NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES

«INTERIOR PENALTY DISCONTINUOUS GALERKIN AND CONTINUOUS INTERIOR PENALTY FINITE ELEMENT METHODS FOR BOUNDARY VALUE PROBLEMS OF STRAIN GRADIENT ELASTICITY AS WELL AS OF PLATE THEORY»

DISSERTATION

Presented on 27-3-2013 in Partial Fulfilment of the Requirements

for the Degree Doctor of Philosophy

in the School of Applied Mathematical and Physical Sciences

of N.T.U.A.

By KONSTANTINOS G. EPTAIMEROS

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ATHENS, March 2013

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ATHENS, March 2013

to my parents,
my brothers
and my friends

Abstract

This dissertation revolves around the development of both h- and hp-version interior penalty discontinuous Galerkin finite element methods for boundary value problems of strain gradient elasticity and of plate theory. It also engages with the design of h- and hp-version continuous interior penalty finite element method for one-dimensional boundary value problems of strain gradient elasticity. Overall, this research endeavor focuses on conducting either a priori error analysis for one-dimensional problems or a posteriori error analysis for higher dimesional problems. To that end, a functional, analytic framework is presented employing broken Sobolev spaces as well as corresponding finite element spaces for the above methods.

Περίληψη

Η συγκεκριμένη διδακτορική διατριβή πραγματεύεται την ανάπτυξη της h- και της hp-εκδοχής των ασυνεχών Galerkin μεθόδων πεπερασμένων στοιχείων εσωτερικής ποινής για τα προβλήματα συνοριακών τιμών της θεωρίας βαθμίδας της παραμόρφωσης, καθώς και της θεωρίας των πλακών. Επιπλέον, πραγματεύεται τον σχεδιασμό της h- και της hp-εκδοχής της συνεχούς μεθόδου πεπερασμένων στοιχείων εσωτερικής ποινής για τα μονοδιάστατα προβλήματα συνοριακών τιμών της θεωρίας βαθμίδας της παραμόρφωσης. Γενικά, η συγκεκριμένη ερευνητική προσπάθεια επικεντρώνεται στη διεξαγωγή ανάλυσης σφάλματος είτε εκ των προτέρων για προβλήματα μίας διάστασης είτε εκ των υστέρων για προβλήματα υψηλότερης διάστασης. Γι’ αυτό τον λόγο, παρουσιάζουμε όλους τους απαραίτητους ορισμούς, καθώς και τα μαθηματικά εργαλεία, των χώρων συναρτήσεων που χρησιμοποιούνται σε αυτές τις μεθόδους, ήτοι τους επονομαζόμενους χώρους Sobolev και τους αντίστοιχους χώρους πεπερασμένων στοιχείων, επίσης.

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Abbreviations

- ***bSs*:** broken Sobolev space.
- ***DGE*:** Dipolar Gradient Elasticity.
- ***SGE*:** Strain Gradient Elasticity.
- ***FEM*:** Finite Element Method.
- ***FEMs*:** Finite Element Methods.
- ***IP*:** Interior Penalty.
- ***IPDG*:** Interior Penalty Discontinuous Galerkin.
- ***CG*:** Continuous Galerkin.
- ***CIP*:** Continuous Interior Penalty.
- ***CIPFEM*:** Continuous Interior Penalty Finite Element Method.
- ***CIPFEMs*:** Continuous Interior Penalty Finite Element Methods.
- ***DG*:** Discontinuous Galerkin.
- ***DGFEM*:** Discontinuous Galerkin Finite Element Method.
- ***IPDGFEM*:** Interior Penalty Discontinuous Galerkin Finite Element Method.
- ***IPDGFEMs*:** Interior Penalty Discontinuous Galerkin Finite Element Methods.
- ***NIPG*:** Non-Symmetric Interior Penalty Galerkin.
- ***SIPG*:** Symmetric Interior Penalty Galerkin.

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Chapter 1

Introduction

The models of biology, chemistry, engineering, finance, mathematics and physics are usually ordinary or partial differential equations equipped with certain boundary and/or initial conditions. It is notable that in most cases of interest there are not known ways of finding the solutions of such differential equations analytically. Ergo, the need to resort to numerical approximation of the solution of differential equations is apparent.

During the last decades, FEMs have been widely accepted as one of the most powerful tools for the numerical approximation of the solutions of partial differential equations. The success of FEMs is mainly thanks to their ability to deal with complicated geometries and non-structured discretizations, as well as thanks to the solid mathematical theory that has been developed for the analysis of their performance.

Various FEMs have been proposed over the years aiming to solve more complicated problems or to improve the performance of existing methods. Important classes of FEMs go under the general terms DGFEMs as well as CIPFEM. The former methods have been proven to be efficient for the solution of equations, due to some favourable properties they share, such as local conservation of the state variable, good performance near possible discontinuities of the solution, great flexibility in the mesh design as well as in the enforcement of the boundary and initial conditions. The latter method has been proven to be efficient for the solution of fourth-order equations, on account of some encouraging properties it shares, for instance combination of the advantages of the CG and DG schemes, involvement of the primary variable only, use of continuous shape functions (i.e. lower computational cost) and great flexibility in the enforcement of the boundary and initial conditions.

The success motivated researchers to consider DGFEMs as attractive contenders for the numerical solution of diverse problems. On the other hand, the CIPFEM counts only eleven years of living (although it was based on the ideas of Baker [16]) and is already considered a superior method for the numerical solution of 4th-order equations. Indeed, we have recently witnessed a revived interest on the derivation of non-conforming discretizations for various kinds of operators, partially based on ideas which appeared in the 1970's and in the early 1980's whose further development had been somewhat latent until the late 1990's, and especially in the early 2000's for the CIPFEM, when the potential of such methods in this context was realised.

This work revolves around the error analysis, either a priori or a posteriori, and performance of the h-version as well as of the hp-version both of the IPDGFEMs and the CIPFEM for boundary value problems (either in SGE or in linear elasticity) on shape-regular meshes.

DG methods can be separated into two main categories: methods which are discontinuous in time, and methods being discontinuous in space. Time-DG methods have been developed for first- and second-order hyperbolic equations, and are often encountered in fully discrete space-time finite element formulations.

In this work, we will focus on methods which are discontinuous in space. The DG method has established itsself as a viable method for solving first-order hyperbolic partial differential equations in fluid mechanics. Discontinuities may be present in the solution, and the DG methods seems to be the natural approach for capturing these numerically.

The first DGFEM was introduced in 1973 by Reed and Hill [173] for the numerical solution of first-order hyperbolic problems, i.e. for the numerical solution of the neutron transport equation, as an alternative to high-order finite difference and finite volume methods. Lesaint and Raviart presented the first analysis for this method in [148] using Fourier techniques. Johnson and Pitkäranta proved error bounds for the L^2-norm error [139], which were improved by Richter [174]. Ten years ago, there had been an increasing interest in their development, see [148, 138, 59, 61] and [175, 63, 93, 94, 95], as well as generalizations of these ideas for hyperbolic systems by Houston, Jensen and Suli [131] and Larson and Barth [147], following the work in [139] on Friedrichs systems.

Simultaneously, but quite independently, Galerkin methods using discontinuous finite elements were proposed as non-standard schemes for the numerical approximation of second-order elliptic problems. These DG methods were traditionally termed as penalty methods. The name is by reason of the presence of certain terms; in those finite element formulations penalise the discontinuities in order to ensure the consistency of the approximation. Nitsche in [163] first proposed the idea of weakly imposing the boundary conditions for elliptic problems, deriving a symmetric finite element formulation. In fact, Nitsche's method features two crucial characteristics, namely the elimination of a Lagrange multiplier by means of an energetically consistent flux weighting function and the introduction of a stabilization term. Dirichlet boundary conditions are built into the weak form. Rather than invoking an unknown Lagrange multiplier, however, Nitsche chose a flux-like weighting function instead. In addition, Nitsche introduced a stabilizing term on the boundary for enforcing the homogeneous Dirichlet boundary condition he considered. This choice for the weighting function and the stabilization on the boundary led to optimal convergence rates for the elliptic Poisson equation. A penalty method, presented by Babuska [8], had the same goal of building the boundary condition into the weak form. However, that penalty method suffered from a consistency error, as the weak formulation proposed therein did not satisfy the original problem.

Douglas and Dupont [82] extended Nitsche's idea to DG methods for linear elliptic and parabolic problems. They applied Nitsche's approach on an element level, and summing all element contributions. It resulted in a modified weak form with flux-weighted and penalized interelement conditions. In particular, jump terms in the unknown variable across interior boundaries arose, which were penalized to approximately enforce continuity across element interfaces. Wheeler [202] analyzed this method, and Percell and Wheeler [168] recognized the significance of this method for *hp*-adaptive finite element strategies. Arnold [3, 4] investigated these interior penalty methods for linear and non-linear parabolic boundary value problems. A method by Delves and Hall [77], which the authors termed the global element method, also features flux-weighted interface conditions. However, it lacks the stabilization (or interior penalty) term, which results in the matrix being indefinite. The most successful attempts to formulate consistent penalty methods, in the spirit of Nitsche's work, were made by Wheeler and Arnold as referred above. These appeared in the 1970's and early 1980's and their development continues until today.

In fact, in the late 1990's until today, a number of methods, belonging to this family, were proposed by various researchers. These include the method of Bassi and Rebay [20]; its variants developed by Brezzi et al. [46, 47], the generalization of these ideas in the context of local DGFEMs due to Cockburn et al. [62, 69, 49, 50], as well as the so-called IP methods by Rivière et al. [177, 178, 179], Houston et al. [130], Suli et al. [188, 189, 190] and Georgoulis [113, 114].

Furthermore, another method, belonging to the above family, is the DG method of Baumann and Oden [21, 164, 15] which is essentially a parameter-free counterpart to the IP methods. A strong influence on the development of the DG method for second-order equations was the work of Baumann [21]. By reversing the sign of the flux-weighted interface conditions in the global element method, which can be traced back to the original work by Nitsche, he obtained a method amenable to a variety of elliptic and hyperbolic problems of fluid mechanics. The sign reversal renders the originally symmetric formulation of Delves and Hall and Nitsche non-symmetric. For the application of Baumanns method to advective-diffusive problems, which was the ultimate goal, symmetry of the system is, however, unimportant, as the contribution of the advective operator leads to a non-symmetric system. In addition, the sign reversal results in advantages for the analysis of the method. Another benefit of Baumann's method is the fact that no auxiliary variables need to be introduced, as was the case for mixed methods; which reduces the number of unknowns. The disadvantage of the method, however, is that stability and convergence could only be established analytically for cubic or higher order interpolation. Babuska et al. [15] provide numerical evidence that the method is also stable for quadratic interpolation by numerically evaluating the inf-sup condition in a one-dimensional setting. An interesting detail worth mentioning is their observation of a loss of accuracy of Baumann's method in the L^2-norm for even orders of interpolation.

Various contributions by Oden et al. [164] as well as Baumann and Oden [22, 23] explore Baumanns method numerically and analytically and apply it to the solution of the Euler and Navier-Stokes equations in an *hp*-adaptive environment. Prudhomme et al. in [170] establish *hp*-version a priori error estimates. Research efforts have been aimed at stabilizing Baumann's method and thereby linking it back to the original development by Nitsche and to the IP methods discussed earlier. Arnold et al. have presented a general framework in [5] for stabilized and non-stabilized DG methods for elliptic equations. Suli and Mozolevski also presented in [190] an *hp*-IPDGEMs for the biharmonic equation, including symmetric and non-symmetric formulations together with their combinations, the semi-symmetric formulation. They additionally established error bounds that were optimal in h and slightly suboptimal in p.

The introduction of non-conforming (or imcompatible) finite elements for bending problems in 1973 can be considered as a precursor to the DG method in structural mechanics. Wilson et al. [204] designed an imcompatible modes finite element which was discontinuous on the interior boundaries between the nodes. This construction resulted in improved bending behavior of the element. The convergence behavior of the non-conforming element was analyzed by Lesaint [149]. Kikuchi and Ando [145] presented a formulation for thin plates and shells with non-conforming normal rotations.

Most of the development of the discontinuous Galerkin method has been motivated by problems from fluid mechanics. Cockburn and Shu [59] introduced a Runge-Kutta discontinuous Galerkin method for the solution of first-order non-linear hyperbolic conservation laws. An *hp*-adaptive discontinuous Galerkin method for first-order problems was developed and analyzed by Bey and Oden [29], and a more detailed analysis was presented by Houston et al. [128]. Bassi and Rebay [18] presented a DG method for the solution of the Euler equations, which paved the way for research efforts aimed at solving the Navier-Stokes equations.

The initial approach for solving second-order hyperbolic equations was to rewrite the equation as a system of first-order equations. Introducing the flux as an auxiliary variable, which is the first derivative of the primary variable, one can numerically solve the first-order system of equations. The advantage of this mixed approach is that one can solve a second-order equation with a method originally designed for first-order problems. The disadvantage, however, is a considerable increase in the unknowns due to the introduction of auxiliary variables. Bassi and Rebay in [18] extended their method to a mixed approach for solving the compressible Navier-Stokes equations. Their method was analyzed by Brezzi et al. in [47] and generalized by Cockburn and Shu in [62].

