Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών
Τομέας Τεχνολογίας Πληροφορικής και Υπολογιστών

Πάχος Γραφήματος και Παραλλαγές, Πάχος Σχεδίασης Γραφήματος, Πολυπλοκότητα

Διπλωματική Εργασία

του

Αλέξανδρου Αγγελόπουλου

Επιβλέπων: Ευστάθιος Ζάχος
Καθηγητής Ε.Μ.Π.

Συνεπιβλέποντες: Δημήτρης Φωτάκης
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Άρης Παγουρτζής
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Εργαστήριο Λογικής και Επιστήμης Υπολογισμών
Αθήνα, Ιούλιος 2012

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Εγκρίθηκε από την τριμελή εξεταστική επιτροπή την 23η Ιουλίου 2012.

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Ευστάθιος Ζάχος	Άρης Παγουρτζής	Δημήτρης Φωτάκης
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Αλέξανδρος Αγγελόπουλος
Διπλωματούχος Ηλεκτρολόγος Μηχανικός και Μηχανικός Υπολογιστών Ε.Μ.Π.

Περίληψη

Θα αναφερθούμε σε προβλήματα που αφορούν τη σχεδίαση γραφημάτων σε επίπεδα, τα οποία συνήθως βρίσκουν εφαρμογές στα VLSI κυκλώματα και την απεικόνιση των γραφημάτων. Παρουσιάζουμε μια εναλλακτική ώθηση για τη μελέτη της σχεδίασης γραφημάτων από το πεδίο της διαχείρησης της εναέριας κυκλοφορίας και ειδικότερα τα πρότυπα που προκύπτουν από (ή φαίνονται ως) πτήσεις σε ευθεία μεταξύ αεροδρομίων ή σημείων διέλευσης. Εισάγουμε το πάχος σχεδίασης (ϑ) της σχεδίασης D ενός γραφήματος ως *τον ελάχιστο αριθμό επιπέδων στα οποία μπορούμε να διαχωρίσουμε τις ακμές μιας (αναλλοίωτης) σχεδίασης γραφήματος, έτσι ώστε μέσα σε ένα επίπεδο οι ακμές να μη διασταυρώνονται*, και συζητάμε για τις καλά μελετημένες έννοιες του πάχους γραφήματος (θ), του γεωμετρικού πάχους ($\bar{\theta}$) και του πάχους εμφύτευσης σε βιβλίο (bt). Εξερευνούμε την ιστορία και σημαντικά αποτελέσματα για το πάχος γραφήματος και το γεωμετρικό πάχος γραφήματος, συμπεριλαμβανομένων των ιδιοτήτων του πάχους του K_n και του $K_{m,n}$. Βασιζόμενοι στον ορισμό του πάχους εμφύτευσης σε βιβλίο που χρησιμοποιεί μια κυρτή εμφύτευση των κορυφών του γραφήματος στο επίπεδο, επικεντρονώμαστε στις κυρτές σχεδιάσεις D_{conv} ενός γραφήματος και παρουσιάζουμε μερικές ιδιότητες, μεταξύ άλλων και για να αποδείξουμε εκ νέου ότι $bt(K_n) = \vartheta(D_{conv}(K_n)) = \lceil \frac{n}{2} \rceil$. Προχωρώντας στις αυθαίρετες σχεδιάσεις του γραφήματος G, ισχυριζόμαστε πως $\vartheta(K_n) \leq \lceil \frac{n}{2} \rceil$ για κάθε σχεδίαση. Στο τελευταίο κεφάλαιο που αφορά στην πολυπλοκότητα, ορίζουμε μια οικογένεια προβλημάτων που αφορούν το πάχος γραφήματος, δίνοντας ιδιαίτερη προσοχή στο αντίστοιχο ως προς τη δουλειά μας πρόβλημα χρωματισμού: "*Δοσμένης μιας σχεδίασης ενός γραφήματος, μπορούν οι ακμές του να διαχωριστούν σε k επίπεδα*"; Χρησιμοποιούμε τα SEG γραφήματα, τα γραφήματα τομής ενός συνόλου ευθυγράμμων τμημάτων στο επίπεδο, και την κλάση-υποσύνολο των circle γραφημάτων για να δείξουμε ότι το πρόβλημα είναι NP-complete τόσο για μια τυχαία $k \geq 3$, όσο και για μια κυρτή σχεδίαση $k = 3$, αποδεικνύοντας στην πορεία ότι τα CROSS γραφήματα, τα γραφήματα διασταύρωσης ευθυγράμμων τμημάτων, συμπίπτουν με τα SEG γραφήματα. Τέλος, αναφερόμαστε σε 3 προβλήματα ύπαρξης τριγωνοποίησης για μια σχεδίαση γραφήματος, *TRI, poly-TRI* και *convex TRI*, που αφορούν κατ΄ αντιστοιχία στην τριγωνοποίηση συνόλου σημείων, την τριγωνοποίηση πολυγώνου και την τριγωνοποίηση κυρτού πολυγώνου (ή κυρτού συνόλου σημείων). Παρουσιάζουμε μια παραλλαγή της κλάσης των SEG γραφημάτων, την SEG_h, για να αναπαράγουμε ότι το πρόβλημα *TRI* είναι NP-complete, ενώ διαμέσου των circle γραφημάτων, το πρόβλημα *convex TRI* είναι στο P.

Abstract

In this thesis, we discuss problems of layered graph drawing which usually apply to VLSI circuits and graph visualizations. We present an alternate motivation to study graph drawing from the area of air traffic management and especially the patterns occurring from (or appearing as) direct-to flights between airports or waypoints. We introduce drawing thickness (ϑ) of a graph drawing D as *the minimum number of layers to which the edges of a (fixed) drawing of a graph are partitioned to, so that within any layer edges do not cross*, and discuss the well studied graph-theoretical thickness (θ), geometric thickness ($\bar{\theta}$) and book thickness (bt). We explore the history and significant results on thickness and geometrical thickness, including the thickness properties of K_n and $K_{m,n}$. Based on the definition of book thickness using a convex placement of the vertices of the graph, we focus in graph's convex drawings D_{conv} and present some properties to -among others- independently show that $bt(K_n) = \vartheta(D_{conv}(K_n)) = \lceil \frac{n}{2} \rceil$. Moving on to arbitrary drawings of graph G, we claim that $\vartheta(K_n) \leq \lceil \frac{n}{2} \rceil$ for any drawing, bound being tight. In the final chapter concerning complexity, we introduce a family of thickness-related problems, paying particular attention on the respective to our work *COLOR* problem: *"Given a drawing of a graph, can its edges be decomposed into k planar layers"*? We use SEG graphs, the intersection graphs of line segments on the plane, and their subclass of circle graphs to show the problem is *NP*-complete both for an arbitrary $k \geq 3$ and a convex drawing $k > 3$, proving alongside that CROSS graphs, the crossing graphs of line segments, coincide with class SEG. Last, we mention 3 triangulation existence problems for a graph drawing, *TRI, poly-TRI* and *convex TRI*, dealing with point set triangulation, polygon triangulation and convex polygon (or convex point set) triangulation respectively. We introduce a variation of the SEG class, SEG_h, to reproduce that *TRI* is *NP*-complete, while through circle graphs we get *convex TRI* is in P.

Keywords

graph thickness, drawing thickness, book thickness, geometrical thickness, convex graph drawing, complexity, SEG graphs, circle graphs, polygon triangulation, point set triangulation

Εν είδει Ευχαριστιών

Σε όλους του μαθηματικούς που κατά καιρούς
με μύησαν στη μαγεία των αριθμών,
αι με μια ιδαίτερη αφιέρωση στον κ. Τ.Μ.

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Chapter 1

Introduction and motivation

The graph is a mathematical structure. Its uniqueness lies on its abstractness: given a set of objects x_i and a pairwise relation R of them, the (undirected) graph $G(V, E)$ is the structure that contains the information on the object set (vertex set) V and the relations it includes: edge set $E \subseteq \binom{V}{2}, (x_i, x_j) \in E \Leftrightarrow R(x_i, x_j)$. Despite being abstract, graph theory way born partially because mathematicians (and later on a wide number of scientists) wanted to actually draw and visualize these relations as it should provide insight in problem solving. So it is natural that a graph's abstractness conflicts with it being drawn.

The question of planarity of graphs indicates this, and it is in the core of all problems presented in this thesis: *A graph is planar if it can be drawn in the plane without its edges to cross.* More on, if a graph is not planar, what is the minimum number of planar layers it can be decomposed to?

It is evident that the applications of what we call "layered graph drawing" include, first of all, graph visualization, in order to help the one to study graphs to interpret the drawing. The second major application is VLSI design. Within a PCB, it is unlikely that all electronic components (vertices) along with the short-circuits between them (edges) can be printed in a single layer, i.e. the corresponding graph is non-planar. So, the decomposition of a graph is of high importance, and indicates the number of layers within a PCB.

From our part, we are going to consider a different motivation to study layered graph drawing: air traffic management (ATM). We believe that both existing and new results may be important for the design of air flows and the analysis of its complexity. We study the case of direct-to flights between airports or waypoints, and define a graph model to work on.

1.1 Motivation

Based on a number of works regarding air flows (we will indicatively cite a few: [4], [42], [27], [46]), we present some terms to fully understand what follows.

Air traffic control (ATC) is a service provided by ground-based controllers who direct aircraft on the ground and in the air. The primary purpose of ATC systems worldwide is to *separate aircraft* (to avoid collisions), to organize and expedite the flow of traffic, and to provide information and other support for pilots when able.

Separation is a term that describes the actions taken to prevent aircraft from coming too close to each other (in conflict) by use of lateral, vertical and longitudinal separation minima.

Airway is a designated route in the air. Airways are laid out between navigational aids such as VORs, NDBs (radio beacons) and Intersections (virtual radio beacons). Airways are of certain width of a few miles (8-10nm), but of no specific altitude. Thus, they are initially defined as 3-dimensional objects. Airways are divided into low-altitude and hi-altitude, and by assigning an aircraft to fly them at a certain *flight level* they obtain their 2-dimensional substance (w.r.t. the earth's surface, i.e. a spherical coordinate system).

Flight Level (FL) is the standard nominal altitude of an aircraft in hundreds of feet. It is calculated using the International Standard Atmosphere (ISA) model. We write *FLxxx* where xxx is the altitude divided by 100.

When airplanes travel, they most commonly use airways to get to their destination. An alternative way to fly is the so called *direct-to* flight: fly the shortest route between any 2 points. When chosen, direct-to flight is probably just a part of the route, and it is mostly used in long transcontinental routes. However, attempts and studies are made to maximize direct-to routes, in order to minimize the cost of the flight (minimization of fuel needed). The drawback of this choice is the increased complexity of air traffic patterns. Let us explain:

The safest way to separate planes is vertical separation. That's easily understood: If the altitude of 2 or more airplanes is different, then they will not collide no matter their route or speed, or time reaching an airway intersection. Therefore vertical separation becomes the AT controller's priority. Only if vertical separation is impossible the controller resorts to lateral (horizontal) and then longitudinal (time) separation. With this in mind, let's give an example to understand what we mean by claiming that *direct-to flight increases the complexity of air traffic patterns.*

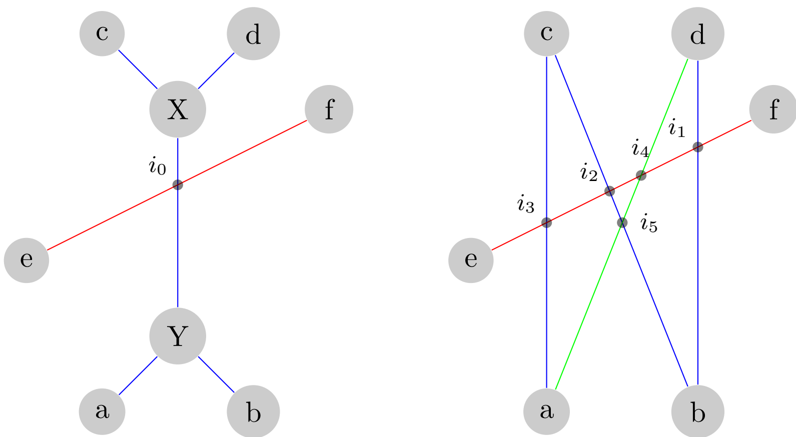


Figure 1.1: 6 airports, 5 routes. Using airway XY vs. using direct-to flight

Above is shown a set of 6 airports (a through f). Let us assume that the following routes need to be flown: ac, ad, bc, bd and ef. Using the airway XY, the routes from the bottom to the top airports become a bit more expensive but they all intersect route ef at the same point i_0. On the contrary, direct-to flights result in optimum traveling distance for every route, but there are 5 points of intersection with i_2, i_4 and i_5 being particularly important. Now assume that we cannot foreknow when aircraft will approach the intersections in their routes[1]. As a result, vertical separation of intersecting routes would be the ultimate trouble-solver.

[1] This is not at all unlikely to occur: The departure of flights depends on the traffic in the airport, not in the air. Initial horizontal separation may be lost due to winds or other parameters affecting a flight. Some routes may be heavily loaded barely keeping up with time separation minima.

In the case where airway XY exists, seeing that aircraft using it will reach X or Y at different or (in general) appropriate times, we only need to assign the same altitude to routes using XY (blue color), and another one to route ef (red color). Thus 2 flight levels (FL) are needed, whereas in the second case 3 FLs (colors) are needed, as a result of ad, bc and ef all intersecting with each other (triangle $i_2 i_4 i_5$).

1.2 Setting the problem and suggesting a model

Given a set of airports and a set of routes, *the minimum number of FLs that can be statically assigned to the routes so that intersecting routes are flown at different FLs* can be used as a *measure* of how complex an air traffic pattern can get. And it seems that extinction of airways will indeed increase complexity in air routes.

In this work, we will only examine patterns occurring from direct-to flights between airports. So it is quite clear that we need a geometric graph model, where nodes (vertices) represent airports, and use only straight lines as (undirected) edges between nodes to represent the direct-to/minimum distance routes. Intersections of edges have a major role in our model, as our goal will be *to determine an optimum edge coloring where every pair of intersecting edges have different colors.*

Explaining the nodes. Aircraft need airspace to gain their flying altitude after take-off and airspace to lose altitude before landing. We assume that a node not only represents the airport, but also the airspace around it used for departure and arrival procedures, which we will simply call *Airport Area*. While over this area, it is pointless for aircraft flying from or to the airport to maintain direct-to flight[2]. Over Airport Area, controllers are responsible to keep aircraft separated (Property 1), guide them through departure and arrival patterns, and make sure that departing flights exit this circular area at the desired altitude and at the point closest to their destination.

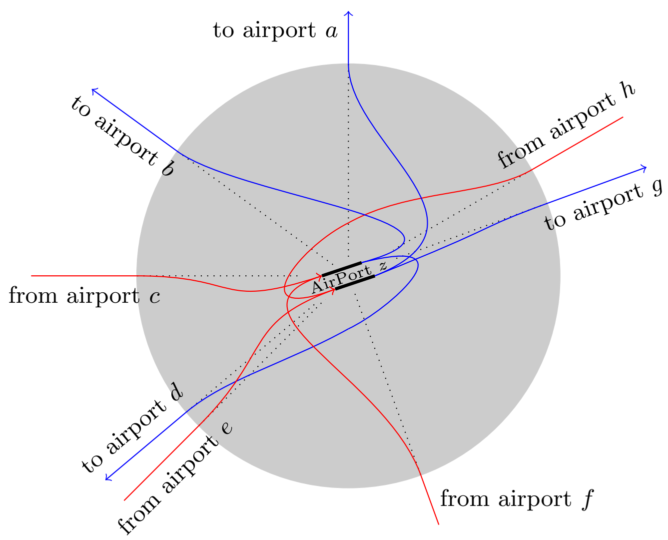


Figure 1.2: "inside" a node. Departure and arrival routes are colored blue and red respectively.

Using Property 1 from above, we can assume that any aircraft passing through an Airport

[2]Aircraft take-off and land using runways defined by the wind direction and need space to line-up with them while landing, some minimum altitude to gain after take-off, not to mention the horizontal separation needed in order for the traffic to flow regularly

Area (neither departing or landing at the airport) will be kept separated from traffic arriving or departing the airport without altering its altitude (or course or speed). Thus, for this work, the 2-dimensional Airport Area can be transformed in a single point (node), along with the generation of another property of our model: *If the intersection point of 2 edges is an endpoint of either of the segments (i.e. a node), then coloring these 2 edges with the same color is allowed.*

Let us also note that in case where 2 or more Airport Areas intersect, a simple approach is to create a larger united Airport Area for all these airports and treat them as a single node. This raises questions of how to determine which airports will be grouped together and which not, setting exact or approximate radii of airport areas etc. However, this is not an objective of this work.

3 or more collinear nodes. When 3 or more airports lie on the same line, all flights between them can share *the same flight level*. This occurs using Property 1 from above. Consider 3 collinear airports x,z,h. Flight $A : x \to h$ passes over z, and there is a departing flight $B : z \to h$. So, the segment zh is common for the 2 flights. All the controller has to do is make sure that when flight B reaches its exiting point, the horizontal separation minima w.r.t. flight A are preserved.

We must note that this is a non-generic case, which we briefly analyze for the model's consistency, but it will not be studied theoretically in this work. Moreover, the possibility of sharing a flight level within a line actually lessens the flight levels needed, so the generic case is enough to give us the upper bounds we seek.

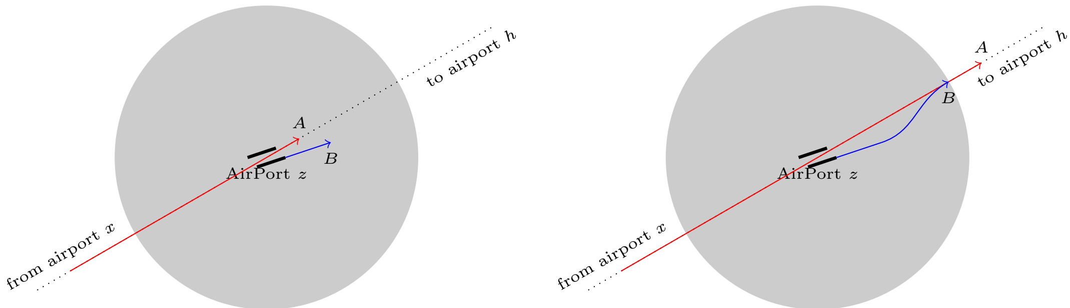
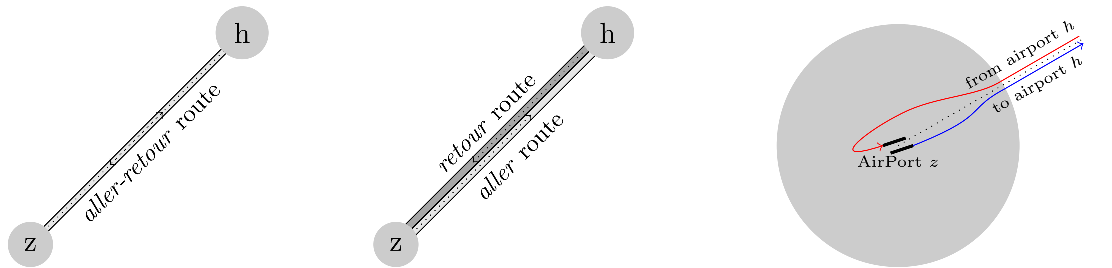


Figure 1.3: Guiding flight B to exiting point behind flight A.

Explaining the edges. Up to this point, we have not mentioned anything about the direction of a flight. But this hides indeed a crucial error: if two airplanes travel the same direct-to route, at the same altitude but in different directions, they will be in conflict as long as they travel around the same time of day. We have two options: either we assign different FLs in opposite direction's flights and consequently need double the colors to properly color the graph (bad!), or we use an idea of horizontal separation and create 2 deviations of a direct-to route.

As we mentioned before, an airway has a few miles' width w. A simple idea is to double the width and add some extra width for safety, thus having $2w + w_s$ as the width of the direct-to route. Half (okay, a bit less!) of it is to be assigned to the departing flights of an airport, and the other half to the arriving ones. It is easy for the on-board aircraft systems (and generally) to determine these 2 deviated routes to follow. It is also simple for the controllers to direct the airplanes from or to the altered entering or exiting points of the airport area.

The advantage of this idea is that our model remains the same, as we either way transform the 2-dimensional routes in simple lines. The collinear-node case raises no problems either, so in a few words, as of the needs of this work, a simple edge between 2 nodes is enough.



(a) Single-corridor route zh. (b) Double-corridor route zh. (c) Flights from and to airport h.

Figure 1.4: Creating deviated routes linking two airports. Notice the altered entering and exiting points regarding airport h of the Figure 1.2.

Areas of direct-to flight. Having a set of routes between airports all flown direct-to may practically be found only in areas with few airports and probably low traffic congestion. However, within specified areas, air traffic may seem as a set of direct-to route legs if there is a local absence of navaids. Such an area may well be congested, and assigned to a certain controller. It is an advantage to theoretically study a conflict-free assignation of Flight Levels to involved legs in order to estimate both the complexity of the local traffic pattern and the controller's work load.

In Figure 1.7 it is illustrated how "zooming-in" a congested area yields a study case similar to what we have previously presented. All entering and exiting points of flights over the area are nodes of our model, as well as all airports within the area[3].

1.3 Our model

Definition 1.1. A drawing D of an undirected graph $G(V, E)$ is an embedding of G into $\mathbb{R}^2$. We will write $D(G)$ to denote a drawing of graph G.

Any drawing (mapping) actually comprises of a "1-1" function $D^V : V \to \mathbb{R}^2$, and a function D^E which maps each $e = vu \in E$ to some curve joining $D^V(v)$ and $D^V(u)$. However, as we have already stated, in this work we only discuss straight-line embeddings, and unless noted otherwise, a drawing D will assume all edges are drawn as straight lines.

This means edge function becomes $D^E : E \to \binom{D^V(V)}{2}$ and D is well defined as soon as D^V is defined. So, we are allowed to use notation $D : V \to \mathbb{R}^2$ to express D^V and completely define D by determining $D(V)$. An extra fact is that D^V being "1-1", is enough to make D^E "1-1", too.

We will not be using any directed graphs in this thesis; from this point on, declaration of a graph G will imply G is undirected.

Definition 1.2. Let D be a drawing of $G(V, E)$. We define the *drawing thickness*, $\vartheta(D(G))$[4], to be the smallest value of k such that we can assign each edge to one of k planar layers so that no two edges on the same layer cross[5] (while keeping all vertices fixed to the points that D obligates).

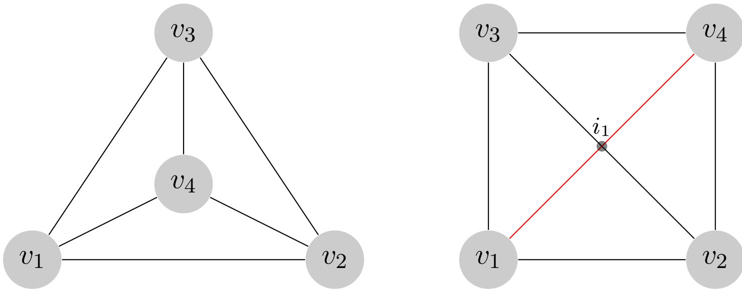
[3]The ellipse shape to zoom-in to is arbitrary and does not reflect any real air traffic pattern.

[4]ϑ appears in [43] as notation for graph (theoretical) thickness; recent bibliography instead, consistently uses θ or t, making ϑ a non-confusing choice.

[5]We will say that two edges cross if they interiorly intersect at exactly one point.

1.3.1 Discussing the model

Let us draw some graphs and comment on them:



(a) $D_1(K_4)$ (b) $D_2(K_4)$

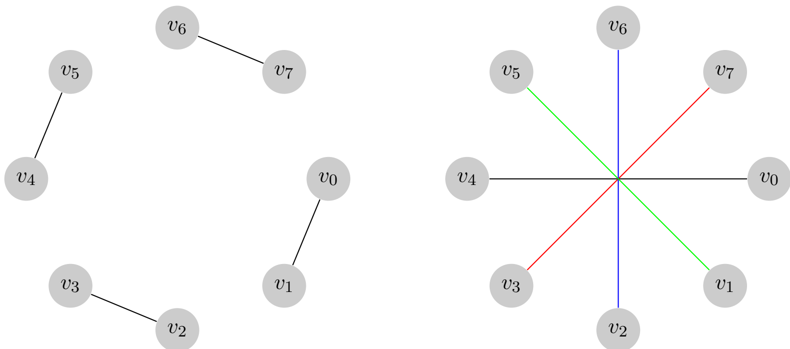
Figure 1.5: 2 different drawings of the K_4. $\vartheta(D_1(K_4)) = 1$, $\vartheta(D_2(K_4)) = 2$.

Figure 1.5a, K_4 is drawn in a way (D_1) it shows it is a planar graph. In this case there are no crossing edges, thus $\vartheta(D_1(K_4)) = 1$.

Figure 1.5b is an alternative drawing D_2 of K_4 as a square with its 2 diagonals. There is 1 point of intersection that leads to a 2-coloring of this graph. Let us see one more example:

Consider the family of the graphs $G(V, E)$ that have $deg(v) = 1$ for all $v \in V$. This is obviously a "perfect matching" graph of $n = 2k$ elements (vertices). Let $D_{adj} : V \to \mathbb{R}^2$ be a mapping of V to a (regular) convex n-gon, such that all adjacent pairs of vertices of G are also adjacent vertices of the n-gon. In this case, we easily get $\vartheta(D_{adj}(G)) = 1$ (Figure 1.6a).

