downarrow 0} w'_\omega(\delta) = 0$.*

Proof. The proposition is an immediate consequense of lemma 1.1 and the subsequent of lemma remark. □

In the case of space of continuous functions, we can visualise the distance of two functions by saying that two functions are close to each other if the one can be enclosed in a "small perturbation of the ordinates of the other" with the abscissas kept fixed. In $\mathbb{D}$ the notion is a bit more tricky. We will need to "stretch" and "squeeze" time and consequently act on the same way on space-dimension, however discontinuities stay unaffected. The above description can be stated in a rigorous form.

Definition 1.3. *Let*

$$\Lambda = \Big\{\lambda : [0,1] \to [0,1] \;:\; \lambda \textit{ is strictly continuous with } \lambda(0) = 0, \lambda(1) = 1\Big\}$$

then we define $d : \mathbb{D} \times \mathbb{D} \to \mathbb{R}_+$

$$d(\omega_1, \omega_2) := \inf\Big\{\varepsilon > 0 \;:\; \exists\lambda \in \Lambda \textit{ s.t. } \sup_{t\in[0,1]} |t - \lambda(t)| < \varepsilon \textit{ and } \sup_{t\in[0,1]} |\omega_1(t) - \omega_2(\lambda(t))| < \varepsilon\Big\}$$

It can be proved, using the properties of strictly increasing functions, that d is a metric. The topology generated by the given metric is called Skorokhod topology. The first impressive result is that the Skorokhod topology relativized to $C\big([0,1]\big)$ coincides with the uniform topology.

Proposition 1.2. *The space* $(\mathbb{D}, d)$ *is separable.*

For the space $\mathbb{D}$ to be Polish we have to prove that it is complete. However this is not the case. On the other hand, the second impressive result is that Skorokhod picked over the suitable elements of Λ and adjusted the metric, so that the new metric, d_0, is equivalent to d but under which $\mathbb{D}$ is complete. The intuition behind the second selection has to do with the fact that we do not want to "stretch" and "squeeze" time too much and stay close to the "natural time scaling", i.e. to the identity function. Thus we need only these elements of Λ whose slopes of each chord stay close to 1, or equivalently

$$\sup_{t_1,t_2\in[0,1]} \Big| \log \frac{\lambda(t_1)-\lambda(t_2)}{t_1-t_2} \Big| \text{ stays close to } 0.$$

We define

$$\|\lambda\| = \sup_{t_1,t_2\in[0,1]} \Big| \log \frac{\lambda(t_1)-\lambda(t_2)}{t_1-t_2} \Big|$$

and

$$d_0(\omega_1,\omega_2) := \inf\big\{\varepsilon>0 \ : \ \exists\lambda\in\Lambda \text{ s.t. } \|\lambda\|<\varepsilon \text{ and } \sup_{t\in[0,1]} |\omega_1(t)-\omega_2(\lambda(t))|<\varepsilon\big\}.$$

Proposition 1.3. *The function* d_0 *is a metric equivalent to* d.

Theorem 1.1. *The space* $(\mathbb{D}, d_0)$ *is complete.*

For the proofs we restate that [4, Chapter 3] will remedy the curiosity of the reader.

Corollary 1.1. *The space* $\mathbb{D}$ *is Polish.*

In the following we will give the characterization of relative compact sets in $\mathbb{D}$, which is a generalization of well-known Arzela-Ascoli theorem.

Theorem 1.2. *A set* $A \subseteq \mathbb{D}$ *has compact closure in the Skorokhod topology if and only if*

(B)

$$\sup_{\omega\in A}\sup_{t\in[0,1]} |\omega(t)| < \infty$$

(UE')

$$\lim_{\delta\downarrow 0}\sup_{\omega\in A} w'_\omega(\delta) = 0$$

Remark 1.3. *The condition (B) is obvious that it can be translated as boundedness and (UE') as uniform equicontinuity with respect to modulus of continuity* $w'_\omega(\delta)$.

For a second characterisation of compactness we will introduce another mudulus of continuity, $w''_\omega(\delta)$ which is weaker than $w'_\omega(\delta)$. We define

$$w''_\omega(\delta) = \sup\big\{|\omega(t_1)-\omega(t)| \wedge |\omega(t)-\omega(t_2)| \ : \ t\in[t_1,t_2] \text{ and } t_2-t_1\le\delta\big\}.$$

Remark 1.4. *The relationship between the three defined moduli of continuity can be given ny the inequalities*

$$w''_\omega(\delta) \le w'_\omega(\delta) \le w_\omega(2\delta).$$

Theorem 1.3. *A set* $A \subseteq \mathbb{D}$ *has compact closure in the Skorokhod topology if and only if*

(B)

$$\sup_{\omega\in A}\sup_{t\in[0,1]}|\omega(t)|<\infty$$

(UE$''$)

$$\lim_{\delta\downarrow 0}\sup_{\omega\in A}w''_{\omega}(\delta)=0$$

(UE0)

$$\lim_{\delta\downarrow 0}\sup_{\omega\in A}w_{\omega}\big([0,\delta)\big)=0$$

(UE1)

$$\lim_{\delta\downarrow 0}\sup_{\omega\in A}w_{\omega}\big([1-\delta,1)\big)=0$$

Remark 1.5. *The condition (UE$''$) can be translated as uniform equicontinuity with respect to modulus of continuity $w''_{\omega}(\delta)$ and (UE0), (UE1) as uniform equicontinuity with respect to modulus of continuity $w_{\omega}(\delta)$ at 0,1 respectively.*

We close the section with a tightness criterion that we will need to prove the corresponding Kolmogorov-Chentsov criterion in capacity theory framework.

Theorem 1.4. *Let $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{D}$. The $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ is tight if and only if the following hold simultaneously:*

(i) for each $t\in\mathcal{T}$ dense in $[0,1]$ and $1\in\mathcal{T}$

$$\lim_{\alpha\to\infty}\limsup_{n}\mathbb{P}_n\big[|\omega(t)|\geq\alpha\big]=0$$

(ii) for each positive ε

$$\lim_{\delta\downarrow 0}\limsup_{n}\mathbb{P}_n\big[w''_{\omega}(\delta)\geq\varepsilon\big]=0$$

$$\lim_{\delta\downarrow 0}\limsup_{n}\mathbb{P}_n\big[|\omega(\delta)|\geq\varepsilon\big]=0$$

$$\lim_{\delta\downarrow 0}\limsup_{n}\mathbb{P}_n[|\omega(T-)-\omega(T-\delta)|\geq\varepsilon]=0$$

1.2 Notation and auxiliary definitions in probability spaces

Let $(\Omega,\mathcal{F})$ a measurable space, a family $X=\{X_t\}_{t\in T}$ of random variables defined on $(\Omega,\mathcal{F})$ with values in $(\mathbb{R}^n,\mathcal{B})$ is called a stochastic process. For any $\omega\in\Omega$, the map $X_{\cdot}(\omega)$ is called a sample path.

If $(\Omega,\mathcal{F})$ is endowed with a probability measure $\mathbb{P}$ and $T\subseteq\mathbb{R}_+$, additional properties are inherited in stochastic processes as well as notions of comparison in $(\Omega,\mathcal{F},\mathbb{P})$ arise. More precisely,

Definition 1.4. *A stohastic process X is* continuous in probability (or stochastically continuous) *if*

$$X_s\xrightarrow[s\to t]{\mathbb{P}}X_t\quad\forall t\in T,$$

where the limit is understood as

$$\lim_{s\to t}\mathbb{P}\big(|X_s-X_t|>\varepsilon\big)=0,\forall\varepsilon>0.$$

Definition 1.5. *A process $X = \{X_t\}_{t\in T}$ is said to be* càdlàg *if*

$$\mathbb{P}\big(\{\omega \in \Omega \ : \ X_{\cdot}(\omega) \ \textit{is càdlàg}\ \}\big) = 1.$$

Definition 1.6. *Two stochastic processes $\{X_t\}_{t\in T}, \{\overline{X}_t\}_{t\in T}$ are said to be* stochastically equivalent *if*

$$\mathbb{P}\big(\{\omega \in \Omega \mid X_t(\omega) = \overline{X}_t(\omega)\}\big) = 1, \quad \forall t \in T.$$

A stronger notion of similarity between stochastic processes than the previous one is the following,

Definition 1.7. *Two stochastic processes $\{X_t\}_{t\in T}, \{\overline{X}_t\}_{t\in T}$ are said to be* indistinguishable *if*

$$\mathbb{P}\big(\{\omega \in \Omega \mid X_t(\omega) = \overline{X}_t(\omega)\ \forall t \in T\}\big) = 1.$$

Remark 1.6. *In the case T is countable, the definitions 1.6 and 1.7 are equivalent. A sufficient condition for two equivalent processes to be indistinguishable is to have càdlàg sample paths.*

Definition 1.8. *In a measurable space $(\Omega, \mathcal{F})$, a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in T}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$, i.e.*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \forall s \le t.$$

Given a filtration $\mathbb{F}$, we can form a new filtration $\mathbb{F}_+ = \{\mathcal{F}_{t+}\}_{t\in T}$ where

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_t.$$

If $\mathbb{F} = \mathbb{F}_+$ we say that $\mathbb{F}$ is right continuous.

Definition 1.9. *A stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the* usual conditions *if $\mathbb{F}$ is right continuous and complete, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.*

Definition 1.10. *A stochastic process X defined on stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ has* independent increments *if, for any $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.*

Definition 1.11. *A stochastic process X defined on stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ has* stationary increments *if, for any $0 \le s < t$, $X_t - X_s \stackrel{d}{=} X_{t-s}$.*

1.3 Lévy processes

Definition 1.12. *Let A stochastic process $X = \{X_t\}_{t\in T}$ on $\mathbb{R}^d$ is a Lévy process in law if the following conditions are satisfied:*

(i) $X_0 = 0 \quad \mathbb{P} - a.s.$

(ii) X has independent increments

(iii) X has stationary increments

(iv) X is stochastically continuous

The following theorem enables us to use space$\mathbb{D}_0 := \{\omega \in \mathbb{D} \ : \ \omega(0) = 0\}$ as path space and consequently omti the term *in law* that accompanies Lévy processes.

Theorem 1.5. *Every Lévy process in law has a unique càdlàg modification which is itself a Lévy process.*

Proof. We refer to [1, Theorem 2.1.7] □

The simplest Lévy processes is the linear drift, a deterministic process. The *only* non-deterministic continuous Lévy process is the Brownian motion. Some other examples of Lévy processes are the Poisson, compound Poisson, compensate Poisson and the compensated compound Poisson. It will be proved that the independent sum of Lévy processes remains Lévy process.

In the following we will define infinite devisible distributions, determine their characteristic functions and show their correspondece with Lévy processes.

1.3.1 Infinitely divisible distributions

Let X a r.v., $\mathbb{P}_X$ its law and ϕ_X its characteristic function. The connection between them can be described by the following equation

$$\phi_X(u) = \mathbb{E}\big[e^{i\langle u,X\rangle}\big] = \int_\Omega e^{i\langle u,X\rangle} d\mathbb{P} = \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} \mathbb{P}_X(dx), \quad \forall u \in \mathbb{R}^n.$$

We will need frequently the characteristic function of a probability measure, let ρ, so we introduce the notation

$$\hat{\rho}(u) = \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} \rho(dx), \quad \forall u \in \mathbb{R}^n.$$

We also remind the definition of convolution which will play a crucial role hereinafter:

Definition 1.13. *Let π, ρ porbability measures on $\mathbb{R}^n$, we define the convolution of π and ρ as*

$$(\pi * \rho)(A) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbb{1}_A(x+y)\pi(dx)\rho(dy), \quad A \in \mathcal{B}(\mathbb{R}^n)$$

Proposition 1.4. *Let π, ρ probability measures on $\mathbb{R}^n$, then*

(i) $\pi * \rho$ *is a probability measure,*

(ii) *for* $f \in B_b(\mathbb{R}^n)$ *holds*

$$\int_{\mathbb{R}^n} f(x)(\pi * \rho)(dx) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x+y)\pi(dx)\rho(dy)$$

(iii) *for X, Y independent r.v. with distributions π, ρ respectively and $f \in B_b(\mathbb{R}^n)$ holds*

$$\mathbb{E}\big[f(X+Y)\big] = \int_{\mathbb{R}^n} f(x)(\pi * \rho)(dx).$$

In particular for the indcator function we have

$$\mathbb{P}(X+Y \in A) = (\pi * \rho)(A).$$

We will denote the n-fold convolution of a measure ρ as

$$\rho^{*n} = \underbrace{\rho * \rho * \ldots * \rho}_{n-times}$$

and we will say that the measure ρ has a *convolution n-th root* if there exists a measure ρ_n s.t.

$$\rho = (\rho_n)^{*n}.$$

Definition 1.14. *Let X a r.v., it will be called* infinitely divisible *(ID) if for each $n \in \mathbb{N}$ there exists i.i.d. r.v. $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$ s.t*

$$X \stackrel{d}{=} X_1^{(n)} + X_2^{(n)} + \ldots + X_n^{(n)}$$

Remark 1.7. *Let $\{X_t\}_{t \in T}$ a Lévy process, then each X_t is ID.*

By Proposition 1.4 and definition of ID r.v. we can immediately obtain the following characterisation:

Proposition 1.5. *The following are equivalent:*

(i) X *is ID,*

(ii) $\mathbb{P}_X$ *has a convolution n-th root for each $n \in \mathbb{N}$,*

(iii) ϕ_X *has an n-th root for each $n \in \mathbb{N}$, that is itself the characteristic function of a r.v., i.e.*

$$\hat{\rho}(u) = \left(\widehat{\rho_n}\right)^n$$

Remark 1.8. *The preceding characterisation enables us to use te notion of infinitely divisible for probability measures, i.e. a probability measure ρ will be called infinitely divisible if it has an n-th convolution root for each $n \in \mathbb{N}$.*

Some of the most common distributions are ID.

Example 1.1 (Normal distribution)**.** *Let $X \sim N(\mu, \sigma^2)$, then for each n we have*

$$\phi_X(u) = e^{iu\mu - \frac{1}{2}u^2\sigma^2} = e^{n(iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n})} =$$

$$= \left(e^{iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}}\right)^n = \left(\phi_{X^{(\frac{1}{n})}}(u)\right)^n$$

where $X^{(\frac{1}{n})} \sim N(\frac{\mu}{n}, \frac{\sigma^2}{n})$ and therefore Normal distributed r.v. are ID.

Example 1.2 (Compound Poisson distribution)**.** *Let $N \sim Poisson(\lambda)$ with $\lambda \geq 0$ and $J = \{J_k\}_{k \in \mathbb{N}}$ i.i.d. r.v. with distribution $\mathbb{P}_J$, independent from N. We define a* compound Poisson *r.v. X (we will denote $X \sim \pi(\lambda, \mathbb{P}_J)$)as*

$$X = \sum_{k=1}^{N} J_k,$$

for which hold

$$\phi_X(u) = \mathbb{E}[e^{iuX}] = \mathbb{E}\Big[\exp(iu\sum_{k=1}^{N} J_k)\Big] =$$

$$= \mathbb{E}\Big[\mathbb{E}\big[\exp(iu\sum_{k=1}^{N} J_k) \,\big|\, N = n\big]\Big] =$$

$$= \sum_{n \in \mathbb{N}} \mathbb{E}\Big[\exp(iu\sum_{k=1}^{N} J_k) \,\big|\, N = n\Big]\mathbb{P}\big[N = n\big] =$$

$$= \sum_{n \in \mathbb{N}} \mathbb{E}\Big[\exp(iu\sum_{k=1}^{n} J_k)\Big]\mathbb{P}\big[N = n\big] =$$

$$= \sum_{n\in\mathbb{N}} \prod_{k=1}^{n} \mathbb{E}\Big[e^{iuJ_k}\Big]\mathbb{P}\big[N=n\big] =$$

$$= e^{-\lambda} \sum_{n\in\mathbb{N}} \Big(\mathbb{E}\big[\exp(iuJ)\big]\Big)^n \frac{\lambda^n}{n!} =$$

$$= e^{-\lambda} \sum_{n\in\mathbb{N}} \frac{(\lambda\phi_J(u))^n}{n!} = \exp\big(\lambda(\phi_J(u)-1)\big) =$$

$$= \exp\Big[\lambda \int_{\mathbb{R}} (e^{iux}-1)\mathbb{P}_J(dx)\Big].$$

Compound Poisson is ID since

$$\phi_X(u) = \Big(\phi_{X^{(1/n)}}(u)\Big)^n$$

where $X^{(1/n)} \sim \pi(\frac{\lambda}{n}, \mathbb{P}_J)$

Other examples of infinitely divisible distributions are Dirac measures, geometric, negative binomial, exponential and Γ-distributions on $\mathbb{R}$. In [28] the interested reader can find an extensive study of infinitely divisible distributions.

For $X \sim \pi(\lambda, \mathbb{P}_J)$ is easy to calculate that $\mathbb{E}[X] = \mathbb{E}[N] \cdot \mathbb{E}[J] = \lambda\mathbb{E}[J] < \infty$. Compensated compound Poisson process $\widetilde{N}$ is defined by

$$\widetilde{N}_t := \sum_{k=1}^{N_t} J_k - t\lambda\mathbb{E}[J]$$

where N is a Poisson process with intensity λ and $J = \{J_k\}_{k\in\mathbb{N}}$ i.i.d. r.v. with distribution $\mathbb{P}_J$, independent from N. Then it is easy to verify that

$$\phi_{\widetilde{N}_t}(u) = \mathbb{E}[e^{iu\widetilde{N}_t}] = exp\Big[\lambda \int_{\mathbb{R}} (e^iux - 1 - iux)\mathbb{P}_J(dx)\Big]$$

and $\widetilde{N}$ is martingale.

Lemma 1.2. *Let π, ρ infinitely divisible probability measures, then $\pi * \rho$ is also infinitely divisible.*

Proof. The proof is immediate because of commutativity of convolution. □

Corollary 1.2. *Let B a Brownian motion with drift b and $\widetilde{N}$ independent compensated compound Poisson, then $B_t + \widetilde{N}_t$ is ID $\forall\ t \geq 0$ with characteristic function*

$$\phi_{B_t+\widetilde{N}_t}(u) = \exp\Big[t\Big(iub - \frac{1}{2}u^2\sigma^2 + \lambda \int_{\mathbb{R}} (e^{iux} - 1 - iux)\mathbb{P}_J(dx)Big\Big)\Big] \tag{1.3}$$

It will turn out that the characteristic function of each infinitely divisible probability measure has a form which resembles to that of equation 1.3. The result is named after Lévy and Khintchine. The former presented in the general case the result and the latter provided a simpler proof. It was B. de Finetti and A. Kolmogorov who first detected the connection between ID probability measures and characteristic function of the form 1.3.

1.3.2 The Lévy-Khintchine formula

In this section we will describe the way we can obtain the Lévy-Khintchine formula. For the proofs we refer to [28].

Definition 1.15 (Lévy measure). *Let ν a Borel measure on $\mathbb{R}^n$, we will call it a* Lévy measure *if it satisfies*

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^n\setminus\{0\}} (|x|^2 \wedge 1)\nu(dx) < \infty$$

Remark 1.9. *The Lévy measures will play the role of distributions of jumps. In that sense, the first restriction can be interpreted as excluding "jumps of height 0". The second one has to do with the decomposition of discontinuous part in the unit ball (square integrable martingale part) and out of the unit ball (compound Poisson part). The above description will be clear after the presentation of Lévy-Itô decomposition in the next subsection.*

Lemma 1.3. *If $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence of infinitely divisible measures and $\rho_n \to \rho$, then ρ is also infinitely divisible.*

For the limit ρ in the previous lemma to be a distribution of a r.v. we need an extra condition to be satisfied. Lévy's continuity theorem describe it precisely.

Theorem 1.6 (Lévy's continuity theorem). *Let $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of probability measues. If $\lim_{n\in\mathbb{N}} \hat{\rho}_n(u)$ exists for each $u \in \mathbb{R}^n$ and $\hat{\rho} : \mathbb{R}^n \to \mathbb{C}$ is continuous at 0, where*

$$\hat{\rho}(u) := \lim_{n\in\mathbb{N}} \hat{\rho}_n(u),$$

then $\hat{\rho}$ is the characteristic function of a probability measure.

For the proof we refer to [31].