The revived interest in discontinuous methods can also be witnessed by the work of many researchers developing DG methods for various boundary value problems. Feng and Karakashian [96, 97] proposed some two-level non-overlapping and non-overlapping additive Schwarz methods for a DG method for solving second order elliptic problems as well as the biharmonic equation. They also presented in [98] fully discrete DG methods, with variable meshes in time, developed for the fourth order Cahn-Hilliard equation, arising from phase transition in material science. Karakashian and Pascal [143] introduced some a posteriori error estimators for a DG approximation of second order, based on the ideas and techniques of domain decomposition. In [180], Rivière extensively analyzed the theoretical foundations of the DG methods for solving elliptic and parabolic equations. What is more, McBride and Reddy, in [156], proposed a DG method for classical and gradient plasticity. The drawback of that method is that the use of low order elements is essential to contain the computational expense of the formulation, but these elements are prone to locking. By using lifting operators, Georgoulis and Houston [114] presented the design and the analysis of *hp*-version DGFEMs for boundary value problems involving the biharmonic operator. In addition, Georgoulis and al. [115] introduced a residual based a posteriori error indicator for DG discretizations of the biharmonic equaton. In [121], Gudi developed a new error analysis of DG methods, by replacing the Galerkin orthogonality with estimates borrowed from a posterior error analysis and by employing a discrete norm that is well defined in H^k. Furthermore, Schötzau et al. [184] proposed a mixed *hp*-DGFEM for the incompressible flows. Bassi, Karakashian and Oden have also developed DG methods for the Navier-Stokes equations in [20, 142, 166]. Depres, in [78, 79], additionally proposed DG methods for the Euler equations. Finally, Warburton [200] presented *hp*-version DG methods for the Maxwell equations, etc.

The similarities between the growing number of new such methods led Arnold et al. to seek a unified framework for deriving and analyzing DG methods [6]. A nice survey of the method mentioned can be found in Cockburn et al. in the important volume [67].

Stabilized methods were proposed in the 1980s to account for the fact that continuous Galerkin methods of certain boundary value problems do not inherit the stability properties of the continuous problem. They were originally introduced for the advection-diffusion equation by Brooks and Hughes [48] and subsequently generalized by Hughes et al. [136]. Stabilized methods have been analyzed extensively, and the theoretical foundations are still being explored. Two frameworks for deriving stabilized methods and stabilization parameters have been established, namely the variational multiscale approach [137] and the residual-free bubble approach [43, 44]. The two approaches have been shown to be equivalent under certain circumstances [45]. Stabilization enhances stability without sacrificing accuracy by changing the weak formulation of the boundary value problem under consideration. Weighted residual terms are added to the variational equation, which involve a mesh-dependent stabilization parameter. These extra terms usually are functions of the Euler-Lagrange equations on an element level to ensure consistency of the method.

As for the CIP method, IP methods were also used in 1977 by Baker [16] for imposing C^1-interelement continuity on C^0-elements for fourth-order problems. In these, of course, it was the jump in the normal derivative that penalized. That method was considered the fundamental step of CIP method. Since the early 1980's, less attention had been paid to the idea of Baker, until 2002, when Engel et al. [86] developed the CIP method (or alternatively C/DG method) for the numerical solution of fourth-order elliptic partial differential equations. In other words, they achieved to combine the advantages of CG, DG and stabilized methods to design a superior finite element formulation for fourth-order elliptic problems.

From that moment, there was a great interest for the CIP method. Brenner and Sung [34] presented CIP methods for fourth-order elliptic boundary value problems on polygonal domains. With exact words, they established new CIP methods which are based on a post-processing procedure that can generate C^1-approximate solutions from C^0-approximate solution. Brenner et al. in [37] proposed a reliable and efficient residual-based a posteriori error estimator for the quadratic CIP method for the biharmonic problem on polygonal domains. Furthermore, Brenner and Neilan [38] developed a CIP method for a fourth-order singular perturbation elliptic problem, in two dimensions, on polygonal domains. Brenner et al. additionally developed a priori as well as a posteriori error estimates for quadratic CIP methos for linear, fourth-order boundary value problems with essential and natural boundary conditions of the Cahn-Hilliard type [39]. Moreover, Hansbo and Larson in [122], presented a posteriori error estimates for C/DG approximations of the Kirchhoff-Love plate. What is more, Eptaimeros and Tsamasphyros [90] presented the h-version CIP method for a sixth-order equation of SGE.

In the sequel, we shall be concerned with IP methods. In particular, we shall be concerned with the h-version and hp-version of both IPDG and CIP methods, following the concepts of Arnold, Georgoulis, Houston, Rivière, Suli, Wheeler [202, 4, 188, 176, 177, 179, 130, 190, 114] in the former case, but concepts of Engel et al. [86] in the latter.

We shall provide a priori error estimates for one-dimensional problems in SGE, and a posteriori error estimates for higher-dimensional problem in linear elasticity as well as in SGE, with assumptions on the shape of the elements. A subdivision $\mathcal{T}$ of Ω into triangular or quadrilateral elements K is said to be shape-regular if there exists a positive constant C such that, for every element $K \in \mathcal{T}$,

$$C^{-1} \leqslant \frac{R_K}{r_K} \leqslant C,$$

where R_K, r_K are the radii of the circumcircle and inscribed circle either of the triangular or the quadrilateral K, respectively. This condition essentially means that all the edges of K are of comparable size and their ratios are uniformly bounded throughout the mesh. Actually, as in certain cases the solution behaves differently in different coordinate directions (e.g., existence of boundary or interior layers, or regularity constraints), the use of shape-regular elements can lead either to prohibitively expensive discretizations or, if the computational resources are limited, to poor accuracy. This is because, in the presence of steep gradients in the solution, a significant number of degrees of freedom may be needed to capture the behaviour in one coordinate direction, whereas a substantially smaller number may be required for the other coordinate directions in which the solution exhibits little variation.

Our theory will cover the h-version and the hp-version of IPDGFEMs and of CIPFEM, respectively. The notion of hp-version FEM was introduced by Babuska in the 1980's, after the p-version FEM was proposed by Babuska et al. [11]. The hp-version admits local variation of both the size of the elements (for instance, by subdividing the existing ones) and of the degree of the (polynomial) basis functions on every element, in order to achieve convergence. Especially, DGFEMs appear to be advantageous in the hp-version context compared to standard (conforming) FEMs; in the sense that they are flexible in terms of using different polynomial degrees on every element without concerns about interelement continuity requirements.

The current use of complex materials in nanotechnology and industrial engineering has led to a number of intricate problems in mechanics as the macroscopic behavior of such materials often depends critically on their substructures. In this context, higher-order continuum theories, i.e. theories being capable of reflecting the effects of inner structure through introduction of proper material constants, have become a very attractive alternative for the mathematical modeling of the behavior of modern technological materials. Such theories have been proven competent to yield more realistic results in several phenomena of solid deformation, when compared to classical elasticity. The problem of dispersion of elastic waves at low frequencies has been treated by Mindlin in 1964 [157]. Within the concept of dipolar (first) strain gradient theory of elasticity, Mindlin produced dispersion curves that closely resembled the experimental ones. Further applications of first strain gradient elasticity involve fracture and dislocation modeling [107, 155].

The success of strain gradient theories is based on the effects of the underlying micro-structure of solids taken into account. A basic drawback is the imperative insertion of a large number of material parameters. For instance, in the case of linear anisotropic elasticity, there are 903 independent material constants, reducing to 18 for centrosymmetric, isotropic materials.

The fundamental characteristic of the gradient elasticity theory (and the main difference from classical elasticity) is that the strain energy density is a positive-definite functional of the standard strain (as in classical elasticity) and either the second gradient of the displacement field (Form I) or the first gradient of strain (Form II). Other forms can be found in the literature [157, 158].

The main idea of generalized continuum theories is that a macroscopic medium (macro-medium) contains either elements or particles (macroparticles), considered a deformable medium, as well. The internal structure of the macro-particles is considered responsible for, as macro-particles consist of sub-particles called micro-media. Owing to this assumption, we reach the conclusion that each macro-particle obtains an internal displacement field. Now, if we consider that the internal field is linear in internal coordinate variables, then dipolar theories or grade-two theories are deduced [157, 30, 158].

The new material constants, which relate generalized stress variables with generalized strains, contain certain characteristic lengths associated with the size and topology or the material micro-structure. In the same fashion, size effects are introduced in the stress analysis. Typical gradient elasticity models are concerned with materials having periodic micro-structure such as crystals (crystal lattices), polycrystal materials (crystallites), polymers (molecules) as well as granular materials (grains); we denote by words into the parentheses the respective micro-media [157].

Various either analytical methods or FEMs have been proposed over the years for the SGE. Amanatidou and Aravas [2] presented mixed finite element formulations for SGE problems in 2002. In addition, Engel et al. developed a C/DG method for fourth-order elliptic problems with application to SGE the same year. In 2003, Papargyri-Beskou et al. [167] analytically solved the problems of bending and stability of Bernoulli-Euler beams, on a basis of a simple linear theory of gradient elasticity with surface energy. In 2006, Georgiadis and Grentzelou [110] also introduced energy theorems for DGE and the year after, Giannakopoulos and Stamoulis [116] carried out the size effects in the problems of cantilever beam bending together with cracked bar tension within the gradient elasticity framework; thus, they managed to deduce analytical solutions for metrics that characterized both the normalized stiffness and toughness. Tsamasphyros et al. [195] that year proposed mixed finite element C^0-continuity formulations, based on the generalizations of the well-known Ciarlet-Raviart mixed method, for the solution of some types of one-dimensional fourth- and sixth-order equations, resulting in axial tension and buckling of gradient elastic beams, respectively. In keeping with these developments, Markolefas et al., additionally presented mixed weak formulations for general multi-dimensional DGE boundary value problems [153]. Three years later in 2010, Tsamasphyros and Vrettos [196] developed a mixed finite volume formulation for the solution of 1D as well as 2D equations in strain gradient elasticity while Filopoulos et al., proposed a dynamic finite element analysis for a gradient elastic bar by including the micro-inertia [100].

1.1 Motivation

The aim of this section is to further motivate the use of DGFEMs as well as the use of CIPFEM.

It is well known that in problems where steep gradients are present in the analytical solution (existence of boundary or interior layers, etc.), standard conforming Galerkin FEMs produce oscillatory solutions, when the number of degrees of freedom is insufficient to resolve the rapid variation. Stabilisation methods (streamline-diffusion stabilisation [134, 140] bubble stabilisation [43, 45]) may be employed in order to counteract the undesirable oscillatory effects in standard conforming Galerkin FEMs. Ergo, an investigation of IPDG methods as well as the CIP method on shape regular meshes emerges to be interesting.

DGFEMs provide great flexibility in terms of mesh design. They can easily handle very general non-matching grids containing hanging nodes, varying polynomial degrees in the local basis functions, without introducing any interelement compatibility conditions that are required by standard FEMs to maintain the conformity of the finite element space. DG methods are, therefore, well suited in the context of adaptivity. Adaptive h-refinement gives rise to hanging nodes in the adapted mesh. For conforming FEMs this difficulty is addressed by also refining elements adjacent to the one which has been refined in order to eliminate the hanging nodes. On the contrary, for DG methods, the presence of hanging nodes does not constitute a problem as meshes containing hanging nodes are perfectly admissible. Wherefore, the additional work of removing hanging nodes becomes redundant.

CIP method is DG method that use standard continuous finite elements. Furthermore, CIPFEM has certain advantages over classical FEMs for fourth order problems. First of all, it is much simpler than C^1-FEMs. Indeed, the lowest order CIP method is as simple as classical non-conforming FEMs. But unlike classical non-conforming FEMs that only use low order polynomials, CIP method comes in a natural hierarchy and higher order CIP method can capture smooth solutions efficiently. Compared with mixed FEMs, the stability of CIP method can be established in a straightforward manner and the symmetric positive definiteness of the continuous problems is preserved by CIP method. Moreover, a noteworthy feature of the CIP method is that, unlike mixed methods and non-conforming methods, the design of the quasi-optimal CIP method is straightforward even for complicated fourth-order problems, using only integration by parts, symmetrization and penalization. Since the underlying finite element spaces are standard spaces for second order problems, multigrid solves for the Laplace operator can be used as natural preconditioners for CIP method [35], and problems on smooth domains can be easily handled by isoparametric CIP method [40]. These are also significant advantages of CIP method over the classical approaches.

The advantages of DG methods become more obvious in the context of p-adaptivity. Indeed, as there is no continuity requirement across element interfaces, every element admits a local basis with arbitrary polynomial degree (or, more generally, basis consisting of functions qualitatively different from the ones of neighbouring elements). In the context of hp-FEM, it is also feasible for neighbouring elements to have local bases with different (polynomial) degrees. This is done by carefully choosing these basis functions; see [11, 14] for details. In a nutshell, the idea is to consider basis functions that vanish on the element edges for the additional degrees. This does not appear to be very efficient though, as the information is transmitted through the generally lower degree basis functions residing on the interfaces.

What is more, we mention that discontinuous methods are very suitable for handling geometrically complicated computational domains. Indeed, as the boundary conditions are imposed in a weak sense rather than pointwise, as is the case for standard conforming FEMs, the flexibility in this context is apparent.

Furthermore, certain discontinuous methods, such as IP methods, admit weaker communication between the elements than standard conforming

FEMs. This is a desirable property as these methods can be easily parallelised and may generally produce linear systems with sparser matrices than the ones obtained from standard FEMs. Moreover, as there are no strong interelement continuity requirements, orthogonal basis functions can be easily constructed, and lead to diagonal mass matrices. This fact is significant for unsteady problems where successive computations of mass matrices may become expensive for long-time simulations.

Moreover, many DG methods admit local conservation of the state variable, which makes them attractive for the numerical approximation of non-linear hyperbolic problems [21, 67, 6]. For problems that may admit discontinuous solutions due to either discontinuous initial conditions or the development of shocks, DG methods seem to be advantageous compared to the standard conforming techniques as, if non-physical oscillations are present in the solution around discontinuities, they are typically more localised than in the continuous FEMs [67].