But we may choose to draw the graph like this: If the edge set is expressed as $E = \{v_k v_{k+n/2}, k = 0, ..., n-1\}$, define $D_{opp}(v_k) = e^{-\frac{k\pi}{n}i}, k = 0, ..., 2n-1$. Now all edges join diametrically opposite points (of the unit circle) and every 2 edges cross each other, dictating drawing thickness to now be $\vartheta(D_{opp}(G)) = \frac{n}{2} = |E|$ (Figure 1.6b).



(a) $D_{adj}(G), |V| = 8$ (b) $D_{opp}(G), |V| = 8$

Figure 1.6: $\vartheta(D_{adj}(G)) = 1$ but $\vartheta(D_{opp}(G)) = 4$, with $n = 8$.

We make some first observations at this point, in order to give an idea of how we are going to move on in the next chapters.

1. Defining drawing thickness as above, the studied variants of thickness come to mind; *(graph-theoretical) thickness (θ), geometric thickness ($\bar{\theta}$), and book thickness (bt)* are all to be examined and give some (mostly lower) bounds for the drawing thickness (ϑ).

2. We can understand from the examples given, that two isomorphic graphs may be easily drawn to have *different drawing thickness*. In fact, the very same graph G may be drawn to have different ϑ. This is in contrast with most of the properties of isomorphic graphs remaining invariant for their respective class (all the above thicknesses included).

3. This difference is attributed to the fact that $\theta(G)$, $\bar{\theta}(G)$ and $bt(G)$ are *minimizations over all allowed (by the respective definitions) drawings* of G and thus characterize G itself, while ϑ characterizes individually each and every drawing D of the graph. Remember, we write $\vartheta(D(G))$, and not $\vartheta(G)$, which is not well defined[6].

Definition 1.3. Two drawings D_1, D_2 of graph $G(V, E)$ are said to be equivalent if and only if any 2 edges that cross when drawn via D_1 are drawn to cross via D_2 and vice versa.

As in this thesis, we are never concerned for drawn edges' endpoints, we will use the $\cap$ symbol to indicate an edge crossing question: for 2 drawn edges $s_a s_b, s_c s_d \in D(G)$, $s_a s_b \cap s_c s_d$ will be regarded as a first-order logic formula, assigned *TRUE* if and only if the edges cross.

Also, throughout this work, equivalence will be denoted by $\sim$, and isomorphism with $\cong$. So, the 3rd observation from the above may be symbolically expressed as:

Lemma 1.1. For any graphs $G \cong H$, there is a $D : V \to \mathbb{R}^2$ such that $\vartheta(D(G)) \neq \vartheta(D(H))$.

Definition 1.3 becomes:

Definition 1.4. Let $G(V, E)$ and $D_1, D_2 : V \to \mathbb{R}^2$ drawings of G. $D_1(G) \sim D_2(G)$ if and only if for any v_a, v_b, v_c, v_d with $D_1(v_i) = s_i^1, D_2(v_i) = s_i^2$ it is $s_a^1 s_b^1 \cap s_c^1 s_d^1 \Leftrightarrow s_a^2 s_b^2 \cap s_c^2 s_d^2$.

Finally, let us expand the operand sets of θ,$\bar{\theta}$ and bt, for completeness, just in case it is needed.

Definition 1.5. Let G be a graph and D be a drawing of G. Then $\theta(D(G)) = \theta(G)$, $\bar{\theta}(D(G)) = \bar{\theta}(G)$ and $bt(D(G)) = bt(G)$.

[6]We may use the notation $\vartheta(G)$ only in cases where a general conclusion is reached, for all drawings D of G.

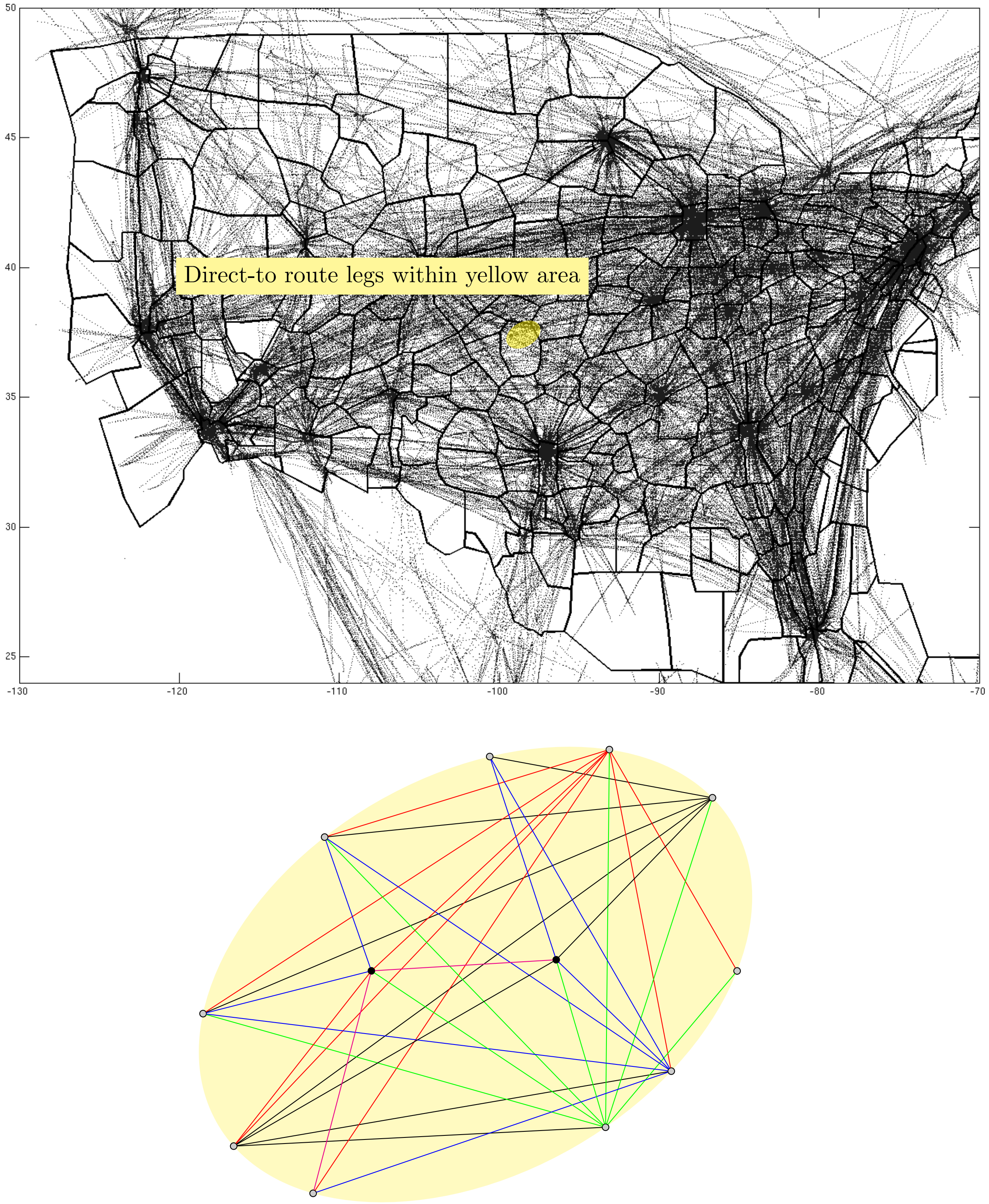


Figure 1.7: An area zoom-in. Nodes on the area boundary are entering and exiting flights' points, and black nodes are airports. Assigning layers to route legs is the same as described before.

Chapter 2

Graph thickness and geometrical thickness

2.1 Graph Thickness

Graph thickness is a term introduced by W.T. Tutte in 1963 ([56]), following the proof of a conjecture: *for any graph G with $|V| = 9$ either G or its complementary graph $\bar{G}$ is not planar*, or, speaking with terms of the time K_9 *is not biplanar*. This was shown by exhaustion, and K_9 is the smallest complete graph having this property. Graph thickness came to expand the idea of biplanar graphs, measuring the number of planar subgraphs a graph may be partitioned to:

Definition 2.1 (Thickness)**.** Graph (theoretical) thickness, $\theta(G)$, is the minimum number of planar graphs into which a graph G can be decomposed.

Equivalently, the graph-theoretical thickness can be defined as the minimum number of planar layers required to embed a graph such that the vertex placements agree on all layers and the edges can be arbitrary curves.

Planar graphs: one of the most significant graph classes, contains all graphs that can be embedded (drawn) in the plane with no edge crossings. By definition we have $\theta(G_{planar}) = 1$. Characterizing (and recognizing) planar graphs is of particular importance for Graph Theory, and the following theorem is well known to anyone introduced to the field:

Theorem 2.1 (Kuratowski, 1930)**.** *A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Also, according to a corollary of the famous Euler's polyhedron-formula, a planar graph $G(V, E), |V| = n$ has at most $3n - 6$ edges (see also Section 5.3); such a graph is called *maximal*

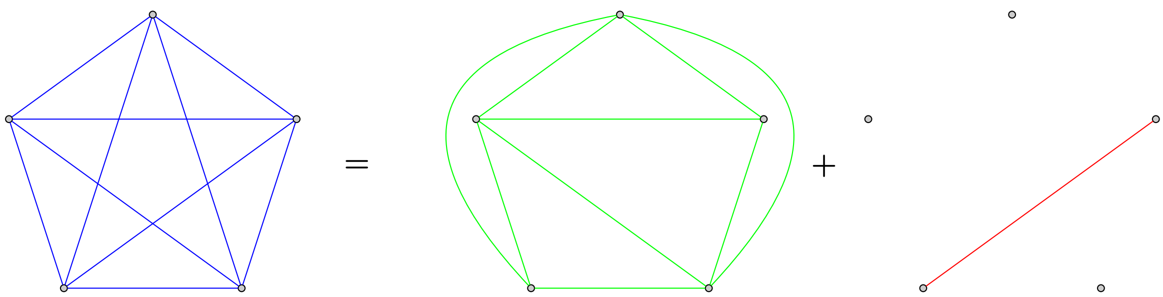


Figure 2.1: Planar decomposition of the K_5: $\theta(K_5) = 2$.

planar graph. This may be used to get a first bound regarding the thickness of a graph: using the pigeonhole principle, we have

$$\theta(G) \geq \left\lceil \frac{|E|}{3n-6|} \right\rceil \tag{2.1}$$

Note that in the same time, characterizing planar graphs led to efficiently recognizing them (Hopcroft, Tarjan, [31]).

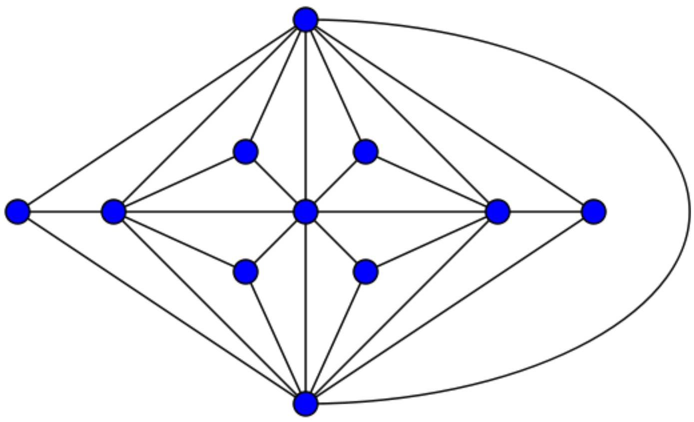


Figure 2.2: The Goldner-Harary graph, with 11 vertices and 27 edges is a maximum planar graph. We will refer to it in Section 3.2.2 as it is the smallest planar graph which is not Hamiltonian (source: http://en.wikipedia.org/wiki/Goldner-Harary_graph).

The thickness of K_n

Starting from the early 1960's, it was believed that the thickness of K_n could be expressed with a "nice", simple formula. Applying $|E| = n(n-1)/2$ in 2.1, it is easy to get that

$$\theta(K_n) \geq \left\lfloor \frac{n+7}{6} \right\rfloor$$

Number $\lfloor \frac{n+7}{6} \rfloor$ appears in the work of L. W. Beineke and F. Harary ([5]) to be the thickness of K_n for $n \neq 3, 4 \mod 6$. Later on, Harary ([30]) conjectures $\theta(K_n) = \lfloor \frac{n+7}{6} \rfloor$ except possibly for $n = 4 \mod 6$. At the time, whether $\theta(K_{16})$ equals 3 or 4, was still an open question (Harary conjectures it is 4), while for every other n up to 45, thickness of K_n was known. The answer came from Jean Mayer, a french literature professor, who showed ([44]) that $\theta(K_{16}) = 3$; Mayer had also given the constructions to show $\theta(K_{34}) = 6$ and $\theta(K_{40}) = 7$. In 1976, Alekseev and Gončakov ([1]) finally gave the desired proof, overcoming Harary's possible exceptions, so in total we have:

$$\theta(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 10 \\ \lfloor \frac{n+7}{6} \rfloor, & n > 10 \end{cases}$$

The thickness of $K_{m,n}$

Beineke and Harary, this time together with J. W. Moon, gave probably the first result on the thickness of bipartite graphs, a result which has not been improved to date ([6]).

Theorem 2.2 (Beineke, Harary, Moon, 1964)**.** *The thickness or the complete bipartite graph is*

$$\theta(K_{m,n}) = \left\lceil \frac{mn}{2(m+n-2)} \right\rceil$$

except possibly when $m < n$, mn is odd, and there exists an integer k such that $n = \left\lfloor \frac{2k(m-2)}{m-2k} \right\rfloor$. It is also

$$\theta(K_{n,n}) = \left\lceil \frac{n+5}{4} \right\rceil$$

The quantity $\left\lceil \frac{mn}{2(m+n-2)} \right\rceil$ for bipartite graphs is actually the analogous quantity to $\left\lceil \frac{|E|}{3n-6|} \right\rceil$ for complete graphs: Euler's polyhedron formula dictates that no bipartite graphs with more than $2(m+n-2)$ edges can be embedded in the plane. It is an interesting fact that the inequality is proved to be an equality.

Determining the thickness of a graph is *NP*-complete

The natural complexity problem one would consider regarding thickness was solved independently by A. Mansfield ([43]) and V. Chvátal in the most convincing way:

Theorem 2.3 (Mansfield, 1983, Chvátal)**.** *Given a graph G, the decision problem whether G can be decomposed into 2 planar layers is* NP*-complete.*

We will refer to the problem above as *2-THICK* (see Definition 5.1). Mansfield reduces planar *3-SAT* to *2-THICK* after explaining why planar *3-SAT* with exactly 3 literals in each clause remains *NP*-complete.

Theorem 2.4 (Lichtenstein, 1982)**.** *Planar* 3-SAT *is* NP-complete *([38]).*

For completeness, we quote the definition of the problem as presented in [43].

Planar *3-SAT* is the satisfiability problem of $\xi = c_1 \wedge c_2 \wedge ...$ in conjunctive normal form, with literals in U, and the following properties:

- The (bipartite) graph $G(V, E)$ with $V = C \cup U$ and $E = (u, c)$ for which either u or $\bar{u}$ is a literal of c.
- Each clause $c \in C$ has 3 literals at most.

2.2 Geometrical Thickness

Geometrical thickness is the last of the three variants of thickness to be defined ([15]) by M. B. Dillencourt, D. Eppstein and D. S. Hirschberg. Graph visualization is stated as a motivation in this work, as it seems natural to draw graphs using straight lines.

Definition 2.2 (Geometrical thickness)**.** *We define $\bar{\theta}(G)$, the geometrical thickness of a graph G, to be the smallest value of k such that we can assign planar point locations to the vertices of G, represent each edge of G as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross.*

Remark 2.1. As geometrical thickness is a restriction over graph-theoretical thickness (straight line segments over fixed points on $\mathbb{R}^2$), it is clear that for any graph G stands $\boldsymbol{\theta(G) \leq \bar{\theta}(G)}$.

To commence our discussion, comparing geometrical thickness with thickness, we note on one hand, that by *Fáry's theorem* ([22]), any planar graph G can be drawn in such a way that all edges are straight line segments, therefore $\bar{\theta}(G_{planar}) = 1$; on the other hand, we know that $K_{6,8}$ has graph-theoretical thickness 2, but geometric thickness 3 ([15]) . So let us begin stating results on the thickness, to reach a result on relation between thicknesses in the last paragraph of the chapter.

Geometrical thickness of K_n

Theorem 2.5 (Dillencourt, Eppstein, Hirschberg, 2000)**.** *For the complete K_n, $n \geq 12$ it is*

$$\left\lceil \frac{n}{5.646} + 0.342 \right\rceil \leq \bar{\theta}(K_n) \leq \left\lceil \frac{n}{4} \right\rceil$$

We will note give the rather long proof of the theorem, but through Figure 2.3 we will illustrate the idea to get the upper bound of $\lceil \frac{n}{4} \rceil$: we consider $n = 4r$ and the key is to draw $K_{2r} \subset K_{4r}$ as a small regular $2r$-gon (inner ring in) and place the remaining $2r$ vertices to form a large convex $2r$-gon (outer ring). Now we can draw zigzag paths[1] for the inner and the outer $2r$-gons (they give $n/4$ layers/colors) in such way that all remaining edges require no new layer to be assigned to.

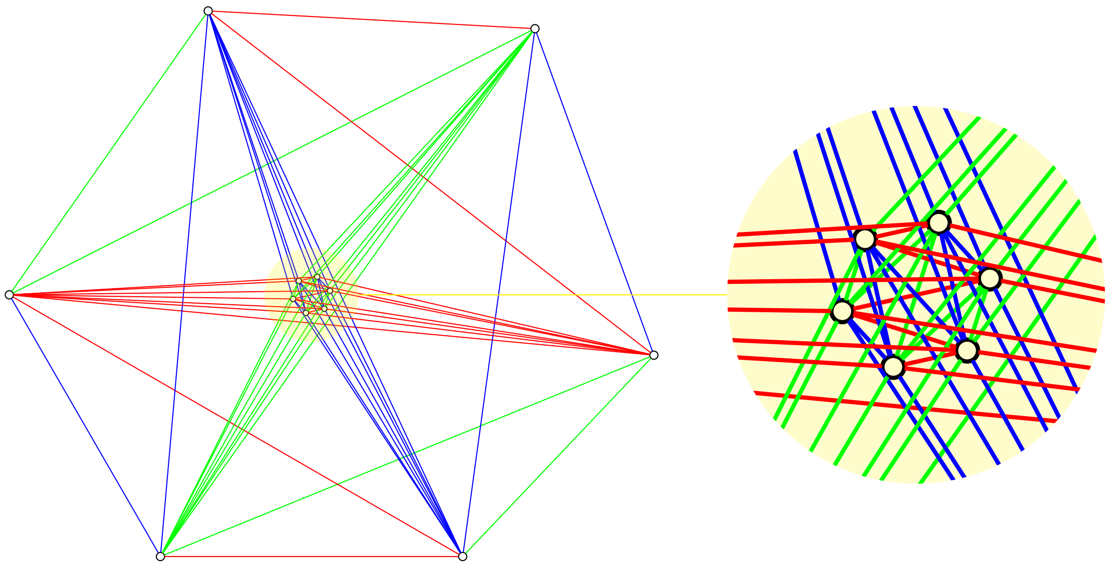


Figure 2.3: Drawing of K_{12} where $\bar{\theta}(K_{12}) = 3$.

In their work, Dillencourt et al. deal also with the special case of K_{15}, and in total they give:

$$\bar{\theta}(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 12 \\ 4, & 15 \leq n \leq 16 \end{cases}$$

For the cases of K_{13} and K_{14}, there is no proof for their exact geometric thickness, which lies obviously between 3 and 4.

[1] see also the proof of Theorem 3.11.

Geometrical thickness of $K_{m,n}$

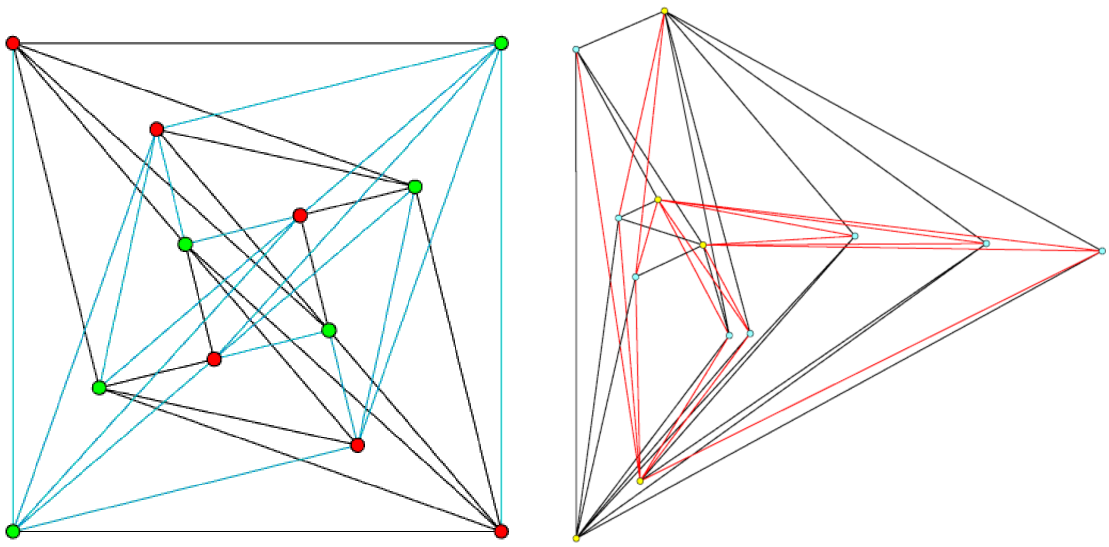
Still in their very same publication, Dillencourt et al. prove some bounds on $\bar{\theta}(K_{m,n})$. We present the result and its proof, as it is easy to follow.

Theorem 2.6 (Dillencourt, Eppstein, Hirschberg, 2000)**.** *For the complete bipartite graph $K_{m,n}$ it is*

$$\left\lceil \frac{mn}{2m+2n-4} \right\rceil \leq \theta(K_{m,n}) \leq \bar{\theta}(K_{m,n}) \leq \left\lceil \frac{\min(m,n)}{2} \right\rceil \tag{2.2}$$

Proof. The first inequality is discussed in the previous section. To establish the final inequality, assume that $m \leq n$ and m is even. Draw n blue vertices in a horizontal line, with $m/2$ red vertices above the line and $a/2$ red vertices below. Each layer consists of all edges connecting the blue vertices with one red vertex from above the line and one red vertex from below. □

Note that when $m << n$ the rightmost and leftmost quantities of 2.2 coincide and $\theta(K_{m,n}) = \bar{\theta}(K_{m,n}) = m/2$. Also, the bounds are not tight (theorem only implies $\bar{\theta}(K_{6,6}) \leq 3$ when it is $\bar{\theta}(K_{6,6}) = 2$. We conclude by presenting some figures of D. Eppstein on his webpage, along with some of his additional notes.



(a) 2 layer straight-line drawing of $K_{6,6}$.

(b) $K_{5,8}$ may be the larger complete bipartite graph having $\bar{\theta} = 2$.

Figure 2.4: Complete bipartite graphs of particular interest (source: http://www.ics.uci.edu/ eppstein/junkyard/thickness/).

Geometrical thickness vs. thickness. We see that geometrical thickness is close to the thickness of a graph. This rises the question of whether there is some particular asymptotic behavior of $\frac{\bar{\theta}(G)}{\theta(G)}$. D. Eppstein answers this question in the negative ([20]) by constructing graphs with thickness 3 and arbitrarily large geometrical thickness, therefore $\frac{\bar{\theta}(G)}{\theta(G)} = \omega(1)$. Eppstein approaches the problem using Ramsey Theory ([29]), but his proof is very long and technical to present here. In the meantime, the question of how large $\bar{\theta}$ can be when $\theta = 2$ remains open.

Geometrical thickness of bounded degree graphs. A similar result to the above is given in the work of J. Barát, J. Matoušek and D. R. Wood ([3]), which is actually the reply to a question posed by Eppstein in [20]:

Theorem 2.7. *For all* $\Delta \geq 9$ *and* $\epsilon > 0$*, for all large* $n > n(\epsilon)$ *and* $n \geq c\Delta$ *there exists a* Δ*-regular* n*-vertex graph with geometric thickness at least*

$$c\sqrt{\Delta}n^{1/2-4/\Delta-\epsilon}$$

for some absolute constant c*.*

Geometrical thickness proved to be significantly different and separated from thickness, and in the next chapter where *book thickness* is introduced, we will find an analogous separation theorem (another work of Eppstein) regarding book thickness and geometrical thickness. This time we will outline the proof, which is based again on Ramsey Theory (see Section 3.5).

Chapter 3

Convex graph drawing and book thickness

3.1 Book embeddings and convex graph drawing

Book embeddings were first studied by L. T. Ollman in 1973. The idea is to embed a graph $G(V, E)$ in a rather common 3-D object. So, let L be a line in $\mathbb{R}^3$ (most commonly when drawn it is the z-axis), and $P = \{P_1, ..., P_k\}$ be a set of (open) half-planes all having L as their boundary. Then a "book" is the union $B = L \cup P$, L representing the spine, and P the pages. We can define:

Definition 3.1. A k-book embedding β of $G(V, E)$ is a placing of all $v \in V$ along the spine L of a book B, and a drawing of all edges $e \in E$ as arbitrary open (Jordan) arcs joining respective vertices, either in L or onto one exactly of k book pages $\{P_1, ..., P_k\}$, such that arcs on the same page do not cross.