Theorem 1.7 (Lévy-Khintchine). *A measure ρ is infinitely divisible if and only if there exists a triplet (b, c, ν) with $b \in \mathbb{R}^n$, c a symmetric, non-negative definite $n \times n$ matrix and ν a Lévy measure such that*

$$\hat{\rho}(u) = \exp\left(i\langle u, b\rangle - \frac{1}{2}\langle u, cu\rangle + \int_{\mathbb{R}} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{|x|\leq 1\}}\right)\nu(dx)\right) \tag{1.4}$$

Proof. We will present the "if" part. For the reverse direction we refer to [28]. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ a decreasing sequence converging to 0. For each $u \in \mathbb{R}^n$ and $k \in \mathbb{N}$ we define

$$\hat{\rho}_k(u) = \exp\left(i\left\langle u, b - \int_{\{\varepsilon_k<|x|\leq 1\}} x\nu(dx)\right\rangle - \frac{1}{2}\langle u, cu\rangle + \int_{\{|x|>\varepsilon_k\}} \left(e^{i\langle u,x\rangle} - 1\right)\nu(dx)\right).$$

It is clear that it is the convolution of a normal with a compound Poisson distribution. Moreover,

$$\lim_{k\to\infty} \hat{r}_k(u) := \hat{\rho}(u)$$

since the limit exists. It suffices to prove that $\hat{\rho}$ is continuous at 0. The only term we have to examine is the integral term. Using Taylor expansion, the Cauchy-Schwarz inequality and the fact that ν is Lévy measure, we have

$$|\int_{\{|x|\leq 1\}} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\right)\nu(dx) + \int_{\{|x|>1\}} \left(e^{i\langle u,x\rangle} - 1\right)\nu(dx)| \leq$$

$$\le \frac{1}{2}\int_{\{|x|\le 1\}} |\langle u,x\rangle|^2\nu(dx) + \int_{\{|x|>1\}} |e^{i\langle u,x\rangle}-1|\nu(dx) \le$$

$$\le \frac{|u|^2}{2}\int_{\{|x|\le 1\}} |x|^2\nu(dx) + \int_{\{|x|>1\}} |e^{i\langle u,x\rangle}-1|\nu(dx) \overset{u\to 0}{\longrightarrow} 0.$$

□

Remark 1.10. *From Lévy-Khintchine formula on each ID probability measure corresponds a triplet (d, c, ν) belonging to the above described spaces and vice versa. The following theorem guarantees that the triplet is unique, so the correspondence is bijective.*

Theorem 1.8. *The representation of $\hat{\rho}$ in 1.4 by (b, c, ν) is unique.*

The remark 1.7 it was obvious in view of definition of infinite divisible random variables and properties of Lévy processes. However, we had no information regarding the form of characteristic functions of each X_t. After the Lévy-Khintchine formula we can completely determine ϕ_{X_t}, which embrace all the information for each random variable.

Let $\{X_t\}_{t\ge 0}$ a Lévy process and $\psi(u)$ the Lévy exponent of X_1, i.e. $Log(\phi_{X_1})$. Then, since X_1 is infinitely divisible, there exists a triplet (b, c, ν) s.t.

$$\psi(u) = i\langle u,b\rangle - \frac{\langle u,cu\rangle}{2} + \int_{\mathbb{R}} (e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbb{1}_{\{|x|\le 1\}})\nu(dx).$$

Theorem 1.9. *Let X be a Lévy process and $\psi(u)$ the Lévy exponent of X_1, then*

$$\mathbb{E}[e^{i\langle u,X_t\rangle}] = e^{t\psi(u)}.$$

Proof. Let $\phi_u(t) = \mathbb{E}[e^{i\langle u,X_t\rangle}]$, then

$$\phi_u(s+t) = \mathbb{E}[e^{i\langle u,X_{s+t}\rangle}] = \mathbb{E}[e^{i\langle u,X_{s+t}-X_t+X_t\rangle}] =$$

$$= \mathbb{E}[e^{i\langle u,X_{s+t}-X_t\rangle}]\mathbb{E}[e^{i\langle u,X_t\rangle}] == \mathbb{E}[e^{i\langle u,X_s\rangle}]\mathbb{E}[e^{i\langle u,X_t\rangle}] = \phi_u(s)\phi_u(t),$$

i.e.

$$\phi_u(s+t) = \phi_u(s)\phi_u(t). \tag{1.5}$$

and by definition $\phi_u(1) = e^{\psi(u)}$. Since $\phi_u(\cdot)$ satisfies Cauchy functional equation 1.5, it is easy top prove that for rational numbers

$$\phi_u(q) = e^{q\psi(u)}.$$

It can be proved that stochastic continuity of X implies continuity of the map $\phi_u(t)$ (see [1, Lemma 1.3.2]) and consequently $\phi_u(\cdot)$ can be extended continuously so that

$$\phi_u(t) = e^{t\psi(u)} \quad \forall\, t \ge 0.$$

□

Corollary 1.3. *The infinite divisible random variable X_t has the Lévy triplet $(bt, ct, \nu t)$.*

1.3.3 The Lévy-Itô decomposition

Until now we have described how a Lévy process can be characterised by a triplet. In this section, the Lévy-Itô decomposition will enable us to follow the inverse track, i.e. given a triplet, how can we construct a Lévy process. Behind the Lévy-Itô decomposition there is a beautiful machinery: Poisson random measures. For a comprehensive presentation of Poisson random measures we refer to [18]. In this subsection we will cite only the necessary definitions and results in order to present the proof of the main theorem of the subsection.

Theorem 1.10 (Lévy-Itô decomposition). *Given a triplet (b, c, ν) where $b \in \mathbb{R}^n$, $c \in \mathbb{S}_+(n)$ and ν is a Lévy measure. Then, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes $X^{(1)}, X^{(2)}, X^{(3)}$ and $X^{(4)}$ exist:*

(i) $\phi_{X^{(1)}}(u) = i\langle u, b\rangle$, *i.e.* $X^{(1)}$ *is a constant drift,*

(ii) $\phi_{X^{(2)}}(u) = -\frac{\langle u, cu\rangle}{2}$, *i.e.* $X^{(2)}$ *is a Brownian motion,*

(iii) $\phi_{X^{(3)}}(u) = \int_{\{|x|>1\}}(e^{i\langle u,b\rangle} - 1)\nu(dx)$, *i.e.* $X^{(3)}$ *is a compound Poisson process with jumps greater than 1 and*

(iv) $\phi_{X^{(4)}}(u) = \int_{\{|x|\leq 1\}}(e^{i\langle u,b\rangle} - 1 - i\langle u, b\rangle)\nu(dx)$, *i.e.* $X^{(4)}$ *is a square integrable martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite interval.*

Setting $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$ *we have that* X *is a Lévy process with Lévy exponent*

$$\psi(u) = i\langle u, b\rangle - \frac{1}{2}\langle u, cu\rangle + \int_{\mathbb{R}} \big(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{|x|\leq 1\}}\big)\nu(dx).$$

Definition 1.16 (Poisson random measures). *Let $(S, \mathcal{S}, \eta)$ be a σ-finite measure space and suppose that $N : \mathcal{S} \to \mathbb{N}_0 \cup \{\infty\}$ be defined such that $\{N(A) : A \in \mathcal{S}\}$ are random variables defined on some probability space with measure $\mathbb{P}$. Then N is called Poisson random measure with intensity η if*

(i) $\mathbb{P}$*-a.s.* N *is a measure*

(ii) for each $A \in \mathcal{S}$, $N(A)$ *is Poisson distributed with parameter* $\eta(A)$

(iii) for mutually disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{S}$, *the random variables* $N(A_1), N(A_2), \ldots, N(A_n)$ *are independent.*

Theorem 1.11. *A Poisson random measure always exists.*

When the intensity η of a Poisson random measure N has no atoms, we can simply define the integral of a positive measurable function$f$ with respect to N by

$$\int_S f(x)N(dx) = \sum_{s \in suppN} f(s).$$

Convergence of such integrals for more general function and related integrals is given by the following result.

Theorem 1.12. *Let N a Poisson random measure with intensity η and f a measurable function. Then $X = \int_S f(x)N(dx)$ is almost surely absolutely convergent if and only if*

$$\int_S \big(1 \wedge |f(x)|\big)\eta(dx).$$

Furthermore, when this is the case, then for any $\beta \in \mathbb{R}$

$$\mathbb{E}[e^{i\beta X}] = \exp\Big(- \int_S (1 - e^{i\beta f(x)})\eta(dx)\Big).$$

Lemma 1.4. *Suppose N is a Poisson random measure with intensity $dt \times \Pi(dx)$ and that $\Pi(B) < \infty$. Then*

$$X_t = \int_{[0,t]} \int_B xN(ds \times dx)$$

is a compound Poisson process with rate $\Pi(B)$ and jump distribution $\frac{\Pi(dx)}{\Pi(B)}$ for $x \in B$.

Proof. Let $0 \leq s \leq t$, then

$$X_t - X_s = \int_{(s,t]} \int_B xN(ds \times dx)$$

is independent from $\{X_u : 0 \leq u \leq s\}$. From the previous theorem

$$\mathbb{E}[e^{i\beta X_t}] = \exp\Big(- t\int_B (1 - e^{i\beta f(x)})\Pi(dx)\Big),$$

i.e. X_t is a compound Poisson r.v. with jump distribution $\frac{\Pi(dx)}{\Pi(B)}$ and intensity t. It is easy to verify stationarity. Thus we have the required result. □

Lemma 1.5. *Suppose that N and B are asin the previous lemma and that*

$$\int_b |x|\Pi(dx) < \infty.$$

Then

$$M_t = \int_{[0,t]} \int_B xN(ds \times dx) - t\int_B x\Pi(dx)$$

is a martingale. If furthermore,

$$\int_B x^2\Pi(dx) < \infty,$$

then M is a square-integrable martingale.

Theorem 1.13. *Suppose N is a Poisson random measure with intensity $dt \times \Pi(dx)$ and that $\Pi(B) < \infty$ and suppose that*

$$\int_{(-1,1)} x^2\Pi(dx) < \infty.$$

For each $\varepsilon \in (0,1)$ we define the martingale

$$M_t^\varepsilon = \int_{[0,t]} \int_{B_\varepsilon} xN(ds \times dx) - t\int_{B_\varepsilon} x\Pi(dx),$$

where $B_\varepsilon = (-\infty, -\varepsilon) \cup (\varepsilon, \infty)$. Then there existsa Lévy process which is also a martingale, with countably many jumps and such that for each fixed $T > 0$ the sequence of martingales $\{M_t^\varepsilon\}_{t\in[0,T]}$ converges uniformly on $[0,T]$ a.s along a subsequence in ε which may depend on T.

Proof. (of Theorem 1.10) For $X^{(1)}$ and $X^{(2)}$ there is a probability space $(\Omega^b, \mathcal{F}^b, \mathbb{P}^b)$ such that have Lévy exponents $\psi^{(1)} = i\langle u, b\rangle$ and $\psi^{(2)} = -\frac{\langle u,cu\rangle}{2}$ respectively.

Given a Lévy measure ν we can construct a probability space, let $(\Omega^{\#}, \mathcal{F}^{\#}, \mathbb{P}^{\#})$, such that we can construct a Poisson random measure μ^L on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^n), Leb\otimes\nu)$. To go on, we define the process $X^{(3)}$ with

$$X_t^{(3)} = \int_{[0,t]}\int_{\{|x|>1\}} x\mu^L(ds, dx).$$

From Lemma 1.4 $X^{(3)}$ is a compound Poisson process with intensity $\lambda = \nu(D^c)$ and jump distribution

$$F(dx) = \frac{\nu(dx)}{\nu(D^c)}\mathbb{1}_{\{|x|>1\}}(x).$$

For each $0 < \varepsilon \leq 1$ we define the compensted compound Poisson process

$$X^{(4),\varepsilon} = \int_{[0,t]}\int_{\{\varepsilon<|x|\leq 1\}} x\mu^L(ds, dx) - t\int_{\{\varepsilon<|x|\leq 1\}} x\nu(dx).$$

By lemma 1.5, $X^{(4)}$ has Lévy exponent

$$\psi^{(4),\varepsilon} = \int_{\{\varepsilon<|x|\leq 1\}} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\nu(dx).$$

Taking into account Theorem 1.13 there exists a Lévy process, let $X^{(4)}$, which is square integrable, pure jump martingale defined on $(\Omega^{\#}, \mathcal{F}^{\#}, \mathbb{P}^{\#})$, such that $X^{(4),\varepsilon}$ converges to $X^{(4)}$ uniformly on $[0, T]$ along an appropriate subsequence as $\varepsilon \downarrow 0$. Clearly, the Lévy exponent of the latter Lévy process is

$$\psi^{(4)} = \int_{\{\|x|\leq 1\}} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\nu(dx).$$

Since sets $\{\|x| \leq 1\}$, $\{\|x| > 1\}$ are disjoint, the processes $X^{(3)}$, $X^{(4)}$ are independent and both of them independent from $X^{(1)}$ and $X^{(2)}$.

The required space for the proof to be completed is

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^b, \mathcal{F}^b, \mathbb{P}^b) \times (\Omega^{\#}, \mathcal{F}^{\#}, \mathbb{P}^{\#}).$$

□

As a last result in Lévy processes will be the characterisation of infinitesimal generator by using Lévy exponent or the associated triplet.

Theorem 1.14. *Let X a Lévy process with Lévy exponent η and characteristic triplet (b, c, ν). Let $(T_t)_{t\geq 0}$be tha associated Feller semigroup and $\mathcal{A}$ be its infinitesimal generator. Then for each $t \geq 0$, $f \in S(\mathbb{R}^n)$ tha Schwartz space, $x \in \mathbb{R}^n$,*

$$(\mathcal{A}_X f)(x) = \langle b, D_x f(x)\rangle + \frac{1}{2}tr[cD_{xx}f(x)] + \int_{\mathbb{R}^n}\big(f(x+y) - f(x) - \langle y, D_x f(x)\rangle\mathbb{1}_{\{|y|\leq 1\}}\big)\nu(dy).$$

The infinitesimal generator connects stochastic processes with parabolic PDEs. By defining $u(t, x) = \mathbb{E}[f(x+X_t)]$ it can be proved that u is the unique solution of $u_t - \mathcal{A}_X u = 0$ with initial condition $u(0, x) = f(x)$. In the case of G-Levy process the corresponding infinitesimal generator will play a crucial role since G-Lévy processes will be defined through parabolic PDEs.

Part II

G-Lévy Processes

Chapter 2

Sublinear Expectations Spaces

2.1 Introductory notions

Definition 2.1. *Let Ω be a given set and $\mathcal{H}$ a linear space of real valued functions defined on Ω. A functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying :*

(i) (Monotonicity) If $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$

(ii) (Constant preserving) For each $c \in \mathbb{R}$, $\hat{\mathbb{E}}[c] = c$

is a (nonlinear) expectation and the triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a nonlinear expectation space. Moreover, if $\hat{\mathbb{E}}$ satisfies :

(iii) (Sub-additivity) For each $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$

(iv) (Positive homogeneity) For each $\lambda \geq 0$, $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$

then $\hat{\mathbb{E}}$ is a sublinear expectation and the triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

A sublinear expectation can be represented as the supremum of linear expectations.

Theorem 2.1. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a sublinear expectation space. Then, there exists a family $\{\mathbb{E}_\mu\}_{\mu \in \mathcal{M}}$ of linear expectations defined on $\mathcal{H}$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[X], \text{ for each } X \in \mathcal{H}.$$

Proof. For each $X \in \mathcal{H}$, we define $\mathbb{R}_X = \{\alpha X | \alpha \in \mathbb{R}\}$ and $I_X : \mathbb{R}_X \to \mathbb{R}$ a real valued linear functional by

$$I_X[\alpha X] = \alpha \hat{\mathbb{E}}[X],$$

then $I_X \leq \hat{\mathbb{E}}|_{\mathbb{R}_X}$. By Hahn-Banach theorem, there exists a linear extension of I_X, let $\mathbb{E}_X$, such that

$$\mathbb{E}_X[Y] \leq \hat{\mathbb{E}}[Y], \text{for each } Y \in \mathcal{H}.$$

By definition, $\mathbb{E}_X[X] = \hat{\mathbb{E}}[X]$, for each $X \in \mathcal{H}$, so it is immediate that

$$\hat{\mathbb{E}}[Y] = \sup_{X \in \mathcal{H}} \mathbb{E}_X[Y], \text{ for each } Y \in \mathcal{H}.$$

We have to prove that $\mathbb{E}_X$ is an expectation. Since it is a linear functional it suffices to prove that $\mathbb{E}_X[Y] \geq 0$ for each $Y \geq 0$. For $Y \geq 0$ we have

$$0 \leq -\hat{\mathbb{E}}[-Y] \leq -\mathbb{E}_X[-Y] = \mathbb{E}_X[Y]$$

and for each $c \in \mathbb{R}$

$$c = -(-c) = -\hat{\mathbb{E}}[-c] \leq -\mathbb{E}[-c] = \mathbb{E}[c] \leq \hat{\mathbb{E}}[c] = c.$$

$\square$

Remark 2.1. *It is important to note that the linear expectations that appear in the above representation theorem are not necessarily induced by a σ-additive measure. Under certain conditions, however, we can determine a unique σ-additive measure μ_X for which $\mathbb{E}_X[Y] = \int_\Omega Y d\mu_X$ holds for each $Y \in \mathcal{H}$.*

2.1.1 Daniell-Stone Integrals

In the current subsection we will present for the convenience of the reader the definitions and the main results regarding the Daniell-Stone Integrals. The idea is that when an operator satisfies a minimal set of properties that an integral has, then there is a $\sigma-$additive measure with indeuced integral equal to the given operator. For a more detailed description of the original ideas we refer to [9]. The proofs of the current subsection can be found in [14].

Definition 2.2. *Let $\mathcal{L}$ a collection of real-valued functions defined on a set Ω. $\mathcal{L}$ is called a* vector lattice *if it is a vector space and for each $X, Y \in \mathcal{L}$,*

$$X \vee Y \in \mathcal{L}, \text{ where } (X \vee Y)(\cdot) = \max\{X(\cdot), Y(\cdot)\}.$$

Definition 2.3. *Given a set Ω and a vector lattice $\mathcal{L}$ of real functions on Ω, a* pre-integral *is a functional $I : \Omega \to \mathbb{R}$ such that*

(i) I is linear

(ii) I is nonnegative, i.e. for each $X \in \mathcal{L}$ such that $X \geq 0$, then $I(X) \geq 0$

(iii) For each $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ such that $X_n \downarrow 0$ pointwise, $I(X_n) \downarrow 0$ holds.

Example 2.1. *Let $(\Omega, \mathcal{T})$ a topological space, the collection $C_b(\Omega, \mathcal{T})$ of all bounded continuous real-valued functions on Ω is a vector lattice. On the other hand, let $C^1 = \{f : \mathbb{R} \to \mathbb{R} | f'$ exists and is continuous$\}$, then C^1 is a vector space but not a vector lattice.*

Remark 2.2. *For any vector lattice $\mathcal{L}$, if $X \in \mathcal{L}$, then $X^+, X^- \in \mathcal{L}$, since $\mathbf{0} \in \mathcal{L}$ (where $\mathbf{0}$ is the constant zero function). However, we cannot conclude the same for other constant functions. This was a main point of* M. H. Stone's *contribution to the theory, since it was him that found that this additional assumption was useful.*

Definition 2.4. *A vector lattice $\mathcal{L}$ is called a* Stone vector lattice *if $X \in \mathcal{L}$ implies that $X \vee \mathbb{1} \in \mathcal{L}$, where $\mathbb{1}(\cdot) = 1$.*

Theorem 2.2. *(Daniell-Stone) Let I be a pre-integral on a Stone vector lattice $\mathcal{L}$. Then there is a measure μ on Ω such that $I(X) = \mathbb{E}_\mu[X]$, for all $X \in \mathcal{L}$. The measure μ is uniquely determined on the smallest $\sigma-$ring $\mathcal{B}$ for which all functions in $\mathcal{L}$ are measurable.*

Remark 2.3. *A sufficient condition to apply Theorem 2.2 on each linear expectation $\mathbb{E}_\mu$ of Theorem 2.1, is to prove that $\mathcal{H}$ is a Stone lattice and for each $\{X_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $X_n \downarrow 0$ pointwise, $\hat{\mathbb{E}}(X_n) \downarrow 0$ holds. In this case it is clear that $\mathbb{E}_\mu[X_n] \downarrow 0$*

2.2 Distribution of a random variable and Independence

In the following we suppose that the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ has the following structure: if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\phi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\phi \in \mathcal{L}$,where $\mathcal{L}$ is a Stone lattice. In this case, $X = (X_1, X_2, \ldots, X_n)$ is called an n-dimensional random vector, which will be denoted by $X \in \mathcal{H}^n$.

There are many Stone lattices on $\mathbb{R}^n$ which are rich enough for our needs, some of which are:

- $\mathbb{L}^\infty(\mathbb{R}^n)$: the space of bounded Borel-measurable functions
- $C_b(\mathbb{R}^n)$: the space of bounded and continuous functions
- $C_{l.Lip}(\mathbb{R}^n)$: the space of locally Lipschitz continuous functions
- $C_{b.Lip}(\mathbb{R}^n)$: the space of bounded Lipschitz continuous functions

In the following, unless otherwise stated, we will use $C_{l.Lip}(\mathbb{R}^n)$, only for some convenience of techniques. We remind the reader that if $\phi \in C_{l.Lip}(\mathbb{R}^n)$ then there exist $C_\phi > 0$, $m_\phi \in \mathbb{N}$ such that

$$|\phi(x) - \phi(y)| \le C_\phi(1 + |x|^{m_\phi} + |y|^{m_\phi})|x - y|, \text{ for each } x, y \in \mathbb{R}^n.$$

Lemma 2.1. *Let $X, Y \in \mathcal{H}$ and $\phi, \psi \in C_{l.Lip}(\mathbb{R}^n)$, then $\phi(X)\psi(Y) \in \mathcal{H}$. In particular, $\hat{\mathbb{E}}[|X|^n] < \infty$.*

Proof. Since $\mathbb{R}^n$ is locally compact metric space, it suffices to prove that $\phi\psi$ is Lipschitz continuous on every compact subset of $\mathbb{R}^n$. Let K compact, then there exist $C_\phi^K, C_\psi^K > 0$ such that they are the Lipschitz constants for $\phi|_K, \psi|_K$. Subsequently we have,

$$\begin{aligned} |(\phi\psi)(x) - (\phi\psi)(y)| &= |\phi(x)\psi(x) - \phi(x)\psi(y) + \phi(x)\psi(y) - \phi(y)\psi(y)| \le \\ &\le |\phi(x)\psi(x) - \phi(x)\psi(y)| + |\phi(x)\psi(y) - \phi(y)\psi(y)| \le \\ &\le \Big(C_\psi^K \max_{x \in K} \phi(x) + C_\phi^K \max_{x \in K} \psi(x)\Big)|x - y| \end{aligned}$$

□

In correspondence to the linear case, we would like to define a sublinear expectation on $(\mathbb{R}^n, \mathcal{L})$, where $\mathcal{L}$ a Stone lattice, given a random variable defined on Ω and a sublinear expectation on $(\Omega, \mathcal{H})$.