In addition, DG methods are particularly suitable when the solution exhibits elementwise regularity only. Hence these methods are applicable, with minor modifications only, to elliptic transmission problems, where discontinuities in the data may lead to discontinuities in the diffusive fluxes of the solution.

Into the bargain, efficient preconditioners have recently been proposed by Houston for the p-version IP method. Actually, it has been observed that block-diagonal preconditioners (block- Jacobi, block-Gauss-Siedel etc.) are robust with respect to the variation of the polynomial degree p used in the local basis. Also, multigrid preconditioning techniques for the h-version DGFEM have been analysed by Gopalakrishnan and Kanschat [117, 118], Kanschat [141] and by Hemker et al. [124, 125].

In a nutshell, we have chosen to work with the IPDG formulation as it has been used widely in the literature. It also introduces sparser linear systems than other DG methods [6], and its analysis is most well understood. Moreover, it has been used in adaptive strategies producing very satisfactory results. On the other hand, we have chosen to work with the CIP method on fourth-order and on sixth-order equations, since its formulation exhibits the subsequent feature such as the involvement of the primary variable only. Especially, in strain gradient theories, CIP method avoids Lagrange multipliers and yields a displacement gradient free formulation. This leads to greater simplicity and reduces the number of unknowns. Withal, the approximation functions only need to satisfy C^0-continuity requirements across interior boundary, leading to discontinuities in first and higher-order derivatives. So, it is imperative the change of the variational formulation in order that the continuity requirements for the derivatives can be enforced weakly.

1.2 Contributions of this Research

The main goals of this dissertation are:

1. The development of both h- and hp-version IPDGFEMs for boundary value problems of SGE and of plate theory, respectively.

2. The development of both h- and hp-version CIPFEM for one-dimensional boundary value problems of SGE.

Our research endeavor focuses on conducting either a priori error analysis for one-dimensional problems or a posteriori error analysis for higher dimensional problems, respectively.

Our analysis will be based on the ideas of Engel et al. in [86], Suli and Mozolevski in [190] as well as Georgoulis and Houston in [114, 115] and our deducing proofs will contain many features presented therein. The results of Engel et al. are tailored for the h-version CIPFEM for one-dimensional problem of SGE, containing a fourth-order equation in tandem with the results of Suli and Mozolevski and Georgoulis and Houston being tailored for the hp-version IPDGFEMs for the biharmonic equation. For that reason, additional care and considerations had to be employed in order to adapt, or in other words extend, their arguments to the problems of that dissertation.

To begin with, the functional analytic setting of standard Hilbert-Sobolev spaces, used widely in the finite element literature, does not seem to be the natural choice to work with in this dissertation. This occurs owing to the fact that the finite element spaces of IPDGFEMs and CIPFEM are not subspaces of the standard Hilbert-Sobolev spaces. We therefore introduce the notion of the broken Sobolev space for both IPDGFEMs and CIPFEM.

The accuracy of a finite element method heavily depends on the approximation properties of the finite element space. This dependence enters the error analysis via the choice of the interpolant or projection of the analytical solution chosen from the finite element space. In the standard h-version FEM literature such approximants are usually referred to as interpolants, as they are Lagrange interpolation polynomials for the analytical solution [55, 36].

Here, we use a type of approximant, based on the L^2-projection operator. This has been used widely in the literature on hp-FEMs. The definitions of lifting operators, contained in Georgoulis and Houston [114], are extended for a higher dimensional boundary value problem of plate theory, supplemented with complicated boundary conditions, as well as for a higher dimensional boundary value problem in SGE (a system of partial differential equations), supplemented with essential boundary conditions.

In one-dimension, new error estimates on regular families of subdivisions are presented in the sequel. In one-dimension, h-optimal error estimates are derived for both fourth-order and sixth-order equations of SGE either the IPDGFEMs or the CIPFEM are applied. In case of hp-version, error estimates are optimal in the meshsize h, but suboptimal in the polynomial degree p.

In higher dimensions, new error estimates are presented in the suquel on shape regular elements. A recovery operator is presented for the IPDG method for the Kirchhoff-Love plate model problem with essential and complicated natural boundary conditions under minimal regularity assumption on the analytical solution. In addition, stability bounds of lifting operators are deduced for that boundary value problem. Also, by using these stability bounds, we prove the coercivity and the continuity of bilinear form. Then, a technical lemma of a recovery operator develops for the Kirchhoff-Love plate model problem employing macro-elements, by generalizing and extending the results from [115]. What is more, a reliable a posteriori error estimate of residual type is established in the energy seminorm for the (symmetric) IPDG method for problem of SGE, with essential boundary conditions under minimal regularity assumptions on the analytical solution. Stability bounds of lifting operators are deduced for a boundary value problem of SGE and by employing these stability bounds, we prove the coercivity and the continuity of bilinear form. As a result, a technical lemma of a recovery operator presents for that boundary value problem using macro-elements, by generalizing the results from [115] in vector spaces. The reliable a posteriori error estimate, referred above, is based on a suitable recovery operator that maps discontinuous finite element spaces to C^1-conforming finite element spaces, consisting of triangles or quadrilateral macro-elements.

IPDG and CIP methods are not parameter free, in the sense that the methods involve a user-defined quantity, the so-called either stabilization parameter or discontinuity-penalization parameter. Recipes to specify this parameter for shape-regular meshes and isotropic polynomial degree have been provided in [179]. This parameter is dependent on the local meshsize h and on the local polynomial degree p. We mention that, in case of SIPG method, this parameter depends on the stabilization constant (since its selected value is critical for the convergence of the method). These choices for the stabilization parameter emerged from the error analysis, and they were made in order to assure the highest possible convergence rates.

Finally, we perform numerical experiments for the SIPG and the NIPG methods on one-dimensional boundary value problem of SGE to explore the potential utility of DG, record their performance and compare them with the mixed methods.

1.3 Overview

This dissertation is structured as follows. Chapter 2 is devoted to developing function spaces used in the variational formulation of differential equations. In section 2.1, we describe some simple function spaces consist of continuously differentiable functions. We will proceed by reviewing the basic concepts of Lebesgue integration theory in section 2.2. In section 2.3, we generalize the Lebesgue norms and spaces to include derivatives. Then, we present the notion of a (standard) Sobolev space, based on the Lebesgue space L^p, and the inclusion relations to provide some sort of ordering among them. Section 2.4 will subsequently introduce the notion of a broken Sobolev space, which is the natural space to work with the DG and CIP methods.

This dissertation is further divided into two parts. Part I revolves around the one-dimensional problems of SGE. On the other hand, Part II focuses on the higher dimensional problems of SGE and of plate theory, as well.

Chapter 3 contains all the necessary, preliminary notions of IPDG and CIP methods in one-dimension.

Chapter 4 deals with the h- and hp-version IPDGFEMs for SGE in 1-D, respectively. In section 4.1, we consider the boundary value problem of SGE in 1-D. That contains a differential equation of fourth-order, sumplemented with essential and natural boundary conditions. Section 4.2 contains the significant definitions of the jump as well as the mean value operator, respectively. In the same section, we also present the series of steps which lead to the DG weak formulation, i.e, to the bilinear form and linear functional. We then establish the energy seminorm associated with the bilinear form. Thereafter, section 4.2.1 contains a technical lemma, about the weak continuity of the fluxes, used to the proof of the consistency of the IPDG methods. In section 4.3 we propose the corresponding finite element spaces for the IPDG methods. At this point, in Section 4.4, we state the IPDGFEMs for the boundary value problem of SGE in 1-D. We prove the coercivity as well as the continuity of the bilinear form for both h- and hp-version SIPG and NIPG method, respectively. In section 4.5, we conduct an a priori error analysis for the above versions of IPDG methods as well as deducing the error estimates for h- and hp-version. The a priori error estimates, deriving, are optimal in h for the h-version NIPG and SIPG method, respectively. However, for hp-version SIPG and NIPG method, we establish a priori error estimates being optimal in h but are p-suboptimal by $\frac{3}{2}$ orders of p. Finally, section 4.6 exhibits some conclusions for the IPDGFEMs applied in this chapter.

In chapter 5, we propose the hp-version CIPFEM for SGE in 1-D. As in chapter 4, the same boundary value problem is considered. Section 5.1 consists of the imperative definitions of the jump and the mean value operator, respectively. In the same section, we present the procedure leading to the CIP weak formulation, followed by the introduction of the bilinear form and the linear functional, respectively. Then, we associate the corresponding energy seminorm with the bilinear form. In section 5.1.1, we develop a technical lemma about the weak continuity of the fluxes and we employ this lemma in order to prove the conistency property of the CIP method. Section 5.2 contains the appropriate finite element spaces for the CIP method and in section 5.3, we state the CIPFEM for the boundary value problem of SGE in 1-D. We proceed with the proofs of coercivity and continuity of the bilinear form. Section 5.4 focuses on a priori error analysis of the hp-version CIP method. We deduce the error estimate of the hp-version CIP method. The error estimate establishing is optimal in h, but is p-suboptimal by $\frac{1}{2}$ orders of p. Eventually, section 5.5 refers to the conclusions of the CIPFEM applied in this chapter.

The content of chapter 6 revolves around the design of h- and hp-version CIPFEM for a 6th-order equation of SGE in 1-D, respectively. In section 6.1, we consider the boundary value problem of SGE in 1-D, consisting of a differential equation of sixth-order. We supplement this equation with essential and natural boundary conditions and the following section 6.2 contains the definitions of the jump together with the mean value operator, respectively. In that section, we also present the series of steps leading to the CIP weak formulation, followed by the definition of the bilinear form and the linear functional. We then establish the corresponding energy seminorm. Afterwards, section 6.2.1 contains a technical lemma about the weak continuity of fuxes. By the use of that lemma, we prove the consistency property of that CIP method. Next, in section 6.3, we introduce the finite element spaces corresponding to the h- and hp-version CIP method, respectively. In section 6.4, we state the CIPFEM for the boundary value problem followed by the proof of coercivity and continuity of the bilinear form for both h- and hp-version CIPFEM, respectively. In section 6.5, our goal is to conduct an a priori error analysis for the above versions of the CIP method. Especially, we establish a priori error estimate that is optimal in h for the h-version CIP method. Nevertheless, when we focus on the hp-version CIP method, a priori error estimate deduced is optimal in h but is p-suboptimal by $\frac{3}{2}$ orders of p. Finally, Section 6.6 exhibits some important conclusions about the CIPFEM applied in this chapter.

Chapter 7 deals with the hp-version IPDGFEMs for a bending plate model in linear elasticity. Next, section 7.1 contains all the necessary, preliminary notions of IPDG methods in higher dimensions. Thereafter, in section 7.2, we consider a boundary value problem consisting of a partial differential equation of fourth-order. We supplement the equation with essential and complicated natural boundary conditions. Afterwards, section 7.3 contains the imperative definitions of the jump and the mean value operator, respectively. We develop, in the same section, the procedure leading to the DG weak formulation, followed by the introduction of the bilinear form and the linear functional, respectively. Then, we establish the energy seminorm associated with the bilinear form. In section 7.4, we introduce the appropriate finite element spaces for IPDG methods. In section 7.5, we state the IPDGFEMs for the boundary value problem and proceed with the proofs of coercivity of the bilinear form in order to define the format of the stabilization parameters. In section 7.6, we introduce the lifting operators as well as the IPDG methods. As a result, the modified bilinear form and the modified linear functional, deriving, contain in their formulations the lifting operators. After that, we establish stability bounds for the trace lifting (alternative name for lifting operators). We employ these bounds in order to prove coercivity and continuity of the symmetric modified bilinear form referred above. In section 7.7, our research endeavor focuses on the introduction of a suitable recovery operator. We eventually present a technical lemma for the recovery operator. By the use of that techincal lemma, someone can establish reliable and efficient a posteriori error estimates.

The content of chapter 8 revolves around the design of hp-version IPDG-

FEMs for a problem of SGE in 2-D and section 8.1, in particular, contains all the imperative, preliminary notions of IPDG methods in two dimensions. In section 8.2, we consider the boundary value problem consisting of system of partial differential equations. We then supplement the equation only with essential boundary conditions. At this point, we note that the fourth order problem is formulated with respect to the vector of displacement. Section 8.3 contains the definitions of the jump as well as the mean value operator, respectively. In the same section, we also develop the series of steps leading to the DG weak formulation, followed by the definition of the bilinear form and the linear functional. We then establish the corresponding energy seminorm associated with the bilinear form. Afterwards, the introduction of the finite element spaces, corresponding to our methods, follows in Section 8.4. In section 8.5, we present the appropriate lifitng operators for our problem and then we propose the IPDG methods. In consequence, the bilinear form and linear functional deriving contain in their formulations the lifting operators. We proceed with the introduction of stability bounds for the trace lifitngs. The above stability bounds will be employed for the proofs of coercivity and continuity of the symmetric bilinear form. Overall, in section 8.6, our research endeavor focuses on the introduction of a suitable recovery operator. We then present a technical lemma, for this recovery operator, which will be used to establish a h-version reliable a posteriori error estimate of residual type for the symmetric IPDG method in the corresponding energy seminorm.

In chapter 9, we test numerically the h- and hp-version IPDG for the problem of SGE in 1-D. In fact, we investigate the convergence of the SIPG and NIPG methods; if the h-version SIPG and NIPG methods are applied, optimal rates of convergence are proven as the mesh size is decreased. Interestingly enough, if the hp-version SIPG and NIPG methods are applied, optimal convergence rates are proven under h-refinement but under p-refinement, exponential convergence is indicated.