Naturally, the first question to pose is to minimize the pages (layers) needed for a book embedding of G; or the equivalent decision problem "Does G have a k-book embedding?"

Definition 3.2 (*Book thickness*)**.** We define $bt(G)$, the book thickness of a graph G, to be the smallest value of k such that G has a k-book embedding.

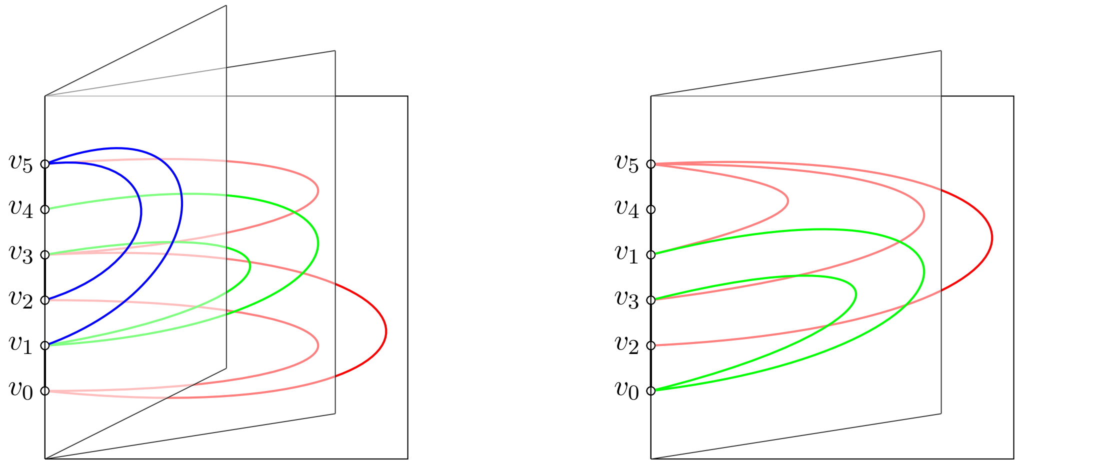
Figure 3.1 shows $G_{ex}(V, E)$ with $|V| = 6$, $|E| = 10$. It can be embedded in a book with 3 pages (3.1a), but there is an optimum book embedding in 2 pages. Thus $bt(G) = 2$. Notice also that book thickness could not be 1 (to be explained later on).

The book thickness of a graph, in opposition to graph theoretical and geometric thickness can be 0, because according to the definition, if we can place all arcs along the spine L of the book, then there is no need for any page to exist. This occurs if and only if G is a path[1]. There is another definition of book thickness, drawing a convex planar embedding of the graph:

Definition 3.3 (*Book thickness alternate definition*)**.** Let G be a connected graph which is not a path. Then the $bt(G)$ is the minimum number k of layers in a drawing of G onto $\mathbb{R}^2$, where the vertices of G are placed in convex position and all edges are drawn as straight line segments such that each edge belongs to a single layer and no two edges in the same layer cross.

Each of these layers may be referred to with the addition of the word *planar*, just to clarify its property of including pairwise non-crossing edges. We have excluded the trivial case of G being a path, because while a path's book thickness is proved to be 0, as soon as we draw an

[1] or a family of pairwise disconnected paths, but as we have stated, there is no need to study disconnected graphs for any of the thicknesses.



(a) A book embedding β of G_{ex} with 3 pages

(b) A book embedding β_{opt} of G_{ex} with the optimum of 2 pages

Figure 3.1: For $G_{ex}(V, E)$ it is $bt(G_{ex})$=2.

edge on the plane, we have already created a layer to assign the edge to. Also, we can state the following:

Remark 3.1. If G is connected and not a path, Definition 3.3 allows us to see that book thickness is a restriction over geometrical thickness (points in convex position); so it is $\bar{\theta}(G) \leq bt(G)$, and in total we have:

$$\boldsymbol{\theta(G) \leq \bar{\theta}(G) \leq bt(G)}$$

The equivalence between the 2 definitions was first introduced in [7]. Though it is not hard to understand, we will present a complete proof of it. But first, we will discuss some properties of convex graph drawings, as we will use them frequently when wondering around drawing thickness, but also to sketch variations of some proofs of [7], [8], in order to have a more consistent notation when trying to link together the presented ideas and propositions.

3.1.1 Convex graph drawing

First, we present a group of definitions regarding convexity.

Definition 3.4.

A set X is said to be convex if for any $a, b \in X$, it is $\boldsymbol{a} + \lambda \boldsymbol{b} \in X, \forall \lambda \in [0, 1]$. In other words, segment ab lies entirely within X.

The *convex hull* of a set X (we write *CH(X)*) is the intersection of all convex sets that cover (include) X.

In the special case of (finite) point sets, the definitions imply that:

For a point set P, its convex hull P is the minimum polygon which covers all points in P.

A point set S is said to be convex if all its points lie on the boundary of *CH(S)*.

We will maintain marking convex point sets with letter S, while usually marking arbitrary points sets by P.

Convex graph drawing. Let $S = \{s_0, s_1, ..., s_{n-1}\}$ be a convex point set on the plane and consider it as the convex n-gon $s_0 s_1 ... s_{n-1} s_0$, drawn in a clockwise direction; also let D_{conv} map V of graph $G(V, E)$ to S. For our own convenience, we use the obvious isomorphisms $S \overset{f(s_i)=i}{\cong} \{0, ..., n-1\}, V \overset{g(s_i)=i}{\cong} \{0, ..., n-1\}$ as it is $|S| = |V| = n$ to handle easily the mapping function. $D_{conv} : V \to S \cong D_{conv} : \{0, ..., n-1\} \to \{0, ...n-1\}$. D_{conv} is "1-1" so it is invertible, and as $|V| = |S|$, it is one of the functions to reveal the isomorphism $V \cong S$. Therefore it is $D_{conv}^{-1} : S \to V \cong D_{conv}^{-1} : \{0, ..., n-1\} \to \{0, ...n-1\}$.

We may well denote any of S, V as $\{0, ..., n-1\}$, write $D_{conv}(i) = k \Leftrightarrow D_{conv}^{-1}(k) = i$. When the graph is drawn, we will usually refer to a vertex placing only as "i" rather than "point s_i" and if $D_{conv}(v_i) = a, D_{conv}(v_j) = b$ with $a, b \in \{0, ..., n-1\}$, the edge $v_i v_j$ will be simply referred to as (a, b). If an operation regarding the enumeration of these vertices gives a result $a \geq n$, then we actually refer to vertex $a \bmod n$. Using this enumeration, and having in mind definition 1.3 we will list a few properties:

Proposition 3.1. Properties of convex graph drawings:

i) Of course, it is $(a, b) \equiv (b, a)$ for all $a, b \in \{0, ..., n-1\}$.

ii) All possible *boundary* edges (or edges of the n-gon) E_{bnd} of $D_{conv}(E)$ are expressed as $(k, k+1)$ for some $k \in \{0, ..., n-1\}$ and do not cross any other edge of the drawing of G. It is $|E_{bnd}| \leq n$.

iii) *Interior* edges (E_{in}) of the drawing are all drawn edges that are not boundary, i.e. they are diagonals of the convex polygon. It is $|E_{in}| \leq \frac{n(n-3)}{2}$.

iv) For any edge $(a, b), a \neq b$ we define its *span* $|(a, b)| = \min((a-b) \mod n, (b-a) \mod n)$ and it is the minimum path length from a to b around the polygon's boundary. It is $|(a, b)| \leq \lfloor \frac{n}{2} \rfloor$.

v) For all edges of the drawing we can assign each edge in a *span class*: $(a, b) \in E_{|(a,b)|}$. All edges belong to exactly one span class $E_k, k = 1, ..., \lfloor \frac{n}{2} \rfloor$, creating a partition of set $D_{conv}(E)$. It is $|E_k| \leq n$ except if $n = 2r$ and $k = r$. In that case $|E_k| \leq \frac{n}{2} = r$. Also $|E_1|$ is the set of all drawn boundary edges, while $E_{\lfloor \frac{n}{2} \rfloor}$ is the set of all drawn maximal diagonals of the polygon[2,3].

vi) A diagonal $s_a s_b$ of a convex polygon crosses diagonal $s_c s_d$ if and only if s_c and s_d lie on either of the 2 smaller polygons defined when drawing $s_a s_b$. For a drawing $D_{conv}(G)$ this is translated to the formula for $a < b, c < d$:

$$\begin{gathered} (a, b) \cap (c, d) \Leftrightarrow \\ (c < a) \wedge (a < d < b) \vee (a < c < b) \wedge (b < d) \ (\textit{crossing-check formula}) \end{gathered} \tag{3.1}$$

The crossing-check formula applies even if we generalize $(a, b), (c, d)$ to be arbitrary drawn edges, but also if some of a, b, c, d are $\geq n$.

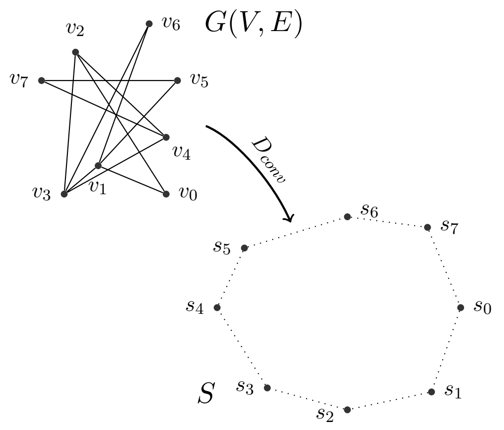
vii) Let $D_{conv}^{(n)}$ the set of all convex mappings $V \to S, |V| = |S| = n$, and $\Pi^{(n)}$ the set of all permutations of $\{0, ..., n-1\}$. It is $D_{conv}^{(n)} \cong \Pi^{(n)}$. This allows us to define $D_{conv} : V \to S$ as a permutation: $D_{conv}(V) \leftrightarrow \Pi_{D_{conv}} = \{l_0, ..., l_{n-1}\} \Leftrightarrow D_{conv}(v_{l_k}) = k, \forall k \in \{0, ..., n-1\}$. The permutation gives us the clockwise order of the vertices places on S.

[2] E_k is a subset of the edges of the star polygon (or star figure) $S_{n/k}$, the drawings one gets when joining with a straight line every k-th vertex of a convex n-gon (see Appendix A).

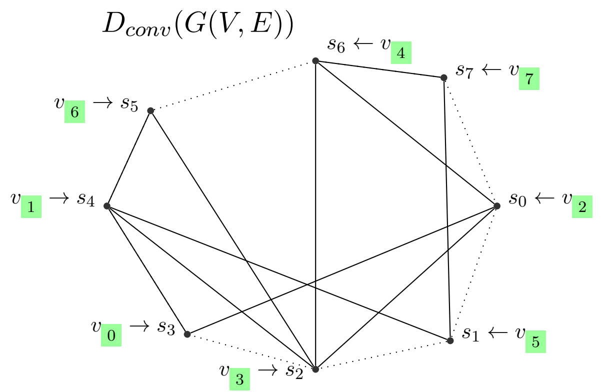
[3] The classification of an edge as boundary, exterior, or belonging to a certain span class E_k depends on the very drawing of the graph, and it is meaningless without some D. Therefore, if we write E_{bnd}, E_{in} or E_k we indicate the existence of a drawing D, and refer to the obvious one of the context.

viii) Let a second mapping with $D'_{conv}(V) = S$ correspond to $\Pi' = \Pi_{D'_{conv}} = \{l'_0, l'_1, ..., l'_{n-1}\}$. If we can get Π' from $\Pi_{D_{conv}}$ using only *reflection*: $f_{ref}(\Pi) = \{l_{n-1}, ..., l_0\}$, *rotation*: $f_{rot_t}(\Pi) = \{l_{n-t}, l_{n-t+1}, ..., l_{n-1}, l_0, l_1, ..., l_{n-t-1}\}$, or any combination of these two transforms, then the graph drawings $D_{conv}(G)$, $D'_{conv}(G)$ are equivalent. A pair of such permutations are said to be equivalent and we will write $\Pi \sim \Pi'$.

ix) Let S' be another set with all properties of S. A convex drawing $D^{S'}_{conv}(G)$, with $D^{S'}_{conv}(V) = S'$, is equivalent to $D_{conv} : V \to S$ if $\Pi_{D_{conv}} \sim \Pi_{D^{S'}_{conv}}$.

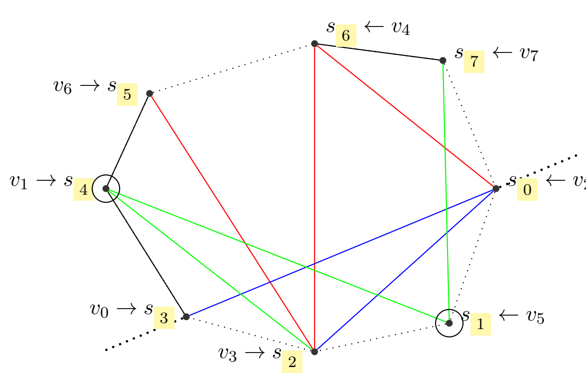


(a) Graph G to be embedded to convex point set S

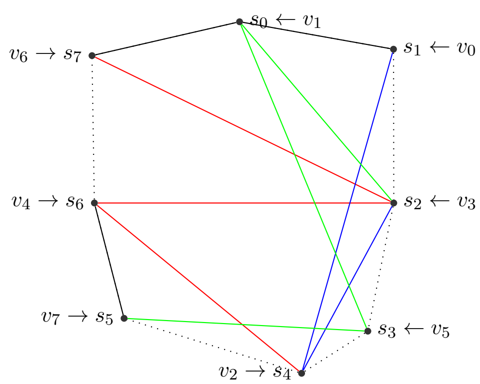


(b) Assignation of each vertex to a point. It is $\Pi_{D_{conv}} = \{2, 5, 3, 0, 1, 6, 4, 7\}$

Figure 3.2: A convex drawing. $v_3v_4 \in E_4, v_3v_6 \in E_3, v_2v_4 \in E_2$ and v_0v_1 is drawn to be boundary.



(a) Crossing checking and color assignments. v_0v_2 crosses v_1v_5 under D_{conv} as crossing-check formula for $(3, 0) \cap (4, 1)$ evaluates "*TRUE*".



(b) New S', with new Π'. The drawing is equivalent to D_{conv} because $\Pi' = f_{ref}(f_{rot_3})(\Pi)$.

Figure 3.3: Crossings and an equivalent drawing of G.

Proof.

i) Trivial, the graph is undirected.

ii) Because of the labeling of S, the n possible edges of the polygon are $(k, k+1), 0 \leq k < n$. They do not cross each other, and no any other edges, because if they did that would imply a point in the exterior of the n-gon, which is false.

iii) Possible diagonals join non-consecutive vertices, i.e. edges $(a, b), b \neq a \pm 1$, which are at most $\binom{n}{2} - n = \frac{n(n-3)}{2}$.

iv) Let WLOG $b > a$ and $b - a = l$. Then $a - b = -l$ but $-l = n - l \mod n$. Therefore it is $b - a = l$ and $a - b = n - l \mod n$, $0 \leq l < n$, so $\min((a-b) \mod n, (b-a) \mod n) = \min(l, n-l) \leq \lfloor \frac{n}{2} \rfloor$.

v) All edges go to exactly one layer because of the span function above. Thus sets E_k are a partition of $D_{conv}(E)$. Consider an edge $E_1 = (a, a+k), 0 \leq a < n \in E_k$, and let $E_2 = (a+i, a+i+k), 1 \leq i < n \in E_k$. We will show that all edges of E_k are distinctive except for the case $n = 2r, k = r$. As $a \neq a+i$ and $a+k \neq a+i+k$ we have:

$$E_1 \equiv E_2 \Leftrightarrow \begin{cases} a = a+i+k \mod n \\ a+i = a+k \mod n \end{cases} \Leftrightarrow \begin{cases} 2i = 0 \mod n \\ i = k \mod n \end{cases} \Leftrightarrow \begin{cases} 2i = n \\ i = k \end{cases}$$

As the solution of the system indicates (with $i \equiv r$) that if $n = 2r, k = r$ the expression $(a, a+k)$ yields identical edge $(a+k, a+2k) \equiv (a+k, a), \forall a \in \{0, ..., k-1\}$. So this is the case where $|E_k| \leq \frac{n}{2}$. In any other case $|E_k| \leq n$.

vi) Let us express our convex point set as $S = \{s_0, ..., s_a, s_{a+1}, ..., s_b, s_{b+1}, ..., s_{n-1}\}$. Diagonal (a, b) defines convex polygons $S_1 = \{s_0, ..., s_a, s_b, s_{b+1}, ..., s_n, s_0\}$ and $S_2 = \{s_a, s_{a+1}, ..., s_b, s_a\}$. $p_c p_d$ crosses $p_a p_b$ if and only if $p_c \in S_1$ and $p_d \in S_2$ or $p_c \in S_2$ and $p_d \in S_1$, with $a, b \neq c, d$. This means $(c < a \vee c > b) \wedge (a < d < b)$ or $(d < a \vee d > b) \wedge (a < c < b)$. As $(a, b) \equiv (b, a), (c, d) \equiv (d, c)$ we convert any edge so that $a < b, c < d$, and get the shorter crossing-check formula:

$$(a, b) \cap (c, d) \Leftrightarrow (c < a) \wedge (a < d < b) \vee (d > b) \wedge (a < c < b)$$

If (a, b) is not a diagonal, then it is a boundary edge and should not cross any other edge. Indeed, if $b = a+1$, then neither of $a < c < b, a < d < b$ can be true, and the crossing-check formula is always false.

vii) Define $\Pi_{D_{conv}} = \{D_{conv}^{-1}(0), ..., D_{conv}^{-1}(n-1)\}$ for any $D_{conv} \in D_{conv}^{(n)}$. For $D_{conv} \neq D'_{conv}$ we have: $\exists k : D_{conv}(k) = i \neq j = D'_{conv}(k) \Rightarrow \Pi_{D_{conv}}(i) = \Pi_{D'_{conv}}(j)$ and therefore it cannot be $\Pi_{D_{conv}} \neq \Pi_{D'_{conv}}$.

For any $\Pi = \{l_0, ..., l_{n-1}\} \in \Pi^{(n)}$ define $D_{conv_\Pi}(v_{l_k}) = k, \forall k \in \{0, ..., n-1\}$. Since l_k is term of a permutation, running through all $k \in \{0, ..., n-1\}$ means D_{conv_Π} as defined gives $V \cong S$. If $\Pi \neq \Pi' \Rightarrow \exists i, j \neq i : l_i = l'_j = y \Rightarrow D_{conv,\Pi}(v_y) = i \neq j = D_{conv,\Pi'}(v_y)$.

viii) Let $e_1, e_2 = (a, b), (c, d)$ under D_{conv}, so $e_1 = v_{D_{conv}^{-1}(a)} v_{D_{conv}^{-1}(b)}, e_2 = v_{D_{conv}^{-1}(c)} v_{D_{conv}^{-1}(d)}$ We will use the crossing-check formula, so we assume $a < b$, $c < d$.

- Reflection. If $D_{conv}^{-1}(i) = l_i \in \Pi_{D_{conv}}$ then $l_i \in f_{ref}(\Pi_{D_{conv}})$, but $l_i = l'_{n-1-i}$, so under D'_{conv} we have to check edges $(n-1-a, n-1-b), (n-1-c, n-1-d)$. Now it is

$n-1-l_b < n-1-l_a, n-1-l_d < n-1-l_c$ and:

$$(n-1-b, n-1-a) \cap (n-1-d, n-1-c)$$
$$\Leftrightarrow$$
$$((n-1-b < n-1-c < n-1-a) \wedge (n-1-d < n-1-b)) \vee$$
$$((n-1-b < n-1-d < n-1-a) \wedge (n-1-a < n-1-c))$$
$$\Leftrightarrow$$
$$((a < c < b) \wedge (b < d)) \vee$$
$$((a < d < b) \wedge (c < a))$$
$$\Leftrightarrow$$
$$(a, b) \cap (c, d)$$

So two edges cross under D'_{conv} if and only if they cross under D_{conv}.

- Rotation. If $D^{-1}_{conv}(i) = l_i$ then as $l'_i = l_{i-t}$ for a rotation $\Pi' = f_{rot_t}(\Pi)$ the two edges under D'_{conv} are $(a+t, b+t), (c+t, d+t)$. Of course this is equivalent to the crossing-check formula for $(a, b), (c, d)$ and we get to the same conclusion for the rotation function.

So, if $\Pi_{D'_{conv}} = f_1 \circ f_2 \circ ... \circ f_k(\Pi_{D_{conv}})$, where k is finite and $\forall k, f_k \in \{f_{ref}, f_{rot_t}\}$, then the 2 drawings of the graph are equivalent. In fact, for any $\Pi \sim \Pi'$ we can get from one the other with at most 1 reflection and 1 rotation.

ix) Let $S' \cong S \cong \{0, ..., n-1\}$. So $\Pi_{D^{S'}_{conv}}$ also defines an equivalent drawing $D'_{conv} \sim D^{S'}_{conv}$ onto S, as S and S' are also isomorphic as polygons. If $\Pi_{D_{conv}} \sim \Pi_{D^{S'}_{conv}}$ then $D_{conv} \sim D'_{conv}$ and therefore $D_{conv} \sim D^{S'}_{conv}$.

□

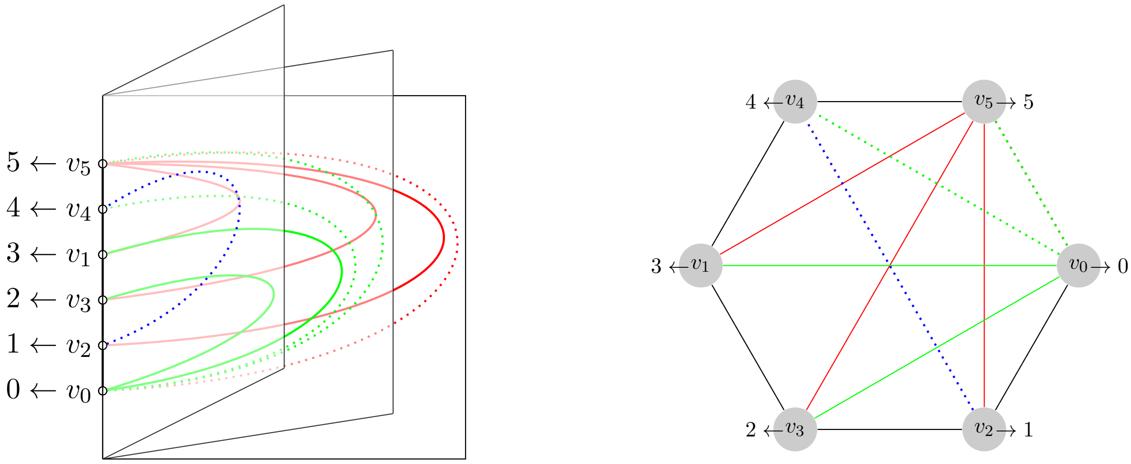
Notes on the above:

- The gist of all the above expressed in a less formal language, is that the *exact points S selected to fix the vertices of G in a convex position does not matter, as long as the order of vertex placings is given, and this order appears around the polygon defined by S, in a clockwise or counterclockwise direction.*
- As the set of all permutations $\Pi^{(n)}$ includes the reflection $f_{ref}(\Pi), \forall \Pi \in \Pi^{(n)}$, we will usually assume a clockwise ordering of the vertices.
- We will mostly draw convex graphs to form regular n-gons. This is not to be regarded as a special convex drawing, but as a convenient drawing, which is, as we proved, equivalent to any non-regular convex drawing.

3.1.2 Book thickness

Theorem 3.1. *If G is connected and not a path, then the two Definitions 3.2, 3.3 of book thickness are equivalent.*

Proof. Let $G(V, E)$ be a graph with $|V| = n$, let book $B = L \cup \{P_1, ..., P_k\}, n \geq 1$ and let $S = \{s_0, ..., s_{n-1}\}$ be a set of points on the plane that form the convex n-gon $s_0 ... s_{n-1} s_0$ clockwise. Choose a placing of vertices of G along spine L and enumerate them according to their order of appearance along L, so it is $V \overset{f_L}{\cong} \{0, ..., n-1\}$, where (using the same approach



(a) The book embedding β_{opt} of G_{ex} (b) Equivalent convex drawing of β_{opt}

Figure 3.4: Adding edges to $\beta_{opt}(G)$ of Figure 3.1b.

as with convex mappings) $f_L(v_k) = l_i \Leftrightarrow f_L^{-1}(l_i) = k$ and $\Pi_\beta = \{f_L^{-1}(0), ..., f_L^{-1}(n-1)\}$ is a permutation of $\{0, ..., n-1\}$. Defining $D_{conv,\beta} = \Pi_\beta$ we have the corresponding convex mapping. As we did for the convex drawing, we will refer to $v_{f_L^{-1}(i)}$ of L as i and to $v_{f_L^{-1}(a)}v_{f_L^{-1}(b)}$ of B as (a, b).

Any edge drawn on the spine of the book, which does not affect book thickness, is equivalent to a boundary edge of the polygon, which does not affect the number of layers either, because it does not cross any other edge. And vice versa: any boundary edge of the polygon is either an edge of the spine of the book, or it is $(n-1, 1)$, an edge that can be assigned to any of the existing layers, as it may be drawn on the "outside" of all edge drawings of an arbitrary page without crossing them.