Definition 2.5. *Let $X \in \mathcal{H}^n$ a random variable on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The functional $\mathbb{F}_X : \mathcal{L} \to \mathbb{R}$ defined by*

$$\mathbb{F}_X[\phi] = \hat{\mathbb{E}}[\phi(X)]$$

is called the distribution of X under $\hat{\mathbb{E}}$.

The triple $(\mathbb{R}^n, \mathcal{L}, \mathbb{F}_X)$ forms a sublinear expectation space.

Definition 2.6. *Let X_1 and X_2 n-dimensional random vectors defined on the sublinear expectation spaces, $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ respectively. They are called identically distributed (denoted by $X_1 \overset{d}{=} X_2$) if*

$$\mathbb{F}_{X_1}[\phi] = \mathbb{F}_{X_2}[\phi], \text{ for each } \phi \in \mathcal{L}.$$

In the case that

$$\mathbb{F}_{X_1}[\phi] \ge \mathbb{F}_{X_2}[\phi], \text{ for each } \phi \in \mathcal{L}$$

we say that the distribution of X_1 is stronger than that of X_2.

Remark 2.4. *From theorem 2.1 we can conclude that identically distributed random variables are characterised by the same set $\mathcal{M}$. We use the term* uncertainty set of (a sublinear expectation) $\mathbb{F}$ *to refer to the set $\mathcal{M}$ of theroem 2.1. Similarly, if the distribution of X_1 is stronger than that of X_2, then $\mathcal{M}_2 \subset \mathcal{M}_1$, where $\mathcal{M}_1, \mathcal{M}_2$ the uncertainty sets of $\mathbb{F}_1$,$\mathbb{F}_2$ respectively.*

In the general case, the distribution of a random variable $X \in \mathcal{H}$ presents uncertainty in its moments. In the following we are mainly interested for the *mean-uncertainty* and the *variance-uncertainty* of X. For our convenience we denote

$$\underline{\mu} := -\hat{\mathbb{E}}[-X],\ \overline{\mu} := \hat{\mathbb{E}}[X],\ \underline{\sigma}^2 := -\hat{\mathbb{E}}[-X^2],\ \overline{\sigma}^2 := \hat{\mathbb{E}}[X^2].$$

It is immediate (from sublinearity of expectation) that $\underline{\mu} \leq \overline{\mu}$ and (from monotonicity and sublinearity) $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2$. Mean-uncertainty and variance-uncertainty are characterised by the intervals $[\underline{\mu}, \overline{\mu}]$ and $[\underline{\sigma}^2, \overline{\sigma}^2]$ respectively.

Proposition 2.1. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $X, Y \in \mathcal{H}$ such that Y has no mean-uncertainty, i.e. $\hat{\mathbb{E}}[Y] = -\hat{\mathbb{E}}[-Y]$. Then*

$$\hat{\mathbb{E}}[X + \alpha Y] = \hat{\mathbb{E}}[X] + \alpha\hat{\mathbb{E}}[Y],\ \textit{for each } \alpha \in \mathbb{R}.$$

Proof. For each real number α we have $\alpha = \alpha^+ - \alpha^-$, where α^+, α^- are positive and at least one of them is zero. So, we have

$$\hat{\mathbb{E}}[\alpha Y] = \hat{\mathbb{E}}[\alpha^+ Y - \alpha^- Y] = \alpha^+\hat{\mathbb{E}}[Y] + \alpha^-\hat{\mathbb{E}}[-Y] = \alpha^+\hat{\mathbb{E}}[Y] - \alpha^-\hat{\mathbb{E}}[Y] = \alpha\hat{\mathbb{E}}[Y]$$

and

$$\hat{\mathbb{E}}[X + \alpha Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[\alpha Y] = \hat{\mathbb{E}}[X] + \alpha\hat{\mathbb{E}}[Y] = \hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[-\alpha Y] \leq \hat{\mathbb{E}}[X + \alpha Y].$$

□

Corollary 2.1. *If $Y \in \mathcal{H}$ has no mean-uncertainty then $\alpha Y \in \mathcal{H}$ has no mean-uncertainty. Thus the set*

$$\mathcal{H}_{nmu} = \{X \in \mathcal{H} \mid X \textit{ has no mean-uncertainty } \}$$

is a linear subspace of $\mathcal{H}$.

Remark 2.5. *If $X \in \mathcal{H}$ has no mean-uncertainty*

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[X^+ - X^-] \geq \hat{\mathbb{E}}[X^+] - \hat{\mathbb{E}}[X^-] = \hat{\mathbb{E}}[\frac{1}{2}(|X| + X)] - \hat{\mathbb{E}}[\frac{1}{2}(|X| - X)] =$$

$$\frac{1}{2}\hat{\mathbb{E}}[|X|] + \frac{1}{2}\hat{\mathbb{E}}[X] - \frac{1}{2}\hat{\mathbb{E}}[|X|] + \frac{1}{2}\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[X],$$

so

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[X^+] - \hat{\mathbb{E}}[X^-].$$

Corollary 2.2. *A random variable $X \in \mathcal{H}$ has no mean-uncertainty if*

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[X^+] - \hat{\mathbb{E}}[X^-] \textit{ and}$$

$$\hat{\mathbb{E}}[-X] = \hat{\mathbb{E}}[(-X)^+] - \hat{\mathbb{E}}[(-X)^-] = \hat{\mathbb{E}}[X^-] - \hat{\mathbb{E}}[X^+].$$

The structure of $\mathcal{L} = C_{l.Lip}(\mathbb{R}^n)$ enables us to prove that in a sublinear expectation space the condition of Remark 2.3 is satisfied.

Lemma 2.2. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space and $X \in \mathcal{H}^n$ given. Then, for each $\{\phi_n\}_{n\in\mathbb{N}} \subset C_{l.Lip}(\mathbb{R}^n)$ satisfying $\phi_n \downarrow 0$, we have $\hat{\mathbb{E}}[\phi_n(X)]$.*

Proof. We denote B_R the closed ball of $\mathbb{R}^n$ with center 0 and radius R. For each $\phi_n \in C_{l.Lip}(\mathbb{R}^n)$ we have

$$\phi_n(x) = \phi_n(x)\mathbb{1}_{B_R} + \phi_n(x)\mathbb{1}_{B_R^c} \leq \max_{x\in B_R} \phi_n(x) + \phi_1(x)\frac{|x|}{R},$$

which implies

$$\hat{\mathbb{E}}[\phi_n(X)] \leq \hat{\mathbb{E}}[\max_{X\in B_R} \phi_n(X)] + \frac{1}{R}\hat{\mathbb{E}}[\phi_1(X)|X|].$$

Letting $n \to \infty$, we have $\max_{x\in B_R} \phi_n(x) \downarrow 0$, since $\phi_n \downarrow 0$ and, moreover, since R can be arbitarily large we have that $\hat{\mathbb{E}}[\phi_n(X)] \downarrow 0$. □

Corollary 2.3. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space and $X \in \mathcal{H}^n$ given. Then, if $\mathcal{L} = C_{l.Lip}(\mathbb{R}^n)$, there exists a family of probability measures $\{\mathbb{P}_\mu\}_{\mu\in\mathcal{M}}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that*

$$\mathbb{F}_X[\phi] = \sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}^n} \phi(x)\mathbb{P}_\mu(dx), \quad \phi \in C_{l.Lip}(\mathbb{R}^n).$$

Due to the last result

$$\overline{\mu} = \sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}} x\mathbb{P}_\mu(dx),$$

$$\underline{\mu} = -\sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}} -x\mathbb{P}_\mu(dx) = \inf_{\mu\in\mathcal{M}} \int_{\mathbb{R}} x\mathbb{P}_\mu(dx),$$

$$\overline{\sigma}^2 = \sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}} x^2\mathbb{P}_\mu(dx),$$

$$\underline{\sigma}^2 = -\sup_{\mu\in\mathcal{M}} \int_{\mathbb{R}} -x^2\mathbb{P}_\mu(dx) = \inf_{\mu\in\mathcal{M}} \int_{\mathbb{R}} x^2\mathbb{P}_\mu(dx).$$

Definition 2.7. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $X_2 \in \mathcal{H}^{n_2}$ is said to be independent from another random vector $X_1 \in \mathcal{H}^{n_1}$ if for each $\phi \in C_{l.Lip}(\mathbb{R}^{n_1+n_2})$ we have*

$$\hat{\mathbb{E}}[\phi(X_1, X_2)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\phi(x_1, X_2)]_{x_1=X_1}].$$

An interpretation that can be given in the above definition is that the uncertainty set of X_2 stays unaffected after the realisation of X_1. It is easy to verify that, in contrast to the classical case, the independence in a sublinear space is not a symmetric property. We present an example ([23]):

Example 2.2. *Let $X, Y \in \mathcal{H}$ identically distributed with no mean-uncertainty and $\underline{\sigma}_X^2 < \overline{\sigma}_X^2$. If Y is independent from X, we have*

$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[xY^2]_{x=X}] = \hat{\mathbb{E}}[x^+\hat{\mathbb{E}}[Y^2] + x^-\hat{\mathbb{E}}[-Y^2]|_{x=X}] =$$

$$= \hat{\mathbb{E}}[x^+\overline{\sigma}^2 - x^-\underline{\sigma}^2|_{x=X}] = \hat{\mathbb{E}}[X^+\overline{\sigma}^2 - X^-\underline{\sigma}^2] =$$

$$= \hat{\mathbb{E}}[X^+\overline{\sigma}^2 - X^+\underline{\sigma}^2 + X\underline{\sigma}^2] \overset{\hat{\mathbb{E}}[X]=0}{=\!=} \hat{\mathbb{E}}[(\overline{\sigma}^2 - \underline{\sigma}^2)X^+] \overset{\overline{\sigma}^2-\underline{\sigma}^2>0}{=\!=}$$

$$= (\overline{\sigma}^2 - \underline{\sigma}^2)\hat{\mathbb{E}}[X^+] > 0$$

On the other hand, if X is independent from Y, we have

$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[Xy^2]_{y=Y}] = \hat{\mathbb{E}}[y^2\hat{\mathbb{E}}[X]|y = Y] = \hat{\mathbb{E}}[Y^2]\hat{\mathbb{E}}[X] = 0$$

2.3 Product Spaces

Definition 2.8. *Let* $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ *sublinear expectation spaces. We define*

$$\mathcal{H}_1^{n_1} \otimes \mathcal{H}_2^{n_2} := \{\phi\big(X_1(\omega_1), X_2(\omega_2)\big) \mid (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, (X_1, X_2) \in \mathcal{H}^{n_1} \times \mathcal{H}^{n_2}, \phi \in C_{l.Lip}(\mathbb{R}^{n_1+n_2})\ \}$$

and

$$(\hat{\mathbb{E}}_1 \otimes \hat{\mathbb{E}}_2)[\phi(X_1, X_2)] := \hat{\mathbb{E}}_1\big[\hat{\mathbb{E}}_2[\phi(x_1, X_2)]_{x_1 = X_1}\big].$$

It is straighforward that $(\Omega_1 \times \Omega_2, \mathcal{H}_1^{n_1} \otimes \mathcal{H}_2^{n_2}, \hat{\mathbb{E}}_1 \otimes \hat{\mathbb{E}}_2)$ forms a sublinear expectation space. We call it the product space of sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$.

In an inductive way we can define the product space of $\{(\Omega_i, \mathcal{H}_i, \hat{\mathbb{E}}_i)\}_{i=1}^{l}$.

$$\Big(\prod_{i=1}^{l} \Omega_i, \bigotimes_{i=1}^{l} \mathcal{H}_i^{n_i}, \bigotimes_{i=1}^{l} \hat{\mathbb{E}}_i\Big).$$

Proposition 2.2. *Let* $(\Omega_i, \mathcal{H}_i, \hat{\mathbb{E}}_i)$*,* $i = 1, 2, \ldots, l$ *sublinear expectation spaces and* $X_i \in \mathcal{H}^{n_i}$*,* $i = 1, 2, \ldots, l$*. We define*

$$Y_i(\omega_1, \omega_2, \ldots, \omega_l) := X_i(\omega_i),\ \ i = 1, 2, \ldots, l.$$

Then $Y_i \in \bigotimes_{i=1}^{l} \mathcal{H}_i^{n_i}$*,* $i = 1, 2, \ldots, l$*,* $Y_i \overset{d}{=} X_i$ *and* Y_{i+1} *is independent from* $(Y_1, Y_2, \ldots, Y_i)$ *for each* $i = 2, \ldots, l$*.*

In the special case that $(\Omega_i, \mathcal{H}_i, \hat{\mathbb{E}}_i) = (\Omega, \mathcal{H}, \hat{\mathbb{E}})$ for each $i = 1, 2 \ldots, l$, we denote the product space $(\Omega^l, \mathcal{H}^{\otimes l}, \hat{\mathbb{E}}^{\otimes l})$. If $X, \overline{X} \in \mathcal{H}^n$ we say that $\overline{X}$ is an independent copy of X if $\overline{X} \overset{d}{=} X$ and $\overline{X}$ is independent from X. Using the last proposition we can construct a sequence of identically distributed random variables $\{Y_k\}_{k \in \mathbb{N}}$, such that Y_k is independent from $(Y_1, Y_2, \ldots, Y_{k-1})$ for each $k \geq 2$.

2.4 Completion of Sublinear Expectation Spaces

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a sublinear expectation space.

Proposition 2.3. *For each* $X, Y \in \mathcal{H}$*,* $1 \leq p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have,*

$$\hat{\mathbb{E}}[|XY|] \leq \Big(\hat{\mathbb{E}}[|X|^p]\Big)^{\frac{1}{p}} \cdot \Big(\hat{\mathbb{E}}[|Y|^q]\Big)^{\frac{1}{q}}$$

$$\Big(\hat{\mathbb{E}}[|X + Y|^p]\Big)^{\frac{1}{p}} \leq \Big(\hat{\mathbb{E}}[|X|^p]\Big)^{\frac{1}{p}} \cdot \Big(\hat{\mathbb{E}}[|Y|^p]\Big)^{\frac{1}{p}}$$

In particular, for $1 \leq p_1 \leq p_2$ *we have*

$$\Big(\hat{\mathbb{E}}[|X|^{p_1}]\Big)^{\frac{1}{p_1}} \leq \Big(\hat{\mathbb{E}}[|X|^{p_2}]\Big)^{\frac{1}{p_2}}$$

The proof is similar to the case of a classical Banach space so it is omitted. As a result of the last proposition, in each sublinear expectation space we can define in a natural way a semi-norm $\|\cdot\|_{\hat{\mathbb{E}},p} : \mathcal{H} \to [0, \infty]$ with $p \geq 1$, by

$$\|X\|_{\hat{\mathbb{E}},p} = \Big(\hat{\mathbb{E}}[|X|^p]\Big)^{\frac{1}{p}}.$$

We will drop the symbol $\hat{\mathbb{E}}$, unless there are more than one sublinear expectations defined on the same space.

Since $\|\cdot\|_p$ is a semi-norm, $\mathcal{H}_0^p = \{X \in \mathcal{H} | \hat{\mathbb{E}}[|X|^p] = 0\}$ is a linear subspace of $\mathcal{H}$. Taking $\mathcal{H}_0^p$ as the null space, we introduce the quotient space $\mathcal{H}/\mathcal{H}_0^p$ and thus $(\mathcal{H}/\mathcal{H}_0^p, \|\cdot\|_p)$ is a linear space with norm. We define $\mathcal{H}^p := \{X \in \mathcal{H}/\mathcal{H}_0^p \quad | \quad \|X\|_p < \infty\}$ and we will denote its completion, which becomes a Banach space, by $\hat{\mathcal{H}}_p$. In particular $\hat{\mathcal{H}} := \hat{\mathcal{H}}_1$.

Proposition 2.4. *The sublinear expectation* $\hat{\mathbb{E}} : (\mathcal{H}^p, \|\cdot\|_p) \to (\mathbb{R}, |\cdot|), p \geq 1$, *can be continuously extended to* $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$*, on which remains a sublinear expectation. We will denote the extension of* $\hat{\mathbb{E}}$ *with the same notation.*

Proof. Let $X, Y \in \mathcal{H}^p$, then from subadditivity of expectation we have

$$\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$$

which implies that

$$|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[X - Y] \vee \hat{\mathbb{E}}[Y - X].$$

From monotonicity

$$\hat{\mathbb{E}}[X - Y] \leq \hat{\mathbb{E}}[|X - Y|] \text{ and } \hat{\mathbb{E}}[Y - X] \leq \hat{\mathbb{E}}[|X - Y|]$$

thus

$$\hat{\mathbb{E}}[X - Y] \vee \hat{\mathbb{E}}[Y - X] \leq \hat{\mathbb{E}}[|X - Y|].$$

So finally we have,

$$|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[|X - Y|] \leq \left(\hat{\mathbb{E}}[|X - Y|^p]\right)^{\frac{1}{p}} = \|X - Y\|_p.$$

Since, $\hat{\mathbb{E}}$ is 1-Lipschitz, it can be extended to the completion of the space and remain 1-Lipschitz function.

It is easy to verify that the extension satisfies the properties of sublinear expectation. □

Remark 2.6. *Although we attained a continuous extension, in the case* $C_{l.Lip}(\mathbb{R}^n)$ *is the Stone lattice space, it has to be replaced by* $C_{b.Lip}(\mathbb{R}^n)$*. Thus, in the definitions of distributions, independence and product spaces on* $(\Omega, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$ *it is assumed that the Stone lattice space is* $C_{b.Lip}(\mathbb{R}^n)$*. Moreover, proposition 2.3 still holds for* $(\Omega, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$*.*

Chapter 3

Maximal Distribution and G-Normal Distribution

In this section are introduced two types of distributions in sublinear expectation spaces, namely, maximal distribution and G-normal distribution. The former corresponds to constants and the latter to normal distribution in classical probability theory. Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a sublinear expectation space.

In the general case we have no information about the set of distribution uncertainty of a random variable or about an identity that characterizes the random variable. A special case is when positive linear combinations of independent equally distributed random variables are equally distributed with the initial random variable.

Definition 3.1. *Let $\eta \in \mathcal{H}^n$ a random variable. If for each $a, b \geq 0$ and $\bar{\eta}$ an independent copy of η*

$$a\eta + b\bar{\eta} \overset{d}{=} (a+b)\eta,$$

then η is maximal distributed*.*

Remark 3.1. *It can be proved that if η is maximal distributed, then there is a compact, convex $\Gamma \subset \mathbb{R}^n$ such that*

$$\mathbb{F}_X[\phi] = \sup_{\gamma \in \Gamma} \phi(\gamma) = \max_{\gamma \in \Gamma} \phi(\gamma).$$

This fact justifies the term maximal.

Definition 3.2. *Let $X \in \mathcal{H}^n$ and $\overline{X}$ an independent copy of X. If*

$$\sqrt{a}X + \sqrt{b}\overline{X} \overset{d}{=} \sqrt{a+b}X, \text{ for each } a, b \geq 0$$

then X is (centralised) normally distributed*.*

Remark 3.2. *The definition is based on a characterisation of normally distributed random variables in classical probability theory. The term centralised is justified by the fact that X has no mean uncertainty. Indeed, for each $a, b \geq 0$, we have :*

(i) from independence of $\overline{X}$ from X,

$$\hat{\mathbb{E}}[\sqrt{a}X + \sqrt{b}\overline{X}] = \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\sqrt{a}x + \sqrt{b}\overline{X}]_{x=X}\big] =$$

$$= \hat{\mathbb{E}}\big[\sqrt{a}x + \sqrt{b}\hat{\mathbb{E}}[\overline{X}]|_{x=X}\big] = \sqrt{a}\hat{\mathbb{E}}[X] + \sqrt{b}\hat{\mathbb{E}}[\overline{X}] = (\sqrt{a} + \sqrt{b})\hat{\mathbb{E}}[X]$$

and

(ii) since $\sqrt{a}X + \sqrt{b}\overline{X} \overset{d}{=} \sqrt{a+b}X$

$$\hat{\mathbb{E}}[\sqrt{a}X + \sqrt{b}\overline{X}] = \hat{\mathbb{E}}[\sqrt{a+b}X] = \sqrt{a+b}\hat{\mathbb{E}}[X]$$

so $\hat{\mathbb{E}}[X] = 0$*. In an analogous way we can prove that* $\hat{\mathbb{E}}[-X] = 0$*.*

Definition 3.3. *Let* (η, X) *a pair of n-dimensional random variables on* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *such that*

$$\big(a\eta + b\bar{\eta}, \sqrt{a}X + \sqrt{b}\overline{X}\big) \overset{d}{=} \big((a+b)\eta, \sqrt{a+b}X\big), \text{ for } a, b \geq 0,$$

where $(\bar{\eta}, \overline{X})$ *is an independent copy of* (η, X)*. The pair* (η, X) *is called* G-distributed.