In conclusion, chapter 10 contains some final comments regarding this dissertation and explores some potential avenues for future research.

Chapter 2

Function Spaces

This chapter is devoted to developing function spaces used in the variational formulation of differential equations. We begin with a review of spaces of continuous functions and of Lebesgue integration theory, upon which our notion of "variational" or "weak" derivative rests. Functions with such "generalized" derivatives make up the spaces commonly referred to as Sobolev spaces. We present only a small fraction of the known theory for these spaces that is at the same time sufficient to establish a foundation for the FEM. Finally, it is imperative to introduce a special kind of Sobolev spaces, mentioned as broken Sobolen spaces which they are appropriate for the development of non-conforming methods.

2.1 Spaces of Continuous Functions

In this section, we describe some simple function spaces which consist of continuously differentiable functions. For the sake of notational convenience, we introduce the concept of multi-index.

Let $\aleph$ denote the set of non-negative integers. An n-tuple

$$a = (a_1, \ldots, a_n) \in \aleph^n$$

is called a multi-index. The non-negative integer $|a| := a_1 + \ldots + a_n$ is referred to as the length of the multi-index $a = (a_1, \ldots, a_n)$. Let

$$D^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{a_n} = \frac{\partial^{|a|}}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}.$$

Let Ω be an open set in $\Re^n$ and let $k \in \aleph$. We denote by $C^k(\Omega)$ the set of all continuous real-valued functions defined on Ω such that $D^a f$ is continuous on Ω for all $a = (a_1, \ldots, a_n)$ with $|a| \leqslant k$. Assuming that Ω is a bounded open set, $C^k(\bar{\Omega})$ will denote the set of all f in $C^k(\Omega)$ such that $D^a f$ can be extended from Ω to a continuous function on $\bar{\Omega}$, the closure of the set Ω, for all $a = (a_1, \ldots, a_n)$, $|a| \leqslant k$. $C^k(\bar{\Omega})$ can be equipped with the norm

$$||f||_{C^k(\bar{\Omega})} := \sum_{|a| \leqslant k} \sup_{x \in \Omega} |D^a f(x)|.$$

In particular, when $k = 0$ we shall write $C(\bar{\Omega})$ instead of $C^0(\bar{\Omega})$ to denote the set of all continuous functions defined on $\bar{\Omega}$, in this case

$$||f||_{C(\bar{\Omega})} = \sup_{x \in \Omega} |f(x)| = \max_{x \in \bar{\Omega}} |f(x)|.$$

The support of a continuous function f defined on an open set $\Omega \subset \Re^n$ is defined as the closure in Ω of the set $\{x \in \Omega : f(x) \neq 0\}$. We shall write supp f for the support of u. Thus, supp u is the smallest closed subset of Ω such that $f = 0$ in $\Omega \backslash \text{supp } f$.

We denote by $C_0^k(\Omega)$ the set of all f contained in C^k whose support is a bounded subset of Ω. Let

$$C_0^\infty = \cap_{k \geqslant 0} C_0^k(\Omega).$$

2.2 Spaces of Integrable Functions

We will now review the basic concepts of Lebesgue integration theory (see [36] for more details). By "domain" we mean a Lebesgue-measurable (usually either open or closed) subset of $\Re^n$ with non-empty interior. We restrict our attention for simplicity to real-valued functions, f, on a given domain, Ω, that are Lebesgue measurable; by

$$\int_\Omega f(x) dx$$

we denote the Lebesgue integral of f (dx denotes Lebesgue measure).

Let p be a real number, $p \geqslant 1$, we denote by $L^p(\Omega)$ the set of all real-valued functions defined on an open subset Ω of $\Re^n$ such that

$$\int_\Omega |f(x)|^p \, dx < \infty$$

Any two functions which are equal almost everywhere (i.e. equal, except on a set of measure zero) on Ω are identified with each other. Thereby, strictly speaking, $L^p(\Omega)$ consists of equivalence classes of functions, still, we shall not insist on this technicality. For $1 \leqslant p < \infty$, $L^p(\Omega)$ is equipped with the norm

$$||f||_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

We shall also consider the space $L^\infty(\Omega)$ consisting of functions f defined on Ω such that $|f|$ has finite essential supremum on Ω (namely, there exists a positive constant M such that $|f(x)| \leqslant M$ for almost every x in Ω, the smallest such number M is called the essential supremum of $|f|$, and we write $M = ess. \sup_{x\in\Omega} |f(x)|$). $L^\infty(\Omega)$ is equipped with the norm

$$||f||_{L^\infty(\Omega)} := ess. \sup_{x\in\Omega} |f(x)|.$$

In either case, we define the Lebesgue spaces as

$$L^p(\Omega) := \left\{ f : ||f||_{L^p(\Omega)} < \infty \right\}.$$

A particularly important case corresponds to taking $p = 2$. The space $L^2(\Omega)$ can be equipped with the inner product

$$(f, g)_\Omega := \int_\Omega f(x) g(x) dx.$$

Clearly

$$||f||_{L^2(\Omega)} = (f, f)_\Omega^{1/2}.$$

The norm of $L^2(\Omega)$ will be denoted by $|| \cdot ||_\Omega$ for brevity.

2.3 Sobolev Spaces

Using the notion of weak derivatives, we can generalize the Lebesgue norms and spaces to include derivatives. Ergo, we start by recalling the notion of a (standard) Sobolev space, based on the Lebesgue space L^p (see [1, 36] for more information).

Definition 2.3.1. *Let k be a non-negative integer, $p \in [1, \infty]$, $a = (a_1, \ldots, a_n)$ a multi-index and Ω an open domain in $\Re^n$. We define the Sobolev space $W_p^k(\Omega)$ on Ω by*

$$W_p^k(\Omega) := \{f \in L^p(\Omega) : D^a f \in L^p(\Omega) \ for |a| \leqslant k\},$$

with the associated norm $|| \cdot ||_{W_p^k(\Omega)}$ and seminorm $| \cdot |_{W_p^k(\Omega)}$

$$||f||_{W_p^k(\Omega)} := \left(\sum_{|a| \leqslant k} ||D^a f||^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, \quad |f|_{W_p^k(\Omega)} := \left(\sum_{|a| = k} ||D^a f||^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

in the case $1 \leqslant p < \infty$, and in the case $p = \infty$

$$||f||_{W_\infty^k(\Omega)} := \max_{|a| \leqslant k} ||D^a f||_{L^\infty(\Omega)}, \quad |f|_{W_\infty^k(\Omega)} := \max_{|a| = k} ||D^a f||_{L^\infty(\Omega)}.$$

We shall also refer to k as the Sobolev index of the function f. Moreover, we shall denote the space W_p^k with $p = 2$ by $W_2^k \equiv H^k$, and we shall use the abbreviated notations $|| \cdot ||_{k,\Omega}$, $| \cdot |_{k,\Omega}$ for the Hilbert-Sobolev norm as well as seminorm, respectively.

Negative and fractional Sobolev spaces are also defined by standard duality and function-space interpolation procedures, respectively (see [150, 1, 36] for more on these techniques).

An important special case corresponds to taking $p = 2$, the space W_2^k is then a Hilbert space with the inner product

$$(f, g)_{W_2^k(\Omega)} := \sum_{|a| \leqslant k} (D^a f, D^a g)_\Omega,$$

For this reason, we shall usually write H^k instead of W_2^k.

Given the number of indices defining Sobolev spaces, it is natural to hope that there are inclusion relations to provide some sort of ordering among them. Wherefore, it is easy to derive the following propositions.

Proposition 2.3.2. *Suppose that Ω is any domain, k and m are non-negative integers satisfying $k \leqslant m$, and p is any real number satisfying $1 \leqslant p \leqslant \infty$. Then*

$$W_p^m(\Omega) \subset W_p^k(\Omega).$$

Proposition 2.3.3. *Suppose that Ω is a bounded domain, k is a non-negative integer, and p and q are real numbers satisfying $1 \leqslant p \leqslant q \leqslant \infty$. Then*

$$W_q^k(\Omega) \subset W_p^k(\Omega).$$

2.4 Broken Sobolev Spaces

Since the discontinuous Galerkin finite element method and the continuous interior penalty finite element method are non-conforming methods, it is necessary to introduce the notion of a broken Sobolev space.

Broken Sobolev spaces are natural spaces to work with the DG and CIP methods. These spaces depend strongly on the partition of the domain.

Let $\mathcal{T}$ be a subdivision of the polygonal domain Ω into disjoint (triangular or quadrilateral) open elements K constructed via mappings either $\mathcal{P}_K \circ F_K$ or $\mathcal{Q}_K \circ F_K$, where $F_K : \hat{K} \to K$ is an affine mapping of the form

$$F_K(\mathbf{x}) = A_K(\mathbf{x}) + \mathbf{b},$$

with non-singular Jacobian, and where $\hat{K}$ is the reference triangle or quadrilateral.

Heuristically, we can say that the affine mapping F_K defines the "magnitude" of the element K and the diffeomorphism either $\mathcal{P}_K$ or $\mathcal{Q}_K$ defines the shape.

In addition, we assume that the intersection of two elements is either empty, a vertex, an edge, or a face. Such a mesh is called conforming. On the other hand, a mesh is called non-conforming if there exists at least one hanging node.

Furthermore, we assume that the subdivision $\mathcal{T}$ is shape regular (see Defintion A.1.7).

The above mappings are constructed in a way to ensure that the union of their closures forms a covering of the closure of Ω, i.e., $\bar{\Omega} = \cup_{K\in\mathcal{T}} \bar{K}$.

Definition 2.4.1. *We define the broken Sobolev space of composite order* $\mathbf{s}$ *on an open set* Ω*, subject to a subdivision* $\mathcal{T}$ *of* Ω*, as*

$$H^{\mathbf{s}}(\Omega, \mathcal{T}) = \left\{ f \in L^2(\Omega) : f|_k \in \mathcal{H}^{s_K}(K) \ \forall K \in \mathcal{T} \right\},$$

in the case of discontinuous Galerkin methods and

$$H^{\mathbf{s}}(\Omega, \mathcal{T}) = \left\{ f \in H^1(\Omega) : f|_k \in \mathcal{H}^{s_K}(K) \ \forall K \in \mathcal{T} \right\},$$

in the case of continuous interior penalty methods, respectively. We denote by s_K *the local Sobolev space index on the element* K *and* $\mathbf{s} := (s_K : K \in \mathcal{T})$*. We also define the associated broken norms and seminorms*

$$||f||_{\mathbf{s},\mathcal{T}} = \left(\sum_{K\in\mathcal{T}} ||f||^2_{\mathcal{H}^{s_K}(K)} \right)^{\frac{1}{2}}, \quad |f|_{\mathbf{s},\mathcal{T}} = \left(\sum_{K\in\mathcal{T}} |f|^2_{\mathcal{H}^{s_K}(K)} \right)^{\frac{1}{2}}.$$

Furthermore, when $s_K = s$ *for all* $K \in \mathcal{T}$, *we shall write* $H^s(\Omega, \mathcal{T})$ *as well as* $||f||_{s,\mathcal{T}}$ *and* $|f|_{s,\mathcal{T}}$.

Clearly, we have

$$H^s(\Omega) \subset H^s(\Omega, \mathcal{T}) \quad \text{and} \quad H^{s+1}(\Omega, \mathcal{T}) \subset H^s(\Omega, \mathcal{T}).$$

In this vein spirit we give the following definition.

Definition 2.4.2. *Let* $f \in H^2(\Omega, \mathcal{T})$ *and* $\mathbf{g} \in [H^2(\Omega, \mathcal{T})]^2$. *We define the broken gradient* $\nabla_{\mathcal{T}} f$ *and the broken Laplacian* $\Delta_{\mathcal{T}} f$ *of* f *as well as the broken divergence* $\nabla_{\mathcal{T}} \cdot \mathbf{g}$, *the broken gradient* $\nabla_{\mathcal{T}} \mathbf{g}$ *and the broken Laplacian* $\Delta_{\mathcal{T}} \mathbf{g}$ *of* $\mathbf{g}$ *by*

$$(\nabla_{\mathcal{T}} f)|_K = \nabla(f|_K), \quad (\Delta_{\mathcal{T}} f)|_K = \Delta(f|_K), \quad K \in \mathcal{T}$$

and

$$(\nabla_{\mathcal{T}} \cdot \mathbf{g})|_K = \nabla \cdot (\mathbf{g}|_K), \quad (\nabla_{\mathcal{T}} \mathbf{g})|_K = \nabla(\mathbf{g}|_K), \quad (\Delta_{\mathcal{T}} \mathbf{g})|_K = \Delta(\mathbf{g}|_K), \quad K \in \mathcal{T}.$$

Part I

One-Dimensional Problems

Chapter 3

Preliminaries of 1-D Problems

Suppose that Ω is an open bounded convex domain in $\Re$ with boundary Γ. Let us consider a family of subdivisions $\{\mathcal{P}(\Omega)\}_{h>0}$ of Ω, parametrized by $h > 0$. That is, for each $h > 0$, $\{\mathcal{P}(\Omega)\}_{h>0}$ is a partition of Ω into disjoint open convex element domains $\Omega_e = \Omega_e^j$ such that

$$\bar{\Omega} = \bigcup_{\Omega_e \in \mathcal{P}(\Omega)} \bar{\Omega}_e,$$

$$\Omega_e^i \cap \Omega_e^j = \emptyset \quad \text{for } i \neq j$$

and the intersection $\bar{\Omega}_e^i \cap \bar{\Omega}_e^j$ is either empty or a vertex. We define a piecewise constant mesh function $h_{\mathcal{P}(\Omega)}$ by

$$h_{\mathcal{P}(\Omega)}(x) = h_e = diam(\Omega_e), \quad x \in \Omega_e, \quad \Omega_e \in \mathcal{P}(\Omega)$$

and put

$$h = \max_{\Omega_e \in \{\mathcal{P}(\Omega)\}_{h>0}} h_e.$$

Let us assume that the family of subdivisions $\{\mathcal{P}(\Omega)\}_{h>0}$ is regular (see Definition A.1.7). Next, in this chapter, we shall use the abbreviated notation $\mathcal{P}(\Omega)$ for the family of subdivisions $\{\mathcal{P}(\Omega)\}_{h>0}$.