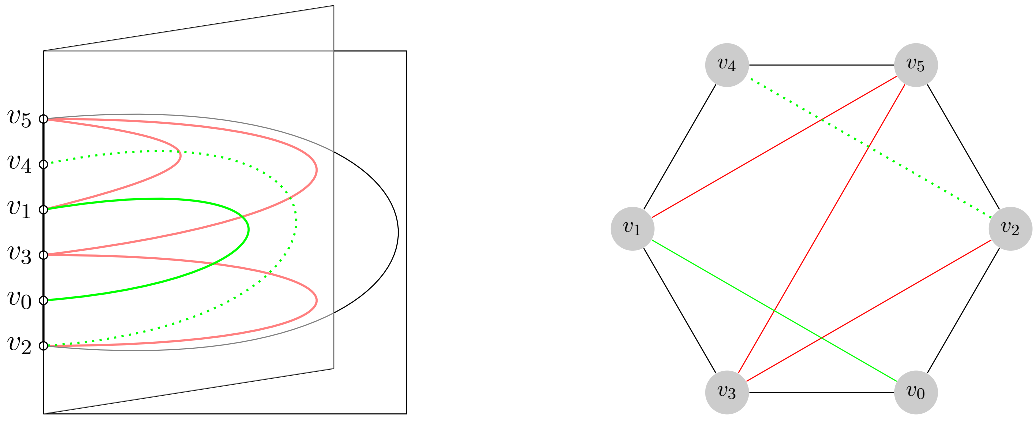
It now remains to show that adding an edge $(a, b), a < b + 1$ to one of the pages P_i of the book is equivalent to adding the diagonal of the polygon (a, b) to the i^{th} layer of the planar drawing.

For edge (a, b) to be placed in P_i there must be no edge $(c, d) \in P_i, c < d$ with $c < a < d, b > d$, or $c < b < d, a < c$. Existence of edge (c, d) leads to the drawing of an arc $\widehat{cd}$ on half-plane P_i. If WLOG $a \in (c, d)$, i.e. $c < a < d$, then the arc to be drawn $\widehat{ab}$ on same half-plane P_i, will cross $\widehat{cd}$ as long as it needs to have a point outside curve $(c, d, \widehat{dc})$. This is true if b lies outside (c, d), or $b > d$ as we assumed $b > a + 1$. Likewise, if $c < b < d$ and $a < c$ we have arcs crossing. To sum up, we must evaluate the formula $(c < a) \wedge (a < d < b) \vee (a < c < b) \wedge (b < d)$ which is the crossing-check formula for the convex drawing.

□

Illustrating the proof with an example. Given the β_{opt} of Figure 3.1b, we have $\Pi_\beta = \{0, 2, 3, 1, 4, 5\}$ and draw the corresponding convex polygon as described. We may add edge v_0v_4 to the green page/layer, but not to the red one because v_4 is a point of the line segment of curve $(v_3, v_5, \widehat{v_5v_3})$, trying to be "reached" by an arc $\widehat{v_0v_4}$ lying on the same half-plane as $(v_3, v_5, \widehat{v_5v_3})$ and needing to have a point outside of it (v_0); v_0v_5 on the other hand, can be added to either page/layer. But edge v_2v_4 cannot be added to the existing layers.

Important note: Adding v_2v_4 does not mean that $G'_{ex} = G_{ex}(V, E) \cup \{v_2v_4\}$ has now $bt(G'_{ex}) = 3$. In fact, reorganizing our layers we get again $bt(G'_{ex}) = 2$ (Figure 3.5).



(a) The new book embedding β'_{opt} of G'_{ex} (b) Equivalent convex drawing of β'_{opt}

Figure 3.5: $bt(G'_{ex}) = 2$. Edge v_2v_5 of the book embedding is drawn black, to show it could be assigned to the green page, too.

Printing cycle and σ-thickness. To prove equivalence of the two definitions, we mapped vertices of the spine of the book to a convex point set on the plane in a certain order, called *printing cycle* ([7]). Two properties of the printing cycle found in [7], state the following:

Lemma 3.2. If G has a k-book embedding $\beta, n \geq 1$, with printing cycle $v_1, ..., v_n$ then it has a n-book embedding with printing cycle $v_2, ..., v_n, v_1$ and printing cycle $v_n, ..., v_1$.

This is actually our Proposition 3.1 (viii) applied on the convex embedding created from β. In the very same work, the term σ-thickness briefly appears:

Definition 3.5. Let G be a graph and σ a listing of its vertices. The σ-thickness $bt(G, \sigma)$ is the smallest k such that G has a k-book embedding with σ as a printed cycle. Then it is simply $bt(G) = \min bt(G, \sigma)$ overall σ.

Let us transform this, using our terminology.

Definition 3.6. Let G be a graph $|V| = n$ and $D_{conv} \in D^{(n)}$ a mapping of its vertices to a convex point set. Then $bt(G) = \min_{D_{conv}} \vartheta(D_{conv}(G))$. Also $bt(G) = \min_{\Pi \in \Pi^n} \vartheta(D_{conv_\Pi}(G))$.

Point-line covering number for a graph. Denoted by $\alpha(G)$, the point-line covering number is the smallest number of vertices of G that are incident with every edge of G.

Lemma 3.3. For any convex drawing D_{conv}, it is $\vartheta(D_{conv}(G)) \leq \alpha(G)$.

Proof. Let $S_\alpha = \{v_1, v_2, ..., v_{\alpha(G)}\}$ cover the edges. We can construct $\alpha(G)$ layers, so that we can assign any edge incident to v_k to layer k. As edges with common endpoint do not cross, and all edges are incident to some $v_k \in S_\alpha$, we get our lemma. □

3.2 Graphs with small book thickness

Theorem 3.4. *Let G be a connected graph. Then:*

i) $bt(G) = 0$ *if and only if G is a path.*

ii) $bt(G) \leq 1$ *if G is outerplanar, and vice versa.*

iii) $bt(G) \leq 2$ *is true if and only if* G *is a subgraph of a Hamiltonian* (subhamiltonian) *planar graph.*

3.4 (i) is trivial to prove. For 3.4 (ii) and 3.4 (iii), we must discuss the terms *outerplanar graph* and *Hamiltonian planar graph.* Also let us state an easy-to-get lemma:

Lemma 3.5. A graph with $bt(G) \leq 2$ is planar. $\Leftrightarrow$ If G is not planar, then $bt(G) \geq 3$.

Proof. Just consider the 2 at most pages as the two half-planes spine L divides the plane to. $\square$

3.2.1 Outerplanar graphs

Definition 3.7. A graph G is outerplanar if it can be drawn on the plane without crossing edges, and in a way so that all vertices lie on the boundary of the exterior (unbounded) region of the drawing; equivalently, no vertex is totally surrounded by edges.

The condition is equivalent with having all vertices on the boundary of the interior region. Chartrand and Harary showed in [10] that a graph G is outerplanar if and only if it contains no minor $H = K_4$ or $K_{2,3}$, two "forbidden" graphs, analogous to the critical graphs of Wagner's theorem for planarity. Another interesting result is the following:

Theorem 3.6. *Graph* G *is outerplanar if and only if* $G + K_1$ *is planar.*

Proof. If $G(V, E)$ is outerplanar, suppose its vertices lie on the boundary of G's exterior. Adding a new vertex w allows us to draw all edges $wv, v \in E$ without crossings. Now, let $G + K_1$ be planar, and w the vertex adjacent to all vertices of G. By deleting w we get that G has its vertices lying on the boundary of the same region, as they were reached from w without edges to cross. $\square$

Theorem 3.6 gives us an easy-to-execute drawing condition to check if G is outerplanar. Now, back to 3.4 (ii).

Proof of 3.4 (ii). If $bt(G) = 1$, G can be drawn with its vertices placed in convex position and with no crossings of edges. Thus, all vertices belong to the boundary of the drawing, and G is outerplanar.

For the converse, let G be outerplanar. As it can be drawn with straight edges ([22]), the isomorphism defined by mapping vertices along the boundary to vertices around a convex polygon, guarantees straight line edges with no crossings. So $bt(G) = 1$. $\square$

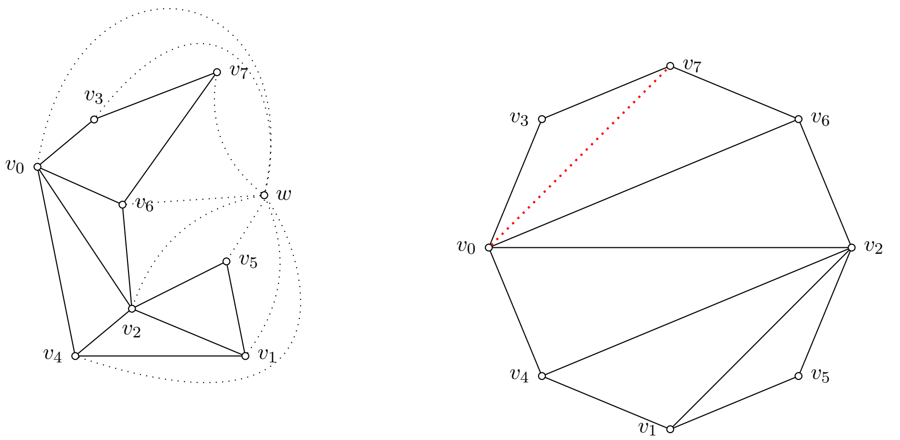
Maximal outerplanar graphs and polygon triangulation

Definition 3.8. An outeplanar graph is maximal if no edge can be added to the drawing without losing outerplanarity.

It is quite clear that a maximal outerplanar graph $G(V, E), |V| = n$ may be embedded as a polygon triangulation (see also [30], [14]). That gives us $|E| = 2n - 3$, the sum of the boundary edges (n) plus the number of diagonals in a triangulation ($n - 3$). This leads to a very useful bound for the book thickness, which we state here and use again later on:

Proposition 3.2. Let $G(V, E)$ with $|V| = n$. It is $bt(G) \geq \left\lceil \frac{|E|-n}{n-3} \right\rceil$.

Proof. Consider the graph embedded via D_{conv}. The boundary edges do not affect graph's thickness, so to get a lower bound let all these edges exist ($|E_1| = n$). But it is $bt(G) \geq \frac{|E|-n}{n-3}$ because if we assume the contrary, it would imply (pigeonhole principle) that we can assign $n - 2$ interior edges (diagonals) to the same layer (page). This is false, as the maximum number of pairwise non-crossing diagonals of a polygon is exactly $n - 3$ (triangulation). $\square$



(a) $G + w$ is planar.

(b) As $|E| = 12 = 2n - 4$ we may add one more edge and keep outerplanarity.

Figure 3.6: Outerplanar graph G.

3.2.2 Hamiltonian graphs

Definition 3.9. A graph is called Hamiltonian if it contains a cycle with all vertices of G (Hamiltonian cycle[4]).

The root of this problem lies in 1855. Reverend Thomas Penyngton Kirkman studied whether it is possible to visit all vertices of a polyhedron exactly once by moving along edges and returning to the starting vertex. He observed that this could be done for some polyhedra but not all. But as Sir William Rowan Hamilton introduced a new game, the *Icosian Game*, one year later, and posed a number of deeper and more varied questions related to what we call now "Hamiltonian cycle", it seems that along with his fame, it was the catalyst to name such cycles after him. At this time, the game was motivated by non-commutative algebras, which Hamilton had developed, but it turned out to evolve to one of graph theory's most significant problems([11]).

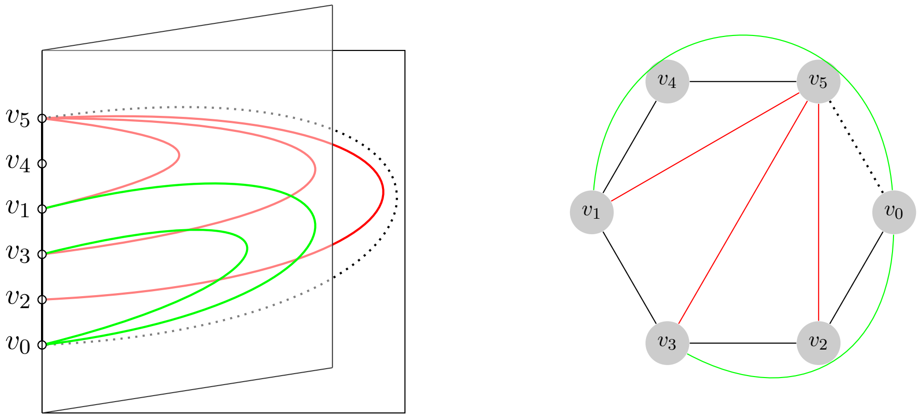
It was not until the 1950's that results on sufficient conditions for Hamiltonian graphs appeared, but never has a necessary condition been found. Karp showed in 1972 ([33]) that the problem of finding whether a graph has a Hamiltonian cycle (HC) is NP-complete. Garey, Johnson and Tarjan showed in 1976 that HC for planar graphs is also NP-complete ([25]).

Let us now discuss the proposition left to prove:

Proof of 3.4 (iii). Let G have $bt(G) \leq 2$. The graph is planar, and consider the cycle going through the vertices of the spine in order of appearance $\{v_{l_0}, ..., v_{l_{n-1}}\}$, adding all missing edges $v_{l_i}v_{l_{i+1}}$, including edge $v_{l_{n-1}}v_{l_0}$, which we proved not to affect book thickness. The graph with the added edges G' is Hamiltonian, and $G \subseteq G'$.

Now assume planar G' has a Hamiltonian cycle. Fix the vertices on the spine in order of appearance in the cycle. All edges drawn in the interior of the cycle may be assigned to a single page because along with the cycle the form an outerplanar graph; so do the edges of the exterior of the cycle. Therefore a Hamiltonian planar graph has $bt(G') = 2$, and so does any $G \subseteq G'$. □

[4]or Hamiltonian circuit. Whichever the choice, the meaning of initials HC is identical.



(a) Creating G' of proof of 3.4 (iii).

(b) Subgraphs *black-red* and *black-green* are outerplanar.

Figure 3.7: Illustrating the proof of Proposition 3.4 (iii).

3.2.3 Non-subhamiltonian planar graphs.

In 1975, A. Golder and F. Harary showed that a maximal planar graph with 11 vertices and 27 edges is the smallest in the class which does not have a Hamiltonian cycle (Figure 2.2); its book thickness is actually 3. Of course, there are arbitrarily large maximal planar graphs that are not Hamiltonian; Wigderson ([57]) and Chvátal independently showed that the maximal planar *HC* problem is *NP*-complete. In particular, Widgerson used the result of Garey et. al ([25]) as his known *NP*-complete problem. This transforms to the following:

Lemma 3.7. Given a graph G, the problem of determining whether it is $bt(G) = 2$ is *NP*-complete.

Proof. The instances of maximal planar graphs are enough to give us the *NP*-completeness as an immediate reduction from maximal planar *HC*. See also table 5.1. □

The fact that Bernhart and Kainen ([7]) could construct infinitely large planar graphs without Hamiltonian cycles led them to make the conjecture that planar graphs' book thickness could be also arbitrarily large. This was soon proved false, and finally M. Yannakakis ([58]) gave the following result:

Theorem 3.8. *For any planar graph G, it is $bt(G) \leq 4$.*

In Yannakakis's publication, along with a very technical proof of the above, a planar graph with book thickness 4 is constructed, making the bound to be tight, and settling once and for all the open to this date problem. Yannakakis's complex example seems to be the only published instance of its kind, leading most to believe that 3 pages are enough to accommodate any relatively small planar graph.

There are many more related results and extensive bibliography on planar graphs and special instances of them, but we shall now retract our limitation for planarity and small book thicknesses, and examine the complete and complete bipartite graphs.

3.3 Book thickness of the complete graph K_n

Let us begin this section by stating the rather obvious observation that all convex embeddings of the complete graph are equivalent. So, the special note on this case is that the book thickness of K_n equals the drawing thickness of any convex drawing of K_n. More formally:

Lemma 3.9. For $G = K_n$, $bt(G) = \vartheta(D_{conv}(G)), \forall D_{conv} \in D_{conv}^{(n)}$.

As a consequence, we may well draw the convex K_n mapping each $v_i \in V$ to point s_i (using our standard notation for convex drawings) without loss of generality, and now i actually denotes vertex v_i. Using the Proposition 3.2 we directly get the following lemma:

Lemma 3.10. If $G = K_n$ then $bt(G) \geq \lceil \frac{n}{2} \rceil$.

Proof. The total of the edges of the complete graph are $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$. Replacing $|E|$ in the mentioned proposition we get $bt(G) \geq \left\lceil \frac{\frac{n(n-1)}{2} - n}{n-3} \right\rceil = \left\lceil \frac{\frac{n(n-3)}{2}}{n-3} \right\rceil = \lceil \frac{n}{2} \rceil$. □

Theorem 3.11. *For $n \geq 4$, $bt(K_n) = \lceil \frac{n}{2} \rceil$.*
As equally, $\vartheta(D_{conv}(K_n)) = \lceil \frac{n}{2} \rceil, \forall D_{conv} \in D_{conv}^{(n)}$.

Proof. First we will prove that for $n = 2r, bt(K_n) = \frac{n}{2}$. Consider the convex embedding of the graph on the plane. We have repeatedly stated that edges that are drawn to be boundary do not affect the layering of the rest of the edges. As the graph is complete we know all possible boundary edges are drawn ($|E_1| = n$), so are all interior edges ($|E_{in}| = \frac{n(n-3)}{2}$). We will now assign all interior edges into exactly $\frac{n}{2}$ edge sets (layers). Construct the sets $L_i = \{(i, i+\frac{n}{2})\} \cup \{(i, i+2), (i, i+3), ..., (i, i+\frac{n}{2}-1)\} \cup \{(i+\frac{n}{2}, i+\frac{n}{2}+2), (i+\frac{n}{2}, i+\frac{n}{2}+3), ..., (i+\frac{n}{2}, i-1)\}$ for all $i = 0, ..., \frac{n}{2} - 1$. The sets have the following properties:

1. No L_i includes any boundary edges.

   This follows from our construction.

2. Within each set L_i, edges do not cross.

   Edge $(i, i + \frac{n}{2})$ is a maximal diagonal of the polygon and does not cross any of $\{(i, i + 2), (i, i+3), ..., (i, i + \frac{n}{2} - 1)\}$, which in turn do not cross each other (common endpoint i). Same stands for the maximal diagonal and $\{(i+\frac{n}{2}, i+\frac{n}{2}+2), (i+\frac{n}{2}, i+\frac{n}{2}+3), ..., (i+\frac{n}{2}, n-1)\}$ (common endpoint $i+\frac{n}{2}$). The two large sets also do not induce crossing edges in between them, because their edges lie on either of the smaller polygons defined by $(i, i + \frac{n}{2})$.

3. For $i \neq j$, it is $L_i \cap L_j = \emptyset$.

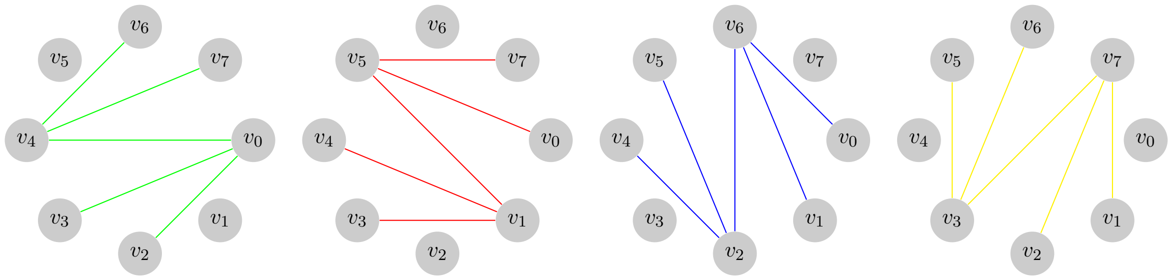
   We will compare edges with the same span. If $i \neq j < \frac{n}{2}$ then $(i, i+\frac{n}{2}) \neq (j, j+\frac{n}{2})$ because $j \neq i + \frac{n}{2}$. For the rest of the edges that belong in pairs in each of $E_k, k = 2, ..., \frac{n}{2} - 1$ we have $\{(i, i+k), (i+\frac{n}{2}, i+\frac{n}{2}+k)\} \cap \{(j, j+k), (j+\frac{n}{2}, j+\frac{n}{2}+k)\} = \emptyset$ for the same reason.

4. $E_{in} = \bigcup L_i$.

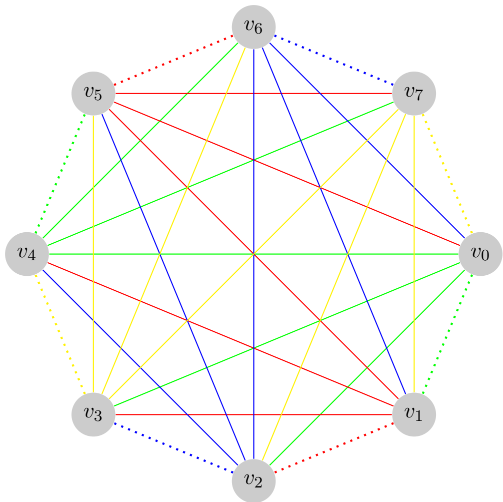
   Of course, all edges of L_i are distinct, thus $|L_i| = n - 3, \forall i \in \{0, ..., \frac{n}{2} - 1\}$. The fact that $|\bigcup L_i| = \frac{n}{2}|L_i| = \frac{n}{2}(n-3) = |E_{in}|$ together with the 1st and 3rd property are enough to verify $E_{in} = \bigcup L_i$.

Since we have assigned all interior edges to one of $\frac{n}{2}$ layers, we have completed our proof for $n = 2r$. Now let $n = 2r - 1$. $K_{2r-1} \subset K_{2r} \Rightarrow bt(K_{2r-1}) \leq bt(K_{2r}) = r$. For the sake of contradiction assume $bt(G) = r - 1 = \lceil \frac{n}{2} \rceil - 1$. The maximum number of edges that may form such layers is $(n-3)(r-1) = (2r-4)(r-1)$, while we have a total of $\frac{(2r-1)(2r-4)}{2} = (2r-4)(r-\frac{1}{2}) = (2r-4)(r-1) + r - 2$ edges, which disproves our assumption and $bt(K_{2r-1}) = r = \lceil \frac{n}{2} \rceil$.
□

Some notes:



(a) L_0, L_1, L_2, L_3 for K_8.



(b) Complete picture of the edge-colored K_8.

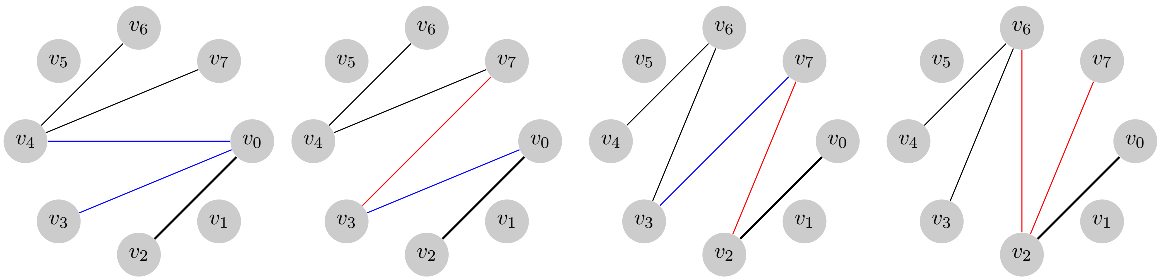
Figure 3.8: Showing that $bt(K_8) = 4$.

- K_2 is a path so $bt(K_2) = 0$, and $bt(K_3) = 1$. K_4 includes edge of form $(i, i+2)$, so it is the smallest complete graph supporting a construction of L_i sets.
- We may assign a pair of boundary edges to each L_i for consistency (see Figure 2.3).
- The sets we constructed are not the only sets that have these properties mentioned in the proof. In fact, there are $2^{\frac{n}{2}-2}$ different set constructions (mirror-symmetrical sets being distinct) that may be used for the very same approach to prove Theorem 3.11.

  Recipe (only for n even):

  1. Choose a vertex i to be your starting point for constructing set L_i.
  2. Place edge $(i, i+2)$ in the set and assign $a \leftarrow i, b \leftarrow i+2$.
  3. Choose a sign $(+)$ or $(-)$. For $(+)$ place edge $(a, b+1)$ in L_i and $b \leftarrow b+1$; for $(-)$ place $(a-1, b)$ in L_i instead and $a \leftarrow a-1$.
  4. Repeat for $\frac{n}{2} - 2$ times. In the end of this procedure an edge of maximal span is placed in L_i.
  5. For every edge (a, b) in L_i (except for the maximal span one), place $(a + \frac{n}{2}, b + \frac{n}{2})$ in L_i.

  Each set construction now can be defined by a sequence of $\frac{n}{2} - 2$ pluses or minuses. For instance, the set we constructed in our proof corresponds to the all-plus set $\{+, +, ..., +\}$ of size $\frac{n}{2} - 2$. It is easy to see that within any L_i edges to not cross; the proof that for any construction it is $L_i \cap L_j = \emptyset$ is analogous to the one we presented, and the proof that all $2^{\frac{n}{2}-2}$ set constructions yield different L_i is equally technical[5], but better understood if one starts building L_i from a maximal span edge.



(a) $\{+,+\}$ (b) $\{+,-\}$ (c) $\{-,+\}$ (d) $\{-,-\}$

Figure 3.9: The 4 different L_0 sets (v_0v_2 as initial edge) we can construct for K_8.