Remark 3.3. *The letter G arises because of the function*

$$G(p, A) := \hat{\mathbb{E}}[\langle p, \eta\rangle + \frac{1}{2}\langle AX, X\rangle], \text{ where } (p, A) \in (\mathbb{R}^n, \mathbb{S}(n))$$

and we denote by $\mathbb{S}(n)$ *the collection of all* $n \times n$ *symmetric matrices. This function plays an important role since it characterizes the distribution of the random variable.*

Remark 3.4. *In fact, if the pair* (η, X) *is G-distributed, it can be proved that* η *is maximal distributed and* X *is (centralised) Normally distributed. In the following, we will say* X *is G-Normally distributed.*

In order to prove the existence of normal distributed and maximal distributed random variables we will use again an inverse direction to that of the classical case. Main role in this procedure plays the theory of viscosity solution of scalar fully nonlinear partial differential equations of second order (see [8]). We remind the reader some basic facts from the classical case so that it will be easier to perceive the inverse direction.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space under which X has the typical normal distribution. Then

$$\mathbb{E}[\phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) e^{-\frac{x^2}{2}} dx.$$

Moreover, if $u^\phi(t, x)$ is the solution of the heat equation

$$\partial_t u - \frac{1}{2}\partial_{xx} u(t, x) = 0$$

with Cauchy condition $u(0, x) = \phi(x)$, it holds

$$u^\phi(1, 0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) e^{-\frac{x^2}{2}} dx.$$

Now going on a reverse direction, given a second order nonlinear PDE, i.e.

$$\partial_t u - G(D_x u, D_{xx} u) = 0$$

with Cauchy condition $u(0, \cdot) = \phi(\cdot)$ and under mild assumptions imposed on G, the aforementioned theory of viscosity solutions guarantees the existence of a unique continuous solution u^ϕ. Under a suitable construction we will provide the space $(\mathbb{R}^n, C_{l.Lip}(\mathbb{R}^n))$ with a sublinear expectation $\mathbb{F}$ such that

$$\mathbb{F}[\phi] = u^\phi(1, 0), \text{ for each } \phi \in C_{l.Lip}(\mathbb{R}^n).$$

For $\omega = x \in \mathbb{R}^n$, the distribution of random variable $X(\omega) = x$ coincides with $\mathbb{F}$.

3.1 A very short description of viscosity solutions theory

In this section we will present some definitions and results regarding the theory of viscosity solutions of PDE. Viscosity solutions of PDE consist of a powerful tool that enables us to handle effectively the arised problem. The interested reader should refer in [8], and, especially for our case, in Appendix C of [23] for Parabolic PDEs.

The theory of viscosity solutions apply to certain PDE of the form

$$F(x, u, D_x u, D_{xx} u) = 0,$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$, $\mathbb{S}(n)$ is the set of symmetric $n \times n$ matrices and $D_x u = [\partial_{x_i} u]_{i=1}^n$ and $D_{xx} u = [\partial_{x_i x_j} u]_{i,j=1}^n$. The fundamental property that the operator F should satisfy is the monotonicity in a specific sense.

Definition 3.4. *Let $(x, r_1, p, A_1), (x, r_2, p, A_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ with $r_1 \leq r_2$ and $A_2 \leq A_1$, where $\mathbb{S}(n)$ is equipped with its usual partial order, the operator is called* proper *if*

$$F(x, r_1, p, A_1) \leq F(x, r_2, p, A_2).$$

In the following it will always be assumed that F is proper and continuous.

Remark 3.5. *We observe that if $(x, r, p, A) \to F(t, x, r, p, A)$ is proper for fixed $t \in [0, T]$, then so is the associated parabolic problem $u_t + F(t, x, u, D_x u, D_{xx} u) = 0$ when considered as an equation in the $n + 1$ independent variables (t, x).*

As it is mentioned, the solution u may be merely continuous, which implies that the first derivative may not even exist. The way that the theory treats $D_x u, D_{xx} u$ can be captured in the next few lines.

First, we suppose that u is $C^2(\mathbb{R}^n, \mathbb{R})$ and

$$F(x, u(x), D_x u(x), D_{xx} u(x)) \leq 0$$

holds for all x, which means that u is a classical subsolution of $F = 0$ or, equivalently, a classical solution of $F \leq 0$. Suppose that ϕ is also $C^2(\mathbb{R}^n, \mathbb{R})$ and x_0 is a local maximum of $u - \phi$. Then, it is well known that

$$D_x u(x_0) = D_x \phi(x_0) \text{ and } D_{xx} u(x_0) \leq D_{xx} \phi(x_0)$$

and so,

$$F(x_0, u(x_0), D_x \phi(x_0), D_{xx} \phi(x_0)) \leq F(x_0, u(x_0), D_x u(x_0), D_{xx} u(x_0)) \leq 0,$$

since F is proper. The extremes of this inequality do not depend on the derivatives of u and so we may consider defining an arbitrary function u to be subsolution of $F = 0$ if

$$F(x_0, u(x_0), D_x \phi(x_0), D_{xx}(x_0)) \leq 0$$

whenever ϕ is $C^2(\mathbb{R}^n, \mathbb{R})$ and x_0 is a local maximum of $u - \phi$. Before going on, we note that $u(x) \leq u(x_0) - \phi(x_0) + \phi(x)$ for x near x_0 and $\phi \in C^2(\mathbb{R}^n, \mathbb{R})$, Taylor approximation imply

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), (x - x_0) \rangle + o(|x - x_0|^2)$$

for $x \to x_0$, where $p = D_x \phi(x_0)$ and $X = D_{xx} \phi(x_0)$. Moreover, if the last equation holds for some $(p, X) \in \mathbb{R}^n \times \mathbb{S}(n)$ and u is twice differentiable at x_0, then $p = D_x \phi(x_0)$ and

$X = D_{xx}\phi(x_0)$. Thus in the case u is a subsolution in the classical sense it follows that $F(x_0, u(x_0), p, X) \leq 0$ when the last inequality holds. When the function u is not twice differentiable, we will use the above properties for $C^2(\mathbb{R}^n)$ test functions "touching" u from above or below to say that u solves the PDE in a weak sense.

For the convenience of the reader who is unfamiliar with the theory of viscosity solutions, we adapt the definitions to the parabolic case. Consider the following parabolic PDE:

$$u_t - G(t, x, u, D_x u, D_{xx} u) = 0 \text{ on } (0, T) \times \mathbb{R}^n, \tag{3.1}$$
$$u(0, x) = \phi(x) \text{ for } x \in \mathbb{R}^n$$

with $G : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ continuous and satisfies the following degenerate elliptic condition

$$G(t, x, r, p, A_1) \leq G(t, x, r, p, A_2), \text{ whenever } A_1 \leq A_2$$

and $\phi \in C(\mathbb{R}^n, \mathbb{R})$.

Remark 3.6. *The reader should notify that the degenerate condition does not conflict with the property that F is proper, since there is a minus sign in front of G in 3.1.*

Definition 3.5. *A* visocsity subsolution *of 3.1, or* G-subsolution, *on $(0, T) \times \mathbb{R}^n$ is an upper semicontinuous function u such that for all $(t, x) \in (0, T) \times \mathbb{R}^n$, $\phi \in C^2((0, T) \times \mathbb{R}^n, \mathbb{R})$ with $u(t, x) = \phi(t, x)$ and $u < \phi$ on $(0, T) \times \mathbb{R}^n \backslash (t, x)$, we have*

$$\partial_t \phi(t, x) - G(t, x, \phi(t, x), D_x \phi(t, x), D_{xx} \phi(t, x)) \leq 0;$$

likewise, a visocsity supersolution *of 3.1, or* G-supersolution, *on $(0, T) \times \mathbb{R}^n$ is a lower semicontinuous function u such that for all $(t, x) \in (0, T) \times \mathbb{R}^n$, $\phi \in C^2((0, T) \times \mathbb{R}^n, \mathbb{R})$ with $u(t, x) = \phi(t, x)$ and $u > \phi$ on $(0, T) \times \mathbb{R}^n \backslash (t, x)$, we have*

$$\partial_t \phi(t, x) - G(t, x, \phi(t, x), D_x \phi(t, x), D_{xx} \phi(t, x)) \geq 0.$$

A viscosity solution *of 3.1 on $(0, T) \times \mathbb{R}^n$ is a function u that is simultaneously a viscosity subsolution and a viscosity supersolution of 3.1 on $(0, T) \times \mathbb{R}^n$.*

Firstly, we are going to present the main results we will use in the special case that equation 3.1 boils down to

$$u_t - G(D_x u, D_{xx} u) = 0 \tag{3.2}$$

where $G : \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ is a continuous sublinear function, monotonic in $A \in \mathbb{S}(n)$, i.e. for each $p, \bar{p} \in \mathbb{R}^n$, $A, \bar{A} \in \mathbb{S}(n)$ and $\lambda \geq 0$

(S) $G(p + \bar{p}, A + \bar{A}) \leq G(p, A) + G(\bar{p}, \bar{A})$

(PH) $G(\lambda \bar{p}, \lambda \bar{A}) = \lambda G(p, A)$

(M) if $A \leq \bar{A}$, then $G(p, A) \leq G(p, \bar{A})$,

Theorem 3.1. *Let $G : \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ is a continuous sublinear function, monotonic in $A \in \mathbb{S}(n)$, then we have*

(i) If u, v have polynomial growth and they are respectively viscosity subsolution and viscosity supersolution of 3.2, then $u \leq v$.

(ii) Let $\phi \in C(\mathbb{R}^n)$ with polynomial growth. Then there exists a viscosity solution of G-equation 3.2 with initial condition ϕ.

(iii) *If $u^\phi \in C([0,T] \times \mathbb{R}^n)$ denotes the polynomial growth solution of 3.2 with initial condition ϕ, then*

(1) $u^{\lambda\phi} = \lambda u^\phi$ *for each* $\lambda > 0$

(2) $u^{\phi+c} = u^\phi + c$, *for* $c \in \mathbb{R}$

(3) $u^{\phi+\psi} \leq u^\phi + u^\psi$

(iv) *If a given function $\widetilde{G} : \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ is dominated by G, i.e.*

$$\widetilde{G}(p,X) - \widetilde{G}(q,Y) \leq G(p-q, X-Y), \text{ for } p,q \in \mathbb{R}^n, X, Y \mathbb{S}(n),$$

then for each $\phi \in C(\mathbb{R}^n)$ satisfying polynomial growth condition, there exists a unique $\widetilde{G}$-solution $\tilde{u}^\phi(t,x)$ on $[0,\infty) \times \mathbb{R}^n$ with initial condition $\tilde{u}^\phi(0,x) = \phi(x)$, i.e.

$$\partial_t \tilde{u}^\phi - \widetilde{G}(D_x \tilde{u}^\phi, D_{xx}\tilde{u}^\phi) = 0$$

$$\tilde{u}^\phi(0,x) = \phi(x).$$

Moreover,

$$\tilde{u}^\phi(t,x) - \tilde{u}^\psi(t,x) \leq u^{\phi-\psi}(t,x), \text{ for } t \geq 0, x \in \mathbb{R}^n.$$

Consequently, the following comparison holds:

$$\psi \geq \phi \Rightarrow \tilde{u}^\psi(t,x) \geq \tilde{u}^\phi(t,x)$$

(v) $u^\phi(t+s,x,y) = u^{u^\phi(t,x+\cdot,y+\cdot)}(s,0,0)$, *for* $\lambda \geq 0$.

3.2 Existence of G-distributed random variables

Let $G : \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ a sublinear function monotonic in $A \in \mathbb{S}(n)$. Moreover, let $u^\phi(t,x,y) : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $\phi \in C_{l.Lip}(\mathbb{R}^{2n})$, be the unique viscosity solution of the equation

$$\partial_t u - G(D_x u, D_{yy} u) = 0, \tag{3.3}$$

$$u|_{t=0} = \phi,$$

where $D_x u = [\partial_{x_i} u]_{i=1}^n$ and $D_{yy} u = [\partial_{y_i y_j} u]_{i,j=1}^n$.

We take $\widetilde{\Omega} = \mathbb{R}^{2n}, \widetilde{\mathcal{H}} = C_{l.Lip}(\mathbb{R}^{2n})$ and $\widetilde{\omega} = (x,y) \in \mathbb{R}^{2n}$. For each $\xi \in \mathcal{H}$ of the form $\xi(\widetilde{\omega}) = \phi(x,y)$, the corresponding sublinear expectation $\widetilde{\mathbb{E}}_G$ is defined by

$$\widetilde{\mathbb{E}}_G[\xi] = u^\phi(1,0,0).$$

Let now $(\eta, X)(\widetilde{\omega}) = (x,y)$ and $\phi \in C_{l.Lip}(\mathbb{R}^{2n})$, we have $\widetilde{\mathbb{E}}_G[\phi(\eta,X)] = u^\phi(1,0,0)$.

In particular, just setting $\psi(x,y) = \langle p, x\rangle + \frac{1}{2}\langle Ay, y\rangle$ it holds

$$u^\psi(t,x,y) = G(p,A)t + \langle p,x\rangle + \frac{1}{2}\langle Ay,y\rangle$$

and consequently

$$\widetilde{\mathbb{E}}_G[\langle p,\eta\rangle + \frac{1}{2}\langle AX,X\rangle] = \widetilde{\mathbb{E}}_G[\psi(\eta,X)] = u^\psi(1,0,0) = G(p,A), \ (p,A) \in \mathbb{R}^n \times \mathbb{S}(n).$$

Remark 3.7. *Let u^ϕ the viscosity solution of equation 3.3 with initial condition ϕ. Then, the function $v(t,x,y) := u^\phi(\lambda t, \bar{x}+\lambda x, \bar{y}+\sqrt{\lambda}y)$ solves the same equation, but with Cauchy condition $\phi(\bar{x}+\lambda\,\cdot, \bar{y}+\sqrt{\lambda}\,\cdot)$, thus*

$$\widetilde{\mathbb{E}}[\phi(\bar{x}+\lambda\eta, \bar{y}+\sqrt{\lambda}X)] = v(1,0,0) = u^\phi(\lambda,\bar{x},\bar{y}).$$

We construct the product space

$$(\Omega,\mathcal{H},\hat{\mathbb{E}}) = (\widetilde{\Omega}\times\widetilde{\Omega}, \widetilde{\mathcal{H}}\otimes\widetilde{\mathcal{H}}, \widetilde{\mathbb{E}}\otimes\widetilde{\mathbb{E}}),$$

and use the pairs of random variables

$$(\eta_1,X_1)(\widetilde{\omega_1},\widetilde{\omega_2}) = \widetilde{\omega_1}, (\eta_2,X_2)(\widetilde{\omega_1},\widetilde{\omega_2}) = \widetilde{\omega_2}, \text{ where } (\widetilde{\omega_1},\widetilde{\omega_2}) \in \Omega.$$

Using proposition 2.2, (η_2, X_2) is an independent copy of (η_1, X_1).

We are ready to prove that the distribution of (η, X) satisfies definition 3.3. For each $s,t>0$ we have

$$\hat{\mathbb{E}}[\phi(s\eta_1+t\eta_2, \sqrt{s}X_1+\sqrt{t}X_2)] = \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\phi(sx+t\eta_2,\sqrt{s}y+\sqrt{t}X_2)]_{(x,y)=(\eta_1,X_1)}\big] =$$

$$= \hat{\mathbb{E}}[u^\phi(t, s\eta_1, \sqrt{s}X)] = u^{u^{\phi(s,\cdot,\cdot)}}(t,0,0) =$$
$$u^\phi(s+t,0,0) = \hat{\mathbb{E}}[\phi((s+t)\eta, \sqrt{s+t}X)].$$

Remark 3.8. *From now on, on each sublinear expectation space we can assume that there exists a pair of random vectors (η, X) such that (η, X) is G-distributed.*

3.3 Characterization of Maximal distributed and G-Normal distributed random variables

Lemma 3.1. *Let $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, (η, X) a G-distributed random variable. We define $u(t,x,y) : [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by*

$$u(t,x,y) := \hat{\mathbb{E}}[\phi(x+t\eta, y+\sqrt{t}X)], \quad \phi \in C_{l.Lip}(\mathbb{R}^n\times\mathbb{R}^n) \tag{3.4}$$

Then

$$u(t+s,x,y) = \hat{\mathbb{E}}[u(t, x+s\eta, y+\sqrt{s}X)], s \geq 0.$$

Proof. Let $(\bar{\eta}, \overline{X})$ an independent copy of (η, X).

$$u(t+s,x,y) = \hat{\mathbb{E}}[\phi(x+(t+s)\eta, y+\sqrt{t+s}X)] =$$
$$= \hat{\mathbb{E}}[\phi(x+s\eta+t\bar{\eta}, y+\sqrt{s}X+\sqrt{t}\overline{X})] =$$
$$= \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\phi(x+s\bar{x}+t\bar{\eta}, y+\sqrt{s}\bar{y}+\sqrt{t}\overline{X})]_{(\bar{x},\bar{y})=(\eta,X)}\big] =$$
$$= \hat{\mathbb{E}}[u(t,x+s\eta, y+\sqrt{s}X)]$$

□

Lemma 3.2. *Under the assumptions of the previous lemma, for each $T>0$, there exist constants $C,k>0$ such that, for all $s,t\in[0,T]$ and $x,\bar{x},y\bar{y}\in\mathbb{R}^n$,*

$$|u(t,x,y)-u(t,\bar{x},\bar{y})| \leq C(1+|x|^k+|y|^k+|\bar{x}|^k+|\bar{y}|^k)(|x-\bar{x}|+|y-\bar{y}|)$$

and

$$|u(t,x,y)-u(t+s,x,y)| \leq C(1+|x|^k+|y|^k)(s+s^{1/2})$$

Proof.

$$|u(t,x,y,)-u(t,\bar{x},\bar{y})| = |\hat{\mathbb{E}}[\phi(x+t\eta, y+\sqrt{t}X)] - \hat{\mathbb{E}}[\phi(\bar{x}+t\eta, \bar{y}+\sqrt{t}X)]| \le$$
$$\le \hat{\mathbb{E}}[|\phi(x+t\eta, y+\sqrt{t}X) - \phi(\bar{x}+t\eta, \bar{y}+\sqrt{t}X)|] \le$$
$$\le \hat{\mathbb{E}}[C_1(1+|X|^k+|\eta|^k+|x|^k+|y|^k+|\bar{x}|^k+|\bar{y}|^k)](|x-\bar{x}|+|y-\bar{y}|) \le$$
$$\le C(1+|x|^k+|y|^k+|\bar{x}|^k+|\bar{y}|^k)(|x-\bar{x}|+|y-\bar{y}|).$$

We will prove the second inequality using the previous lemma,

$$|u(t+s,x,y)-u(t,x,y)| = |\hat{\mathbb{E}}[u(t,x+s\eta, y+\sqrt{s}X)] - u(t,x,y)| =$$
$$|\hat{\mathbb{E}}[u(t,x+s\eta, y+\sqrt{s}X) - u(t,x,y)]|$$
$$\le \hat{\mathbb{E}}[C_1(1+|x|^k+|y|^k+|X|^k+|\eta|^k)(\sqrt{s}|X|+s|\eta|)]$$

and since each moment is bounded, we conclue the required result. □

Proposition 3.1. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, (η, X) a G-distributed random variable. We define $u(t,x,y): [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by*

$$u(t,x,y) := \hat{\mathbb{E}}[\phi(x+t\eta, y+\sqrt{t}X)], \quad \phi \in C_{l.Lip}(\mathbb{R}^n\times\mathbb{R}^n) \tag{3.5}$$

Then u is the unique continuous (in the sense of the previous lemma) viscosity solution of PDE 3.3, i.e.

$$\partial_t u - G(D_x u, D_{yy} u) = 0,$$
$$u|_{t=0} = \phi,$$

where $D_x u = [\partial_{x_i} u]_{i=1}^n$ and $D_{yy} u = [\partial_{y_i y_j} u]_{i,j=1}^n$.