The union of element interiors can be defined as

$$\tilde{\Omega} = \bigcup_{e=1}^{N_{el}} \Omega_e. \tag{3.1}$$

For the L^2-inner product on element interiors, we adopt the notation

$$(a,b)_{\tilde{\Omega}} = \sum_{e=1}^{N_{el}} (a,b)_{\Omega_e}, \tag{3.2}$$

for suitaby defined a, b.

The union of interior boundaries can be expressed as the intersection of the boundaries $\partial\Omega_e$ of individual elements Ω_e by

$$\tilde{\Gamma} = \bigcup_{r,s=1:s>r}^{N_{el}} \left(\partial\Omega_e^r \cap \partial\Omega_e^s\right).$$

One can alternatively write the interior boundaries as

$$\tilde{\Gamma} = \bigcup_{i=1}^{N_i} \Gamma_i, \tag{3.3}$$

where N_i is the number of interior boundaries Γ_i. Analogous to (3.2), we define the L^2-inner product on interior boundaries as

$$(a,b)_{\tilde{\Gamma}} = \sum_{i=1}^{N_i} (a,b)_{\Gamma_i}. \tag{3.4}$$

We define the norms on element interiors and interior boundaries as

$$||\cdot||^2_{\tilde{\Omega}} = \sum_{e=1}^{N_{el}} ||\cdot||^2_{\Omega_e}, \tag{3.5}$$

and

$$||\cdot||^2_{\tilde{\Gamma}} = \sum_{i=1}^{N_i} ||\cdot||^2_{\Gamma_i} \tag{3.6}$$

respectively.

Only for the boundary value problem of gradient elastic beam in bending, let $\tilde{\Gamma}_1 = \tilde{\Gamma} \cup \Gamma_q$ and $\tilde{\Gamma}_2 = \tilde{\Gamma} \cup \Gamma_r$. We define for $u,w \in L^2(\tilde{\Gamma}_1)$ and for $u,w \in L^2(\tilde{\Gamma}_2)$, the inner products

$$uw_{\tilde{\Gamma}_1} = uw_{\tilde{\Gamma}} + uw|_{\Gamma_q}, \tag{3.7}$$

$$uw_{\tilde{\Gamma}_2} = uw_{\tilde{\Gamma}} + uw|_{\Gamma_r} \tag{3.8}$$

with associated norms $||\cdot||_{\tilde{\Gamma}_1}$ and $||\cdot||_{\tilde{\Gamma}_2}$. So, it will hold as well

$$||u||^2_{\tilde{\Gamma}_1} = ||u||^2_{\tilde{\Gamma}} + ||u||^2_{\Gamma_q} \tag{3.9}$$

or

$$||u||^2_{\tilde{\Gamma}_1} = \sum_{i=1}^{N_i} ||u||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||u||^2_{\Gamma_j} \tag{3.10}$$

and

$$||u||^2_{\tilde{\Gamma}_2} = ||u||^2_{\tilde{\Gamma}} + ||u||^2_{\Gamma_r}. \tag{3.11}$$

or

$$||u||^2_{\tilde{\Gamma}_2} = \sum_{i=1}^{N_i} ||u||^2_{\Gamma_i} + \sum_{s=1}^{N_r} ||u||^2_{\Gamma_s}. \tag{3.12}$$

Chapter 4

IPDGFEMs for SGE in 1-D

4.1 Model Problem

Toupin and Mindlin included higher-order stresses and strains in the theory of linear elasticity, which serves today as the foundation of more advanced strain gradient elasticity and plasticity formulation [192, 157, 102], respectively. Let us introduce a one-dimensional model problem following their concepts.

Let $\Omega \subset \Re$ be an open, bounded domain and Γ its boundary. Let Γ_c, Γ_q, Γ_R and Γ_P denote the (axial) displacement, displacement gradient, double force and (axial) force boundaries, respectively.

We consider the equation:

$$\sigma_{,x} - \bar{\sigma}_{,xx} + \frac{\bar{f}}{A} = 0 \quad \text{in } \Omega. \tag{4.1}$$

We supplement the equation with the following boundary conditions

$$\begin{aligned} u &= c \quad \text{on } \Gamma_c, \\ u_{,x} \cdot n &= q \quad \text{on } \Gamma_q, \\ \bar{\sigma} A &= R \quad \text{on } \Gamma_R, \\ (\sigma A - \bar{\sigma}_{,x} A) \cdot n &= P \quad \text{on } \Gamma_P, \end{aligned} \tag{4.2}$$

where n is the unit normal vector to the boundary, exterior to Ω.

Note that we have the relationships

$$\overline{\Gamma_c \cup \Gamma_P} = \Gamma, \tag{4.3}$$
$$\Gamma_c \cap \Gamma_P = \emptyset, \tag{4.4}$$
$$\overline{\Gamma_q \cup \Gamma_R} = \Gamma, \tag{4.5}$$
$$\Gamma_q \cap \Gamma_R = \emptyset, \tag{4.6}$$

between the different parts of the boundary. The constitutive equations for the stress (or Cauchy stress) σ and the higher-order (or double stress) $\bar{\sigma}$ can be expressed as

$$\sigma = Eu_{,x}, \tag{4.7}$$
$$\bar{\sigma} = Eg^2 u_{,xx}, \tag{4.8}$$

where E is a material parameter (the modulus of elasticity) and g a length scale (which represents material length related to the volumetric elastic strain energy). We can rewrite (4.1) and (4.2) with (4.7) and (4.8) as:

$$(g^2 u_{,xx} - u)_{,xx} = \frac{\bar{f}}{AE} = f \quad \text{in } \Omega, \tag{4.9}$$

$$\begin{aligned} u = c \quad & \text{on } \Gamma_c, \\ u_{,x} \cdot n = q \quad & \text{on } \Gamma_q, \\ AEg^2 u_{,xx} = R \quad & \text{on } \Gamma_R, \\ AE(u - g^2 u_{,xx})_{,x} \cdot n = P \quad & \text{on } \Gamma_P, \end{aligned} \tag{4.10}$$

where $f \in L^2(\Omega)$. In the above, u denotes the (axial) displacement, A is a cross-section, AE is the (axial) stiffness, $\bar{f}$ is a given (axially) distributed load and c, q, R and P denote the prescribed boundary displacement, displacement gradient, double force and (axial) force, respectively.

We mention that the first two boundary conditions are called essential and the other two are called natural, respectively. Specifically, the last one is called a Robin boundary condition, as well.

Under suitable conditions on Ω and on the data f, c, q, R and P, the boundary value problem (4.9), (4.10), possesses a unique solution $u \in H^4(\Omega)$ that depends continuously on the data of the problem.

4.2 Weak Formulation

We are ready to derive the weak formulation for the problem (4.9) – (4.10), which will lead to the discontinuous Galerkin finite element method. We shall suppose for the moment that the solution u of the problem is a sufficiently smooth function.

For each face $\Gamma_i \subseteq \tilde{\Gamma}$, let k and l be such indices that $k > l$ and the elements $\Omega_e := \Omega_e^k$ and $\Omega_{e'} := \Omega_e^l$ share the face Γ_i. Let us define the jump across Γ_i and the mean value on Γ_i of $u \in H^1(\Omega, \mathcal{P}(\Omega))$ by

$$[\![u]\!]_{\Gamma_i} := u|_{\partial\Omega_e \cap \Gamma_i} - u|_{\partial\Omega_{e'} \cap \Gamma_i} \quad \text{and} \quad \langle u \rangle_{\Gamma_i} := \frac{1}{2}\left(u|_{\partial\Omega_e \cap \Gamma_i} + u|_{\partial\Omega_{e'} \cap \Gamma_i} \right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and of the mean value to $\Gamma_r \subseteq \Gamma_c$, $\Gamma_j \subseteq \Gamma_q$ that belong to the boundary Γ by letting:

$$\begin{aligned} [\![u]\!]_{\Gamma_r} &= u|_{\Gamma_r} \quad \text{and} \quad \langle u \rangle_{\Gamma_r} = u|_{\Gamma_r}, \\ [\![u]\!]_{\Gamma_j} &= u|_{\Gamma_j} \quad \text{and} \quad \langle u \rangle_{\Gamma_j} = u|_{\Gamma_j}. \end{aligned}$$

In these definitions, the subscripts Γ_i and $\Gamma_{r:j}$ will be supressed when no confusion is likely to occur. With each face $\Gamma_i \subseteq \tilde{\Gamma}$, we associate the unit normal vector $n = n_{\Omega_e^k}$, pointing from element Ω_e^k to Ω_e^l when $k > l$, and we choose $n = n_{\Omega_e}$ to be the unit outward normal when a node belongs to the boundary Γ.

Since the method will be non-conforming, we shall use the broken Sobolev space $H^4(\Omega, \mathcal{P}(\Omega))$ as trial space. We multiply the equation, (4.9), by a test function $w \in H^4(\Omega, \mathcal{P}(\Omega))$ and integrate over Ω

$$\int_\Omega (g^2 u_{,xx} - u)_{,xx} w dx = \int_\Omega f w dx.$$

Afterwards, we split the integrals

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} (g^2 u_{,xx} - u)_{,xx} w dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx,$$

and applying integration by parts on every elemental integral, so we get

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\ - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} g^2 u_{,xx} w_{,x} \cdot nds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx,$$

where n denotes the outward normal to each element boundary.

Now, we split the boundary terms as follows

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\ + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot nwds + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_P} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\ - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot nds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds \\ - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_R} g^2 u_{,xx} w_{,x} \cdot nds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx,$$

and hence we have

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\ + \int_{\Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot nwds + \int_{\Gamma_P} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\ - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot nds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds - \int_{\Gamma_R} g^2 u_{,xx} w_{,x} \cdot nds \\ + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx. \tag{4.11}$$

Using the natural boundary conditions, (4.10), on the fourth and on the seventh term respectively, on the left-hand side of (4.11) and moving it to the right-hand side, we obtain

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot n w ds$$
$$+ \int_{\Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot n ds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds$$
$$+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx + \int_{\Gamma_P} \frac{P}{AE} w ds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot n ds. \tag{4.12}$$

The second and the fourth term respectively on the left-hand side of (4.12) contain the boundary integrals over the interior element boundaries, i.e. the interior boundaries $\Gamma_i \subseteq \tilde{\Gamma}$. Consequently, in this sum of boundary integrals, we have two integrals over every interior boundary.

Remark 4.2.0.1. *With each face $\Gamma_i \subseteq \tilde{\Gamma}$, we associate the unit normal vector $n = n_{\Omega_e^k} = -n_{\Omega_e^l}$, pointing from element Ω_e^k to Ω_e^l when $k > l$, and with each $\Gamma_r \subseteq \Gamma_c$ as well as $\Gamma_j \subseteq \Gamma_q$ we associate the external unit normal vector $n = n_{\Omega_e}$, where $\Gamma_r, \Gamma_j \subset \partial\Omega_e$.*

Let us note that, for a given interior boundary, Γ_i, shared by two adjacent elements Ω_e^k and Ω_e^l $(k > l)$, we can write

$$u_{,x}|_{\Omega_e^k} \cdot n_{\Omega_e^k} w|_{\Omega_e^k} + u_{,x}|_{\Omega_e^l} \cdot n_{\Omega_e^l} w|_{\Omega_e^l} = u_{,x}|_{\Omega_e^k} \cdot n w|_{\Omega_e^k} - u_{,x}|_{\Omega_e^l} \cdot n w|_{\Omega_e^l},$$

Hence, by analogy with the formula

$$ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d) \quad \forall a, b, c, d \in \Re,$$

we get

$$u_{,x}|_{\Omega_e^k} \cdot n_{\Omega_e^k} w|_{\Omega_e^k} + u_{,x}|_{\Omega_e^l} \cdot n_{\Omega_e^l} w|_{\Omega_e^l} = \langle u_{,x} \rangle [\![w]\!] + [\![u_{,x}]\!] \langle w \rangle \quad \forall u, w \in H^1(\Omega, \mathcal{P}(\Omega)). \tag{4.13}$$

In order to evaluate the integrals on interior boundaries, we always use the interior trace of the test function w. Taking into account the Remark 4.2.0.1 and applying (4.13), we can see that the second and the fourth term respectively, on the left-hand side of (4.12), can be rewritten as follows

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Gamma}} \langle (g^2 u_{,xx} - u)_{,x} \rangle [\![w]\!] ds + \int_{\tilde{\Gamma}} [\![(g^2 u_{,xx} - u)_{,x}]\!] \langle w \rangle ds$$
$$+ \int_{\Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot nwds - \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle ds$$
$$- \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx$$
$$= \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx + \int_{\Gamma_P} \frac{P}{AE} wds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot nds. \tag{4.14}$$

By noting that the fluxes $(g^2 u_{,xx} - u)_{,x} \cdot n$ and $g^2 u_{,xx}$ are continuous across the interelement boundaries Γ_i (e.g., when the exact solution $u \in H^4(\Omega)$), we have

$$\int_{\tilde{\Gamma}} [\![(g^2 u_{,xx} - u)_{,x}]\!] \langle w \rangle ds = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)),$$
$$\int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle ds = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)).$$

Then, (4.14) reduces to

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx + \int_{\tilde{\Gamma}} \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!] ds$$
$$+ \int_{\Gamma_c} (g^2 u_{,xx})_{,x} \cdot nwds - \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds$$
$$- \int_{\tilde{\Gamma}} \langle u_{,x} \rangle [\![w]\!] ds - \int_{\Gamma_c} u_{,x} \cdot nwds$$
$$= \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx + \int_{\Gamma_P} \frac{P}{AE} wds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot nds. \tag{4.15}$$

Next, we multiply the boundary condition $u = c$, on Γ_c, by $-\theta(g^2 w_{,xx})_{,x} \cdot n + \alpha_c w$ and by $\theta w_{,x} \cdot n + \delta_c w$ as well. Then, integrating over Γ_c, we get

$$- \int_{\Gamma_c} \theta u (g^2 w_{,xx})_{,x} \cdot nds + \int_{\Gamma_c} \alpha_c uwds = - \int_{\Gamma_c} \theta c (g^2 w_{,xx})_{,x} \cdot nds + \int_{\Gamma_c} \alpha_c cwds, \tag{4.16}$$

and

$$\int_{\Gamma_c} \theta u w_{,x} \cdot n ds + \int_{\Gamma_c} \delta_c u w ds = \int_{\Gamma_c} \theta c w_{,x} \cdot n ds + \int_{\Gamma_c} \delta_c c w ds, \tag{4.17}$$

where θ is the symmetrization parameter. We restrict ourselves to the case $\theta \in \{-1, 1\}$. The non-negative piecewise continuous functions α_c and δ_c, defined on Γ_c, are referred to as the stabilization parameters.