3.4 Book thickness of complete bipartite graphs

Within this section, the bipartite $K_{m,n}$ will have $m \geq n$, following the main work presented ([17]).

Coming to the class of complete bipartite graphs, publications are rather scarce. First results are given in by Bernhart and Kainen ([7]). The use of Lemma 3.3 gives an easy first bound $bt(K_{m,n}) \leq \alpha(K_{m,n}) = \min(m,n)$, and the use of Proposition 3.2 applied for the $m+n$ vertices and $m \cdot n$ edges of $K_{m,n}$ gives $bt(K_{m,n}) \geq (mn - m - n)/(m + n - 3)$. For $K_{n,n}$ the latter inequality becomes $bt(K_{n,n}) \geq \frac{n(n-2)}{2n-3} \geq \frac{n}{2}$. This left a large gap $[\frac{n}{2}, n]$ to place $bt(K_{n,n})$ in, and the only improvement presented in [7] is that $K_{n,n} \leq n - 1$. Finally, we get the following Ramsey-type theorem :

Theorem 3.12. *For $m \geq n^2 - n + 1$ it is $bt(K_{m,n}) = n$.*

Proof. Consider a circle to place n black vertices and m red ones. Using the pigeonhole principle we get that at least one of the n arcs defined by two consecutive black vertices contains n red vertices. Enumerate the black vertices clockwise $1, ..., n$ such that the arc containing the n red ones is the $\overset{\frown}{n,1}$, and enumerate these red vertices clockwise again, with $n+1, ..., 2n$. As there are all n^2 edges between black and red vertices, we consider the subset $N = \{(1, n+1), (2, n+2), ..., (n, 2n)\}$ and observe that for any $e_1, e_2 \in N$ it is $e_1 \cap e_2 = TRUE$ (crossing-check formula). Thus n layers are necessary to accommodate the edges of N and since we know $bt(K_{m,n}) \leq n$ we have our theorem. □

Those bounds were improved by the work of Muder, Weaver and West (see [47]). For $m \geq n$ it was now $bt(K_{m,n}) \leq \min(\lceil (2n+m)/4 \rceil, n)$. Their proof was based on *2-bucket orderings*, that is placing black and red vertices around the circle forming 2 groups (blocks) of consecutive vertices for each color.

In 1997, H. Enomoto, T. Nakamigawa and K. Ota from Keio University in Yokohama, published an excellent work *on the pagenumber of complete bipartite graphs* ([17]). They adopted a more natural approach to the problem, which improved all existing results on $bt(K_{m,n})$, while leading to a one-of-a-kind technique (comparing to relevant publications): the definition of a function f such as the bound occurs from a minimization overall $r \in \mathbb{N}$ and not directly:

[5]If we account for symmetrical sets, i.e. turn all (+) to (−) and all (−) to (+) in the sequence, we may say the distinct sets are a little less, $2^{\frac{n}{2}-3}$.

Theorem 3.13. *Let $f(m, n, r) = \lceil (nr^2 + r + m)/(r^2 + r + 1) \rceil$. For $m \geq n \geq 3$ it is $bt(K_{m,n}) \leq \min_{r \in \mathbb{N}} f(m, n, r)$.*

As a lemma we may easily get:

Lemma 3.14. $bt(K_{n,n}) \leq \lfloor \frac{2n}{3} \rfloor + 1$, $bt(K_{\lfloor n^2/4 \rfloor, n}) \leq n - 1$.

As with the geometrical thickness, we see that if $m >> n$ the book thickness remains $O(n)$. The long -yet more elegant compared to [47]- proof (of Proposition 3.3) features an ordering of n black and m red vertices around a circle in t blocks for each color. Each ordering can be regarded as the set $A = \{b_1, r_1; b_2, r_2; ...; b_t, r_t\}$, $\sum b_i = n$, $\sum r_i = m$, which actually means: *"place in clockwise order around the circle b_1 consecutive black vertices, then r_1 consecutive red vertices, and so on"*. It can be $b_i, r_i = 0$, so a set A can describe any placing of vertices around the circle. For instance, in Figure 3.10 the ordering applied to $K_{9,6}$ is $\{4, 3; 1, 3; 1, 3\}$.

This particular ordering we mentioned, actually derives from the ordering used to prove the critical proposition, from which Theorem 3.13 is then established. We state and consider true that:

Proposition 3.3. If n, r, s are all positive integers with $n \geq (r + 1)s + 1$, then for $m = (r + 1)(n - s) - r^2 s - r$ we have $bt(K_{m,n}) \leq n - s$.

For ordering $A = \{n - rs, n - (r + 1)s; ((1, 1)^{s-1}; 1, n - (r + 1)s)^r\}$, where the power $s - 1$ indicates a repetition of the pattern of the base for $s - 1$ times, it is easy to see that A covers all vertices of the graph. With the appropriate assignment of edges to layers, the proposition is verified. For the example of Figure 3.10 we must set $n = 6, s = 1$ and $r = 2$.

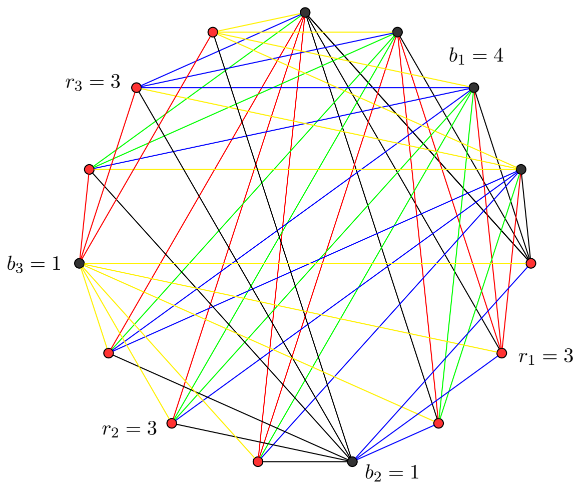


Figure 3.10: Ordering $A = \{4, 3; 1, 3; 1, 3\}$ for the bipartite $K_{9,6}$ shows its thickness is at most 5.

Proof of Theorem 3.13. Let $m \geq n \geq 3$ and r positive, and $t = f(m, n, r)$. We will show that $bt(K_{m,n}) \leq t, \forall r \in \mathbb{N}$. If $n \leq t$ then $bt(K_{m,n}) \leq n \Rightarrow K_{m,n} \leq t$. Now if $t \leq n - 1$, from $t = \lceil (nr^2 + r + m)/(r^2 + r + 1) \rceil \geq (nr^2 + r + m)/(r^2 + r + 1)$ we obtain:

1. $m \leq (r^2 + r + 1)t - nr^2 - r$ and substituting $t = n - s$ we get $m \leq (r + 1)(n - s) - r^2 s - r$.

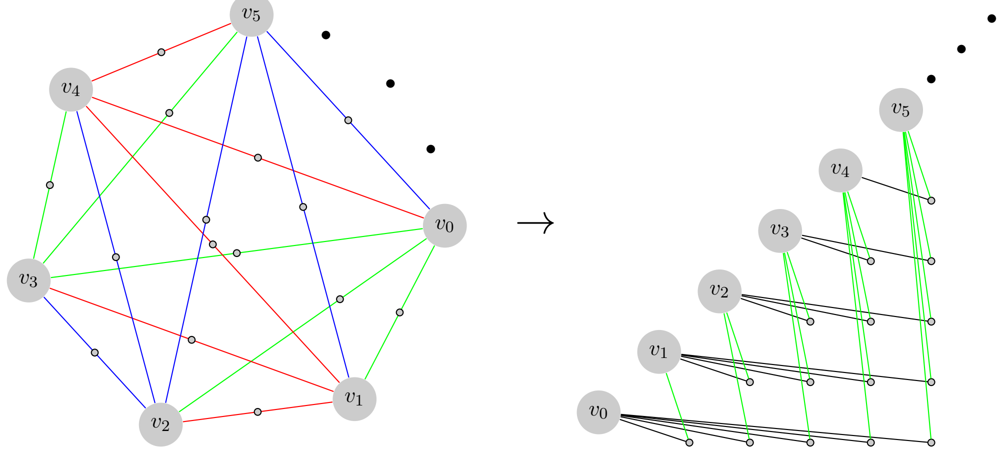
2. $t \geq (nr^2 + r + m)/(r^2 + r + 1) > (nr + 1)/(r + 1)$, and using the same substitution we obtain $n > (r + 1)s + 1$.

But for $n > (r+1)s+1$ and $m = (r+1)(n-s) - r^2 s - r$ it is $bt(K_{m,n}) \leq n - s$ from Proposition 3.3. So for $m \leq (r+1)(n-s) - r^2 s - r$ it is also true that $bt(K_{m,n}) \leq n - s = t$. All stand for any $r \in \mathbb{N}$. □

We suspect that no other published to this date work gives any better bounds on the book thickness of $K_{m,n}$.

3.5 Book thickness vs. geometrical thickness

As we promised, we will now present the work of D. Eppstein ([19]) on the asymptotic behavior of $\frac{bt(G)}{\bar{\theta}(G)}$ to show that book thickness and geometrical thickness are not asymptotically equivalent ($\frac{bt(G)}{\bar{\theta}(G)} = \omega(1)$). First we will show that a family of graphs G_n has bounded geometrical thickness $\bar{\theta}(G_n) = 2$. The construction is based upon a complete graph K_n and the graph denoted G_n, which is the graph we get if we replace all edges of K_n with a path of length 2. Figure 3.11 shows this transformation, and how it leads to a placing of all $n + \binom{n}{2}$ vertices of G_n so we can easily see that $\bar{\theta}(G_n) = 2$. For large n (≥ 5) it cannot be $\bar{\theta}(G_n) = 1$ as the non-planar K_5 is a minor of G_n.



(a) Splitting every edge into a 2-edge path (b) Drawing to show $\bar{\theta}(G) = 2$.

Figure 3.11: Towards showing $\frac{bt(G)}{\bar{\theta}(G)} = \omega(1)$.

Lemma 3.15. For any integers c,l there is an integer $R_c(l)$ such that if we create a partition of the edge set of the complete $K_{R_c(l)}$ into c sets, then at least one of the corresponding c subgraphs contains the complete K_l as subgraph.

The lemma comes from Ramsey theory ([29]) and will be the key to prove the following theorem which verifies our claim.

Theorem 3.16. *For any k there is a graph G_n that has $bt(G_n) \geq k$.*

Proof. We choose $c = \binom{k-1}{2}$, $l = 5$ and let $n = R_c(5)$. Consider the complete K_n, and the G_n formed from K_n. Now suppose, for the sake of contradiction, that there is a book embedding β that gives $bt(G_n) = k - 1$. We use this particular β to create a partition of the edges of K_n into c sets.

Let the 2 edges that connect v_i with v_j in G_n be assigned to different layers x and y under embedding β. As an unordered pair, $\{x, y\}, 1 \leq x, y \leq k-1$ corresponds to one of the wished $c = \binom{k-1}{2}$ edge sets, so we can assign $v_i v_j$ of K_n to the particular subgraph. If the 2 edges belong to the same layer z under β, then we may arbitrarily choose to put $v_i v_j$ into one of the $k-2$ subgraphs involving layer z.

Using the lemma, we know that at least one of the c subgraphs we created has K_5 as a subgraph, the edges of which correspond to all edges of a G_5 embedded under β onto just 2 pages. This is impossible as K_5 is a minor of G_5 making G_5 non-planar, thus $bt(G_5) \geq 3$.

Let us now state our claim as a lemma, deriving directly from Theorem 3.16.

Lemma 3.17. Book thickness and geometrical thickness are not asymptotically equivalent: for any given k we may construct a graph G_n such that $bt(G_n) = k$ while $\bar{\theta}(G_n) = 2$.

□

Chapter 4

Drawing thickness of arbitrary graph drawings

4.1 The drawing thickness of sparse graphs

In the first chapter, Figure 1.6 gave an example of how we can draw a graph $G(V, E)$ with only $|E| = O(n)$ edges such as its drawing thickness is exactly $|E|$. In fact, we can generalize to a simple observation:

Lemma 4.1. For every n there is a graph $G_{sp}(V, E)$ with $|V| = n$, $|E| \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$, and a drawing D that yields $\vartheta(D(G_{sp})) = |E|$.

Proof. Create a perfect matching of $|E|$ vertices, and label them so it is v_k adjacent to $v_{k+\lfloor n/2 \rfloor}$, for all $k \in \{0, ..., |E| - 1\}$. Using the convex mapping function $D_{opp}(v_k) = e^{-\frac{k\pi}{n}i}$ we get the desired result as the drawn edges build set $O = \{(0, \lfloor n/2 \rfloor), (1, 1 + \lfloor n/2 \rfloor), ..., (|E| - 1, |E| - 1 + \lfloor n/2 \rfloor)\}$, a set of $|E|$ diagonals of maximal span, thus a set of pairwise intersecting edges. The floor function appears just because n can be odd and define at most $\lfloor \frac{n}{2} \rfloor$ pairs of vertices. □

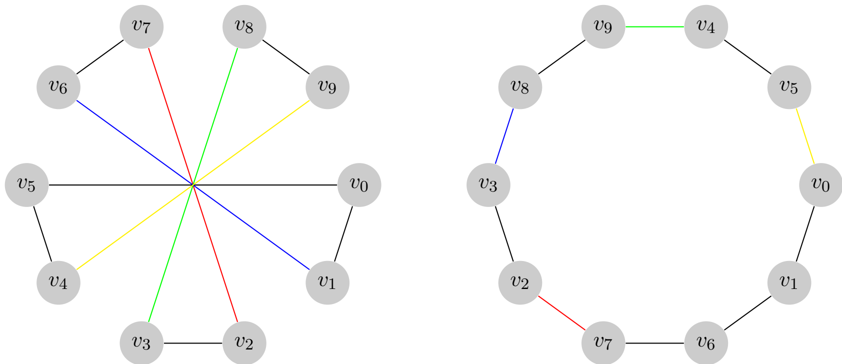
Drawing thickness vs. book, geometrical and graph-theoretical thickness. The previous observation leads directly to the following statement, analogous to the ones of previous chapters:

Proposition 4.1. Ratio between drawing thickness and book, geometric or graph-theoretic thickness is not bounded by any constant factor.

Proof. (For $\bar{\theta}, \theta$): Use the previous perfect matching graph $G_{sp}(V, E)$ with $n = 2r$, $|E| = r$ with $deg(v) = 2, \forall v \in V$. We showed it is $\vartheta(D_{opp}(G_{sp})) = \frac{n}{2}$, while it is easy to see that all edges can be mapped without crossings, for instance as boundary edges of a convex polygon (see D_{adj} of Figure 1.6) and so $\bar{\theta}(G_{sp}), \theta(G_{sp}) = 1$.

(For bt): G_{sp} is a family of mutually disconnected paths, so we must expand its edge set to make it at least outerplanar for the fraction $\frac{\vartheta}{bt}$ to be defined. Now $G'_{sp}(V, E)$ has $|V| = n = 4r+2$, $E = \{v_k v_{k+n/2}, k = 0, ..., 2r\} \cup \{v_k v_{k+1}, k = 0, 2, ..., n - 2\}$. D_{opp} now maps G'_{sp} to a convex n-gon with half its boundary edges plus all maximal span diagonals. It is still $\vartheta(D_{opp}(G'_{sp})) = \frac{n}{2}$ as no boundary edge intersects any other edge. Meanwhile, G'_{sp} is a cycle and its book thickness is 1 (see Figure 4.1). Of course it is also $\bar{\theta}(G'_{sp}), \theta(G'_{sp}) = 1$. □

We observe that sparse graphs ($|E| = O(n)$) may be drawn to have $\vartheta = O(|E|)$, which is not be true for dense graphs as we have shown in the previous chapter. We will proceed with lifting the limitation to convex drawings, and show that any drawing of any graph needs at most $\lceil |V|/2 \rceil$ layers to properly assign its edges to.



(a) For $n = 10$: $\vartheta(D_{opp}(G'_{sp})) = 5$ (b) G'_{sp} is a cycle, $bt(G'_{sp}) = 1$

Figure 4.1: Instance of G of order n and drawing D_{opp} where $\vartheta_{D_{opp}} = \frac{n}{2}bt$

4.2 Drawing thickness of the complete graph

We have shown (Theorem 3.11) that for a convex drawing of K_n it is exactly $\vartheta(D_{conv}(K_n)) = \lceil \frac{n}{2} \rceil$. We will now generalize for any drawing on $\mathbb{R}^2$ with points in general position, i.e. no 3 points are co-linear (we will assume any point set to be in general position unless noted otherwise).

Theorem 4.2. *Let K_n be drawn onto $\mathbb{R}^2$. Then $\bar{\theta}(K_n) \leq \vartheta(K_n) \leq \lceil \frac{n}{2} \rceil$.*

Lemma 4.3. Let $G(V, E)$ be drawn onto $\mathbb{R}^2$. It is $\bar{\theta}(G) \leq \vartheta(G) \leq \min\left(|E|, \left\lceil \frac{|V|}{2} \right\rceil\right)$.

The first part of the inequalities is trivial to show, due to the definition of geometrical thickness and our straight-line embeddings. We can get the Lemma 4.3 from Theorem 4.2 by observing that $G(V, E) \subseteq K_{|V|} \Rightarrow \vartheta(D(G)) \leq \vartheta(D(K_{|V|})) \leq \left\lceil \frac{|V|}{2} \right\rceil$ and combining with Lemma 4.1.

Let us now focus on the proof of Theorem 4.2.

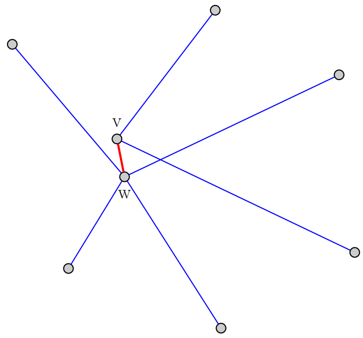
4.2.1 Plane spanning double stars

Definition 4.1. A *double star* is a tree with at most 2 non-leaf vertices. The edge joining these 2 vertices is called *root* of the star.

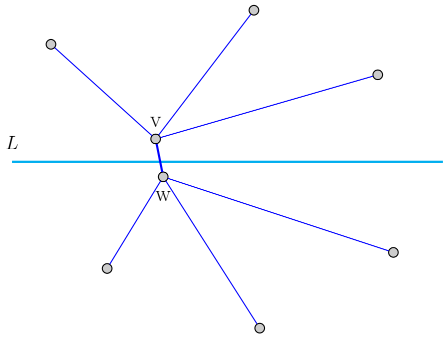
Consequently, a plane spanning double star is a connected acyclic graph drawn on the plane without edge crossings and with at most 2 vertices with degree greater than 1. In [8], P. Bose et al. use this precise object to partition a set of $n = 2r$ points on the plane in $n/2$ mutually edge disjoint sets, thus proposing (w.r.t. our terminology) that any complete graph of order $n = 2r$ can be drawn having drawing thickness at most $n/2$. In order to do so, they construct plane double stars on existing point sets using a few parameters:

Proposition 4.2. Let P be a set of points, L a line dividing plane in H_1 and H_2 half-planes such that $L \cap P = \emptyset$, and $v \in P \cap H_1$,$w \in P \cap H_2$. $T(P, L, v, w)$ is the graph with vertex set P and edge set $E = vw \cup \{vx : x \in (P \backslash \{v\}) \cap H_1\} \cup \{wy : y \in (P \backslash \{w\}) \cap H_2\}$. Then $T(P, L, v, w)$ is a plane spanning double star with root uw.

Proof. Edges adjacent to v are all leaves, do not cross and lie entirely on H_1. Same stands for edges adjacent to w and H_2, so none of the vx edges crosses none of the wy, leaving only $d(v), d(w) > 1$ (see Figure 4.2b). □



(a) A spanning double star with root vw.



(b) A plane spanning double star $T(P, L, v, w)$.

As a spanning tree, a plane spanning double star on n vertices has size of $n-1$ and, of course, it can define one of the planar layers a graph drawing may be decomposed to. Now our goal is to find $n/2$ such double stars that are pairwise edge disjoint and cover all edges of a complete graph. Let us define two plane double stars with the help of two non-parallel lines:

Proposition 4.3. Let $L_1, L_2 \cap P = \emptyset$, $L_1 \nparallel L_2$, and 4 points v, w, x, y that each lie on distinct quarter-planes, with pairs (v, w),(x, y) in opposite quarter-planes. Then, for the double stars $T_1(P, L_1, v, w)$ and $T_2(P, L_2, x, y)$ it is $E(T_1) \cap E(T_2) = \emptyset$.

Proof. Consider, for the sake of contradiction, that T_1 and T_2 have a common edge e. As all points are distinct, e cannot be one of the roots vw, xy and it should be $e \in \{vx, vy, wx, wy\}$ as e is adjacent to (v or w) and (x or y). Edge e crosses one of L_1,L_2 because all v, w, x, y lie on different quarter-planes defined by L_1 and L_2. Without loss of generality let e cross L_1. But the only edge of T_1 which crosses L_1 is $e = vw$ and that is a contradiction. □

Pairs of points like (v, w) and (x, y) of the proposition above will be referred to as *opposite points*: a pair of points lying on the 2 opposite open unbounded regions defined by a pair of intersecting lines.

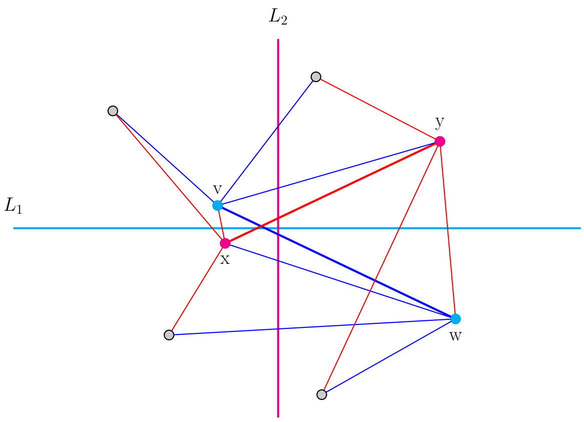


Figure 4.2: $T_1(P, L_1, v, w)$, $T_2(P, L_2, x, y)$ are edge-disjoint.

In order to complete the proof of Theorem 4.2, we would like to partition the $2r$-point set in a way that Proposition 4.3 guarantees the desired result. Formally our demand is described as follows:

Definition 4.2 (Λ family)**.** Let P be a set of $2r$ points in general position on the plane. Let Λ be a family $\{L_0, L_1, ..., L_{r-1}\}$ of r lines with the following properties:

1. all r lines are bisecting lines
2. exactly one point lies in each open unbounded region formed by the Λ family

A Bisecting line of a $2r$-point set P is a line that divides the plane into two half-planes each containing r points of P (see Appendix B).

The Λ family guarantees that

- all lines create distinct partitions of the set and therefore have distinct slopes
- lines create r pairs of opposite points
- for any 2 pairs of opposite points, there are 2 lines of the family w.r.t. which the 2 pairs lie on distinct quarter-planes (a prerequisite for applying Proposition 4.3)

Then, the proof of Theorem 4.2 has as follows:

Proof. Let P be the arbitrary $2r$-point set of the mapping of $V(G)$ through D. Consider the r lines of the Λ family on P: $\Lambda = \{L_0, L_1, ..., L_{r-1}\}$ labeled in ascending order of their angles (with an x axis), and the $2r$ points with $\{0, ..., 2r-1\}$ labeled in a way that the pair of opposite points p_i, p_{i+r} lie in the opposite open regions formed by L_i and L_{i+1}. Consider also the plane spanning double stars $T_i(P, L_i, p_i, p_{i+r})$ (indexes are modulo n). For every $i \neq j$ it is $E(T_i) \cap E(T_j) = \emptyset$ (Proposition 4.3), and all double stars cover $n(n-1)/2$ edges, which are all the edges of the complete K_n. □

In [8], P. Bose et al. appear confident that such a Λ family always exists, and relate it with halving lines ([41],[18],[52]). In fact, we will see that all lines of the Λ family relate to halving lines of P (see Appendix B), however, even though we, too, suspect that such family always exists (Claim B.1), the proof seems far from being trivial. In Appendix B we deal also with the relation of the halving lines with the desired Λ family, and we are led to believe that there is no quick greedy algorithm to prove the family's existence by constructing it.

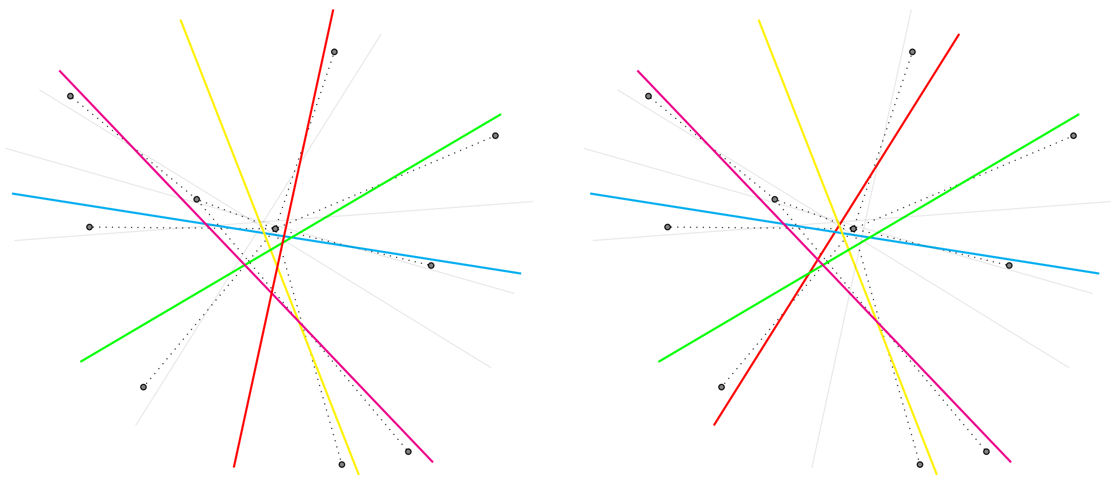


Figure 4.3: Two Λ families of a 10 point set. Dotted lines are halving lines, and shaded lines are more bisecting lines corresponding to halving lines (see Appendix B).