Proof. Initially, we need to prove that for each $(t,x,y) \in (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$ and each $\psi \in C_b^{2,3}([0,\infty),\mathbb{R}^n,\mathbb{R}^n)$ such that $\psi(t,x,y) = u(t,x,y)$ and $u \le \psi$,

$$[\partial_t\psi - G(D_x\psi, D_{yy}\psi)](t,x,y) \le 0.$$

Thus u is a viscosity subsolution, and following an analogous way it can be proved that it is a supersolution. For $\delta \in (0,t)$

$$0 = u(t,x,y) - u(t,x,y) =$$
$$\hat{\mathbb{E}}[u(t-\delta, x+\delta\eta, y+\sqrt{\delta})] - u(t,x,y) \le$$
$$\le \hat{\mathbb{E}}[\psi(t-\delta, x+\delta\eta, y+\sqrt{\delta})] - \psi(t,x,y) \le$$
$$\le \hat{\mathbb{E}}[\psi(t-\delta, x+\delta\eta, y+\sqrt{\delta}) - \psi(t,x,y)] \le$$
$$\le -\partial_t\psi(t,x,y)\delta + \hat{\mathbb{E}}[\delta\cdot\langle D_x\psi(t,x,y),\eta\rangle + \sqrt{\delta}\cdot\langle D_y\psi(t,x,y),X\rangle + \frac{\delta}{2}\langle D_{yy}\psi(t,x,y)X,X\rangle] + C_1(\delta^{3/2}+\delta^2) =$$
$$= -\partial_t\psi(t,x,y)\delta + \hat{\mathbb{E}}[\langle D_x\psi(t,x,y),\eta\rangle + \frac{1}{2}\langle D_{yy}\psi(t,x,y)X,X\rangle]\delta + C_1(\delta^{3/2}+\delta^2) =$$
$$= -\partial_t\psi(t,x,y)\delta + \delta G(D_x\psi, D_{xx}\psi)(t,x,y) + C_1(\delta^{3/2}+\delta^2)$$

Since

$$\partial_t\psi(t,x,y) - G(D_x\psi, D_{xx}\psi)(t,x,y) \le C_1(\delta^{1/2}+\delta), \text{ for each } \delta \in (0,t)$$

we have the required inequality.

□

Corollary 3.1. *Let (η, X) and (ζ, Y) be G-distributed. We emphasize that it is the same G, i.e.*

$$\hat{\mathbb{E}}[\langle p, \eta\rangle + \frac{1}{2}\langle AX, X\rangle] = G(p, A) = \hat{\mathbb{E}}[\langle p, \zeta\rangle + \frac{1}{2}\langle AY, Y\rangle].$$

Then $(\eta, X) \stackrel{d}{=} (\zeta, Y)$.

Proof. We have already proved that $u(t,x,y) := \hat{\mathbb{E}}[\phi(x + t\eta, y + \sqrt{t}X)]$ and $v(t,x,y) := \hat{\mathbb{E}}[\phi(x + t\zeta, y + \sqrt{t}Y)]$ are both solutions of G-equation. On the other hand, equation 3.3 has unique solution. So, $u \equiv v$ and for each $\phi \in C_{l.Lip}(\mathbb{R}^n)$

$$\mathbb{F}_{(\eta,X)}[\phi] = \hat{\mathbb{E}}[\phi(x + t\eta, y + \sqrt{t}X)] = \hat{\mathbb{E}}[\phi(x + t\zeta, y + \sqrt{t}Y)] = \mathbb{F}_{(\zeta,Y)}[\phi].$$

Thus we have the result. □

Corollary 3.2. *Let (η, X) be G-distributed. For each $\psi \in C_{l.Lip}(\mathbb{R}^n)$, the unique continuous viscosity solution of*

$$\partial_t u - G(D_z u, D_{zz} u) = 0, \qquad (3.6)$$
$$u(0, \cdot) = \psi(\cdot)$$

is the $v(t,z) := \hat{\mathbb{E}}[\psi(z + t\eta + \sqrt{t}X)], (t,z) \in [0,\infty) \times \mathbb{R}^n$.

Remark 3.9. *Here, we regard the derivatives under the same variable. The connection with the general case is described by the following relationship*

$$v(t, x + y) \equiv u(t, x, y)$$

where u is the solution of PDE 3.3 and v the solution of PDE 3.6 with initial condition $\psi(x + y) = u(t, x, y)$.

Let (η, X) be G-distributed, if we "drop" η, the random variable X is G-Normally distributed.

Corollary 3.3. *Let $(0, X)$ be G-distributed. Then X is G-Normally distributed. The distribution of X, $\mathbb{F}_X$, is given from the relationship*

$$\mathbb{F}_X[\phi] = u^{\phi}(1, 0)$$

where u^{ϕ} is the unique continous viscosity solution of

$$\partial_t u - G(D_{xx} u) = 0 \qquad (3.7)$$
$$u(0, \cdot) = \phi(\cdot), \quad \phi \in C_{l.Lip}(\mathbb{R}^n).$$

The above equation is a special case of 3.3, so u^{ϕ} is immediately obtained from the relationship $u^{\phi} = \hat{\mathbb{E}}[\phi(x + \sqrt{t}X)]$. Moreover, the operator $G = G_X$ can be characterised by $G_X(A) = \hat{\mathbb{E}}[\frac{1}{2}\langle AX, X\rangle]$, where $A \in \mathbb{S}(n)$.

The parabolic PDE 3.7 is called *G-heat equation.*

In the case we "drop" random variable X form a G-distributed pair, the remaining is maximal distributed.

Corollary 3.4. *Let $(\eta, 0)$ is G-distributed, then η is maximal distributed. The distribution of η is characterised by the solution of*

$$\partial_t u - G(D_x u) = 0$$
$$u(0, \cdot) = \phi(\cdot), \phi \in C_{l.Lip}(\mathbb{R}^n).$$

In this case, the operator $G : \mathbb{R}^n \to \mathbb{R}$ is characterised by $G(p) = \hat{\mathbb{E}}[\langle p, \eta\rangle]$.

Chapter 4

G-Brownian Motion

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a sublinear expectation space.

Definition 4.1. *The family $(X_t)_{t\geq 0}$ is called an n-dimensional* stochastic process *if for each $t \geq 0$, $X_t \in \mathcal{H}^n$.*

Let $G : \mathbb{S}(n) \to \mathbb{R}$ a monotonic, sublinear function. From the previous section, there exists a G-Normal random variable, let X.

Definition 4.2. *An n-dimensional process $(B_t)_{t\geq 0}$ is called a* G-Brownian motion *if the following are satisfied:*

(i) $B_0(\omega) = 0$

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ for each $m \in \mathbb{N}$ and $0 = t_1 < t_2 < \ldots < t_m \leq t$.

Remark 4.1. *If $B_{t+s} - B_t \overset{d}{=} \sqrt{s}X$ and X is G-Normally distributed, then the following hold:*

$$G_{B_{t+s}-B_t}(A) = \hat{\mathbb{E}}[\frac{1}{2}\langle AB_{t+s} - B_t, B_{t+s} - B_t\rangle] =$$
$$= \hat{\mathbb{E}}[\frac{1}{2}\langle A(\sqrt{s}X), (\sqrt{s}X)\rangle] = \hat{\mathbb{E}}[\frac{1}{2}\langle AB_s, B_s\rangle] =$$
$$= G_{B_s}(A) = sG(A) = sG_{B_1}(A).$$

Theorem 4.1 (Characterization of G-Brownian motion)**.** *Let $(B_t)_{t\geq 0}$ n-dimensional process with the following properties:*

(i) $B_0(\omega) = 0$

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \overset{d}{=} B_s$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ for each $m \in \mathbb{N}$ and $0 = t_1 < t_2 < \ldots < t_m \leq t$.

(iii) $\hat{\mathbb{E}}[B_t] = -\hat{\mathbb{E}}[-B_t] = 0$ *and*
$$\lim_{t\downarrow 0} \frac{\hat{\mathbb{E}}[|B_t|^3]}{t} = 0$$

Then $(B_t)_{t\geq 0}$ is a G-Brownian motion with $G(A) = \hat{\mathbb{E}}[\frac{1}{2}\langle AB_1, B_1\rangle]$, $A \in \mathbb{S}(n)$.

Proof. It is sufficient to prove that B_1 is G-Normally distributed and $G_{B_t}(A) = t \cdot G_{B_1}(A)$ for each $A \in \mathbb{S}(n)$.

For each $A \in \mathbb{S}(n)$ we have

$$|\langle AB_t, B_t\rangle| \leq |A| \cdot |B_t|^2.$$

If we set $g(t) = \hat{\mathbb{E}}[\langle AB_t, B_t\rangle]$, it holds $g(0) = 0$ and

$$|g(t)| \leq \hat{\mathbb{E}}[|A| \cdot |B_t|^2] \leq |A| \cdot \hat{\mathbb{E}}[|B_t|^3]^{\frac{2}{3}} \xrightarrow{t \to 0} 0.$$

Also,

$$g(t+s) = \hat{\mathbb{E}}[\langle AB_{t+s}, B_{t+s}\rangle] = \hat{\mathbb{E}}[\langle A(B_{t+s} - B_t + B_t), (B_{t+s} - B_t + B_t)\rangle] =$$

$$= \hat{\mathbb{E}}[\langle A(B_{t+s}-B_t), (B_{t+s}-B_t)\rangle + \langle AB_t, B_t\rangle + 2\langle A(B_{t+s}-B_t), B_t\rangle] = (B_t \text{ has no mean-uncertainty})$$

$$= \hat{\mathbb{E}}[\langle A(B_{t+s} - B_t), (B_{t+s} - B_t)\rangle] + \hat{\mathbb{E}}[\langle AB_t, B_t\rangle] + 2\hat{\mathbb{E}}[\langle A(B_{t+s} - B_t), B_t\rangle] =$$

$$= \hat{\mathbb{E}}[\langle AB_s, B_s\rangle] + \hat{\mathbb{E}}[\langle AB_t, B_t\rangle] =$$

$$= g(s) + g(t)$$

since

$$\hat{\mathbb{E}}[\langle A(B_{t+s} - B_t), B_t\rangle] = \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\langle A(B_{t+s} - B_t), x\rangle]_{x=B_t}\big] = 0.$$

Since g satisfies Cauchy's functional equation $g(t+s) = g(t) + g(s)$ we can prove that it is linear on rational times and, moreover, due to continuity of g, it can be extended so that remain linear. So, $g(t) = tg(1) = t\hat{\mathbb{E}}[\langle AB_1, B_1\rangle]$, or equivalently $G_{B_t} = tG_{B_1}$.

To prove that B_1 is G_{B_1}-Normal we have to prove that $u(t,x) = \hat{\mathbb{E}}[\phi(x + \sqrt{t}B_1)] = \hat{\mathbb{E}}[\phi(x + B_t)]$ is a viscosity solution of

$$u_t - G_{B_1}(D_{xx}u) = 0, \quad u(0, \cdot) = \phi(\cdot) \tag{4.1}$$

where $\phi \in C_{b.Lip}(\mathbb{R}^n)$. We prove that u is Lipschitz in x and $\frac{1}{2}$-Hölder in t.

For each $t \geq 0$ and x, y

$$|u(t,x) - u(t,y)| = |\hat{\mathbb{E}}[\phi(x + B_t)] - \hat{\mathbb{E}}[\phi(y + B_t)]| \leq$$

$$\leq \hat{\mathbb{E}}[|\phi(x + B_t) - \phi(y + B_t)|] = C_\phi|x - y|.$$

For $\delta \in (0,t)$ and due to the independence of $B_\delta, B_t - B_\delta$

$$u(t,x) = \hat{\mathbb{E}}[\phi(x + B_t)] = \hat{\mathbb{E}}[\phi(x + B_\delta + B_t - B_\delta)] = \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\phi(z + B_t - B_\delta)]_{z=x+B_\delta}\big] =$$

$$= \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\phi(z + B_{t-\delta})]_{z=x+B_\delta}\big] = \hat{\mathbb{E}}[u(t-\delta, x + B_\delta)]$$

and easily now we can deduce the Hölder condition in t since,

$$|u(t,x) - u(t-\delta,x)| = |\hat{\mathbb{E}}[u(t-\delta,x) - u(t-\delta,x)]| \leq$$

$$\leq \hat{\mathbb{E}}[|u(t-\delta, x + B_\delta) - u(t-\delta,x)|] \leq \hat{\mathbb{E}}[C_\phi|B_\delta|] \leq$$

$$\leq C_\phi\hat{\mathbb{E}}[|B_\delta|^2]^{\frac{1}{2}} = C_\phi\hat{\mathbb{E}}[\langle B_\delta, B_\delta\rangle]^{\frac{1}{2}} = C_\phi\delta^{\frac{1}{2}}\sqrt{G(I)}.$$

Now we can prove that u is a viscosity solution of 4.1. For a fix $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and for every $v \in C_b^{2,3}\big([0,\infty) \times \mathbb{R}^n\big)$ such that $u(t,x) = v(t,x)$ and $v \geq u$ we have

$$0 = u(t,x) - v(t,x) = \hat{\mathbb{E}}[u(t-\delta, x + B_\delta)] - v(t,x) =$$

$$= \hat{\mathbb{E}}[u(t-\delta, x+B_\delta) - v(t,x)] \leq \hat{\mathbb{E}}[v(t-\delta, x+B_\delta) - v(t,x)] =$$
$$= \hat{\mathbb{E}}\big[v(t-\delta, x+B_\delta) - v(t, x+B_\delta) + v(t, x+B_\delta) - v(t,x)\big] =$$
$$= \hat{\mathbb{E}}\big[\partial_t v(t,x)(-\delta) - I_1 + \langle D_x v(t,x), B_\delta\rangle + \frac{1}{2}\langle D_{xx} v(t,x) B_\delta, B_\delta\rangle + I_2\big] \leq$$
$$\leq -\partial_t v(t,x)\delta + \frac{1}{2}\hat{\mathbb{E}}[\langle D_{xx} v(t,x) B_\delta, B_\delta\rangle] + \hat{\mathbb{E}}[I_1 + I_2] =$$
$$= -\partial_t v(t,x)\delta + G_{B_\delta}\big(D_{xx} v(t,x)\big) + \hat{\mathbb{E}}[I_1 + I_2] =$$
$$= -\partial_t v(t,x)\delta + \delta G_{B_1}\big(D_{xx} v(t,x)\big) + \hat{\mathbb{E}}[I_1 + I_2]$$

where

$$I_1 = \delta \int_0^1 \partial_t v(t-\beta\delta, x+B_\delta) - \partial_t v(t, x+B_\delta)] d\beta,$$
$$I_2 = \int_0^1 \int_0^1 \langle \big(D_{xx}(t, x+\alpha\beta B_\delta) - D_{xx}(t,x)\big) B_\delta, R_\delta\rangle \alpha d\beta d\alpha$$

and

$$\hat{\mathbb{E}}[\langle D_x v(t,x), B_\delta\rangle] = 0 \text{ since } \hat{\mathbb{E}}[B_\delta] = 0.$$

Using the limit assumption it can be proved that $\lim_{\delta\downarrow 0} \frac{\hat{\mathbb{E}}[|I_1+I_2|]}{\delta} = 0$ and as a result we have

$$\partial_t v(t,x) - G_{B_1}\big(D_{xx} v(t,x)\big) \leq 0,$$

so u is a viscosity solution. In an analogous way, it can be proved that u is a viscosity supersolution, thus u is a viscosity solution of 4.1 and $B.$ is a G-Brownian motion. □

Remark 4.2. *The assumption that B_t has not mean uncertainty facilitate in proving that B_1 is $G_{B_1}-$Normally distributed. If we omit this assumption, and enhance in a specific way assumption (ii), the stochastic process $\{B_t\}_{t\geq 0}$ becomes* generalised G-Brownian motion.

Let $G : \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ a continuous,sublinear monotonic in $A \in \mathbb{S}(n)$ function. From the previous section, there exists a G-distributed random variable, let (η, X).

Definition 4.3. *An n-dimensional process $(B_t)_{t\geq 0}$ is called a* generalised G-Brownian motion *if the following are satisfied:*

(i) $B_0(\omega) = 0$

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \overset{d}{=} s\eta + \sqrt{s}X$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ for each $m \in \mathbb{N}$ and $0 = t_1 < t_2 < \ldots < t_m \leq t$.

Theorem 4.2 (Characterization of generalised G-Brownian motion)**.** *Let $(B_t)_{t\geq 0}$ n-dimensional process with the following properties:*

(i) $B_0(\omega) = 0$

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \overset{d}{=} B_s$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ for each $m \in \mathbb{N}$ and $0 = t_1 < t_2 < \ldots < t_m \leq t$.

(iii)

$$\lim_{t\downarrow 0} \frac{\hat{\mathbb{E}}[|B_t|^3]}{t} = 0$$

Then $(B_t)_{t\geq 0}$ is a generalised G-Brownian motion with

$$G(p, A) = \lim_{\delta\to 0} \frac{\hat{\mathbb{E}}[\langle p, B_\delta\rangle + \frac{1}{2}\langle A B_\delta, B_\delta\rangle]}{\delta}, \quad (p, A) \in \mathbb{R}^n \times \mathbb{S}(n).$$

4.1 Existence of G-Brownian motion

As in the classical case, the metric space $\big(C_0(\mathbb{R}_+,\mathbb{R}^n),\rho\big)$, where

$$C_0(\mathbb{R}_+,\mathbb{R}^n) := \{\omega_\cdot : \mathbb{R}_+ \to \mathbb{R}^n \mid \omega_0 = 0 \text{ and } \omega \text{ is continuous}\}$$

and

$$\rho(\omega^1,\omega^2)\sum_{k=1}^{\infty}\frac{1}{2^k}\big\{\max_{t\in[0,k]}|\omega_t^1-\omega_t^1|\wedge 1\big\},$$

will be path space Ω and the canonical process is $B_t(\omega) := \omega_t$, $\omega\in\Omega$.

We define

$$\begin{aligned} Lip(\Omega) = \{\phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}}) :\\ k\in\mathbb{N},\ 0=t_0<t_1<\ldots<t_k,\ \phi\in C_{l.Lip}(\mathbb{R}^{n\times k})\big\}. \end{aligned}$$

and

$$\begin{aligned} Lip(\Omega_T) = \big\{\phi(B_{t_1\wedge T}-B_{t_0}, B_{t_2\wedge T}-B_{t_1\wedge T},\ldots,B_{t_k\wedge T}-B_{t_{k-1}\wedge T}) :\\ k\in\mathbb{N},\ 0=t_0<t_1<\ldots<t_k,\ \phi\in C_{l.Lip}(\mathbb{R}^{n\times k})\big\}. \end{aligned}$$

It is immediate that for $t\le T$, $Lip(\Omega_t)\subset Lip(\Omega_T)$ and

$$Lip(\Omega) = \bigcup_{k\in\mathbb{N}} Lip(\Omega_k).$$

Let $G(\cdot) : \mathbb{S}(n) \to \mathbb{R}$ be a given sublinear and monotonic function. From section 2.3 there is a sublinear expectation space on which we can construct a sequence of random variables $\{\xi_k\}_{k\in\mathbb{N}}$ such that ξ_k is G-Normal distributed and independent from $\{\xi_1,\xi_2,\ldots,\xi_{k-1}\}$, for each $k\in\mathbb{N}$. Let $(\widetilde{\Omega},\widetilde{\mathcal{H}},\widetilde{\mathbb{E}})$ the aforementioned space. We want to construct a sublinear expectation, let $\hat{\mathbb{E}}_G$, on $\big(\Omega, Lip(\Omega)\big)$ such that $\{B_t\}_{t\ge0}$ is G-Brownian motion.

Let $X\in Lip(\Omega)$, then there exist $0=t_0<t_1<\ldots<t_k$ and $\phi\in C_{l.Lip}(\mathbb{R}^{k\times n})$ such that

$$X = \phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}}),$$

we define

$$\begin{aligned} \hat{\mathbb{E}}_G[X] &= \hat{\mathbb{E}}_G[\phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}})]\\ &:= \widetilde{\mathbb{E}}[\phi(\sqrt{t_1-t_0}\xi_1,\sqrt{t_2-t_1}\xi_2,\ldots,\sqrt{t_k-t_{k-1}}\xi_k)] \end{aligned}$$

It is immediate to verify that $\hat{\mathbb{E}}_G$ is a sublinear expectation, which will be called *G-expectation*, under which the canonical process $B_\cdot$ is G-Brownian motion.

According to section 2.4, we can continuously extend the G-expectation on $\big(\Omega, L_G^p(\Omega)\big)$, $p\ge1$, where

$$L_G^p(\Omega) := \mathcal{L}_G^p(\Omega)/\mathcal{L}_{G,0}^p(\Omega),$$

$$\mathcal{L}_G^p(\Omega) := \overline{Lip(\Omega)}^{\|\cdot\|_{G,p}},$$

$$\mathcal{L}_{G,0}^p(\Omega) := \{X\in L_G^p(\Omega) \mid \|X\|_{G,p}=0\}$$

with

$$\|X\|_{G,p} := \big(\hat{\mathbb{E}}_G[|X|^p]\big)^{\frac{1}{p}}.$$

Analogously we define $L_G^p(\Omega_t)$.

The related conditional expectation of $X = \phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}_G[X|\Omega_{t_j}] = \hat{\mathbb{E}}_G[\phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1},\ldots,B_{t_k}-B_{t_{k-1}})|\Omega_{t_j}] :=$$

$$\widetilde{\mathbb{E}}[\phi(x_1,x_2,\ldots,x_{t_j},\sqrt{t_{j+1}-t_j}\xi_{j+1},\ldots,\sqrt{t_k-t_{k-1}}\xi_k)]_{(x_1,x_2,\ldots,x_j)=(B_{t_1},B_{t_2}-B_{t_1},\ldots,B_{t_j}-B_{t_{j-1}})}.$$

4.2 Some useful results related to capacity theory

In this section we will present the results that will enable us to end up with the pathwise analysis of G-Brownian motion in the next section. The effective tool is capacity theory, for which a detailed presentation is given in [7]. For our needs we will slightly modify the definition of capacity.