Furthermore, since we have an elliptic boundary value problem, elliptic regularity ensures us that u will be continuous in Ω. So the jump $[\![u]\!]$ vanishes, i.e. $[\![u]\!] = 0$. If we choose $-\theta\langle (g^2 w_{,xx})_{,x}\rangle + \alpha [\![w]\!]$, $\theta\langle w_{,x}\rangle + \delta[\![w]\!]$ as test functions and integrate over $\tilde{\Gamma}$, we deduce

$$-\int_{\tilde{\Gamma}} \theta [\![u]\!] \langle (g^2 w_{,xx})_{,x}\rangle ds + \int_{\tilde{\Gamma}} \alpha [\![u]\!][\![w]\!] ds = 0, \tag{4.18}$$

and

$$\int_{\tilde{\Gamma}} \theta [\![u]\!] \langle w_{,x}\rangle ds + \int_{\tilde{\Gamma}} \delta [\![u]\!][\![w]\!] ds = 0, \tag{4.19}$$

where α and δ are non-negative piecewise continuous functions, defined on $\tilde{\Gamma}$, which are referred to as stabilization parameters.

Moreover, from the boundary condition $u_{,x} \cdot n = q$, on Γ_q, upon multiplying by $\theta g^2 w_{,xx} + \beta_q w_{,x} \cdot n$ and integrating over Γ_q, we have

$$\int_{\Gamma_q} \theta u_{,x} \cdot n g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q u_{,x} \cdot n w_{,x} \cdot n ds = \int_{\Gamma_q} \theta q g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q q w_{,x} \cdot n ds. \tag{4.20}$$

The non-negative piecewise continuous function β_q, defined on Γ_q, is referred to as the stabilization parameter.

In addition, as mentioned above, elliptic regularity ensures us that $u_{,x}$ will be continuous in Ω. In that case the jump $[\![u_{,x}]\!]$ vanishes, i.e. $[\![u_{,x}]\!] = 0$. If we choose $\theta\langle g^2 w_{,xx}\rangle + \beta[\![w_{,x}]\!]$ as test function and integrate over $\tilde{\Gamma}$, it gives

$$\int_{\tilde{\Gamma}} \theta [\![u_{,x}]\!] \langle g^2 w_{,xx}\rangle ds + \int_{\tilde{\Gamma}} \beta [\![u_{,x}]\!][\![w_{,x}]\!] ds = 0, \tag{4.21}$$

where β is a non-negative continuous function, defined on $\tilde{\Gamma}$, which is referred to as the stabilization parameter.

Now adding (4.15) – (4.21), we get the discontinuous Galerkin weak formulation of the problem

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx + \int_{\tilde{\Gamma}} \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!] ds \\
&- \int_{\tilde{\Gamma}} \theta [\![u]\!] \langle (g^2 w_{,xx})_{,x} \rangle ds - \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds + \int_{\tilde{\Gamma}} \theta [\![u_{,x}]\!] \langle g^2 w_{,xx} \rangle ds \\
&- \int_{\tilde{\Gamma}} \langle u_{,x} \rangle [\![w]\!] ds + \int_{\tilde{\Gamma}} \theta [\![u]\!] \langle w_{,x} \rangle ds + \int_{\tilde{\Gamma}} \alpha [\![u]\!][\![w]\!] ds + \int_{\tilde{\Gamma}} \beta [\![u_{,x}]\!][\![w_{,x}]\!] ds \\
&+ \int_{\tilde{\Gamma}} \delta [\![u]\!][\![w]\!] ds + \int_{\Gamma_c} (g^2 u_{,xx})_{,x} \cdot n w ds - \int_{\Gamma_c} \theta u (g^2 w_{,xx})_{,x} \cdot n ds \\
&- \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds + \int_{\Gamma_q} \theta u_{,x} \cdot n g^2 w_{,xx} ds - \int_{\Gamma_c} u_{,x} \cdot n w ds \\
&+ \int_{\Gamma_c} \theta u w_{,x} \cdot n ds + \int_{\Gamma_c} \alpha_c u w ds + \int_{\Gamma_q} \beta_q u_{,x} \cdot n w_{,x} \cdot n ds + \int_{\Gamma_c} \delta_c u w ds \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx + \int_{\Gamma_P} \frac{P}{AE} w ds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot n ds - \int_{\Gamma_c} \theta c (g^2 w_{,xx})_{,x} \cdot n ds \\
&+ \int_{\Gamma_q} \theta q g^2 w_{,xx} ds + \int_{\Gamma_c} \theta c w_{,x} \cdot n ds + \int_{\Gamma_c} \alpha_c c w ds + \int_{\Gamma_q} \beta_q q w_{,x} \cdot n ds \\
&+ \int_{\Gamma_c} \delta_c c w ds.
\end{aligned}
\tag{4.22}
$$

The bilinear form is defined as

$$
\begin{aligned}
B_{sb}(u,w) \;:=\; & (g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} + (u_{,x}, w_{,x})_{\tilde{\Omega}} \\
& + \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} - \theta [\![u]\!] \langle (g^2 w_{,xx})_{,x} \rangle_{\tilde{\Gamma}} \\
& - \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} + \theta [\![u_{,x}]\!] \langle g^2 w_{,xx} \rangle_{\tilde{\Gamma}} \\
& - \langle u_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} + \theta [\![u]\!] \langle w_{,x} \rangle_{\tilde{\Gamma}} \\
& + \alpha [\![u]\!][\![w]\!]_{\tilde{\Gamma}} + \beta [\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} + \delta [\![u]\!][\![w]\!]_{\tilde{\Gamma}} \\
& + (g^2 u_{,xx})_{,x} \cdot n w|_{\Gamma_c} - \theta u (g^2 w_{,xx})_{,x} \cdot n|_{\Gamma_c} \\
& - g^2 u_{,xx} w_{,x} \cdot n|_{\Gamma_q} + \theta u_{,x} \cdot n g^2 w_{,xx}|_{\Gamma_q} \\
& - u_{,x} \cdot n w|_{\Gamma_c} + \theta u w_{,x} \cdot n|_{\Gamma_c} \\
& + \alpha_c u w|_{\Gamma_c} + \beta_q u_{,x} \cdot n w_{,x} \cdot n|_{\Gamma_q} + \delta_c u w|_{\Gamma_c}.
\end{aligned}
\tag{4.23}
$$

We introduce the linear functional $L_{sb}(\cdot)$ on $H^4(\Omega, \mathcal{P}(\Omega))$

$$\begin{aligned} L_{sb}(w) \;&:=\; (f,w)_{\tilde{\Omega}} + \frac{P}{AE} w|_{\Gamma_P} + \frac{R}{AE} w_{,x} \cdot n|_{\Gamma_R} \\ &- \theta c(g^2 w_{,xx})_{,x} \cdot n|_{\Gamma_c} + \theta q g^2 w_{,xx}|_{\Gamma_q} + \theta c w_{,x} \cdot n|_{\Gamma_c} \\ &+ \alpha_c c w|_{\Gamma_c} + \beta_q q w_{,x} \cdot n|_{\Gamma_q} + \delta_c c w|_{\Gamma_c}. \end{aligned} \tag{4.24}$$

The stabilization parameters, $\alpha, \alpha_c, \beta, \beta_q, \delta, \delta_c$, depend on the discretization parameter h_e for the h-method, and on the discretization parameters h_e, p_e for the hp-method respectively, in a manner that will be specified later in the text.

Then the broken weak formulation of the problem (4.9) – (4.10) reads as follows:

$$\text{Find } u \in bSs \text{ such that } B_{sb}(u,w) = L_{sb}(w) \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)), \tag{4.25}$$

where by bSs we denote the following function space

$$\begin{aligned} bSs \;=\; &\{u \in H^4(\Omega, \mathcal{P}(\Omega)) : u, u_{,x} \cdot n, g^2 u_{,xx}, (g^2 u_{,xx} - u)_{,x} \cdot n \\ &\text{are continuous across } \Gamma_i\}. \end{aligned}$$

Note that for $\theta = -1$ the bilinear form $B_{sb}(\cdot,\cdot)$ is symmetric, whereas for $\theta = 1$ it is not symmetric.

A norm which can be derived from the bilinear of a method is also referred to as the energy norm of the method. We notice that energy norm is mesh-dependent.

We shall associate with the bilinear form $B_{sb}(\cdot,\cdot)$ the energy seminorm, $|||\cdot|||_{sb}$, defined by

$$\begin{aligned} |||u|||_{sb} \;=\; &\Big(||(g^2)^{1/2} u_{,xx}||^2_{\tilde{\Omega}} + ||u_{,x}||^2_{\tilde{\Omega}} + ||\alpha^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + ||\alpha_c^{1/2} u||^2_{\Gamma_c} \\ &+ ||\beta^{1/2}[\![u_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\beta_q^{1/2} u_{,x}||^2_{\Gamma_q} + ||\delta^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + ||\delta_c^{1/2} u||^2_{\Gamma_c}\Big)^{1/2}, \\ &u \in H^2(\Omega, \mathcal{P}(\Omega)). \end{aligned} \tag{4.26}$$

Proposition 4.2.0.2. *If α, α_c, β, β_q, δ, $\delta_c > 0$, then $|||\cdot|||_{sb}$ is a seminorm on $H^2(\Omega, \mathcal{P}(\Omega))$.*

We note in passing that since $H^4(\Omega, \mathcal{P}(\Omega)) \subset H^2(\Omega, \mathcal{P}(\Omega))$, then $|||\cdot|||_{sb}$ is also a seminorm on $H^4(\Omega, \mathcal{P}(\Omega))$.

4.2.1 Consistency

We shall now show that a strong solution to the boundary value problem for the strain gradient bar in tension equation, which is smooth enough at the interelement boundaries, is the solution to the problem in the broken weak formulation. Let us start by demonstrating weak continuity of fluxes across the element faces Γ_i.

Lemma 4.2.1.1. *Suppose that $u \in H^4(\Omega)$; then, for any Γ_i, we have*

$$\begin{aligned}\int_{\Gamma_i} [\![u]\!] w ds &= \int_{\Gamma_i} [\![u_{,x}]\!] w ds = \int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w ds \\ &= \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] w ds = 0 \quad \forall w \in L^2(\Gamma_i).\end{aligned}$$

Proof. We follow the ideas of [181], where the first two integrals were shown to be equal to zero for all w in $L^2(\Gamma_i)$, when $u \in H^2(\Omega)$.

To establish the last equality, let Γ_i be an interior boundary and let $\Omega_{e'}$ and Ω_e be the elements sharing the face Γ_i. Let $\tilde{\Omega}_e = int(\overline{\Omega_{e'} \cup \Omega_e})$. Subsequently, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, after integrating by parts, we have

$$\begin{aligned}\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\tilde{\Omega}_e} (g^2 u_{,xx})_{,x} w_{,x} dx \\ &\quad + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\ &= -\int_{\tilde{\Omega}_e} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx. \qquad (4.27)\end{aligned}$$

Then, we also split the left-hand side integral and apply the integration

by parts formula in each of $\Omega_{e'}$, Ω_e. As a result, we deduce

$$\begin{aligned}
\int_{\tilde{\Omega}_e}(g^2u_{,xx}-u)_{,xx}wdx &= \int_{\Omega_{e'}}(g^2u_{,xx}-u)_{,xx}wdx \\
&\quad + \int_{\Omega_e}(g^2u_{,xx}-u)_{,xx}wdx \\
&= \int_{\partial\Omega_{e'}}(g^2u_{,xx}-u)_{,x}\cdot nwds \\
&\quad - \int_{\Omega_{e'}}(g^2u_{,xx})_{,x}w_{,x}dx + \int_{\Omega_{e'}}u_{,x}w_{,x}dx \\
&\quad + \int_{\partial\Omega_e}(g^2u_{,xx}-u)_{,x}\cdot nwds \\
&\quad - \int_{\Omega_e}(g^2u_{,xx})_{,x}w_{,x}dx + \int_{\Omega_e}u_{,x}w_{,x}dx \\
&= -\int_{\Omega_{e'}}(g^2u_{,xx})_{,x}w_{,x}dx + \int_{\Omega_{e'}}u_{,x}w_{,x}dx \\
&\quad - \int_{\Omega_e}(g^2u_{,xx})_{,x}w_{,x}dx + \int_{\Omega_e}u_{,x}w_{,x}dx \\
&\quad + \int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds \\
&= -\int_{\tilde{\Omega}_e}(g^2u_{,xx})_{,x}w_{,x}dx + \int_{\tilde{\Omega}_e}u_{,x}w_{,x}dx \\
&\quad + \int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds.
\end{aligned} \tag{4.28}$$

Now, from the identities (4.27) and (4.28), it entails that

$$\int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{4.29}$$

Ergo,

$$\int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required.