Proposition 4.4. A complete graph $G = K_{n=2r-1}$, has $\vartheta(D(G)) \leq \lceil \frac{n}{2} \rceil$, for every drawing D. The bound is once again tight.

Proof. $G = K_{n=2r-1}$ is a subgraph of K_{2r} and the convex case is enough to make bound tight. □

A note on halving lines and perfect cross-matchings: Borrowing the title of work of J. Pach and J. Solymosi ([52]), we explain the application of their results to our work.

Definition 4.3. A $2r$-point set P in general position on the plane is said to admit a perfect cross-matching if there are exactly r pairwise crossing segments that cover all $2r$ points. We will denote the class of such point sets by P_{pcm}.

Pach and Solymosi proved that such a point set P admits a perfect cross-matching *if and only if* $h(P) = r$ (in general it is $h(P) \geq n$), and as a consequence they constructed an $O(n \log n)$-time ($O(n)$-space) algorithm that decides if P has this particular property, and if so, computes the perfect cross-matching.

Lemma 4.4. Let D be an arbitrary drawing of the complete graph $K_{n=2r}$. If $D^V(K_n) \in P_{pcm}$ then it is $\vartheta(D(K_{2r})) = r$. The recognition problem whether $P \in P_{pcm}$ is polynomial-time.

Proof. Simply observe that r pairwise crossing edges force that $\vartheta(D(K_{2r})) \geq r$ while it is $\vartheta(D(K_{2r})) \leq r$. □

What we gain from this lemma will be better appreciated after we present the complexity of the problems regarding drawing thickness which are generally *NP*-hard. Yet we have a quick algorithm whose "YES"-instance is a definite answer to the decision problem whether K_{2r} is drawn to have drawing thickness r.

Chapter 5

Complexity

In this final chapter, we will present a number of results on the computational complexity of problems regarding the drawing thickness of a graph, and mention the not so widely studied "side" problem which we will call *"triangulation existence problem"*. As we have already discussed the complexity of determining the graph thickness and book thickness of graphs, once again we will adjust some of the terminology in favor of uniformity within this thesis:

Definition 5.1. We will denote by *k-(variant)-THICK* the decision problem whether an embedded on the plane graph G allows for a partition of its edges to k sets, edges within the same set pairwise not crossing, for each of the variants of thickness:

- Graph-theoretical thickness: *k-THICK*, Input: Graph G
- Geometrical thickness: *k-G-THICK*, Input: Graph G
- Book thickness: *k-B-THICK*, Input: Graph G
- Drawing thickness: *k-D-THICK*, Input: Graph G, drawing D

In the following table we have gathered all the existing results on complexity problems mentioned in previous chapters and examined throughout this very chapter.

Problem	**Class**	**Reduction**	
2-THICK	*NP*-complete	$\geq_p$ planar *3-SAT*	(2.3, [43])
2-B-THICK	*NP*-complete	$\geq_p$ max. planar *HC*	(Lemma 3.7, [57])
2-D-THICK	*P*	$\leq_p$ *2-COLOR*	(Lemma 5.2)
3-D-THICK	*NP*-complete	$\geq_p$ planar *3-COLOR*	(Lemma 5.2, [16], [21])
convex D-THICK	*NP*-complete	$\geq_p$ circle *COLOR*	(Lemma 5.5, [24])
3-convex D-THICK	*P*	$\leq_p$ circle *3-COLOR*	(Section 5.2.1, [54])
TRI	*NP*-complete	$\geq_p$ *3-SAT*, SEG *IND. SET*	(Theorems 5.7, 5.9, 5.10, [39], [37])
convex TRI	*P*	$\leq_p$ circle *IND. SET*	(Theorem 5.11,[26],[49])

Table 5.1: Complexity of thickness problems

A first observation is the absence of results regarding the geometrical thickness of graphs. We would expect that in general *G-THICK* is *NP*-complete, but in this case we do not have any clue for some k that places *k-D-THICK* in the class of *NP*-complete problems.

5.1 The class of SEG graphs and the *k-D-THICK* problem

Definition 5.2. Let $S = \{A_1, ..., A_n\}$ be a family of n geometrical objects. An intersection graph of the family is a graph $G(V, E)$ with $V = \{v_1, ..., v_n\}$ and $v_i v_j \in E \Leftrightarrow A_i \cap A_j \neq \emptyset$.

Definition 5.3 (SEG graphs)**.** SEG graphs are are the intersection graphs of line segments on the plane.

We will write I^S to denote the intersection graph of a set S of line segments. Graphs in the SEG class and its subclasses are easy to construct from the given S ($O(V + |E|^2)$-time) and will prove particularly useful to us, as it is convenient to transform a geometric object (a graph drawing $D(G)$) to an abstract graph (intersection graph) and study this graph instead. But, there is a minor problem: we are concerned for what we would call a "crossing graph", as our drawing thickness is only based on the crossing line segments' structure and in a general positioning of vertices which yields no parallel intersecting segments (see also definition 1.2).

Definition 5.4. Let $S = \{S_1, ..., S_n\}$ be a family of line segments on the plane. The crossing graph of S is the graph $G(V, E)$ with $V = \{v_1, ..., v_n\}$ and $v_i v_j \in E \Leftrightarrow A_i$ crosses A_j.

So, we will write respectively C^S to denote a crossing graph, and suppose all such graphs form the class CROSS. Of course, for a set S it is $I^S \ncong C^S$, but we will eventually show that SEG=CROSS. The proof has two points to focus our attention on: the endpoints of the segments, and the case of intersecting parallel segments. We assume always $\epsilon > 0$. See Figure 5.1 for an illustration of the proof.

Proof.

$G \in$ CROSS $\Rightarrow G \in$ SEG: Let S be the set of segments $v_i v_j$ of which G is the crossing graph. Replace every $v_i v_j$ with the shrunk $\boldsymbol{v_i} + \lambda \boldsymbol{v_j}, \lambda \in [\epsilon_{ij}, 1 - \epsilon_{ji}]$, and then separate all pairs of parallel segments that intersect by some gap ϵ_k to form set S'. No matter the density of the edge set, there will be always be a family of sufficiently small ϵ so that the crossing graph of S' is still G. But now G is also the intersection graph of S', so $G \in SEG$.

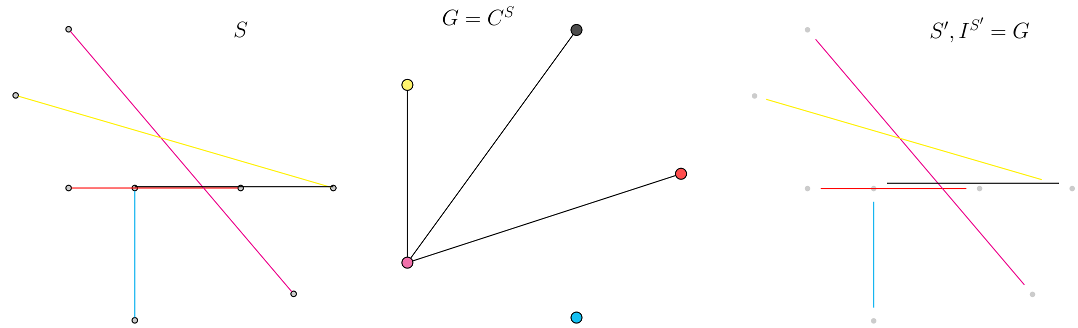
$G \in$ SEG $\Rightarrow G \in$ CROSS: If S is now the set $w_i w_j$ of which G is the intersection graph, first replace every $w_i w_j$ with the extended $\boldsymbol{w_i} + \lambda \boldsymbol{w_j}, \lambda \in [-\epsilon_{ij}, 1 + \epsilon_{ji}]$. Now, to transform all intersections of parallel segments to crossings, there is some more tweaking needed:

Consider each family of segments which are lying on the same line L_i and form a connected subgraph of G, G_{L_i}. If L_i is treated as the real line we have arrange the segments: $\{[a_1, b_1], ..., [a_n, b_n]\}$, $a_1 < a_2 < ... < a_n$ and of course G_{L_i} is an interval graph. Its maximum clique $\omega(G_{L_i})$ can be computed in polynomial time ([53]) and so is the problem of finding and ordering every distinct *maximal* clique, which can easily be solved in $O(n)$ time using a sweep line algorithm (see Appendix C.1). If the set of maximal cliques in G_{L_i} is $\{C_1, C_2, ..., C_k\}$, we can associate some common point c_i of intervals of maximal clique C_i to the clique itself, and have $c_1 < c_2 < ... < c_k$. The adjustment procedure of the segments is the following:

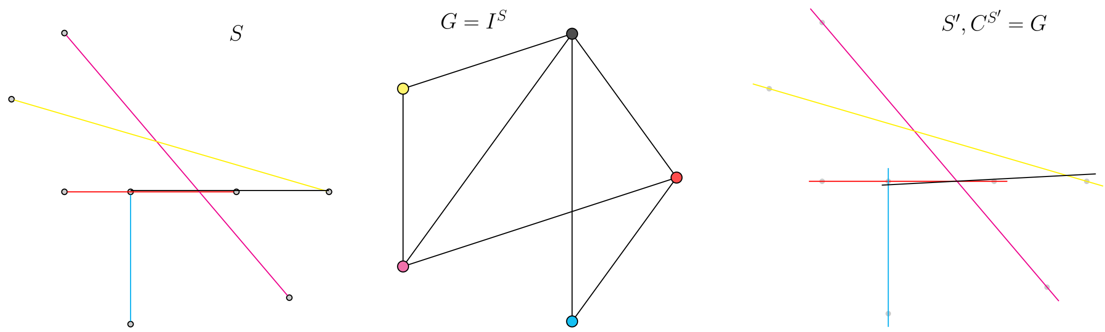
For all maximal cliques C_i from C_1 to C_k, leave their leftmost segment intact, rotate all other segments in C_i *that are not already rotated as members of another maximal clique* by distinct angles $\epsilon_{i,r}$ around c_i, and rotate all segments to the right of C_i along with the rightmost segment of the clique, around c_i. In the end of the procedure, the segments that intersected are now crossing (see Figure 5.2). Again, there is a family of sufficiently small ϵ, such that any intersection of segments on L_i with segments out of the L_i is not affected, and thus we have completed the proof.

□

We suspect that some works involving intersection graphs consider the above to be rather trivial, but it seems important to clearly show that the core of the properties and the complexity



(a) $G \in$ CROSS $\Rightarrow G \in$ SEG



(b) $G \in$ SEG $\Rightarrow G \in$ CROSS

Figure 5.1: Transforming S to S' to prove CROSS=SEG

of the problems regarding SEG graphs lies neither on whether intersection occurs at an endpoint of the segments, nor on the non-generic case of parallel intersecting segments.

From now on, we shall not distinguish a CROSS class, a term which is in fact is absent from bibliography, and refer only to SEG graphs. But, we will keep using both terms "crossing graph" and "intersection graph" of a set of segments, to be specific on the procedure constructing the desired graph. The following lemma summarizes the essence of the proof above.

Lemma 5.1. For any graph $H \in$ SEG there is some graph G_H and a drawing D_H such that H is the crossing graph of $D_H(G_H)$. As explained in the proof above, given some $H(V, E)$ it is polynomial-time to construct $D_H(G_H)$ $(O(V + |E|^2))$.

Recognizing SEG graphs is *NP*-hard. J. Kratochvíl showed in [35] that recognizing SEG graphs is *NP*-hard and so is the problem of recognizing *string graphs*, the intersection graphs of curves on the plane. The class of string graphs is obviously a superset of SEG graphs. A little later, along with J. Matoušek ([36]), they showed that SEG recognition is in *PSPACE*. We note also that M. Schaefer et al. ([55]) proved the recognition of string graphs to be in *NP*, so it is also an *NP*-complete problem.

5.1.1 Determining the drawing thickness of a graph drawing

Remark 5.1. For any graph drawn on the plane as $D(G)$, determining its drawing thickness is equivalent to determining the chromatic number of the drawing's crossing graph $C^{D(G)}$. Of course, $C^{D(G)}$ is a SEG graph.

There are two results that cover the *D-THICK* decision problem.

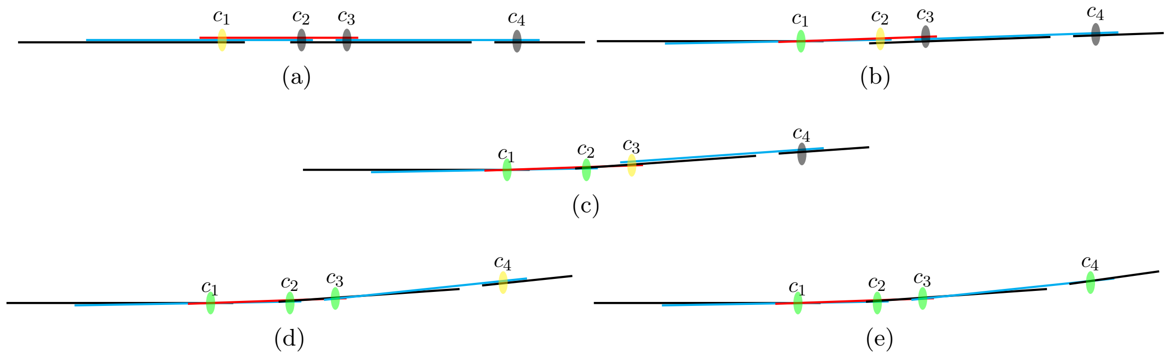


Figure 5.2: Steps of tweaking segments within a line to show SEG⇒CROSS. Segments of yellow maximal cliques are processed in each step, while all segments within green cliques are in their final position.

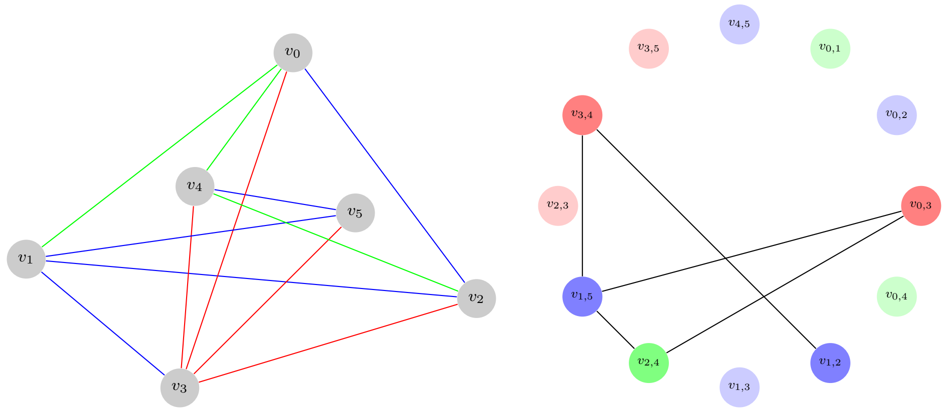


Figure 5.3: A drawing of G and its $C^{D(G)}$. $\vartheta(D) \sim \chi(C^D)$

Lemma 5.2. For any graph G and drawing D, determining whether it is $\vartheta(D(G)) = 2$ is polynomial time, while determining if $\vartheta(D(G)) = 3$ is *NP*-complete.

Proof. Checking if any graph is bipartite is polynomial time, so apply for $C^{D(G)}$. The *NP*-completeness of the *3-D-THICK* problem derives immediately from the following theorem together with Lemma 5.1:

Theorem 5.3 (G. Ehrlich et al., 1976)**.** *Given a set of line segments on the plane, it is* NP*-complete to determine if the intersection graph of its edges is 3-colorable. In other words,* 3-COLOR *is* NP*-complete in SEG graphs.*

Proof. For completeness, we mention that of course the problem is in *NP*: standard certificate of coloring all edges with k colors, our certifier is to check for every pair of edges ($O(|E|^2)$) that they intersect ($O(1)$) if and only if they are assigned different color.

Ehrlich et al. ([16]) but also Eppstein ([21]) more recently, reduce the problem of 3-colorability of planar graphs to what we call *3-D-THICK*, using a rather simple transformation (we present Eppstein's construction): taking a specific (planar) embedding of G, transform every vertex v to a small segment s_v centered on the point v is mapped to, and every embedded edge vw into 3 smaller line segments p_{vw}, q_{vw}, r_{vw} that all share the midpoint of vw as one endpoint, having their other on either of s_v, s_w, but not all three on the same one. With a little attention, *selecting a slope for s_v such that it does not lie on any incident edge to v, choosing sufficiently small s_v and placing the endpoints on s_v and s_w in a way that is consistent with* *the cyclic order of the edges around v and w in the embedding*, our construction of the set S of line segments has no intersections except for the ones involving p_{vw}, q_{vw}, r_{vw} and the recpective edges s_v, s_w. It is not hard to see that G is 3-colorable *if and only if* S is 3-colorable. □

□

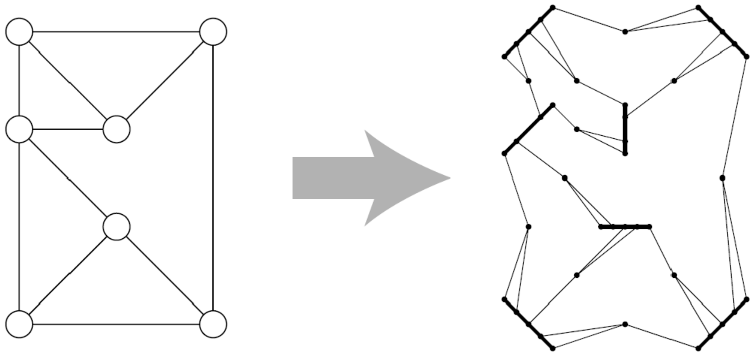


Figure 5.4: (D. Eppstein, [21]) Illustrating the construction to prove Theorem 5.3.

5.2 The class of circle graphs and the *convex D-THICK* problem

Definition 5.5 (Circle graphs)**.** A graph is a circle graph if it is the intersection graph of chords in a circle. CIRCLE $\subset$ SEG.

Again, if instead of the intersection graph of chords we construct their crossing graph, an element of a supposed CROSS-CIRCLE class, it is even easier that before to slightly adjust the chord's endpoints (slightly extend of shrink the chord) to prove CROSS-CIRCLE = CIRCLE.

Lemma 5.4. It is polynomial-time ($O(|V|)$), given any circle graph H to construct a family of chords $S_H = D_{H,circ}(G_H)$ such that $H = C^S$.

Recognizing circle graphs is polynomial time. The first proofs were published by W. Naji, [50],1985 and later on by C.P. Gabor et al., [23], 1989, to reach the very recent work of E. Gioan, C. Paul, M. Tedder, D. Corneil ([28]) who produce an $O(n+m)\alpha(n+m)$-time recognition algorithm (α stands for the inverse Ackermann function).

5.2.1 Determining the drawing thickness of a convex graph drawing

A flashback to Chapter 3 and Property 3.1 (ix) of convex drawings allows us to refresh that every convex drawing of a graph can be transformed to a (convex) drawing with all vertices on a circle, all edges becoming chords of the circle. In fact, it would be equivalent to say that circle graphs are intersection graphs of segments on the plane with their endpoints in convex position.

Remark 5.2. For any D_{conv} of G, $C^{D_{conv}(G)}$ is a circle graph and determining the drawing thickness of $D_{conv}(G)$ is equivalent to determining the chromatic number of the circle graph $C^{D_{conv}(G)}$.

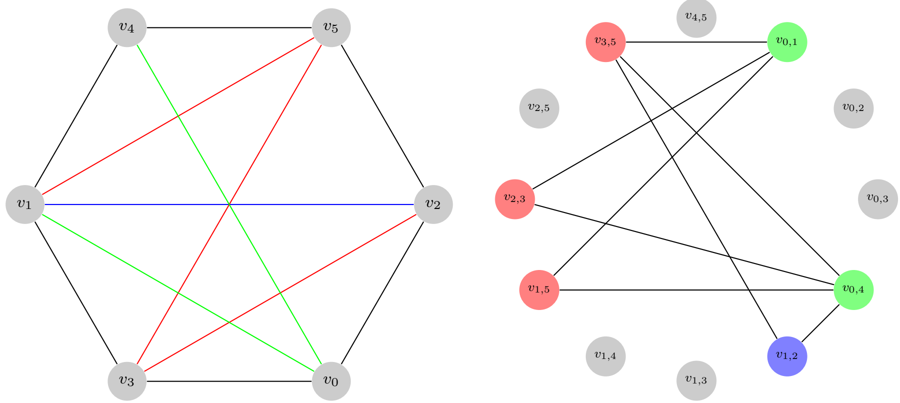


Figure 5.5: A convex drawing of G and its $I^{D_{conv}(G)}$.

Lemma 5.5. For any graph G and convex drawing $D_{conv}(G)$, *k-convex D-THICK* is *NP*-complete for $k > 3$.

We get this lemma, combining Lemma 5.4 with:

Theorem 5.6 (Garey, Johnson, Miller and Papadimitriou, 1980, ([24])). COLOR *in circle graphs is* NP*-complete.*

Garey et al. show the above using reduction from the *circular arc COLOR* problem, proved to be *NP*-complete in the same work. In fact, the full chain of reductions in this very work is the following:

Directed Disjoint Connecting Paths $\leq_p$ *Word Problem for Products of Symmetric Groups* $\leq_p$ *circular arc COLOR* $\leq_p$ *circle COLOR.*

Let us mention the definitions of each of the problems above:

> *DDCS*: Given a directed acyclic graph $G(V, A)$, $s_1, ..., s_n$ having in-degree 0, $t_1, ..., t_n$ having out-degree 0, does there exist a set of n mutually disjoint paths $s_1 - t_1, ..., s_n - t_n$?
>
> *WPPSG*: Given K, $X_1, ..., X_m \subseteq \{1, ..., K\}$ and a permutation $\Pi \in \Pi^{(K)}$. Defining $\Pi^{(X_i)} \subset \Pi^{(n)}$ as the set of permutations of $\{1, ..., K\}$ that leave all elements outside X_i fixed, does $\Pi = \Pi_1 \cdot \Pi_2 \cdot ... \cdot \Pi_m, \forall \Pi_i \in \Pi^{(X_i)}$?
>
> *circular arc COLOR*: Given a family of arcs of a circle and integer k, can they be colored using k colors such that intersecting arcs are colored differently?

***3-colorability* of circle graphs.** This particular problem is stated as *polynomially solvable* in [54], and we characterize *3-convex D-THICK* accordingly.

At this point, we need to note the work of F. R. K. Chung et al., [12], where this exact result (Lemma 5.5) is also presented. Again, there is a description of our D_{conv} mappings of a graph as a book embedding with specific vertex ordering on the spine of the book (also equivalent to the σ-thickness presented on [7]).

Finally, it is simple to see that there is an alternate route to prove *k-D-THICK* is *NP*-complete for $k > 3$, constructing a reduction from *k-convex D-THICK* (Figure 5.6). The idea is to plant an extra triangle on the outside of the existing convex n-gon, which does not add any new edge layer, but makes the new $(n + 2)$-gon concave. Formally:

Reduction: *k-D-THICK* $\in$ *NP*. For any convex instance $D_{conv}(G)$, create $G'(V', E'), V' = V \cup \{w\}, E' = E \cup \{wv_{l_k}, wv_{l_{k+1}}\}$ with $l_k, l_{k+1} \in V$ being the two neighbors of w, $D(G \subset G') = D_{conv}(G)$ and *CH(D(G))* $\neq$ *CH(D(G'))*. Triangle $D(w)D(v_{l_k})D(v_{l_{k+1}})$ is therefore outside $D(G)$ and $D(w)$ can be placed to make D concave. It is easy to see that $\vartheta(D(G')) = k \Leftrightarrow \vartheta(D_{conv}(G)) = k$.

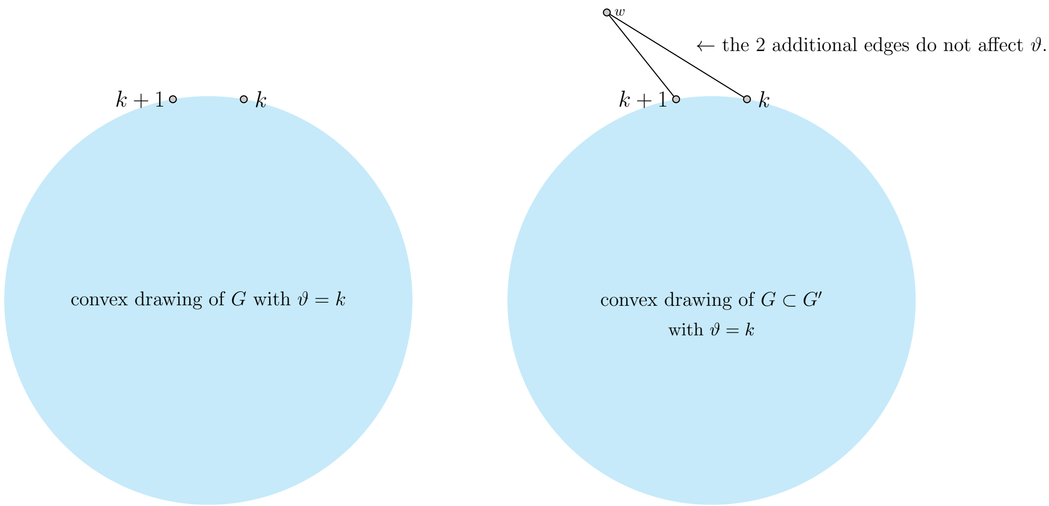


Figure 5.6: *D-THICK* reduction from *convex D-THICK*.