In the current section, Ω is a Polish space with metric d. As usual, $\mathcal{B}(\Omega)$ denotes the Borel σ-algebra of Ω and $\mathcal{M}$ the collection of all probability measures on $\big(\Omega, \mathcal{B}(\Omega)\big)$. A theorem that will play a hiden, but important, role is the following:

Theorem 4.3. *Let $\big(\Omega, \mathcal{B}(\Omega), \mu\big)$ a finite measure space. Then μ is regular measure, i.e. for each $A \in \mathcal{B}(\Omega)$*

$$\mu(A) = \sup\{\mu(F) \mid F \text{ is compact and } F \subseteq A\}$$

and

$$\mu(A) = \inf\{\mu(G) \mid G \text{ is open and } G \supseteq A\}$$

We will define the capacity function having as domain a $\sigma-$algebra.

Definition 4.4. *Let $c : \mathcal{F} \to [0, \infty]$ a set function, where $\mathcal{F}$ a $\sigma-$algebra, c is a* Choquet capacity *if*

(M) for each $A, B \in \mathcal{F}$ with $A \subseteq B$, $c(A) \leq c(B)$.

(UC) for each increasing sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,

$$c(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} c(A_n),$$

(DC) for each decreasing sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,

$$c(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} c(A_n).$$

Remark 4.3. *Properties (UC) and (DC) can be interpreted as Upwards Continuity and Downwards Continuity respectively. Moreover, we have to observe that there is no information reagarding (sub)additivity.*

Let $\mathcal{P} \subseteq \mathcal{M}$, we can construct a Choquet capacity with additional properties so that resemble to that of an outer measure.

Proposition 4.1. *Let $\mathcal{P} \subseteq \mathcal{M}$, the set function $c : \mathcal{B}(\Omega) \to [0, \infty]$ defined by*

$$c(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A), \tag{4.2}$$

then c is a Choquet capacity. Moreover,

(i) $0 \leq c(A) \leq 1$

(ii) for each sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\Omega)$,

$$c(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\to\infty} c(A_n)$$

(iii) $c(A) = \sup\{c(K) : K \text{ is compact and } K \subseteq A\}$

In the setting of capacity theory vocabulary, a property holds *quasi-surely* if there is a set $N \in \mathcal{F}$ with $c(N) = 0$ such that the property holds on $\Omega \backslash N$. Every element $A \in \mathcal{B}(\Omega)$ with $c(a) = 0$ is called *polar set*.

Remark 4.4. *In the following, we regard the case that capacity is defined by 4.2. It is obvious that in this setting, a set is polar if and only if $\mathbb{P}(A) = 0$ for each $\mathbb{P} \in \mathcal{P}$.*

In a trivial way we can derive an analogous to Borel-Cantelli lemma:

Lemma 4.1. *Let a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ such that*

$$\sum_{n\in\mathbb{N}} c(A_n) < \infty,$$

then

$$c(\limsup_{n\in\mathbb{N}} A_n) = 0.$$

We remind the following important theorem.

Theorem 4.4 (Prokhorov)**.** *Let Ω a Polish space, then $\mathcal{P} \subseteq \mathcal{M}$ is relatively compact if and only if $\mathcal{P}$ is tight.*

For a complete proof we refer to [4].

In our case, where capacity c is defined by 4.2, Prokhorov's theorem can be restated as:

Corollary 4.1. *Let Ω a Polish space, then $\mathcal{P}$ is relatively compact if and only if for each $\varepsilon > 0$ there exists $K \subset \Omega$ compact such that $c(\Omega \backslash K) < \varepsilon$.*

Having a set $\mathcal{P} \subseteq \mathcal{M}$ we can define a sublinear expectation. It will be proved that each G-expectation (see section 4.1) can be determined by a relatively compact subset of $\mathcal{M}$.

Definition 4.5. *Let $\mathcal{P} \subseteq \mathcal{M}$, the* upper expectation of $\mathcal{P}$*, $\mathbb{E}^{\mathcal{P}}$, is defined by*

$$\mathbb{E}^{\mathcal{P}}[X] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[X], \quad X \in L^0_{\mathcal{P}},$$

where

$$L^0_{\mathcal{P}} := \{X : \Omega \to \mathbb{R} \mid X \text{ is Borel measurable and } \mathbb{E}_{\mathbb{P}}[X] \text{ exists for each } \mathbb{P} \in \mathcal{P}\}$$

Theorem 4.5. *The upper expectation $\mathbb{E}^{\mathcal{P}}$ is a sublinear expectation on $B_b(\Omega)$, where*

$$B_b(\Omega) := \{X : \Omega \to \mathbb{R} \mid X \text{ is Borel measurable and bounded}\}.$$

It is obvious that $B_b(\Omega) \subset L^0_{\mathcal{P}}$.

Remark 4.5. *The previous theorem stays valid if we replace $B_b(\Omega)$ with $C_b(\Omega)$, where*

$$C_b(\Omega) := \{X : \Omega \to \mathbb{R} \mid X \text{ is continuous and bounded}\}.$$

Definition 4.6. *The upper expectation $\mathbb{E}^{\mathcal{P}}$ is called* regular *if for each $\{X_n\}_{n\in\mathbb{N}} \subset C_b(\Omega)$ such that $X_n \downarrow 0$ pointwise, $\mathbb{E}^{\mathcal{P}}[X_n] \downarrow 0$ holds.*

Before we give a characterization of regular upper expectations, we remind a well-known extension lemma for metric spaces and a lemma for characterisation of relatively compact sets of $\mathcal{M}$ in the capacity theory framework, both of which will be used in the proof of the next theorem

Lemma 4.2 (Urysohn)**.** *Let (Ω, d) metric space and F_0, F_1 closed disjoint sets. Then there exists a continuous function $f : \Omega \to [0,1]$ such that $f(F_0) = \{0\}$ and $f(F_1) = \{1\}$.*

Lemma 4.3. *Let $\mathcal{P} \subseteq \mathcal{M}$, then $\mathcal{P}$ is relatively compact if and only if for each decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed sets with $F_n \downarrow 0$, we have $c(F_n) \downarrow 0$.*

Proof. For the direct way, let $\varepsilon > 0$ and $K \subset \Omega$ compact such that $c(\Omega \backslash K) < \varepsilon$. For the sequence $\{K \cap F_n\}_{n\in\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $K \cap F_n = \emptyset$ for each $n \geq n_0$, otherwise since the sequence is decreasing, it would have the finite intersection property and taking into account that K is compact $\emptyset \neq \cap_{n\in\mathbb{N}}(K \cap F_n) \subseteq \cap_{n\in\mathbb{N}} F_n = \emptyset$ which is a contradiction. As a result, for each $n \geq n_0$ $F_n \subset \Omega \backslash K$. Therefore from the monotonicity of c, for the given ε there exists $n_0 \in \mathbb{N}$ such that $c(F_n) < \varepsilon$ for each $n \geq n_0$.

For the opposite way, let $\varepsilon > 0$. From the separability of Ω there exists $\{x_n\}_{n\in\mathbb{N}}$ dense, then for each $k \in \mathbb{N}$ $\{B(x_n, \frac{1}{k})\}_{n\in\mathbb{N}}$ is an open covering. Since $\cap_{n=1}^{m} B(x_n, \frac{1}{k})^c \downarrow \emptyset$, from our hypothesis $c(\cap_{n=1}^{m} B(x_n, \frac{1}{k})^c) \downarrow 0$ holds, so there exists m_k such that $c(\cap_{n=1}^{m} B(x_n, \frac{1}{k})^c) \leq \frac{\varepsilon}{2^k}$. We set

$$K = \overline{\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{m_k} B(x_n, \frac{1}{k})}.$$

K is complete as a closed subset of complete space Ω and from construction, every sequence has a Cauchy subsequence. Therefore, K is compact and

$$c(\Omega \backslash K) = c(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{m_k} B(x_n, \frac{1}{k})^c) \leq \sum_{n\in\mathbb{N}} c(\bigcap_{n=1}^{m_k} B(x_n, \frac{1}{k})^c) < \varepsilon$$

□

Remark 4.6. *In the previous lemma we used the fact that a metric space Ω is compact if and only if Ω is complete and totally bounded (equivalently for totally boundedness, every sequence has a Cauchy subsequence).*

Theorem 4.6. *The upper expectation $\mathbb{E}^{\mathcal{P}}$ is regular if and only if $\mathcal{P}$ is relatively compact.*

Proof. Let $\{F_n\}_{n\in\mathbb{N}}$ a decreasing sequence of closed nonempty sets with $F_n \downarrow \emptyset$. To each F_n we associate a $g_n \in C_(\Omega)$ which is constructed using Urysohn lemma in the following way: let $K_n := \{\omega \in \Omega : d(\omega, F_n) \geq \frac{1}{n}\}$, then F_n, K_n are disjoint closed sets, so there is $g_n \in C_b(\Omega)$ such that $g_n(\Omega) = [0,1]$, $g_n(F_n) = \{1\}$ and $g_n(K_n) = \{0\}$. Since $C_b(\Omega)$ is a vector lattice, $f_n = \wedge_{i=1}^{n} g_i \in C_b(\Omega)$ and by construction $1_{F_n} \leq f_n \downarrow 0$. The regularity of $\mathbb{E}^{\mathcal{P}}$ implies that

$$0 \leq c(F_n) = \mathbb{E}^{\mathcal{P}}[1_{F_n}] = \mathbb{E}^{\mathcal{P}}[f_n] \downarrow 0.$$

By lemma 4.3 we can conclude $\mathcal{P}$ is relatively compact.

For the inverse direction, for each $\{f_n\}_{n\in\mathbb{N}} \subset C_b(\Omega)$ such that $f_n \downarrow 0$ we have that $[f_n \geq t]$ is closed and $[f_n \geq t] \downarrow \emptyset$, where

$$[f_n \geq t] := \{\omega \in \Omega : f_n(\omega) \geq t\}.$$

Then we have

$$\mathbb{E}^{\mathcal{P}}[f_n] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[f_n] = \sup_{\mathbb{P}\in\mathcal{P}} \int_0^{\infty} \mathbb{P}\big([f_n \geq t]\big) dt \leq \int_0^{\infty} c\big([f_n \geq t]\big) dt \downarrow 0$$

again by lemma 4.3. □

From theorem 4.5, $\mathbb{E}^{\mathcal{P}}$ is sublinear expectation and by using the results of section 2.4,

$$\|X\|_{\mathbb{E}^{\mathcal{P}},p} := \mathbb{E}^{\mathcal{P}}[|X|^p]^{\frac{1}{p}}$$

is a norm. Following the same route we define

$$\mathcal{L}^p_{\mathcal{P}} := \{X \in L^0_{\mathcal{P}} : \mathbb{E}^{\mathcal{P}}[|X|^p] < \infty\};$$

$$\mathcal{N}_{\mathcal{P}} := \{X \in L^0_{\mathcal{P}} : X = 0 \ \mathcal{P} - q.s.\} = \bigcap_{\mathbb{P}\in\mathcal{P}} \{X \in L^0 : X = 0 \ \mathbb{P} - a.s.\}$$

and

$$\mathbb{L}^p_{\mathcal{P}} := \mathcal{L}^p_{\mathcal{P}}/\mathcal{N}_{\mathcal{P}}.$$

Moreover,

$$\mathcal{L}^\infty_{\mathcal{P}} := \{X \in L^0_{\mathcal{P}} : \exists M > 0 \text{ such that } |X| \le M \ \mathcal{P} - q.s.\};$$

and

$$\mathbb{L}^\infty_{\mathcal{P}} := \mathcal{L}^\infty_{\mathcal{P}}/\mathcal{N}_{\mathcal{P}}.$$

The next results are presented without proof, since they are similar to the classical case. An immediate result is the analogous of Markov inequality:

Lemma 4.4. *Let $X \in \mathbb{L}^p_{\mathcal{P}}$, then for each $\alpha > 0$ and $p \ge 1$*

$$c\big([|X| > \alpha]\big) \le \frac{\mathbb{E}^{\mathcal{P}}[|X|^p]}{\alpha^p}.$$

Lemma 4.5. *Let $1 \le p \le \infty$ and $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathbb{L}^p_{\mathcal{P}}$ which converges to X in $\mathbb{L}^p_{\mathcal{P}}$. Then there is a subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$ which converges to X $\mathcal{P}$-q.s..*

Proposition 4.2. *For each $p \ge 1$, $\big(\mathbb{L}^p_{\mathcal{P}}, \|X\|_{\mathbb{E}^{\mathcal{P}},p}\big)$ is a Banach space.*

Proposition 4.3. *The space $\big(\mathbb{L}^\infty_{\mathcal{P}}, \|X\|_{\mathcal{P},\infty}\big)$ is a Banach space, where*

$$\|X\|_{\mathcal{P},\infty} := \inf\{M \ge 0 : |X| \le M \ \mathcal{P} - q.s.\}.$$

Remark 4.7. *Until now we used in our notation the subset $\mathcal{P}$ to underline the role it plays in each definition. For the following, we suppose we fix a certain subset of $\mathcal{M}$. For our conveniece we will drop $\mathcal{P}$ from notation.*

Let $\big(\Omega, \mathcal{B}(\Omega), \mu\big)$ satisfy the conditions of theorem 4.3. Then, it is well-known that $C_b(\Omega)$ is dense in $\mathbb{L}^p$. However, in our case we will see that, in general, we have an inclusion and not equality for the following spaces

$$\overline{C_b(\Omega)}^{\|\cdot\|_p} \subset \overline{B_b(\Omega)}^{\|\cdot\|_p} \subset \mathbb{L}^p.$$

In the following we will use the notation

$$\mathbb{L}^p_b := \overline{B_b(\Omega)}^{\|\cdot\|_p}$$

and

$$\mathbb{L}^p_c := \overline{C_b(\Omega)}^{\|\cdot\|_p}.$$

Some results that modify in the current framework that of the classical theory are the following:

Proposition 4.4. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $X \in \mathbb{L}^p, Y \in \mathbb{L}^q$, then $XY \in \mathbb{L}^1$ and*

$$\|XY\|_1 \leq \|X\|_p \cdot \|Y\|_q$$

Remark 4.8. *The previous result stays valid if $\mathbb{L}^p$ is replaced by $\mathbb{L}_c^p$.*

Proposition 4.5. *Let $1 \leq p < q \leq \infty$, then*

(i) $\mathbb{L}^q \subset \mathbb{L}^p$

(ii) $\mathbb{L}_b^q \subset \mathbb{L}_b^p$

(iii) $\mathbb{L}_c^q \subset \mathbb{L}_c^p$

In classical theory, the notion of uniform integrability of a family of random variables, $\{X_i\}_{i \in I}$, on $(\Omega, \mathcal{F}, \mathbb{P})$ is used to describe the situation in which

$$\lim_{a \to \infty} \sup_{i \in I} \int_{[|X_i| > a]} |X_i| d\mathbb{P} = 0.$$

In our case, turns out to be important the notion of *uniform integrability* of a random variable X *under the family of probability measures* $\mathcal{P}$, i.e.

$$\lim_{a \to \infty} \sup_{\mathbb{P} \in \mathcal{P}} \int_{[|X| > a]} |X| d\mathbb{P} = \lim_{a \to \infty} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[|X| 1_{[|X| > a]} \right] =$$

$$= \lim_{a \to \infty} \mathbb{E}^{\mathcal{P}} \left[|X| 1_{[|X| > a]} \right] = 0$$

It will turn out that this property characterises the random variables which belong in $\mathbb{L}_b$.

Theorem 4.7. *Let $p \geq 1$, then*

$$\mathbb{L}_b^p = \{X \in \mathbb{L}^p \ : \ \lim_{a \to \infty} \mathbb{E}\left[|X|^p 1_{[|X| > a]}\right] = 0\}$$

Proof. Let $X \in \mathbb{L}^p$ such that $\lim_{a \to \infty} \mathbb{E}\left[|X|^p 1_{[|X| > a]}\right] = 0$. Then, the sequence $\{X_n\}_{n \in \mathbb{N}}$ with $X_n := (-n) \vee X \wedge n \in B_b(\Omega)$ and

$$\mathbb{E}[|X - X_n|^p] \leq \mathbb{E}[|X|^p 1_{[|X| > n]}] \overset{n \to \infty}{\longrightarrow} 0$$

thus $X \in \overline{B_b(\Omega)}^{\|\cdot\|_p} =: \mathbb{L}_b^p$.

For the inverse direction, let $X \in \mathbb{L}_b^p$, then there exists a sequence $\{X_n\}_{n \in \mathbb{N}} \subset B_b(\Omega)$ such that $X_n \overset{\|\cdot\|_p}{\longrightarrow} X$. We define $x_n := \sup_{\omega \in \Omega} |X_n(\omega)|$ and $X_n' := (-x_n) \vee X_n \wedge x_n$. It is immediate that $|X - X_n'| \leq |X - X_n|$, hence $X_n' \overset{\|\cdot\|_p}{\longrightarrow} X$. For the required property we have

$$\begin{aligned} \mathbb{E}[|X|^p 1_{[|X| > n]}] &= \mathbb{E}[(|X| - n + n)^p 1_{[|X| > n]}] \leq \\ &\leq \left(\mathbb{E}[(|X| - n)^p 1_{[|X| > n]}] + n^p c(|X| > n)\right) \to 0 \end{aligned}$$

since

$$\mathbb{E}[(|X| - n)^p 1_{[|X| > n]}] \to 0 \text{ is equivalent to } (-n) \vee X \wedge n \overset{\|\cdot\|_p}{\longrightarrow} X$$

and

$$\frac{n^p}{2^p} c(|X| > n) = \frac{n^p}{2^p} \mathbb{E}(1_{[|X| > n]}) \leq \mathbb{E}\left[(|X| - \frac{n}{2})^p 1_{[|X| > n]}\right] \leq \mathbb{E}\left[(|X| - \frac{n}{2})^p 1_{[|X| > \frac{n}{2}]}\right] \to 0.$$

$\square$

The additional property an element of $\mathbb{L}_c^p$ should have -we remind that $\mathbb{L}_c^p \subset \mathbb{L}_b^p$- is that it has a quasi-continuous version.

Definition 4.7. *A random variable X is said to be* quasi-continuous *if $\forall \varepsilon > 0 \ \exists \ O \subset \Omega$ open with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.*

Definition 4.8. *Let X a random variable. We say that X has a* quasi-continuous version *if there exists a quasi-continuous random variable Y such that $X = Y$ q.s.*

Proposition 4.6. *Let $p \geq 1$ and $X \in \mathbb{L}_c^p$, then X has a quasi-continuous version.*

Proof. Let $\{X_n\}_{n\in\mathbb{N}} \subset C_b(\Omega)$ an $\|\cdot\|_p$ −Cauchy sequence. Using a diagonal argument we can extract a subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\left\|X_{n_k} - X_{n_{k+1}}\right\|_p^p \leq \frac{1}{2^{2k}}, \quad \forall k \geq 1.$$

We define

$$A_k := \bigcup_{i=k}^{\infty} \{|X_{n_{i+1}} - X_{n_i}|^p > \frac{1}{2^i}\}$$

which is open set and for which holds

$$c(A_k) \leq \sum_{i=k}^{\infty} c(\{|X_{n_{i+1}} - X_{n_i}|^p > \frac{1}{2^i}\}) \leq \sum_{i=k}^{\infty} 2^i \mathbb{E}[|X_{n_{i+1}} - X_{n_i}|^p] \leq$$

$$\leq \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}},$$

thus for $A := \cap_{k\in\mathbb{N}} A_k$

$$c(A) = \lim_{k\to\infty} c(A_k) = 0.$$

Since on each A_k^c $\{X_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly, the limit is continuous on each A_k^c and consequently on A^c. □

Before we continue to the characterisation of elements of $\mathbb{L}_c^p$ we remind the well-known Tietze's extension theorem.

Theorem 4.8 (Tietze)**.** *Let (Ω, d) a metric space and $X : F \to \mathbb{R}$ continuous, where F closed subset. Then there exists a continuous extension of X, let $\tilde{X} : \Omega \to \mathbb{R}$, such that* $\sup_{\omega\in\Omega} |\tilde{X}(\omega)| = \sup_{\omega\in F} |X(\omega)|$.