Moreover, we shall use similar series of steps so as to establish the equality $\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w ds = 0$. Employing integration by parts formula twice, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, we get

$$
\begin{aligned}
\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\tilde{\Omega}_e} g^2 (u_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\partial\tilde{\Omega}_e} g^2 u_{,xx} w_{,x} \cdot n ds \\
&\quad + \int_{\tilde{\Omega}_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&= \int_{\tilde{\Omega}_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx. \qquad (4.30)
\end{aligned}
$$

If we subsequently split the left-hand side integral and perform integration

by parts twice in each of $\Omega_{e'}$ and Ω_e, we conclude

$$\begin{aligned}
\int_{\tilde{\Omega}_e}(g^2u_{,xx}-u)_{,xx}wdx &= \int_{\Omega_{e'}}(g^2u_{,xx}-u)_{,xx}wdx \\
&\quad + \int_{\Omega_e}(g^2u_{,xx}-u)_{,xx}wdx \\
&= \int_{\partial\Omega_{e'}}(g^2u_{,xx}-u)_{,x}\cdot nwds - \int_{\partial\Omega_{e'}}g^2u_{,xx}w_{,x}\cdot nds \\
&\quad + \int_{\Omega_{e'}}g^2u_{,xx}w_{,xx}dx + \int_{\Omega_{e'}}u_{,x}w_{,x}dx \\
&\quad + \int_{\partial\Omega_e}(g^2u_{,xx}-u)_{,x}\cdot nwds \\
&\quad - \int_{\partial\Omega_e}g^2u_{,xx}w_{,x}\cdot nds + \int_{\Omega_e}g^2u_{,xx}w_{,xx}dx \\
&\quad + \int_{\Omega_e}u_{,x}w_{,x}dx \\
&= \int_{\Omega_{e'}}g^2u_{,xx}w_{,xx}dx + \int_{\Omega_{e'}}u_{,x}w_{,x}dx \\
&\quad + \int_{\Omega_e}g^2u_{,xx}w_{,xx}dx + \int_{\Omega_e}u_{,x}w_{,x}dx \\
&\quad + \int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds \\
&\quad - \int_{\Gamma_i}[\![g^2u_{,xx}]\!]w_{,x}\cdot nds \\
&= \int_{\tilde{\Omega}_e}g^2u_{,xx}w_{,xx}dx + \int_{\tilde{\Omega}_e}u_{,x}w_{,x}dx \\
&\quad + \int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds \\
&\quad - \int_{\Gamma_i}[\![g^2u_{,xx}]\!]w_{,x}\cdot nds.
\end{aligned} \tag{4.31}$$

The identities (4.30), (4.31), entail that

$$\int_{\Gamma_i}[\![g^2u_{,xx}]\!]w_{,x}\cdot nds = \int_{\Gamma_i}[\![(g^2u_{,xx}-u)_{,x}]\!]\cdot nwds. \tag{4.32}$$

By substituting (4.29) into the equation (4.32), we reach to conclusion

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{4.33}$$

As a consequence,

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required. □

Proposition 4.2.1.2. *The broken weak formulation* (4.25) *of the boundary value problem* (4.9) – (4.10) *is consistent in the space $H^4(\Omega)$ in the sense that any solution u to the boundary value problem, such that $u \in H^4(\Omega)$, solves* (4.25) *as well.*

Proof. To begin with, from (4.25) and the defining expressions for $B_{sb}(\cdot,\cdot)$, $L_{sb}(\cdot)$, for $u \in bSs$, we have

$$\begin{aligned}
0 &= B_{sb}(u,w) - L_{sb}(w) \\
&= (g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} + (u_{,x}, w_{,x})_{\tilde{\Omega}} + \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} - \theta [\![u]\!] \langle (g^2 w_{,xx})_{,x} \rangle_{\tilde{\Gamma}} \\
&\quad - \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} + \theta [\![u_{,x}]\!] \langle g^2 w_{,xx} \rangle_{\tilde{\Gamma}} - \langle u_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} + \theta [\![u]\!] \langle w_{,x} \rangle_{\tilde{\Gamma}} \\
&\quad + \alpha [\![u]\!][\![w]\!]_{\tilde{\Gamma}} + \beta [\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} + \delta [\![u]\!][\![w]\!]_{\tilde{\Gamma}} + (g^2 u_{,xx})_{,x} \cdot n w|_{\Gamma_c} \\
&\quad - \theta u (g^2 w_{,xx})_{,x} \cdot n|_{\Gamma_c} - g^2 u_{,xx} w_{,x} \cdot n|_{\Gamma_q} + \theta u_{,x} \cdot n g^2 w_{,xx}|_{\Gamma_q} \\
&\quad - u_{,x} \cdot n w|_{\Gamma_c} + \theta u w_{,x} \cdot n|_{\Gamma_c} + \alpha_c u w|_{\Gamma_c} + \beta_q u_{,x} \cdot n w_{,x} \cdot n|_{\Gamma_q} + \delta_c u w|_{\Gamma_c} \\
&\quad - (f,w)_{\tilde{\Omega}} - \frac{P}{AE} w|_{\Gamma_P} - \frac{R}{AE} w_{,x} \cdot n|_{\Gamma_R} + \theta c (g^2 w_{,xx})_{,x} \cdot n|_{\Gamma_c} \\
&\quad - \theta q g^2 w_{,xx}|_{\Gamma_q} - \theta c w_{,x} \cdot n|_{\Gamma_c} - \alpha_c c w|_{\Gamma_c} - \beta_q q w_{,x} \cdot n|_{\Gamma_q} - \delta_c c w|_{\Gamma_c}.
\end{aligned} \tag{4.34}$$

Next, performing integration by parts in $\int_{\tilde{\Omega}} u_{,x} w_{,x} dx$ and twice in

$\int_{\tilde{\Omega}} g^2 u_{,xx} w_{,xx} dx$ respectively, we obtain

$$\begin{aligned} \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx &= \int_{\tilde{\Gamma}} \langle u_{,x} \rangle [\![w]\!] ds + \int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle w \rangle ds + \int_{\Gamma_c} u_{,x} \cdot nw ds \\ &\quad + \int_{\Gamma_P} u_{,x} \cdot nw ds - \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,xx} w dx \end{aligned}$$

or else

$$(u_{,x}, w_{,x})_{\tilde{\Omega}} = \langle u_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} + [\![u_{,x}]\!] \langle w \rangle_{\tilde{\Gamma}} + u_{,x} \cdot nw|_{\Gamma_c} + u_{,x} \cdot nw|_{\Gamma_P} - (u_{,xx}, w)_{\tilde{\Omega}}, \quad (4.35)$$

and

$$\begin{aligned} \sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx &= \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds + \int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle ds \\ &\quad + \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds + \int_{\Gamma_R} g^2 u_{,xx} w_{,x} \cdot n ds \\ &\quad - \int_{\tilde{\Gamma}} \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!] ds - \int_{\tilde{\Gamma}} [\![(g^2 u_{,xx})_{,x}]\!] \langle w \rangle ds \\ &\quad - \int_{\Gamma_c} (g^2 u_{,xx})_{,x} \cdot nw ds - \int_{\Gamma_P} (g^2 u_{,xx})_{,x} \cdot nw ds \\ &\quad + \sum_{e=1}^{N_{el}} \int_{\Omega_e} (g^2 u_{,xx})_{,xx} w dx \end{aligned}$$

or else

$$\begin{aligned} (g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} &= \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} + [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle_{\tilde{\Gamma}} + g^2 u_{,xx} w_{,x} \cdot n|_{\Gamma_q} \\ &\quad + g^2 u_{,xx} w_{,x} \cdot n|_{\Gamma_R} - \langle (g^2 u_{,xx})_{,x} \rangle [\![w]\!]_{\tilde{\Gamma}} - [\![(g^2 u_{,xx})_{,x}]\!] \langle w \rangle_{\tilde{\Gamma}} \\ &\quad - (g^2 u_{,xx})_{,x} \cdot nw|_{\Gamma_c} - (g^2 u_{,xx})_{,x} \cdot nw|_{\Gamma_P} \\ &\quad + ((g^2 u_{,xx})_{,xx}, w)_{\tilde{\Omega}}. \qquad (4.36) \end{aligned}$$

Then, by substituting the mathematical formulas (4.35) and (4.36) into

(4.34), we deduce that

$$\begin{aligned} 0 \;=\;& ((g^2u_{,xx}-u)_{,xx}-f,w)_{\tilde{\Omega}} + [\![(u-g^2u_{,xx})_{,x}]\!]\langle w\rangle_{\tilde{\Gamma}} + [\![g^2u_{,xx}]\!]\langle w_{,x}\rangle_{\tilde{\Gamma}} \\ &+ \theta[\![u_{,x}]\!]\langle g^2w_{,xx}\rangle_{\tilde{\Gamma}} + \theta[\![u]\!]\langle (w-g^2w_{,xx})_{,x}\rangle_{\tilde{\Gamma}} \\ &+ \theta(u-c)(w-g^2w_{,xx})_{,x}\cdot n|_{\Gamma_c} + \theta(u_{,x}\cdot n-q)g^2w_{,xx}|_{\Gamma_q} \\ &+ \left(g^2u_{,xx}-\frac{R}{AE}\right)w_{,x}\cdot n|_{\Gamma_R} + \left((u-g^2u_{,xx})_{,x}\cdot n-\frac{P}{AE}\right)w|_{\Gamma_P} \\ &+ \alpha[\![u]\!][\![w]\!]_{\tilde{\Gamma}} + \beta[\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} + \delta[\![u]\!][\![w]\!]_{\tilde{\Gamma}} \\ &+ \alpha_c(u-c)w|_{\Gamma_c} + \beta_q(u_{,x}\cdot n-q)w_{,x}\cdot n|_{\Gamma_q} + \delta_c(u-c)w|_{\Gamma_c}. \end{aligned} \tag{4.37}$$

Now, the mathematical equation, (4.37), is identical to zero for all w, when

$$[\![u]\!]=0 \quad \text{on } \tilde{\Gamma}, \tag{4.38}$$

$$[\![u_{,x}]\!]=0 \quad \text{on } \tilde{\Gamma}, \tag{4.39}$$

$$[\![AEg^2u_{,xx}]\!]=0 \quad \text{on } \tilde{\Gamma}, \tag{4.40}$$

$$[\![AE(u-g^2u_{,xx})_{,x}]\!]=0 \quad \text{on } \tilde{\Gamma}, \tag{4.41}$$

and

$$AE(g^2u_{,xx}-u)_{,xx}-\bar{f}=0 \quad \text{in } \tilde{\Omega}, \tag{4.42}$$

$$u=c \quad \text{on } \Gamma_c, \tag{4.43}$$

$$u_{,x}\cdot n=q \quad \text{on } \Gamma_q, \tag{4.44}$$

$$AEg^2u_{,xx}=R \quad \text{on } \Gamma_R, \tag{4.45}$$

$$AE(u-g^2u_{,xx})_{,x}\cdot n=P \quad \text{on } \Gamma_P. \tag{4.46}$$

We note that (4.38) – (4.41) ensure the continuity (see Lemma (4.2.1.1)) of the (axial) displacement, of the displacement gradient, of the double force and of the (axial) force across interior boundaries. We also notice that (4.42) denotes the enforcement of the governing partial differential equation on element interiors and (6.47) – (4.46) account for the enforcement of the boundary condtions.

Wherefore, we conclude that any solution $u \in H^4(\Omega)$ to the boundary value problem (4.9) – (4.10) is a weak discontinuous solution of (4.25). □

An immediate consequence of consistency is the Galerkin orthogonality property

$$B_{sb}(u - u_{wkDG}, w) = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)), \tag{4.47}$$

where $u \in H^4(\Omega)$ is a strong solution to the boundary value problem (4.9) – (4.10) and $u_{wkDG} \in bSs$ is a solution to the broken weak formulation.

For the sake of simplicity, we shall suppose in what follows that the solution u to the boundary value problem (4.9) – (4.10) is sufficiently smooth, that is $u \in H^4(\Omega)$, and for that reason, the broken weak formulation (4.25) of the boundary value problem admits a (unique) solution.

4.3 Finite Element Spaces

In this section, we will consider the finite-dimensional subspace of the broken Sobolev space $H^4(\Omega, \mathcal{P}(\Omega))$ which is used in the finite element approximation of the problem.