5.3 *Triangulation existence*: a "side" problem

We have mentioned the term "triangulation" in Chapter 3, while discussing maximal outerplanar graphs and associated them to *polygon triangulation*. However, there is also the concept of *point set triangulation*. To sum up:

Polygon triangulation is a decomposition of a polygon into a set of area-disjoint triangles that cover all polygonal area. Equivalently, a triangulation is a decomposition of the polygon into triangles by a *maximal set of non-crossing diagonals*. Every triangulation of a n-gon on the plane requires exactly $n-3$ diagonals, and creates exactly $n-2$ triangles that cover the polygonal area ([14]).

Point set triangulation is a triangulation of the convex hull of the point set P with exactly all points of P being vertices of the triangulation. If we consider the point set as the embedded vertices V of a graph G, a triangulation is a set of edges $E \subset \binom{V}{2}$ such that $G(V, E)$ is planar, the outer face of the drawing is bounded by the boundary of *CH(P)* and all other faces of the graph are bounded by 3 edges.

The two definitions coincide when the polygon or the point set are convex. If, though, a point set P is not convex, we may want to distinguish a *triangulation of P* from a *triangulation of a polygon defined on P* and define 2 different triangulation existence problems. Note that by "polygon defined *on P*" we will mean that all points of P are vertices of the polygon.

Definition 5.6.

> ***TRI***: Given a graph $G(V, E)$ and a drawing D of G onto $\mathbb{R}^2$, does there exist some $E' \subseteq E$ so that $D(G(V, E'))$ is a triangulation of point set $D(V)$?
>
> ***poly-TRI***: Given a graph $G(V, E)$ and a drawing D of G onto $\mathbb{R}^2$, does there exist some $E' \subseteq E$ so that $D(G(V, E'))$ is a triangulated polygon on point set $D(V)$?
>
> ***convex TRI***: Given a graph $G(V, E)$ and a convex drawing D_{conv} of G onto $\mathbb{R}^2$, does there exist some $E' \subseteq E$ so that $D_{conv}(G(V, E'))$ is a triangulation of point set $D_{conv}(V)$ (of the convex n-gon on $D_{conv}(V)$)?

In this last part of our thesis, we will explore *TRI* and *convex TRI*, leaving *poly-TRI* for future research.

5.3.1 *TRI* is *NP*-complete: a new approach

Theorem 5.7 (Lloyd, 1977)**.** *For an arbitrary drawing D of G,* TRI *is* NP*-complete.*

In his work ([39]), Lloyd uses a direct reduction from *3-SAT*, producing a hard-to-follow proof of the above. From our part, we will try to take advantage of the crossing graph of the drawing, to approach *TRI* in a more elegant way. In order to do so, we must define the following number:

***h(P)*:** We will denote by $h(P)$ the number of the points of P that lie on the boundary of the convex hull *CH(P)*. It is $3 \leq h(P) \leq |P|$ and $h(P) = |P|$ if and only if point set P is convex (see also Definition 3.4).

Theorem 5.8. *Let $D(G(V, E))$ be a triangulation of point set $D(V)$, with $|V| = n, |E| = e$ and $h(P)$ points on the boundary of its convex hull. Then $e = 3n - h(P) - 3$.*

Proof. We compare the triangulation to a maximal planar graph on n vertices, which has exactly $3n - 6$ edges and all its faces, including the outer face, are bounded by 3 edges. But in a triangulation, the outer face is bounded by $h(P)$ edges, and $h(P) - 3$ edges are "missing" from what would be a maximal planar graph. This is because any n-gon requires $n - 3$ diagonal to be triangulated (or every face of a graph bounded by n edges requires an additional $n - 3$ edges to be decomposed in triangular faces). So $e = 3n - 6 - (h(P) - 3) = 3n - h(P) - 3$. □

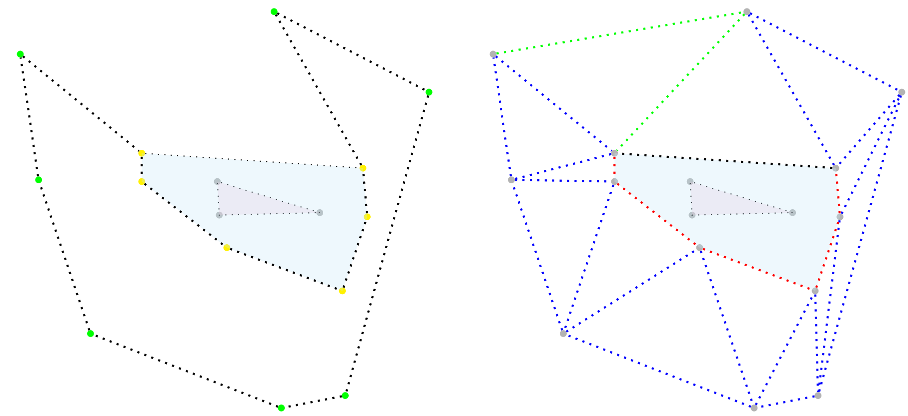
We note that a maximal planar graph may also be referred to as a *(plane) triangulation*. We shall now present a partitioning of a point set P in mutually disjoint convex layers and the occurring term *onion depth* (see [2]), in order to give an alternate proof of Theorem 5.8 and indicate similarities and possible links between different ideas.

Onion depth: Let P be our point set and *CH(P)* its convex hull. Let P_1 be the set of all $h(P)$ points on the boundary of *CH(P)*: P_1 is a convex point set. Using induction, we define P_{i+1} to be the points on the boundary of the convex hull of the non-empty set $P \setminus (P_1 \cup ... \cup P_i)$; if the set becomes empty, the procedure stops. It is $P_i \cap P_j = \emptyset$ and $P_i, P_j \neq \emptyset, \forall i \neq j$. If $P = P_1 \cup ... \cup P_k$, we define k to be the *onion depth* of P.

Following the partitioning in convex sets, we may express the number of points in P as $|P| = n = p_1 + ... + p_k$, where $p_i = |P_i| = h(P_i)$. It is true that $p_i \geq 3, \forall i \in \{1, ..., k-1\}$, but $p_k \geq 1$.

Alternate proof of Theorem 5.8. To build up our theorem, we will start from the center of the "onion" to create a triangulation of P (every other triangulation of P uses exactly the same number of edges): For $p_k \geq 2$, convex P_k needs $2p_k - 3$ edges to be triangulated (we can include the degenerate case of $p_k = 2$ as P_k needs $2 \cdot 2 - 3 = 1$ edges to be "triangulated"). Now, for every new P_i, adding its points to the drawing, we can consider a new polygon (non-convex) on the vertices of P_i and P_{i+1}. For its triangulation it demands $2(p_i + p_{i+1}) - 3$ edges, however $p_{i+1} - 1$ of them are already existing, plus, we need to add 2 more edges to triangulate the quadrilateral whose interior is not part of the $(p_i + p_{i+1})$-gon (see Figure 5.7). This means an addition of $2p_i + p_{i+1}$ new edges to keep $P_k \cup P_{k-1} ... \cup P_i$ triangulated. In the end, there is a total of $2p_k - 3 + 2p_{k-1} + p_k + ... + 2p_1 + p_2$ edges, or $e = 3(p_k + ... + p_2) + 2p_1 - 3 = 3(p_k + ... + p_1) - p_1 - 3$. But $p_1 = h(P_1) = h(P)$ and $e = 3n - h(P) - 3$.

Finally, note that in the case of $p_k = 1$, we can start the procedure considering P_k and P_{k-1} at once. A triangulation of a p_{k-1}-gon with one point in its interior requires $2p_{k-1}$ edges which is equal to $2p_k - 3 + 2p_{k-1} + p_k$ for $p_k = 1$. Therefore, our result needs no modification. ☐



(a) Adding the new layer (7 green points) outside the previous layer (6 yellow points) and forming the thick dotted 13-gon.

(b) New edges are the edges of the 13-gon minus the existing red ones (this is the blue set), plus the 2 green edges that triangulate the quadrilateral.

Figure 5.7: Building a triangulation for point set P with onion depth 3.

Now we are ready to transform the triangulation problem via the crossing graph of a drawing, having also in mind that a point set triangulation features the largest possible set of pairwise non-crossing edges.

Definition 5.7 (*TRI* alternative definition)**.** Given a graph $G(V, E), |V| = n$ and a drawing D of G onto $\mathbb{R}^2$ which maps h vertices to the boundary of the convex hull of $D(V)$, does the maximum independent set of the crossing graph $C^{D(G)}$ have size of $3n - h - 3$?

As our crossing graphs are SEG graphs, let us note that the maximum independent set (*IND. SET*) problem is *NP*-complete in SEG graphs, as it is *NP*-complete in subclasses of SEG:

Theorem 5.9 (J. Kratochvíl, J. Nešetřil, 1990, [37])**.** IND. SET *is polynomially solvable for 1-DIR and PURE-2-DIR, but it is* NP*-complete when restricted to 2-DIR and PURE-3-DIR.*

k-DIR is the class of intersection graphs of segments lying in at most k directions in the plane and PURE-k-DIR is the subset of k-DIR which complies with the additional condition that any two parallel segments are disjoint. In all, we are getting closer to the desired alternative proof for *TRI*, however, there are still some important details missing, because adding a property for the convex hull of a drawing of a SEG graph does not a priori guarantee that the *IND. SET* of such a graph is *NP*-complete as well (although it seems natural).

Computing the convex hull. The convex hull of a point set can be computed in polynomial-time; there are numerous publications and algorithms regarding the problem, from the "Jarvis' march" ([32], $O(nh)$-time) to the most recent output-sensitive algorithm of Chan ([9]) running in $O(n \log h)$ time (again, h is the number of points on the boundary of the convex hull). So, we have an efficient way to get the h number of our point set $D(V)$ of a graph drawing. In fact, in the meanwhile we can check if all h edges defining the convex hull polygon all belong to the drawing; one edge missing means no triangulation of the point set.

Defining SEG$_h$ graphs. It is evident that the convex hull plays no big role in the complexity of *TRI* and we will choose to omit these h edges when constructing the crossing graph. Now we ask for a maximum independent set of size $3n - 2h - 3$ for the crossing graph $C^{D(G) \setminus CH(D(G))}$. We have (accidentally) created another class of graphs at this point, let us call it $SEG_h \subseteq SEG$, including all graphs of SEG that are the crossing (intersection) graphs of segments whose convex hull is a h-gon, but none of the h-gon's edges are drawn. In may be SEG$_h$=SEG, for any $h \geq 3$, but for our work we will just prove *IND. SET* in SEG$_h$ graphs to be *NP*-complete, and complete the proof.

The reduction. We will use the *IND. SET* of SEG graphs as our know *NP*-complete problem. We let $C^{D(G)} \in$ SEG, with maximum independent set of size k. For any $h \geq 3$ we will alter the drawing $D(G)$ adding h more edges, pairwise non-crossing and outside $D(G)$ as follows:

Find an equilateral triangle so it entirely contains $D(G)$; consider also its circumscribed circle[1]. Draw 3 of the h new edges to be line segments between each of the vertices of the triangle and appropriate points on the boundary of *CH*($D(G)$ so they do not cross. Placing the remaining $h - 3$ edges can be done in several ways, but the simplest valid placement would be to create $h - 3$ segments between a point p_i on the circle and the point on the boundary of *CH*($D(G)$) linked to the triangle vertex closer to p_i. This way it is easy to see that the h new points on the circle define the new convex hull.

We have created some $D'(G')$ with $C^{D'(G')} \in$ SEG$_h$, and it is clear that the maximum independent set of $C^{D(G)}$ is k if and only if the maximum independent set of $C^{D'(G')}$ is $k + h$, because we did not invoke any new crossings. *IND. SET* of SEG$_h$ graphs is *NP*-complete, indeed. To conclude:

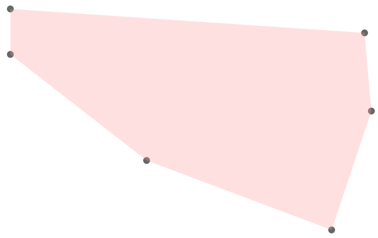
Theorem 5.10. *SEG* IND. SET $\leq_p$ TRI. TRI *is* NP*-complete.*

5.3.2 *Convex TRI* and *IND. SET* in circle graphs

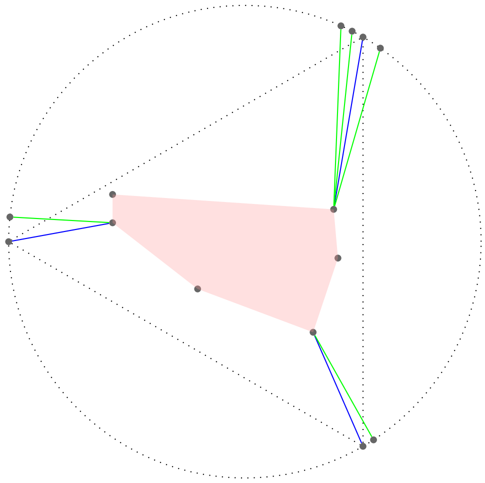
Restricting our triangulation existence problem in convex graph drawings, the analysis becomes simpler: triangulation always requires the maximal of $2n - 3$ mutually disjoint edges on n vertices. Meanwhile, the crossing graph of a convex drawing is shown to be a circle graph. We immediately get the following:

Definition 5.8 (*Convex TRI* alternative definition)**.** Given a graph $G(V, E), |V| = n$ and a convex drawing D_{conv} of G onto $\mathbb{R}^2$, does the circle graph $C^{D_{conv}(G)}$ have independent set of size $2n - 3$?

[1] this can be computed in linear time, for instance finding $x_{min}, x_{max}, y_{min}, y_{max}$ coordinates of all points in $D(G)$ is enough to determine the triangle and circle center and radius.



(a) Arbitrary SEG graph drawn, with its convex hull marked.



(b) Creating a SEG_h graph ($h = 8$). The 3 first edges to be placed, following the reduction, are colored blue.

Figure 5.8: Illustrating the reduction SEG *IND. SET* $\leq_p$ SEG_h *IND. SET*. No new segment crossings (intersections) appear.

***IND. SET* of circle graphs can be computed in polynomial time.** F. Gavril ([26]) described a first polynomial algorithm ($O(n^3)$), while recently, N. Nash and D. Gregg ([49]) presented an $O(n \min(d, \alpha))$-time output sensitive algorithm, d being the density of the graph and α being its independence number (see Appendix C.2 for a simpler $\Theta(n^2)$ algorithm).

Theorem 5.11. Convex TRI *can be solved in polynomial time.*

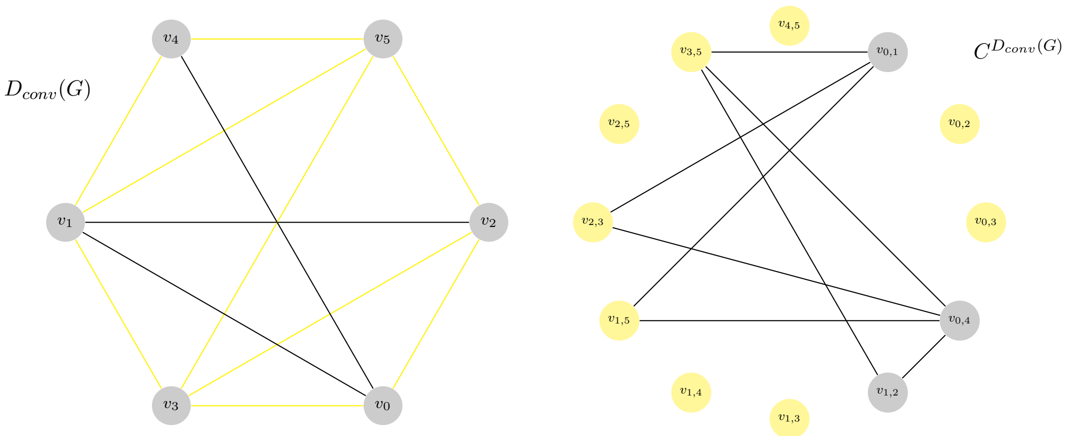


Figure 5.9: Maximal set of $9 = 2 \cdot 6 - 3$ pairwise non-crossing edges for a convex drawing and the corresponding crossing graph with max. ind. set of size 9.

5.4 Open problems

Ratio between $\bar{\theta}$ and θ if $\theta = 2$. This derives directly from the approach of D. Eppstein ([20]) in showing that $\frac{\bar{\theta}(G)}{\theta(G)} = \omega(1)$ using graphs with thickness 3, unlike the work showing $\frac{bt(G)}{\bar{\theta}(G)} = \omega(1)$, which is based on graphs with geometrical thickness 2.

K_{13}, K_{14}. What is the geometrical thickness of these two complete graphs? We suspect that there is no given result to date.

Efficient algorithm to check if a convex drawing of $G, |V| = n$ has $\vartheta < \lceil n/2 \rceil$. This is a problem proposed by us, based on our discussion of sparse graphs and the fact that a graph with $|E| = n/2$ can have equal drawing thickness to a graph with $|E| = \binom{n}{2}$. Star polygons (Appendix A) give some information on structures (spanning subgraphs of some convex K_n) that demand a specific number of planar layers to be decomposed. Does this information lead to efficient recognition of convexly drawn graphs with $\vartheta < \lceil n/2 \rceil$?

Efficient algorithm to check if a non-convex drawing of K_n has $\vartheta < \lceil n/2 \rceil$. We saw that if a complete graph is drawn so its vertex set admits a perfect cross-matching, then its drawing thickness is exactly $\lceil n/2 \rceil$ (Section 4.2.1). The opposite is not true, but it would be very interesting if there is a polynomial-time "recognition" algorithm of point sets that imply that K_n on that point set has $\vartheta < \lceil n/2 \rceil$. In that case, the algorithm's relation with halving lines again would be intriguing, although it will be of no surprise if no such algorithm exists.

For a graph G, find some k for which *k-G-THICK* is *NP*-complete. Geometrical thickness problems are absent from our Table 5.1, as well as from bibliography; it is very natural to expect that *2-G-THICK* is *NP*-complete, as both *2-THICK* and *2-B-THICK* are *NP*-complete.

Complexity of *poly-TRI*. This is an unexplored triangulation problem. Within a point set with several edges, the probability that a triangulated n-gon appears is strictly higher than the probability that a n-point set triangulation exists. Can this increased probability put *poly-TRI* in *P*? A certain fact is that *poly-TRI* $\in$ *NP*.

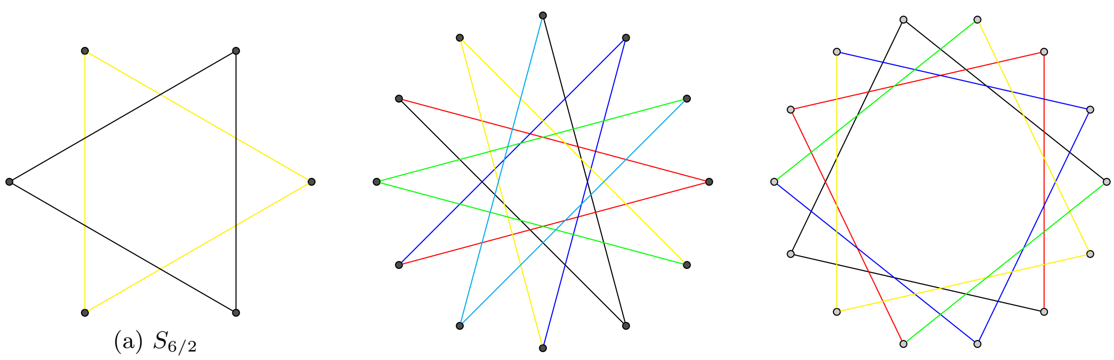
Appendix A

Drawing thickness of star polygons and star figures

A star polygon $\{n/k\}$, with n, k positive integers, is a figure formed by connecting with straight lines every k^{th} point out of n regularly spaced points lying on a circumference ([13]). The number k is called the density of the star polygon. Without loss of generality, take $k \leq \lfloor n/2 \rfloor$. Originally, for a star polygon we have $\gcd(n, k) = 1$, and we can draw it without lifting our pen. If $\gcd(n, k) > 1$ we often come across the term "star figure", but as in this work we deal with both cases, we will definitely not distinguish our notation; we adjust the definition once again, to comply with our terminology and focus on drawing thickness:

Definition A.1 (Star polygon drawing)**.** A graph $G(V, E)$ with $|V| = n$ is drawn as a star polygon (star figure) $S_{n/k}$ if there is some D_{conv} such that all edges $e \in E$ are drawn so they belong to span class E_k and the span class is full (see also Property 3.1 (v)). We may write $D(G) = S_{n/k}$ (and imply that D is a convex mapping).

Note that the case of $n = 2r$ and $|E_r| = n/2$, the only span class including no n edges, $S_{2r/r}$ is regarded as a degenerate star polygon; however, it does not require special attention in what follows.



(a) $S_{6/2}$ (b) $S_{12/5}$ (c) $S_{14/4}$

Figure A.1: Star polygons and their edge layering.

Proposition A.1. For the drawn star polygon (star figure) $S_{n/k}$, no more than $\lfloor n/k \rfloor$ edges may be assigned to the same layer. We will call this quotient $\lfloor n/k \rfloor = q$, and if a layer is assigned some $\lfloor n/k \rfloor$ non-crossing edges, we will call it *full*.

Proof. We can assume, without loss of generality (Property 3.1 (ix)), that the n vertices form a regular n-gon. Every edge (chord) in E_k can be associated with the respective arc, measuring $a = 360 \cdot \frac{k}{n}$ degrees. q edges of E_k can be assigned to the same layer if and only if $q \cdot a \leq 360^o$. Therefore $q \leq \frac{360}{360\frac{k}{n}}$ and as we seek for a maximum integer satisfying the inequality we get $q = \lfloor n/k \rfloor = q_{n/k}$. □

Theorem A.1. *The drawing thickness of $S_{n/k}$ is $\vartheta(S_{n/k}) = \left\lceil \frac{n}{\lfloor \frac{n}{k} \rfloor} \right\rceil = k + \left\lceil \frac{r}{q} \right\rceil$, the integers satisfying the Euclidean division: $n = k \cdot q + r$, $0 \leq r < k$. In addition, for $k_1 > k_2$ it is $\vartheta(S_{n/k_1}) \geq \vartheta(S_{n/k_2})$.*

Note that if $n = k \cdot q + r$, $0 \leq r < k$ we have $\left\lceil \frac{n}{\lfloor \frac{n}{k} \rfloor} \right\rceil = \left\lceil \frac{k \cdot q + r}{q} \right\rceil = \left\lceil k + \frac{r}{q} \right\rceil = k + \left\lceil \frac{r}{q} \right\rceil$. In addition it is true that:

$$\min l : l \cdot k = 0 \mod n \Leftrightarrow l = n/\gcd(n,k) \tag{A.1}$$

Proof. We will refer to *consecutive* edge drawings, drawings without lifting our pen, i.e. paths of the form $i \to i + k \to i + 2k \to ....$

Star polygon: $\gcd(n,k) = 1$. In this case it is always $|E_k|_{max} = n$ (Property 3.1 (v)), and as A.1 suggests, starting from some vertex i we get back to i only after drawing all n edges. Edge set is expressed as $e_j = (jk, (j+1)k) \mod n$, for $j = 0, ..., n-1$. A layer is created per $q = \lfloor n/k \rfloor$ drawn edges (see Figure A.1b), and we are easily led to $\vartheta(S_{n/k}) = \left\lceil \frac{n}{\lfloor \frac{n}{k} \rfloor} \right\rceil$.

Star figure: $\gcd(n,k) = d > 1$. We express $n = d \cdot n_d$, $k = d \cdot k_d$. Now, starting from vertex i we are brought back to i after drawing n/d edges (A.1), creating a minor star polygon isomorphic to S_{n_d/k_d}, with edge set $e^i_j = (jk + i, (j+1)k + i) \mod n$ for $j = 0, ..., n/d - 1$. To complete the star figure draw another $d - 1$ similar star polygons. We observe that for the minimum vertex $i + x, x \in [0, n-1]$ an edge of e^i_j reaches we have:

$$jk + i = i + x + cn \Rightarrow x = jk - cn = d(jk_d - cn_d)$$

which indicates there is a "gap" of size at least d between each of the vertices we reach drawing the star polygon[1]. So we can take all other $d - 1$ stars with mutually disjoint edge sets $e^l_j, 0 \leq j < d, j \neq i$ and we denote by $S^i_{n_d/k_d}$ the star polygon drawn starting from i with edge span k, for $i = 0, ..., d-1$. $S_{n/k} = \bigcup_{i=0}^{d-1} S^i_{n_d/k_d}$.