Theorem 4.9. *Let $p \geq 1$, then*

$$\mathbb{L}_c^p = \{X \in \mathbb{L}_b^p \ : \ X \textit{ has a quasi-continuous version}\}$$

Proof. From proposition 4.6 the direct is immediate. For the inverse direction, let $X \in \mathbb{L}_b^p$ quasi-continuous (otherwise we can choose the q.c.-version of X). We define

$$X_n := (-n) \vee X \wedge n,$$

then $\|X - X_n\|_p \to 0$. By construction, X_n is quasi-continuous, therefore $\forall n \in \mathbb{N} \ \exists O_n$ open with $c(O_n) < \frac{1}{n^{p+1}}$ and X_n is continuous on O_n^c. By Tietze's extension theorem, we continuously extend X_n, let $\tilde{X}_n$ the extension. Thus, $\tilde{X}_n \in C_b(\Omega)$ with $|\tilde{X}_n| \leq n$. It follows that

$$\left\|X_n - \tilde{X}_n\right\|_p \leq 2n \ c(O_n)^{\frac{1}{p}} \leq \frac{2}{n^{\frac{1}{p}}}$$

and therefore
$$\left\|X-\tilde{X}_n\right\|_p \leq \|X-X_n\|_p + \left\|X_n-\tilde{X}_n\right\|_p \to 0$$
and the proof is complete. $\square$

A last result that will be needed in the following is the next:

Lemma 4.6. *Let $\{\mathbb{P}_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$ a weakly convergent sequence, with $\mathbb{P}_n \to \mathbb{P} \in \mathcal{P}$. Then for $X \in \mathbb{L}_c^1$,*
$$\mathbb{E}_{\mathbb{P}_n}[X] \longrightarrow \mathbb{E}_{\mathbb{P}}[X].$$

Proof. Let X quasi-continuous and $\varepsilon > 0$. Then, there exists
$$n_0 \in \mathbb{N} \text{ such that } \mathbb{E}[|X|1_{[|X|>n_0]}] < \frac{\varepsilon}{2}.$$
We define $X_b := (-n) \vee X \wedge n$, which is quasi-continuous. Thus, there is an open set O with $c(O) < \frac{\varepsilon}{4n_0}$ and X_b is continuous on O^c. We extend (by using Tietze's extension theorem) X_b, let X_c. Then for each $\mathbb{P} \in \mathcal{P}$
$$|\mathbb{E}_{\mathbb{P}}[X] - \mathbb{E}_{\mathbb{P}}[X_c]| \leq \mathbb{E}_{\mathbb{P}}[|X-X_b|] - \mathbb{E}_{\mathbb{P}}[|X_b - X_c|] \leq \varepsilon,$$
thus
$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}[X] &\leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}[X_c] + \varepsilon = \\ &= \mathbb{E}_{\mathbb{P}}[X_c] + \varepsilon = \mathbb{E}_{\mathbb{P}}[X] + 2\varepsilon.\end{aligned}$$
Analogously, we can get
$$\liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}[X] \geq \mathbb{E}_{\mathbb{P}}[X] - 2\varepsilon$$
and consequently we have the required result. $\square$

Definition 4.9. *Let I a set of indices and $\{X_t\}_{t\in I}, \{Y_t\}_{t\in I}$ two processes. We say that Y is a* quasi-modification *of X if for all $t \in I$, $X_t = Y_t$ quasi-surely.*

Theorem 4.10 (Kolmogorov)**.** *Let $p \geq 1$ and $X = \{X_t\}_{t\in[0,1]^d}$ be a stochastic process such that $X_t \in \mathbb{L}^p$ for each $t \in [0,1]^d$. If there exist $\varepsilon, C > 0$ such that*
$$\mathbb{E}\big[|X_t - X_s|^p\big] \leq C|t-s|^{d+\varepsilon},$$
then X admits a modification $\tilde{X}$ such that
$$\mathbb{E}\left[\left(\sup_{t\neq s} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\right)\right] < \infty,$$
for every $\alpha \in (0, \varepsilon/p)$.

The proof is almost immediate due to the representation of $\mathbb{E} = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}$ and the classical Kolmogorov's continuity theorem.

Remark 4.9. *In fact, the last relationship implies that paths of $\tilde{X}$ are quasi-surely $\alpha-$Hölder continuous.*

4.3 G-expectation as an upper uxpectation and path analysis of G-Brownian Motion

In the current section it will be proved that for the G-expectation there is a weakly compact $\mathcal{P}_G \subset \mathcal{M}$ such that

$$\hat{\mathbb{E}}_G[X] = \sup_{\mathbb{P} \in \mathcal{P}_G} \mathbb{E}_{\mathbb{P}}[X]$$

for each $X \in L^0_{\mathcal{P}_G}$.

We retain the notation for $\Omega = C_0(\mathbb{R}^n, \mathbb{R})$, $\mathcal{B}(\Omega)$ the Borel $\sigma-$algebra generated by all $\rho-$open sets and

$$\rho(\omega^1, \omega^2) \sum_{k=1}^{\infty} \frac{1}{2^k} \{ \max_{t \in [0,k]} |\omega^1_t - \omega^1_t| \wedge 1 \},$$

and we introduce $\bar{\Omega} := (\mathbb{R}^n)^{\mathbb{R}_+} = \{\bar{\omega} : \mathbb{R}_+ \to \mathbb{R}^n\}$, $\mathcal{B}(\bar{\Omega})$ the $\sigma-$algebra generated by the finite dimensional cylinder sets and the canonical process $\bar{B}_t(\bar{\omega}) = \bar{\omega}_t$. The space $Lip(\Omega)$ has already been introduced in 4.1 and in a completely analogous way we define $Lip(\bar{\Omega})$. According to section 4.1, for a given continuous monotonic and sublinear function $G : \mathbb{S}(n) \to \mathbb{R}$ we constructed the G-expectation on $Lip(\Omega)$. Naturally we can construct a sublinear expectation $\bar{\mathbb{E}}$ on $\big(\bar{\Omega}, Lip(\bar{\Omega})\big)$ such that $\{\bar{B}_t(\bar{\omega})\}_{t \geq 0}$ is a G-Brownian Motion.

The following lemma and corollary are in fact the analogous of lemma 2.2 and corollary 2.3.

Lemma 4.7. *Let $0 \leq t_0 < t_1 < \ldots < t_m$ and $\{\phi_n\}_{n \in \mathbb{N}} \subset C_{l.Lip}(\mathbb{R}^{n \times m})$ such that $\phi_n \downarrow 0$. Then*

$$\bar{\mathbb{E}}\big[\phi_n(\bar{B}_{t_1}, \bar{B}_{t_2}, \ldots, \bar{B}_{t_m})\big] \downarrow 0.$$

Corollary 4.2. *Let $\mathbb{E}$ a* finitely additive *linear expectation* dominated *by $\bar{\mathbb{E}}$ on $Lip(\bar{\Omega})$. Then there exists a unique probability measure $\mathbb{P}$ on $\big(\bar{\Omega}, \mathcal{B}(\bar{\Omega})\big)$ such that*

$$\mathbb{E}[X] = \mathbb{E}_{\mathbb{P}}[X], \quad \forall X \in Lip(\bar{\Omega}).$$

Lemma 4.8. *For the sublinear expectation $\bar{\mathbb{E}}$ there exists a set $\mathcal{P}_{\bar{\mathbb{E}}} \subset \mathcal{M}\big(\bar{\Omega}, \mathcal{B}(\bar{\Omega})\big)$ such that*

$$\bar{\mathbb{E}}[X] = \max_{\mathbb{P} \in \mathcal{P}_{\bar{\mathbb{E}}}} \mathbb{E}_{\mathbb{P}}[X], \quad \forall X \in Lip(\bar{\Omega}).$$

The associated capacity to $\mathcal{P}_{\bar{\mathbb{E}}}$, let $\tilde{c}$ is

$$\tilde{c}(A) := \sup_{\mathbb{P} \in \mathcal{P}_{\bar{\mathbb{E}}}} \mathbb{P}[A], \quad A \in \mathcal{B}(\bar{\Omega})$$

and the upper expectation is

$$\tilde{\mathbb{E}}[X] := \sup_{\mathbb{P} \in \mathcal{P}_{\bar{\mathbb{E}}}} \mathbb{E}_{\mathbb{P}}[X], \quad \forall X \in L^0_{\mathcal{P}_{\bar{\mathbb{E}}}}\big(\bar{\Omega}, \mathcal{B}(\bar{\Omega})\big).$$

Theorem 4.11. *The canonical proces $\{\bar{B}_t\}_{t \geq 0}$ has a continuous modification $\{\tilde{B}_t\}_{t \geq 0}$ such that $\tilde{B}_0 = 0$.*

Proof. Since $\bar{\mathbb{E}}|_{Lip(\bar{\Omega})} = \tilde{\mathbb{E}}|_{Lip(\bar{\Omega})}$ we get

$$\bar{\mathbb{E}}\big[|\bar{B}_t - \bar{B}_s|^4\big] = \tilde{\mathbb{E}}\big[|\bar{B}_t - \bar{B}_s|^4\big] = d|t-s|^2, \quad s, t \in [0, \infty).$$

The last equation holds because $\phi(x) = x^4$ is convex. By theorem 4.10 there exists a continuous modification of $\bar{B}$, let $\tilde{B}$. Moreover, $\tilde{c}([\tilde{B}_0 \neq 0]) = 0$, so we can set $\tilde{B}_0 = 0$. $\square$

It remains to prove that $\overline{\mathcal{P}}_{\bar{\mathbb{E}}}$ is weakly compact. To prove the tightness of $\mathcal{P}_{\bar{\mathbb{E}}}$ we will make use of the next important criterion, the proof of which can be found in [27].

Theorem 4.12 (Kolmogorov-Chentsov)**.** *Let $\{X^k\}_{k\in\mathbb{N}}$ a sequence of $\mathbb{R}^n$-valued continuous processes defined on probability spaces $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$ respectively, such that*

(i) the family $\{\mathbb{P}_{X_0^k}\}_{k\in\mathbb{N}}$ of initial laws is tight in $\mathbb{R}^n$

(ii) $\exists\ \alpha, \beta, \gamma > 0$ such that for each $s, t \in \mathbb{R}_+$ and each $k \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{P}^k}\left[|X_s^k - X_t^k|^\alpha\right] \leq \beta|s-t|^{\gamma+1},$$

then the set $\{\mathbb{P}_{X^k}\}_{k\in\mathbb{N}}$ of distribution of X_k is weakly relatively compact.

Theorem 4.13. *Let $G : \mathbb{S}(n) \to \mathbb{R}$ a sublinear, continuous and monotonic function and $\hat{\mathbb{E}}_G$ the corresponding G-expectation on $(\Omega, Lip(\Omega))$. Then there exists a weakly compact set $\mathcal{P}_G \subset \mathcal{M}(\Omega, \mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}_G[X] = \max_{\mathbb{P}\in\mathcal{P}_G} \mathbb{E}_{\mathbb{P}}[X], \quad X \in Lip(\Omega)$$

Proof. By lemma 4.8 there exists a set $\mathcal{P}_{\bar{\mathbb{E}}}$ with the above described property. We define

$$\mathcal{P} = \{\mathbb{P} \circ \tilde{B}^{-1}\ :\ \mathbb{P} \in \mathcal{P}_{\bar{\mathbb{E}}}\}.$$

Using the Theorem 4.11, we conclude that $\mathcal{P}$ is tight and by denoting $\mathcal{P}_G = \overline{\mathcal{P}}$ we have that it is weakly compact. By lemma 4.7, for each $X \in Lip(\Omega)$, we get $\hat{\mathbb{E}}[|X - (-n) \vee X \wedge n|] \downarrow 0$ and consequently the required result. □

Now we can obtain almost immediately the characterization of G-Brownian paths. Firstly, we need a technical result, for the complete proof of which we refer to [23].

Proposition 4.7. *Let $X \in C_b(\Omega)$ and $\varepsilon > 0$, then there exists $Y \in Lip(\Omega)$ such that $\|X - Y\|_{\bar{\mathbb{E}},1} \leq \varepsilon$.*

Corollary 4.3. *Let $G : \mathbb{S}(n) \to \mathbb{R}$ a sublinear, continuous and monotonic function. Then*

$$L_G^1 = \mathbb{L}^1_{\mathcal{P}_G,c}$$

Chapter 5

G-Lévy Processes

In the current chapter we present the development of the theory of Lévy processes under sublinear expectations. In the cases that proofs are not presented, the interested reader should refer to [15].

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a sublinear expectation space.

Definition 5.1. *A stochastic process* $\{X_t\}_{t\geq 0}$

(i) is called càdlàg *if for each* $\omega \in \Omega$

$$\exists\ X_{t-}(\omega) := \lim_{\delta\downarrow 0} X_{t-\delta}(\omega),\ \ X_t(\omega) = \lim_{\delta\downarrow 0} X_{t+\delta}(\omega)$$

(ii) has independent increments *if for each* $n \in \mathbb{N}$ *and* $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n$, *the increment* $X_{t_n} - X_{t_{n-1}}$ *is independent from* $X_{t_1} - X_{t_0}, \ldots, X_{t_{n-1}} - X_{t_{n-2}}$

(iii) has stationary increments *if for each* s, t *holds* $X_{t+s} - X_t \overset{d}{=} X_s$.

Definition 5.2. *Let* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *a sublinear expectation space and* $X_t \in \mathcal{H}^n$ *for each* $t \geq 0$. *The stochastic process* $\{X_t\}_{t\geq 0}$ *is called* G-Lévy process *if*

(i) $X_0 = 0$,

(ii) is càdlàg,

(iii) has independent increments and

(iv) has stationary increments.

(v) there exists a decomposition $X_t \overset{d}{=} X_t^c + X_t^d$ *for each* $t \geq 0$ *such that*

$$\lim_{t\downarrow 0} \frac{\hat{\mathbb{E}}[|X_t^c|^3]}{t} = 0. \tag{5.1}$$

We will call $\{X_t^c\}_{t\geq 0}$ *the continuous part and* $\{X_t^d\}_{t\geq 0}$ *the jump part.*

Remark 5.1. *In view of theorem 4.2, the assumption imposed for the continuous part of a Lévy process enables us to treat it as a generalised G-Brownian motion.*

Remark 5.2. *In our setting, we will need an extra assumption for the jump part* $\{X_t^d\}_{t\geq 0}$ *to be satisfied, that is*

$$\exists C > 0 \quad : \quad \hat{\mathbb{E}}[|X_t^d|] \leq Ct,\ \textit{for each } t \geq 0.$$

This assumption implies that the jump part is of finite variation.

Until now, we have extensively used the closed connection between stochastic processes and parabolic PDEs. Of course, it was viscosity solutions theory that enabled us to take advantage of solution properties of fully nonlinear second order parabolic PDEs. Having in our mind the classical case, we would like to determine a "G-infinitesimal generator" which will indicate the connection between G-Lévy processes on sublinear expectation spaces and fully nonlinear second order parabolic PDEs

$$\partial_u(t,x) - G(x, u(t,x), D_x u(t,x), D_{xx} u(t,x)) = 0. \tag{5.2}$$

We expect the operator to be nonlocal and, comparing to the form of infinitesimal generator of a Lévy process, end up with an integro-partial differential equation. For that reason we will use the following notation to indicate that on $u(t,x)$ acts the nonlocal part of the G operator.

$$\partial_t u(t,x) - G(u(t,x+\cdot), D_x u(t,x), D_{xx} u(t,x)) = 0.$$

In the current chapter we will make use of the Stone lattice $C_{b.Lip}(\mathbb{R}^n)$, instead of $C_{l.Lip}(\mathbb{R}^n)$. It is obvious that the results of previous chapters stay valid for $C_{b.Lip}(\mathbb{R}^n)$.

Definition 5.3. *Let $\{X_t\}_{t\geq 0}$ a G-Lévy process. The G-infinitesimal generator is defined by*

$$G_X[f] := \lim_{t\downarrow 0} \frac{\hat{\mathbb{E}}[f(X_t)]}{t}, \text{ for } f \in C_b^3(\mathbb{R}^n) \text{ with } f(0) = 0,$$

where $C_b^3(\mathbb{R}^n)$ is the set of bounded and 3-times continuously differential real valued functions defined on $\mathbb{R}^n$ with bounded derivatives of all orders equal to or less than 3.

Lemma 5.1. *For each given $f \in C_b^3(\mathbb{R}^n)$ with $f(0) = 0$ it holds*

$$\hat{\mathbb{E}}[f(X_t)] = \hat{\mathbb{E}}[f(X_t^d) + \langle D_x f(0), X_t^c\rangle + \frac{1}{2}\langle D_{xx} f(0) X_t^c, X_t^c\rangle] + o(t)$$

Proof. It is an application of Taylor approximation, using the assumptions the continuous and jump part satisfy. □

Lemma 5.2. *Let $(p, A) \in \mathbb{R}^n\mathbb{S}(n)$ and $f \in C_b^2(\mathbb{R}^n)$ with $f(0) = 0$. Then the limit*

$$\lim_{t\downarrow 0} \hat{\mathbb{E}}\big[f(X_t^d) + \langle p, X_t^c\rangle + \frac{1}{2}\langle A X_t^c, X_t^c\rangle\big]$$

exists.

Proof. We define

$$h(t) := \hat{\mathbb{E}}\big[f(X_t^d) + \langle p, X_t^c\rangle + \frac{1}{2}\langle A X_t^c, X_t^c\rangle\big],$$

then $h(0) = 0$ and for each $\delta > 0$

$$|h(t+\delta) - h(t)| = \Big|\hat{\mathbb{E}}\big[f(X_{t+\delta}^d) + \langle p, X_{t+\delta}^c\rangle + \frac{1}{2}\langle A X_{t+\delta}^c, X_{t+\delta}^c\rangle\big] - \hat{\mathbb{E}}\big[f(X_t^d) + \langle p, X_t^c\rangle + \frac{1}{2}\langle A X_t^c, X_t^c\rangle\big]\Big| \leq$$

$$\leq \hat{\mathbb{E}}\Big[\big|f(X_{t+\delta}^d) + \langle p, X_{t+\delta}^c\rangle + \frac{1}{2}\langle A X_{t+\delta}^c, X_{t+\delta}^c\rangle - f(X_t^d) - \langle p, X_t^c\rangle - \frac{1}{2}\langle A X_t^c, X_t^c\rangle\big|\Big] \leq$$

$$\leq \hat{\mathbb{E}}\Big[\big|f(X_{t+\delta}^d) - f(X_t^d)\big|\Big] + \hat{\mathbb{E}}\Big[\big|\langle p, X_{t+\delta}^c - X_t^c\rangle + \frac{1}{2}\langle A X_{t+\delta}^c - X_t^c + X_t^c, X_{t+\delta}^c - X_t^c + X_t^c\rangle - \frac{1}{2}\langle A X_t^c, X_t^c\rangle\big|\Big] \leq$$

$$\leq \hat{\mathbb{E}}\Big[\big|f(X_{t+\delta}^d) - f(X_t^d)\big|\Big] + \hat{\mathbb{E}}\Big[\big|\langle p + A X_t^c, X_{t+\delta}^c - X_t^c\rangle + \frac{1}{2}\langle A(X_{t+\delta}^c - X_t^c, X_{t+\delta}^c - X_t^c\rangle\big|\Big] \leq$$

$$\leq C_d \hat{\mathbb{E}}[|X_\delta^d|] + C_c\delta = C\delta.$$

Lipschitz condition ensures that h is almost everywhere differentiable. Let $t_0 < 1$ such that $h'(t_0)$ exists, it can be proved that

$$|\limsup_{d\downarrow 0} \frac{h(\delta)}{\delta} - \liminf_{d\downarrow 0} \frac{h(\delta)}{\delta}| \leq 2C\sqrt{t_0}.$$

by letting $t_0 \downarrow 0$ we get the required result. □

The two preceding lemmata ensure that operator G_X is well-defined. It is also immediate that G_X satisfy monotonicity, subadditivity and positive homogeneity. For the parabolic PDE 5.2 we update the definition of viscosity solution.

Definition 5.4. *A bounded upper (resp. lower) semicontinuous function u is called a viscosity subsolution (resp. supersolution) of the equation 5.2 if*

$$u(0,x) \leq \phi(x) \quad \big(resp.\ u(0,x) \geq \phi(x)\big)$$

and $\forall\ (t,x) \in (0,\infty)\times\mathbb{R}^n$, $\forall\ \psi \in C_b^{2,3}$ such that $\psi \geq u$ (resp. $\psi \leq u$) $\psi(t,x) = u(t,x)$ it holds

$$\partial_t\psi(t,x) - G\big(\psi(t,x+\cdot) - \psi(t,x)\big) \leq 0 (resp. \geq 0).$$

A bounded continuous function u is called a viscosity solution of the equation 5.2 if it is both viscosity solution and viscosity subsolution.