The key difference now is that when completing the drawing of a minor star polygon we must lift our pen, and, in the case the last created layer is not full, complete it in order to minimize the total layers. For $0 < r' < q$, the last drawn edges of S^0 are: $(-r' \cdot k, (-r'+1) \cdot k), ..., (-k, 0)$. "Lifting the pen", we can add the first $q - r'$ edges $(1, k+1), ..., ((q - r'_d)k + 1, (q - r'_d + 1)k + 1)$ of S^1 to place along with remainder edges of S^0: this can be done because the total span of the edges of a full layer $k \cdot q$ is not greater that $n - d$ (or else the gap would not have been d).

We can repeat the procedure after drawing the last edges of each S^i (ending at i), as the gap between vertices $i + 1$ and $d - 1$ permits it. See also Figure A.2.

So once again our drawing, even with $d - 1$ "pen lifts", dictates a full layer every $q = \lfloor n/k \rfloor$ edges and $\vartheta(S_{n/k}) = \left\lceil \frac{n}{\lfloor \frac{n}{k} \rfloor} \right\rceil$. For the degenerate case where $n = 2r, k = r$, the proof is straightforward.

[1] In fact, the gap is exactly d, something it can either be proven using Béjout's identity ($jk_d - cn_d = 1$) and selecting a solution where $0 \leq j < n/d$, or by noticing that when d minor stars (all n edges with span k) are drawn, there is no vertex without edge.

$\vartheta(S_{n/k})$ **is an increasing function of** k**:** $k_1 > k_2 \Rightarrow \lfloor n/k_1 \rfloor \leq \lfloor n/k_2 \rfloor \Rightarrow \left\lceil \frac{n}{\lfloor \frac{n}{k_1} \rfloor} \right\rceil \geq \left\lceil \frac{n}{\lfloor \frac{n}{k_2} \rfloor} \right\rceil$. $\square$

We conclude with a simple lemma:

Lemma A.2. Let $D(G) \subseteq S_{n/k}$, it is $\vartheta(D(G)) \geq \left\lceil \frac{|E_k|}{\lfloor \frac{n}{k} \rfloor} \right\rceil$.

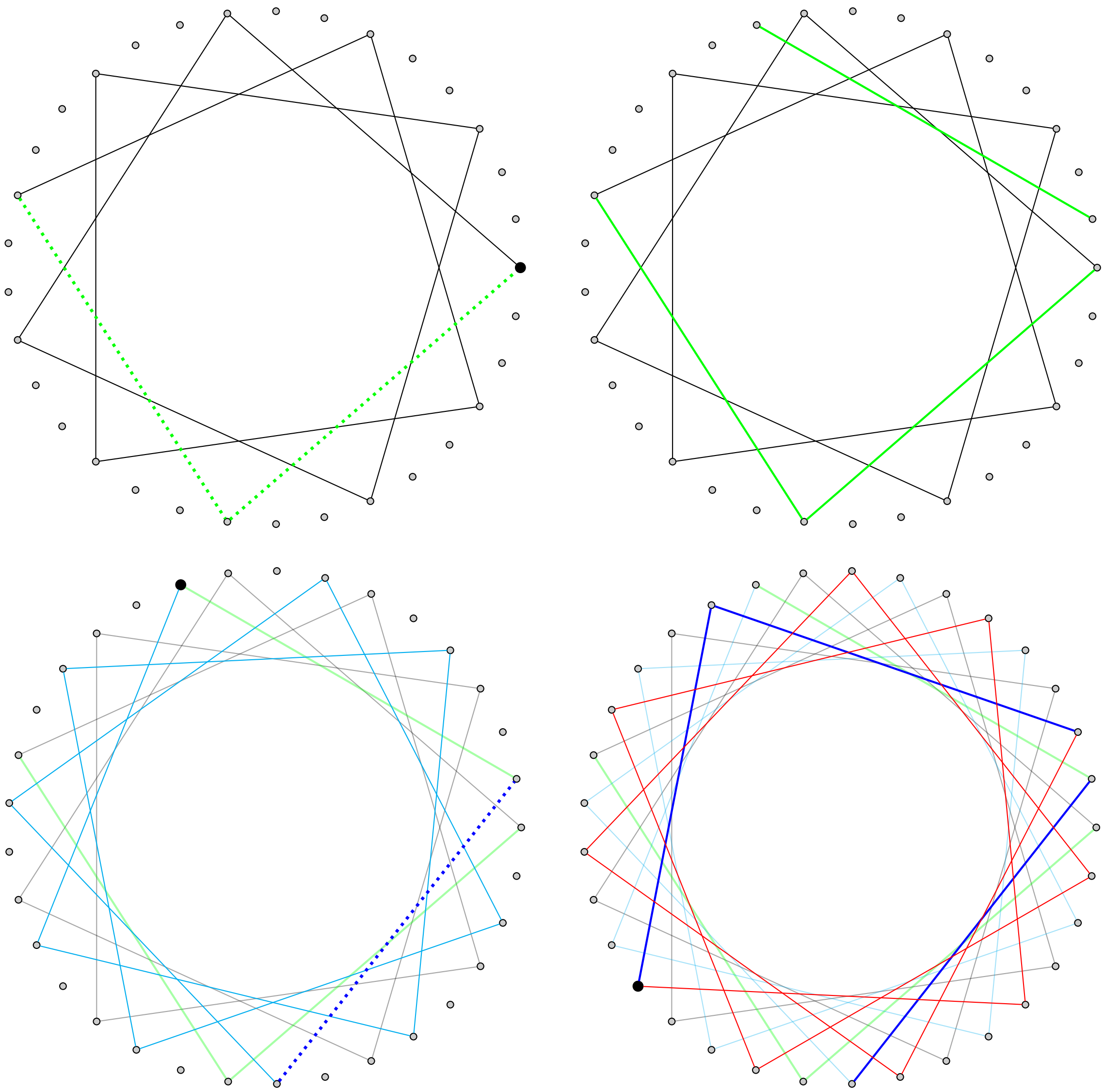


Figure A.2: $S_{33/9}$. Focusing on the remainder edges of each of the minor $S_{11/3}$, and following the algorithmic procedure described in the proof. Of course $\vartheta(S_{33/9}) = \left\lceil \frac{33}{\lfloor \frac{33}{9} \rfloor} \right\rceil = 11$.

Appendix B

Bisecting lines and Λ family

We refresh that we seek for the following:

Definition B.1 (Λ family)**.** Let P be a set of $2r$ points in general position on the plane. Let Λ be a family $\{L_0, L_1, ..., L_{r-1}\}$ of r lines with the following properties:

1. all r lines are bisecting lines
2. exactly one point lies in each open unbounded region formed by the Λ family

Remember also that by the term *opposite points* we refer to a pair of points lying on the 2 opposite open unbounded regions defined by a pair of intersecting lines.

B.1 Halving lines and partitioning lines of point sets

Definition B.2 (Halving line)**.** Let P be a $2r$-point set onto $\mathbb{R}^2$, in general position. A line L through 2 points of P is a halving line if exactly $r-1$ points lie on either of the half-planes L defines. The set of all halving lines of P is denoted by $H(P)$ and the number of halving lines by $h(P) = |H(P)|$.

Halving lines are well studied ([41],[18],[52] and many more), and prove a useful tool for computational geometry. It is easy to observe that there is a halving line through any point of P, and consequently for a $2r$-point set it is $h(P) \geq r$.

Partitioning line is a general term used for a line that, unlike a halving line, includes no point of the point set P and separates it into half-planes that both contain points of P. We will use the term "slope of line" (λ) to indicate the *convex* angle of the line with the Ox axis of a Cartesian coordinate system; consequently, it is $0 \leq \lambda < 180°$ for any line. For a set of lines or line segments, the corresponding set of slopes has a minimum (λ_{min}) and a maximum element (λ_{maw}), for which we will define:

- The minimum slope greater than λ_{max} is λ_{min}
- The maximum slope smaller than λ_{min} is λ_{max}

Definition B.3 (Bisecting line)**.** A (partitioning) line L is a bisecting line of a set of points if at most half of the points lie on either side of L. If the number of points of the set is odd, then one point must lie on the bisecting line L.

The *Ham-sandwich theorem* ([45], [34], [40]) guarantees the existence of a bisecting line, but in this special case (only 1 grouped set of points) any sweeping line can be stopped at a position where it is bisecting[1].

[1] To be exact, if a sweeping line has slope equal to one of the slopes of the line segments defined on set P, it may not become a bisecting line.

Definition B.4. Let L_1, L_2 be 2 bisecting lines of some point set P. If both lines partition the points of P into identical pair of sets, then we will say they belong to the same bisecting class A. We will write $B(P)$ to indicate the set of all bisecting classes of P and $b(P) = |B(P)|$ to indicate their number.

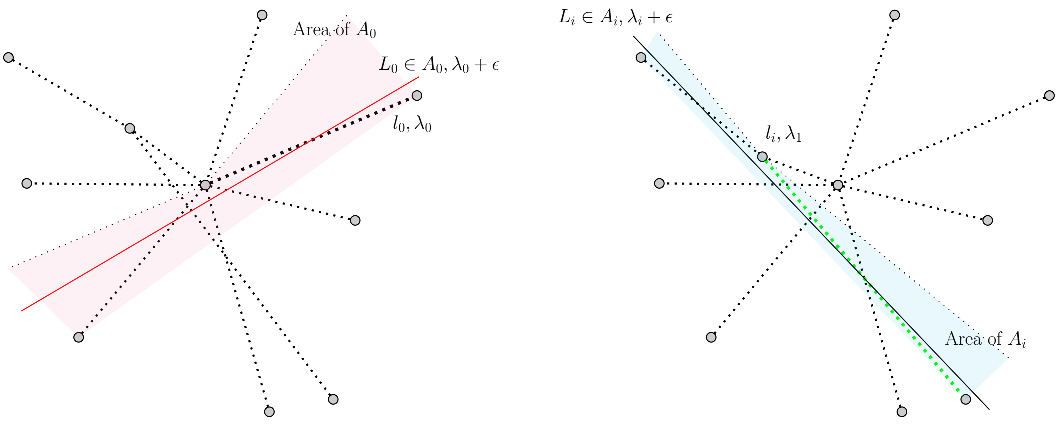
Obviously, every bisecting line belongs to exactly one bisecting class, so $A_1 \neq A_2 \Leftrightarrow A_1 \cap A_2 = \emptyset$.

Proposition B.1. Let P be a $2r$-point set. There is an isomorphism between the set of halving lines $\{l_0, ..., l_{h(P)-1}\}$ and the set of bisecting classes $\{A_0, ..., A_{b(P)-1}\}$ of P.

Proof. Let l_0 be a halving line through points $p_0, p_1 \in P$, with slope λ_0. "Tweak" this halving line by rotating it counterclockwise around any of the interior points of p_0p_1 (around the midpoint, for instance) to generate line L_0 with slope $\lambda_0 + \epsilon, \epsilon > 0$. It is easy to understand that there is a sufficiently small ϵ such that L_0 is a bisecting line, and this ϵ depends on the *minimum slope greater than* λ_0 which the line segments intersecting l_0 define.

Now consider L_0 to be in bisecting class A_0, and let $l_1 \not\equiv l_0$ be a halving line through $p_2, p_3 \in P$ with slope λ_1 which yields bisecting line $L_1 \in A_1$ when tweaked as before. We will show that $A_0 \neq A_1$:

Without loss of generality, let $p_3 \not\equiv p_1$, $\lambda_1 > \lambda_0$. If $p_0 \equiv p_2$, then line L_0 leaves p_2, p_3 on the same half-plane, while L_1 splits them apart. If $p_0 \not\equiv p_2$ then L_1 leaves p_1, p_3 on the same half-plane, while L_0 splits them.



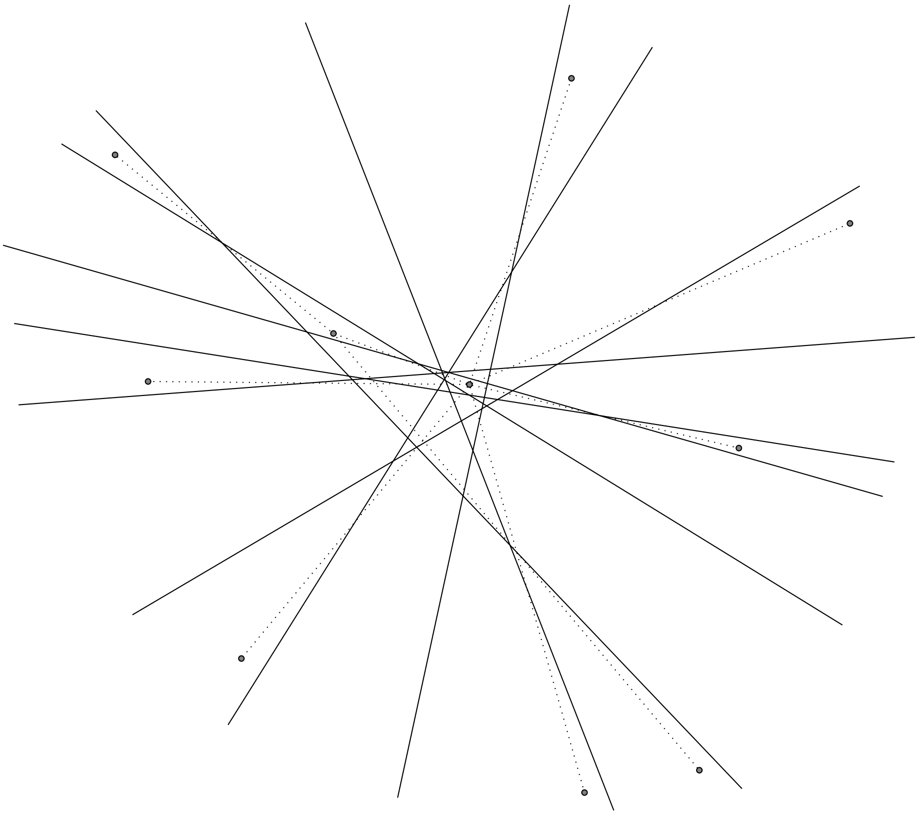
(a) Set P with its halving lines and area of bisecting class A_0 defined by l_0.

(b) Set P with a bisecting line L_i and corresponding area of A_i and halving line l_{i-1}.

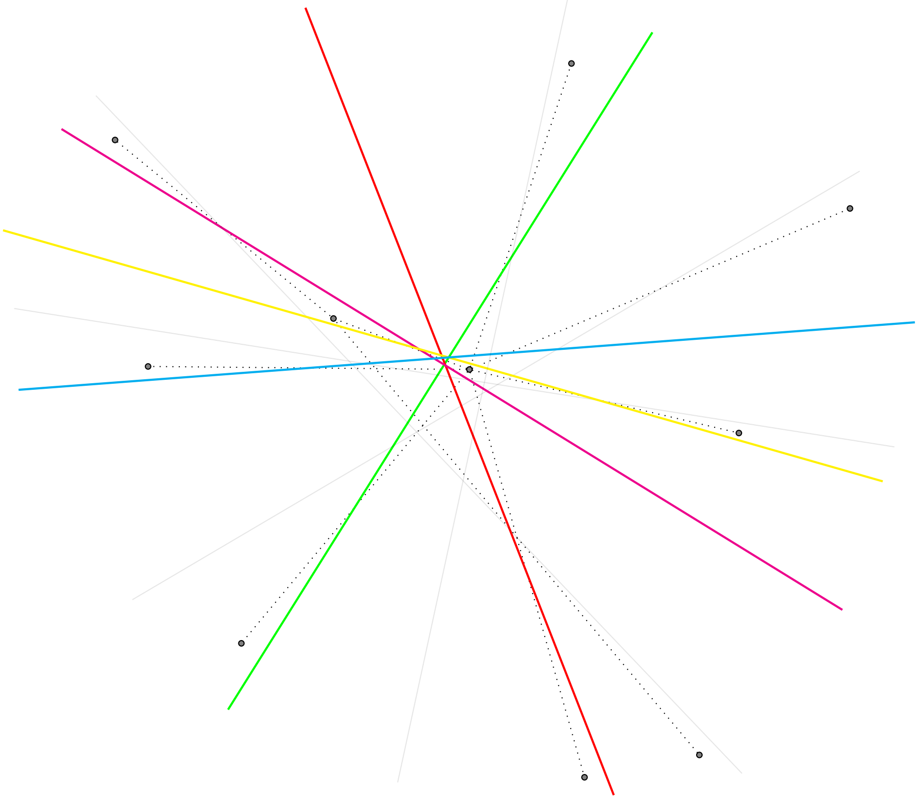
Figure B.1: Illustrating that $H(P) \cong B(P)$.

For the opposite, we can claim that for every bisecting line $L_i \in A_i$ with slope λ_i we search for the line segment $p_{2i-2}p_{2i-1}$ with the *maximum slope smaller than* l_i that also intersects L_i. As there are no 3 co-linear points in P, this segment is unique, and defines a halving line of P. Of course, if $L_i \in A_i, L_j \in A_j \neq A_i$ then for the 2 corresponding segments it is $p_{2i-2}p_{2i-1} \not\equiv p_{2j-2}p_{2j-1}$.

Thus said, the proof is complete and $h(P) = b(P)$. □



(a) Bisecting lines defining all bisecting classes A_i.

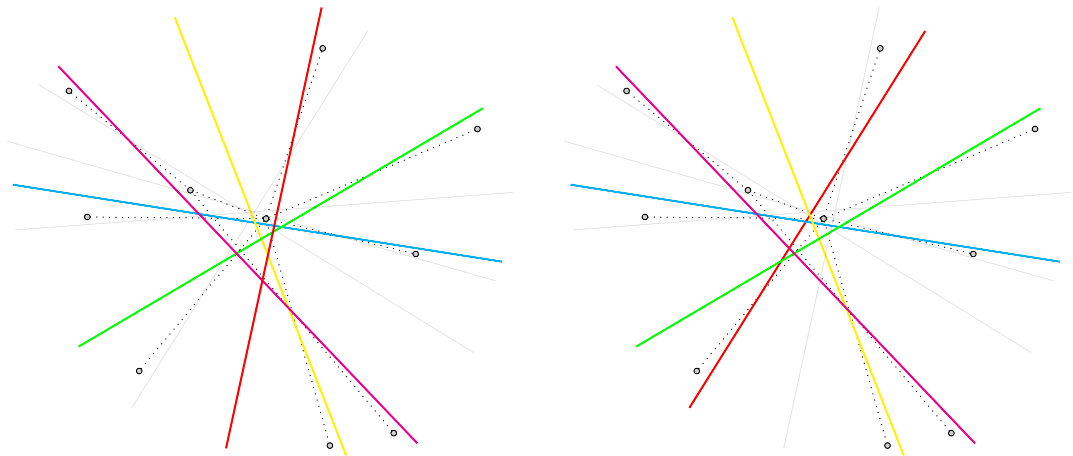


(b) Valid Λ family.

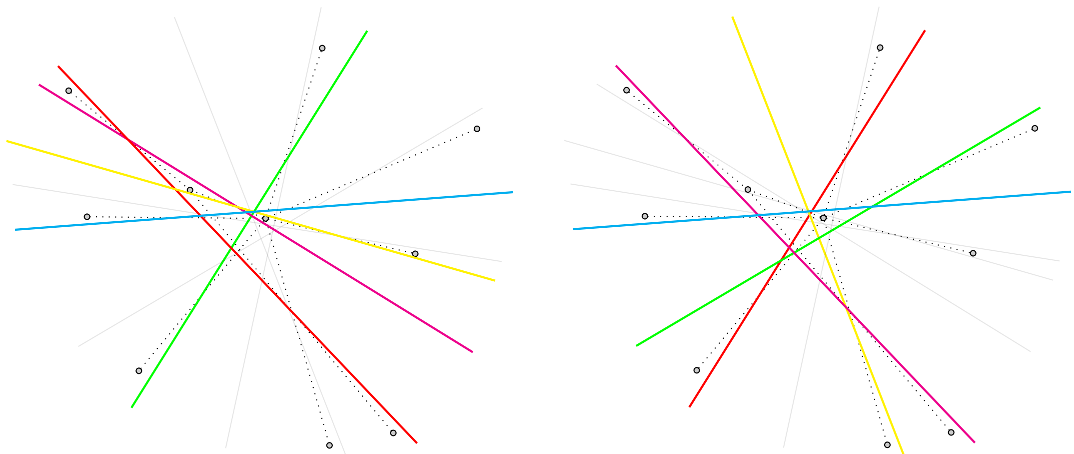
Figure B.2: Bisecting lines of P.

Notice that we need intersecting segments to get our ϵ when moving from a halving to a bisecting line, as two halving lines may share one point (see Figure B.1a below). But, as a bisecting line does not contain any points of P, when searching for intersecting line segments to get to the corresponding halving line (Figure B.1b), we are sure to get a segment that crosses (interiorly intersects) our bisecting line.

Claim B.1. Let P be a set of $2r$ points in general position on the plane. Then there is a set $\{A_0, A_1, ..., A_{r-1}\}, \in 2^{B(P)}$ so that $\Lambda = \{L_0, L_1, ..., L_{r-1}\}$ with $\forall L_i \in A_i$.



(a) More valid Λ families.



(b) Invalid Λ families.

Figure B.3: Selecting families of bisecting lines

Additional observations for Λ. As we already stated in Chapter 4, greedy algorithms seem not to be the tool for an algorithmic proof of our claim, or at least not unless we are able to find a certificate that one of the $b(P)$ bisecting classes has necessarily a line in the Λ family.

Another interesting property is that for every point p not in the boundary of *CH(P)*, the bisecting lines in Λ must associate with halving segments[2] which include p that may all belong to the same half-plane (w.r.t. a line through p). It is clear to see (invalid families of Figure B.3b) that if this is not true, point p is totally surrounded by lines and therefore lies in a bounded region.

[2] We consider as *halving segment* the segment of a halving line between the 2 points of P it includes.

Appendix C

Algorithms

C.1 Maximal cliques of interval graphs

This is a simple greedy algorithm that computes a point of every maximal clique of an interval graph. It may be regarded as a sweep line algorithm: a line parallel to the y-axis running along x-axis. Having a point in a maximal clique, it is easy to find all members of it. Procedure "mergeflag" is the standard merging of two sorted lists with the added task of labeling (flag: *start* or *finish*) members of merged list according to whether they are start points or end points of the intervals. The algorithm is easily understood (we added a counter to get the maximum clique size simultaneously).

Algorithm 1: $\Theta(n)$ algorithm for maximal cliques of interval graphs

```
input  : A set of n intervals [a_i, b_i] sorted a_i < a_{i+1}
output: A point in each of the maximal cliques and maximum clique size
C[1..2n][2] ← mergeflag([a_i],[b_i]);
counter,max.clique ← 0;
for i = 1 to 2n do
    if C[i,2] = start then
        counter ← counter+1;
    else
        selectpointin(C[i − 1,1],C[i,1]);
        if counter > max.clique then
            max.clique ← counter;
        counter ← counter-1;
```

C.2 Independent set of circle graphs

To construct the desired algorithm, given a set of n chords, we need to them to an interval representation where the endpoints of the intervals are the pairs (l_{2k-1}, l_{2k}), for $1 \leq k \leq n$. All l_i may be chosen to be a permutation of $\{1, ..., 2n\}$, and this can be done in polynomial time, so we form the set of intervals I with distinct endpoints. Now chords cross if and only if their respective intervals *ovrelap*, i.e. do not properly contain one another. Introducing some notation:

- $I_{q,m} \subseteq I$: the set of intervals contained in $[q, m]$.

- $MIS_{q,m}$= the set of maximum independent sets of $I_{q,m}$ and $|MIS_{q,m}|$ its cardinality.
- $CMIS_{i=[a,b]} = MIS_{a+1,b-1}$.

The recursion on which the algorithm will be based is the following properties:

- If q is the right endpoint of an interval, then $|MIS_{q,m}| = |MIS_{q+1,m}|$.
- If q is the left endpoint of an interval $i = [q, r]$ then

$$|MIS_{q,m}| = \begin{cases} |MIS_{q+1,m}| & ,\text{if } r > m \\ \max(|MIS_{q+1,m}|, 1 + |CMIS_i| + |MIS_{r+1,m}|) & ,\text{if } r \leq m \end{cases}$$

Proof. The first property is trivial to prove. For the second property, the case where $r > m$ implies that $i = [q, r] \notin I_{q,m}$ so $|MIS_{q,m}| = |MIS_{q+1,m}|$. Focusing on the case where $r \leq m$, notice that $I_{q,m} \backslash i = I_{q+1,m}$ and in the case where i does not belong to the maximum independent set V of $I_{q,m}$ it is again $|MIS_{q,m}| = |MIS_{q+1,m}|$. If now $i \in V$, for any other $j \in I_{q,m}$ that does not overlap i it is $j \in I_{q+1,r-1}$ or $j \in I_{r+1,m}$ and $|V| = |MIS_{q,m}| = 1 + |CMIS_i| + |MIS_{r+1,m}|$. □

The algorithm now has as follows (quoted from [49]):

Algorithm 2: $\Theta(n^2)$ algorithm for *IND. SET* of circle graphs

```
input  : A set of n intervals [a_i, b_i] sorted a_i < a_{i+1} representing a circle graph
output: The size of the maximum independent set of the circle graph
M[1..2n] ← 0;
C[1..n] ← 0;
for m = 1 to 2n do
    if m is the right endpoint of an interval i = [l, m] then
        C[i] ← M[l+1];
    for q = m − 1 downto 1 do
        M[q] ← M[q+1];
        if q is the left endpoint of an interval j[q, r] and r ≤ m then
            M[q] ← max(M[q+1], 1 + C[j] + M[r+1]);
```

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