Theorem 5.1. *Let $\{X_t\}_{n\in\mathbb{N}}$ an n-dimensional G-Lévy process. Then*

$$u(t,x) := \hat{\mathbb{E}}[\phi(x+X_t)]$$

is a viscosity solution of

$$\partial_t u(t,x) - G_X\big(u(t,x+\cdot) - u(t,x)\big) = 0 \tag{5.3}$$

with initial condition

$$u(0,x) = \phi(x)$$

Proof. We will show that u is continuous. It suffices to prove that u is Lipschitz in x and $\frac{1}{2}-$Hölder in t. For the Lipschitz condinuity in x we have,

$$|u(t,x) - u(t,y)| = \big|\hat{\mathbb{E}}[\phi(x+X_t)] - \hat{\mathbb{E}}[\phi(y+X_t)]\big| \leq \hat{\mathbb{E}}\big[|\phi(x+X_t) - \phi(y+X_t)|\big] \leq C_\phi|x-y|.$$

To prove the $\frac{1}{2}-$Hölder in t we firstly prove a useful equality

$$u(t+s,x) = \hat{\mathbb{E}}\big[\phi(x+X_{t+s})\big] = \hat{\mathbb{E}}\big[\phi(x+X_{t+s} - X_t + X_t)\big] =$$

$$= \hat{\mathbb{E}}\Big[\hat{\mathbb{E}}\big[\phi(x+\zeta+X_{t+s}-X_t)\big]_{\zeta=X_t}\Big] = \hat{\mathbb{E}}\Big[\hat{\mathbb{E}}\big[\phi(x+\zeta+X_s)\big]_{\zeta=X_t}\Big] =$$

$$= \hat{\mathbb{E}}\Big[\hat{\mathbb{E}}\big[\phi(x+X_s+X_t)\big]\Big] = \hat{\mathbb{E}}\big[u(t,x+X_s)\big],$$

then

$$|u(t+s,x) - u(t,x)| = |\hat{\mathbb{E}}\big[u(t,x+X_s)\big] - u(t,x)| =$$

$$= \big|\hat{\mathbb{E}}\big[u(t,x+X_s) - u(t,x)\big]\big| \leq \hat{\mathbb{E}}\big[|u(t,x+X_s) - u(t,x)|\big] \leq$$

$$\leq C_\phi\hat{\mathbb{E}}[|X_s|] = C_\phi\hat{\mathbb{E}}[|X_s^c + X_s^d|] \leq C_\phi\big(\hat{\mathbb{E}}[|X_s^c|] + \hat{\mathbb{E}}[|X_s^d|]\big)$$

and for $s \leq 1$ we take

$$|u(t+s,x)-u(t,x)| \leq C_\phi \sqrt{s}.$$

for each fixed $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and $\psi \in C_b^{2,3}$ such that $u \leq \psi$ and $u(t,x) = \psi(t,x)$, we have

$$\psi(t,x) = u(t,x) = \hat{\mathbb{E}}[u(t-\delta, x+X_\delta)] \leq \hat{\mathbb{E}}[\psi(t-\delta, x+X_\delta)]$$

and therefore

$$\begin{aligned}
0 \leq \hat{\mathbb{E}}[\psi(t-\delta,x+X_\delta)] - \psi(t,x) = \hat{\mathbb{E}}[\psi(t-\delta,x+X_\delta) - \psi(t,x)] = \\
= \partial_t\psi(t,x)\delta + \hat{\mathbb{E}}[\psi(t,x+X_\delta) - \psi(t,x) + I_\delta] \leq \\
\leq \partial_t\psi(t,x)\delta + \hat{\mathbb{E}}[\psi(t,x+X_\delta) - \psi(t,x)] + \hat{\mathbb{E}}[|I_\delta|]
\end{aligned}$$

where

$$I_\delta = \delta \int_0^1 [\partial_t\psi(t,x) - \partial_t\psi(t-\beta\delta, x+X_\delta)]d\beta.$$

It holds that

$$\hat{\mathbb{E}}[|I_\delta|] \leq C\hat{\mathbb{E}}[\delta(\delta + |X_\delta|)] = o(\delta).$$

Since $\forall \delta > 0$

$$0 \leq \partial_t\psi(t,x)\delta + \hat{\mathbb{E}}[\psi(t,x+X_\delta) - \psi(t,x)] + o(\delta)$$

and taking into account the definition of G_X, we get

$$\partial_t\psi(t,x)\delta + G_X[\psi(t,x+\cdot) - \psi(t,x)] \leq 0.$$

Hence, u is a viscosity subsolution of 5.3. In an analogous way we can prove that u is a viscosity supersolution and consequently, u is a viscosity solution. □

Remark 5.3. *A careful reader may have noticed that in the previous theorem we did not mentioned that the solution is unique. The uniqueness is obtained due to the corresponding Lévy-Khintchine representation of G-infinitesimal operator.*

Theorem 5.2. *Let $\{X_t\}_{n\in\mathbb{N}}$ an n-dimensional G-Lévy process. Then G_X has the Lévy-Khinchine representation*

$$G_X[f] = \sup_{(v,q,A)\in\mathcal{U}} \left\{ \int_{\mathbb{R}^n\setminus\{0\}} f(z)v(dz) + \langle D_x f(0), q\rangle + \frac{1}{2}tr[D_{xx}f(0)A] \right\} \tag{5.4}$$

for a suitable $\mathcal{U} \subset \mathcal{M} \times \mathbb{R}^n \times \mathbb{S}_+(n)$ satisfying

$$\sup_{(v,q,A)\in\mathcal{U}} \left\{ \int_{\mathbb{R}^n\setminus 0} |z|v(dz) + |q| + tr[A] \right\} < \infty, \tag{5.5}$$

where $\mathcal{M}$ is the set measures on $\Big(\mathbb{R}^n\setminus\{0\}, \mathcal{B}(\mathbb{R}^n\setminus\{0\})\Big)$.

The conditions 5.5 imposed on 5.4 satisfy the hypotheses for the uniqueness of viscosity solution. We restate that the interested reader should consult [15, Appendix] for a comprehensive presentation of viscosity solutions of fully nonlinear second order parabolic PDEs.

Corollary 5.1. *Let $\{X_t\}_{n\in\mathbb{N}}$ an n-dimensional G-Lévy process. Then*

$$u(t,x) := \hat{\mathbb{E}}[\phi(x+X_t)]$$

is a viscosity solution of

$$\partial_t u(t,x) - \sup_{(v,q,A)\in\mathcal{U}} \Big\{ \int_{\mathbb{R}^n\setminus\{0\}} \big(u(t,x+z)-u(t,x)\big)v(dz) + \langle D_x f(0), q\rangle + \frac{1}{2} tr[D_{xx} f(0)A] \Big\} \tag{5.6}$$

with initial condition

$$u(0,x) = \phi(x)$$

where $\mathcal{U}$ represents G_X.

5.1 Existence of G-Lévy processes

In the current chapter we denote $\Omega = \mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n)$, where $\mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued càdlàg functions $(\omega_t)_{t\geq 0}$ with$\omega_0 = 0$. The space $\mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n)$ under the Skorokhod topology is a Polish space. The canonical process is $B_t(\omega) = \omega_t$, for $\omega \in \Omega$.

Following an analogous construction with that of section 4.1, we define

$$Lip(\Omega) = \{\phi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1}, \ldots, B_{t_k}-B_{t_{k-1}}) : \\ k\in\mathbb{N},\ 0=t_0<t_1<\ldots<t_k,\ \phi\in C_{b.Lip}(\mathbb{R}^{n\times k})\big\}.$$

and

$$Lip(\Omega_T) = \big\{\phi(B_{t_1\wedge T}-B_{t_0}, B_{t_2\wedge T}-B_{t_1\wedge T}, \ldots, B_{t_k\wedge T}-B_{t_{k-1}\wedge T}) : \\ k\in\mathbb{N},\ 0=t_0<t_1<\ldots<t_k,\ \phi\in C_{b.Lip}(\mathbb{R}^{n\times k})\big\}.$$

It is immediate that for $t\leq T$, $Lip(\Omega_t)\subset Lip(\Omega_T)$ and

$$Lip(\Omega) = \bigcup_{k\in\mathbb{N}} Lip(\Omega_k).$$

For the case of G-Lévy processes we need to verify the existence and uniqueness of the viscosity solution of second order fully nonlinear parabolic integro-PDEs of the form

$$\partial_t u(t,x) - G(u(t,x+\cdot), D_x u(t,x), D_{xx}u(t,x)) = 0, \quad (t,x)\in(0,T)\times\mathbb{R}^n,$$

$$u(0,\cdot) = \phi(\cdot)$$

with $\phi\in C_{b.Lip}(\mathbb{R}^n)$ and $G : C_b^{1,2}\times\mathbb{R}^n\times\mathbb{S}(n)\to\mathbb{R}$, for which we need to enhance the hypotheses to obtain the required results. We refer to [15, Appendix] for a complete presentation of the necessary hypotheses and desirable results. We have also to point out [15, Remark 43] which justifies the modifications of hypotheses for our needs. This is crucial since we need to include the case the partaking measures be singular. However, we inform the reader that the results are essentially the same with theorem 3.1. Since in our setting G operator satisfies the required hypotheses, we can follow an analogous construction for a sublinear expectation $\hat{\mathbb{E}}_G$ on $(\mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n), C_{b.Lip}(\mathbb{R}^n))$. Firstly, we remind the required results by presenting the enhanced theorem for equation 5.6.

Theorem 5.3. *Let u^ϕ denote the unique viscosity solution of 5.6 with initial condition $\phi\in C_{b.Lip}(\mathbb{R}^n)$. Then we have*

(i) $u^{\lambda\phi} = \lambda u^\phi$ *for each* $\lambda > 0$

(ii) $u^{\phi+c} = u^{\phi} + c$, *for* $c \in \mathbb{R}$

(iii) $u^{\phi+\psi} \leq u^{\phi} + u^{\psi}$

(iv) *If* $\phi \leq \psi$, *then* $u^{\phi} \leq u^{\psi}$

(v) $u^{\phi}(t+s,x,y) = u^{u^{\phi}(t,x+\cdot,y+\cdot)}(s,0,0)$, *for* $\lambda \geq 0$.

Initially, let $X \in Lip(\Omega)$ of the form $X = \phi(B_{t+s} - B_t)$, with $t,s \geq 0$ and $\phi \in C_{b.Lip}(\mathbb{R}^n)$. Let u^{ϕ} the unique viscosity solution of 5.6 with initial condition $u(0,x) = \phi(x)$. We define

$$\hat{\mathbb{E}}_G[X] := u^{\phi}(s,0).$$

Going one step further, if $X \in Lip(\Omega)$ of the form $X = \phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_2})$, with $0 = t_0 < t_1 < t_2$ and $\phi \in C_{b.Lip}(\mathbb{R}^n \times \mathbb{R}^n)$. We define

$$\phi_1(x_1) := \hat{\mathbb{E}}_G\big[\phi(x_1, B_{t_2} - B_{t_1})\big] = u^{\phi(x_1,\cdot)}(t_2 - t_1, 0)$$

where $u^{\phi(x_1,\cdot)}$ the unique viscosity solution of 5.6 with initial condition $u(0,\cdot)(x_1) = \phi(x_1,\cdot)$ and

$$\hat{\mathbb{E}}_G\big[X\big] = \hat{\mathbb{E}}_G\big[\phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1})\big] := u^{\phi_1}(t_1,0) = u^{u^{\phi(\star,\cdot)}(t_2-t_1,0)}(t_1,0)$$

It is immediate that

$$\hat{\mathbb{E}}_G\big[X\big] = \hat{\mathbb{E}}_G\big[\phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1})\big] = \hat{\mathbb{E}}_G\big[\hat{\mathbb{E}}_G[\phi(x_1, B_{t_2} - B_{t_1})]_{x_1 = B_{t_1} - B_{t_0}}\big].$$

We continue by defining $\hat{\mathbb{E}}_G$ for the case $X = \phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}})$, with $0 = t_0 < t_1 < \ldots < t_k$ and $\phi \in C_{b.Lip}(\mathbb{R}^{n \times k})$, recursively in the following sense:

$$\phi_1(x_1, x_2, \ldots, x_{k-1}) = \hat{\mathbb{E}}_G[\phi(x_1, x_2, \ldots, x_{k-1}, B_{t_k} - B_{t_{k-1}})]$$

$$\phi_2(x_1, x_2, \ldots, x_{k-2}) = \hat{\mathbb{E}}_G[\phi_1(x_1, x_2, \ldots, x_{k-2}, B_{tk_1} - B_{t_{k-2}})]$$

$$\vdots$$

$$\phi_{k-1}(x_1) = \hat{\mathbb{E}}_G[\phi_{k-2}(x_1, B_{t_2} - B_{t_1})]$$
$$\hat{\mathbb{E}}_G = \hat{\mathbb{E}}_G[\phi_{k-1}(B_{t_1})]$$

The related conditional expectation of X under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}_G[X|\Omega_{t_j}] = \hat{\mathbb{E}}_G\big[\phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}})|\Omega_{t_j}\big] :=$$

$$\phi_{k-j}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}}) =$$

$$= \hat{\mathbb{E}}_G\big[\phi_{k-j-1}(x_1, x_2, \ldots, x_{j-1}, B_{t_{j+1}} - B_{t_j}, \ldots, B_{t_k} - B_{t_{k-1}})_{(x_1, x_2, \ldots, x_{j-1}) = (B_{t_1} B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}})}\big]$$

By theorem 5.3 it is immediate that $\hat{\mathbb{E}}_G$ defines consistenly a sublinear expectation on $Lip(\Omega)$. Under the norm $\|X\| := \hat{\mathbb{E}}_G[|X|]$ we can continuously extend $\hat{\mathbb{E}}_G$ and $\hat{\mathbb{E}}_G[\cdot|\Omega_t]$ on $\big(\Omega, L_G^p(\Omega)\big)$, $p \geq 1$, such that $\hat{\mathbb{E}}_G : L_G^p(\Omega) \to \mathbb{R}$ and $\hat{\mathbb{E}}_G[\cdot|\Omega_t] : L_G^p(\Omega) \to L_G^p(\Omega_t)$ where

$$L_G^p(\Omega) := \mathcal{L}_G^p(\Omega)/\mathcal{L}_{G,0}^p(\Omega),$$

$$\mathcal{L}_G^p(\Omega) := \overline{Lip(\Omega)}^{\|\cdot\|_{G,p}},$$
$$\mathcal{L}_{G,0}^p(\Omega) := \{X \in L_G^p(\Omega) \mid \|X\|_{G,p} = 0\}$$

with

$$\|X\|_{G,p} := \big(\hat{\mathbb{E}}_G[|X|^p]\big)^{\frac{1}{p}}.$$

Analogously we define $L_G^p(\Omega_t)$.

Theorem 5.4. *The canonical process $\{B_t\}_{t\geq 0}$ is a G-Lévy process.*

Proof. We conseider $\mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n)$ and the canonical process $(\bar{B}_t,\tilde{B}_t)_{t\geq 0}$. Due to the decompoistion of canonical process and the Lévy-Khintchine formula in our setting, it can be proved that $(\bar{B}_t)_{t\geq 0}$ is generalised G-Brownian motion and $(\tilde{B}_t)_{t\geq 0}$ satisfies the

$$\partial_t u(t,x) - \sup_{v\in\mathcal{V}}\Big\{\int_{\mathbb{R}^n\setminus\{0\}}\big(u(t,x+z)-u(t,x)\big)v(dz)\Big\} \tag{5.7}$$

and the required hypothesis. □

Example 5.1. *Let $[\lambda_1,\lambda_2],[1-a,1+b]\subset\mathbb{R}_+$ and*

$$\frac{\partial u}{\partial t} - G_{[\lambda_1,\lambda_2]\times[a,b]}(u(t,x+\cdot)-u(t,x)) = 0, \quad u(0,x)=\phi(x),$$

where

$$G_{[\lambda_1,\lambda_2]\times[1-a,1+b]}(f) = \sup_{(\lambda,\nu)\in\{[\lambda_1,\lambda_2]\times\{\delta_\theta\}_{\theta\in[1-a,1+b]}\}} \lambda\int_{\mathbb{R}^n\setminus\{0\}} f(z)\nu(dz).$$

Under $\hat{\mathbb{E}}_G$ the canonical process $B.$ is called G-Poisson. *We have to observe that we have uncertainty in intensity as well as in jumps.*

5.2 G-expectation as an upper uxpectation

In this section, we will use $\bar{\Omega}$ and related notation (e.g. $\mathcal{B}(\bar{\Omega}), Lip(\bar{\Omega}), \bar{\mathbb{E}}, \tilde{c}, \tilde{\mathbb{E}}$ etc.) as introduced in section 4.3. However, Kolmogorov-Chentsov's tightness criterion need to be modofied since the path space Ω is $\mathbb{D}_0(\mathbb{R}_+,\mathbb{R}^n)$ with Skorokhod topology, as well as a generalisation of Kolmogorov-Chentsov's criterion for càdlàg modification with respect to capacity need to be obtained. The proof of latter criterion is presented in [26].

The structure of the current section is similar to 4.3.

Theorem 5.5 (Kolmogorov-Chentsov). *Let $\{X_t\}_{t\in[0,1]}$ a stochastic process such that, $X_t\in\mathbb{L}^1$ for each t. If the following are satisfied*

(i) $\forall t\in[0,1]$, $\exists\alpha>0$ *such that* $\lim_{s\to t}\hat{\mathbb{E}}[|X_s-X_t|^\alpha]=0$

(ii) $\exists C,r>0,p,q\geq 0$ *with* $p+q>0$ *and for all* $0\leq s\leq u\leq t\leq 1$, *it holds*

$$\hat{\mathbb{E}}[|X_t-X_u|^p|X_u-X_t|^q]\leq C|t-s|^{1+r},$$

then X admits a càdlàg modification.

Theorem 5.6. *The canonical process $\bar{B}$ admits a càdlàg modification, let $\tilde{B}$, such that $\tilde{B}_0=0$ $\tilde{c}$-q.s.*

Proof. The canonical process $\bar{B}$ can be decomposed to $\bar{B}=\bar{B}^c+\bar{B}^d$, in continuous and discontinuous part. The continuous part has a continuous modification $\tilde{B}^c$. Discontinuous part, taking into account the hypothesis of remark 5.2 and the representation of $\bar{\mathbb{E}}$, we have that

$$\tilde{\mathbb{E}}[|\bar{B}^d_t-\bar{B}^d_u|^p|\bar{B}^d_u-\bar{B}^d_t|^q] = \bar{\mathbb{E}}[|\bar{B}^d_t-\bar{B}^d_u|^p|\bar{B}^d_u-\bar{B}^d_t|^q]\leq C|t-s|^2,$$

where $C>0$. Kolmogorov-Chentsov's criterion ensures that $\bar{B}^d$ admits a càdlàg modification, let $\tilde{B}^d$. Thus,

$$\tilde{B}=\tilde{B}^c+\tilde{B}^d$$

is the desired càdlàg modification. □

Theorem 5.7 (Kolmogorov-Chentsov)**.** *Let $\mathcal{P} \subset \mathcal{M}\big(\mathbb{D}_0([0,T],\mathbb{R}), \mathcal{B}(\mathbb{D}_0([0,T],\mathbb{R}))\big)$ and $\mathbb{E}^{\mathcal{P}}$ the related upper expectation. If the following are satisfied*

(i) $\exists \alpha > 0$ *such that* $\mathbb{E}^{\mathcal{P}}[|\omega_s - \omega_t|^\alpha] \leq C|t-s|^\alpha$, $\forall s,t \in [0,T]$

(ii) $\exists C, r > 0, p, q \geq 0$ *with* $p+q > 0$ *and for all* $0 \leq s \leq u \leq t \leq 1$*, it holds*

$$\mathbb{E}^{\mathcal{P}}[|\omega_t - \omega_u|^p |\omega_u - \omega_t|^q] \leq C|t-s|^{1+r},$$

then $\mathcal{P}$ is tight.

Lemma 5.3. *The subset $\mathcal{P} \subset \mathcal{M}\big(\Omega, \mathcal{B}(\Omega)\big)$ defined by*

$$\mathcal{P} := \{\mathbb{P} \circ \tilde{B}^{-1} \ : \ \mathbb{P} \in \mathcal{P}_{\bar{\mathbb{E}}}\}$$

is tight.

Proof. The following inequalities

$$\tilde{\mathbb{E}}[|\tilde{B}^c_t - \tilde{B}^c_s|]^4 = \bar{\mathbb{E}}[|\bar{B}^c_t - \bar{B}^c_s|]^4 \leq C|t-s|^2,$$

$$\tilde{\mathbb{E}}[|\tilde{B}^d_t - \tilde{B}^d_u|^p|\tilde{B}^d_u - \tilde{B}^d_s|^q] = \bar{\mathbb{E}}[|\bar{B}^d_t - \bar{B}^d_u|^p|\bar{B}^d_u - \bar{B}^d_s|^q] \leq C|t-s|^2,$$

in conjuction with tightness criterion 5.7 give the required result. □

Theorem 5.8. *Let X a G-Lévy process and G_X its G-infitesimal generator. Then there exists a weakly compact set $\mathcal{P}_G$ such that*

$$\hat{\mathbb{E}}_{G_X}[X] = \max_{\mathbb{P} \in \mathcal{P}_G} \mathbb{E}_{\mathbb{P}}[X], \quad X \in Lip(\Omega).$$

Comments

Our main target in the current thesis was to compare the generalisation of Lévy processes in sublinear expectation spaces with that of classical case. For that reason we followed a fast track. It would be omission, however, not to mention other results that can be generalised in the sublinear expectation spaces framework. These include Law of Large Numbers (in which the limit is a maximal distributed r.v.), Central Limit Theorem, Stochastic Integration with respect to (centralised) G-Brownian Motion, Itô's formula and Martingale Representation Theorem. It is really interesting the fact that the quadratic variation of centralised G-Brownian motion is a stochastic process (instead of a deterministic function of classical case) whose distribution is maximal distributed. To describe in an intuitive way the connection between G-Brownian Motion (G-BM) and its quadratic variation we suppose we are in one-dimensional space. In that case, the interval that describes the uncertainty of G-BM coincides with the uncertainty interval of quadratic variation. For a complete presentation of the aforementioned results we refer to [23].

On the other hand, the most results that have been obtained until now are restricted to the corresponding continuous case. Moreover, what is known regarding G-Lévy processes is restricted in the corresponding bounded variation case of classical framework. An explanation that can be given is that the hypothesis of remark 5.2 plays a crucial role in proving the well-posedness of G-infinitesimal generator. Moreover, there is no analogous of Doob-Meyer decomposition theorem, so we cannnot define a stochastic integral with respect to a "right-continuous Martingale". However, it seems that, since the framework of sublinear expectation spaces can be regarded as a special case of Aggregation of Stochastic Processes, the aforementioned obstacles may be overcomed. In the framework of Aggregation of Stochastic Processes can be obtained the analogous theorems on which classical Stochastic Calculus is based. For the interested reader of Aggregation of Stochastic Processes, we refer to [30, 17].

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