Moreover, let k be a positive integer. We can now define

$$\mathcal{V}^h = \left\{ v^h \in L^2(\Omega) |\; v^h \in P_k(\Omega_e) \; \forall \Omega_e \in \mathcal{P}(\Omega) \right\} \tag{4.48}$$

as the finite-dimensional space (for the h-version). We denote by $P_k(\Omega_e)$ the finite-dimensional space of all polynomials of degree less than or equal to k defined on Ω_e. Then, to each $\Omega_e \in \mathcal{P}(\Omega)$ we assign a non-negative integer p_e (the local polynomial index). We also remind that $h_e = diam(\Omega_e)$ is the element characteristic length. We will also refer to the functions in $\mathcal{V}^h$ as test functions, being discontinuous across interior boundaries of the mesh.

For the hp-version DGFEMs, we denote the finite-dimensional space by $\mathcal{V}^{hp}$.

4.4 DG Finite Element Method

We are ready to present the numerical method whose analysis we shall investigate in this chapter. Making use of the weak formulation derived in Section 4.2 and the finite element spaces constructed in the previous section, we state the discontinuous Galerkin finite element method for the problem (4.9) – (4.10):

$$\text{Find } u_{DG} \in \mathcal{V}^{hp} \text{ such that } \; B_{sb}(u_{DG}, w) = L_{sb}(w) \quad \forall w \in \mathcal{V}^{hp}, \tag{4.49}$$

where the functions α, α_c, β, β_q, δ, δ_c contained in $B_{sb}(\cdot,\cdot)$ and $L_{sb}(\cdot)$, will be defined in the coercivity property. We shall allude to the discontinuous Galerkin finite element method with $\theta = -1$ as the symmetric interior penalty Galerkin (SIPG), whereas for $\theta = 1$ the discontinuous Galerkin finite element method will be referred to as the non-symmetric interior penalty Galerkin (NIPG).

One can see from the definition of the bilinear form, (4.23), that the DG method has non-local character. In addition, to element contributions we encounter terms on interior boundaries to the two elements adjacent to the respective interfaces.

The approximation $u_{DG} \in \mathcal{V}^{hp}$ to the solution will be generally discontinuous, since there is no continuity requirement in the finite element space.

What's more, we shall suppose throughout that the strong solution u to the boundary value problem satisfies the smoothness assumption $u \in H^4(\Omega)$, so as to ensure that u is a solution to (4.25) and ergo to (4.49). Consequently, the Galerkin orthogonality property

$$B_{sb}(u - u_{DG}, w) = 0 \quad \forall w \in \mathcal{V}^{hp}, \tag{4.50}$$

where u is the analytical solution of the problem and u_{DG} is the discontinuous Galerkin approximation to u defined by the method (4.49). Sufficient conditions for ensuring Galerkin orthogonality are: $u \in H^4(\Omega, \mathcal{P}(\Omega))$ and that u, $u_{,x} \cdot n$, $g^2 u_{,xx}$, $(g^2 u_{,xx} - u)_{,x} \cdot n$ are continuous across the element interfaces Γ_i. Note that the continuity of u, $u_{,x} \cdot n$, $g^2 u_{,xx}$, $(g^2 u_{,xx} - u)_{,x} \cdot n$ in Ω is immediate if u is the weak solution of the problem with $f \in L^2(\Omega)$. Thus, no additional assumptions are posed for the Galerkin orthogonality to hold, because these are already subsumed in the definition of the space bSs.

Clearly, the number of degrees of freedom of $\mathcal{V}^{hp}$ is greater than that of the corresponding finite element space for a conforming *hp*-FEM, as continuity is imposed weakly by the method and not through the choice of shared inter-element degrees of freedom as in a continuous finite element space. Moreover, since typically all basis functions used in DGFEM have non zero-trace on the element interfaces, no static condensation of degrees of freedom can be performed to reduce degrees of freedom.

On the other hand, the weak imposition of inter-element continuity may give rise to sparser linear systems, being easier to solve. Furthermore, DG-FEMs allow greater flexibility in the choice of polynomial degree p on every element. Indeed, as no continuity requirements are imposed across the element interfaces, in practice polynomial degree may vary almost arbitrarily across adjacent elements (cf. also the bounded local variation condition in Remark A.3.5). Thereby *hp*-DGFEM is a very attractive contender in the context of *hp*-adaptivity [15, 188, 130, 112, 190, 114]. Considering also that *hp*-adaptation is superior to *h*-adaptive mesh refinement techniques, particularly when the approximating solutions admit high local regularity, DGFEM offers a very suitable framework for adaptivity.

It is well known that in problems where steep gradients are present in the analytical solution (for instance, the presence of boundary or interior layers, etc.), standard conforming FEMs produce oscillatory approximations, when the degrees of freedom are insufficient to resolve the rapid variation in the solution. In such instances stabilization methods (stramline-diffusion stabilization, bubble stabilization) are often employed to counteract the undesirable oscillatory effects. However, it appears that such stabilizations are unnecessary for the *hp*-DG method [130], as numerical dissipation introduced by the discontinuities in the numerical solution stabilises the numerical solution and reduces the oscillations. This fact was indicated theoretically in [188, 130] as it was shown therein that it is not necessary to include streamline-diffusion stabilization terms to prove meaningful error bounds.

4.4.1 Coercivity of Bilinear Form

Stability 4.4.1.1. *A method is stable when its bilinear form induces a norm which can be bounded from below.*

The choice $\theta = 1$ gives rise to the non-symmetric interior penalty Galerkin (NIPG) formulation, analogous to the one that was considered by Rivière et al. [176, 177, 179] and by Houston et al. [188, 130] for second-order elliptic equations. It is straightforward to show that the corresponding bilinear form is coercive.

Proposition 4.4.1.2. *Let $\theta = 1$, $\alpha, \alpha_c > 0$, $\beta, \beta_q > 0$, $\delta, \delta_c > 0$, then the h-version NIPG method (4.49) has a unique solution $u^h \in \mathcal{V}^h$.*

Proof. As it is easy to see from the bilinear form (4.23), by substituting u^h

for w^h and for $\theta = 1$, we have

$$\begin{aligned} B_{sb}(u^h, u^h) &= (g^2 u^h_{,xx}, u^h_{,xx})_{\tilde{\Omega}} + (u^h_{,x}, u^h_{,x})_{\tilde{\Omega}} \\ &\quad + \alpha [\![u^h]\!] [\![u^h]\!]_{\tilde{\Gamma}} + \beta [\![u^h_{,x}]\!] [\![u^h_{,x}]\!]_{\tilde{\Gamma}} + \delta [\![u^h]\!] [\![u^h]\!]_{\tilde{\Gamma}} \\ &\quad + \alpha_c u^h u^h|_{\Gamma_c} + \beta_q u^h_{,x} \cdot n u^h_{,x} \cdot n|_{\Gamma_q} + \delta_c u^h u^h|_{\Gamma_c} \\ &= ||(g^2)^{1/2} u^h_{,xx}||^2_{\tilde{\Omega}} + ||u^h_{,x}||^2_{\tilde{\Omega}} \\ &\quad + ||\alpha^{1/2} [\![u^h]\!]||^2_{\tilde{\Gamma}} + ||\beta^{1/2} [\![u^h_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\delta^{1/2} [\![u^h]\!]||^2_{\tilde{\Gamma}} \\ &\quad + ||\alpha_c^{1/2} u^h||^2_{\Gamma_c} + ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_q} + ||\delta_c^{1/2} u^h||^2_{\Gamma_c} \\ &\equiv |||u^h|||^2_{sb} \quad \forall u^h \in \mathcal{V}^h. \end{aligned} \tag{4.51}$$

We showed earlier that $|||\cdot|||_{sb}$ is a seminorm on the space $H^4(\Omega, \mathcal{P}(\Omega))$, thereby, since $\mathcal{V}^h \subset H^4(\Omega, \mathcal{P}(\Omega))$, we get that $|||\cdot|||_{sb}$ is also a seminorm on $\mathcal{V}^h$.

Therefore, $B_{sb}(\cdot,\cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{V}^h$, and hence the problem (4.49) has a unique solution in this space. □

Proposition 4.4.1.3. *Let $\theta = 1$, $\alpha, \alpha_c > 0$, $\beta, \beta_q > 0$, $\delta, \delta_c > 0$, then the hp-version NIPG method has a unique solution $u_{DG} \in \mathcal{V}^{hp}$.*

Proof. We use the same arguments to prove the stability of $hp-$version NIPG as in the $h-$version. So, it derives

$$B_{sb}(u, u) = |||u|||^2_{sb} \quad \forall u \in \mathcal{V}^{hp}. \tag{4.52}$$

We showed earlier that $|||\cdot|||_{sb}$ is a seminorm on the space $H^4(\Omega, \mathcal{P}(\Omega))$, thus, since $\mathcal{V}^{hp} \subset H^4(\Omega, \mathcal{P}(\Omega))$, we get that $|||\cdot|||_{sb}$ is also a seminorm on $\mathcal{V}^{hp}$.

Wherefore, $B_{sb}(\cdot,\cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{V}^{hp}$, and as a result the problem (4.49) has a unique solution in this space. □

Setting $\theta = -1$ yields the symmetric interior penalty Galerkin (SIPG) formulation with a symmetric bilinear form. Unfortunately, this bilinear form is non-coercive unless the stabilization parameters are chosen sufficiently large. This formulation was introduced by Arnold [4] and Wheeler [202] for second-order elliptic equations.

Let us now prove that the bilinear form $B_{sb}(\cdot,\cdot)$ of the SIPG method is coercive, and in consequence the problem (4.49) will have a unique solution.

Proposition 4.4.1.4. *The h-version SIPG method* (4.49) *is stable in the energy seminorm* (4.26)*, that is, there exists a positive constant m such that*

$$B_{sb}(u^h, u^h) \geqslant m|||u^h|||_{sb}^2 \quad \forall u^h \in \mathcal{V}^h. \tag{4.53}$$

Proof. Substituting u^h for w^h in (4.23) and for $\theta = -1$, we obtain

$$\begin{aligned}
B_{sb}(u^h, u^h) \geqslant\ & ||(g^2)^{1/2} u^h_{,xx}||^2_{\tilde{\Omega}} + ||u^h_{,x}||^2_{\tilde{\Omega}} \\
& + 2\Big(\langle (g^2 u^h_{,xx})_{,x}\rangle [\![u^h]\!]_{\tilde{\Gamma}} + (g^2 u^h_{,xx})_{,x} \cdot n u^h|_{\Gamma_c}\Big) \\
& - 2\Big(\big|\langle g^2 u^h_{,xx}\rangle [\![u^h_{,x}]\!]_{\tilde{\Gamma}}\big| + \big|g^2 u^h_{,xx} u^h_{,x} \cdot n|_{\Gamma_q}\big|\Big) \\
& - 2\Big(\big|\langle u^h_{,x}\rangle [\![u^h]\!]_{\tilde{\Gamma}}\big| + \big|u^h_{,x} \cdot n u^h|_{\Gamma_c}\big|\Big) \\
& + ||\alpha^{1/2} [\![u^h]\!]||^2_{\tilde{\Gamma}} + ||\beta^{1/2} [\![u^h_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\delta^{1/2} [\![u^h]\!]||^2_{\tilde{\Gamma}} \\
& + ||\alpha_c^{1/2} u^h||^2_{\Gamma_c} + ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_q} + ||\delta_c^{1/2} u^h||^2_{\Gamma_c}.
\end{aligned} \tag{4.54}$$

To bound the bilinear form, we have employed the triangle inequality. Thus, to complete the proof, it only remains to estimate each of the terms appearing into the parentheses on the right-hand side of (4.54).

So we can write the terms into the first parenthesis by using the Cauchy-Schwarz inequality (A.12), as well as the Young inequality (A.17)

$$\begin{aligned}
& \langle (g^2 u^h_{,xx})_{,x}\rangle [\![u^h]\!]_{\tilde{\Gamma}} + (g^2 u^h_{,xx})_{,x} \cdot n u^h|_{\Gamma_c} \\
& \leqslant ||\langle (g^2 u^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}} ||[\![u^h]\!]||_{\tilde{\Gamma}} + ||(g^2 u^h_{,xx})_{,x}||_{\Gamma_c} ||u^h||_{\Gamma_c} \\
& \leqslant \left(\frac{\varepsilon_1}{2} ||\langle (g^2 u^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{2\varepsilon_1} ||[\![u^h]\!]||^2_{\tilde{\Gamma}}\right) + \left(\frac{\varepsilon_1}{2} ||(g^2 u^h_{,xx})_{,x}||^2_{\Gamma_c} + \frac{1}{2\varepsilon_1} ||u^h||^2_{\Gamma_c}\right) \\
& = \sum_{i=1}^{N_i} \left(\frac{\varepsilon_1}{2} ||\langle (g^2 u^h_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon_1} ||[\![u^h]\!]||^2_{\Gamma_i}\right) \\
& + \sum_{r=1}^{N_c} \left(\frac{\varepsilon_1}{2} ||(g^2 u^h_{,xx})_{,x}||^2_{\Gamma_r} + \frac{1}{2\varepsilon_1} ||u^h||^2_{\Gamma_r}\right),
\end{aligned} \tag{4.55}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

The above terms can be bounded by invoking the mean value inequality

(A.19) in (4.55), then we deduce

$$\begin{aligned}
&\sum_{i=1}^{N_i}\left(\frac{\varepsilon_1}{2}||\langle(g^2u^h_{,xx})_{,x}\rangle||^2_{\Gamma_i}+\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}\right)\\
&+\sum_{r=1}^{N_c}\left(\frac{\varepsilon_1}{2}||(g^2u^h_{,xx})_{,x}||^2_{\Gamma_r}+\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}\right)\\
&\leqslant\sum_{i=1}^{N_i}\left(\frac{\varepsilon_1}{2}\left(||(g^2u^{h+}_{,xx})_{,x}||^2_{\Gamma_i}+||(g^2u^{h-}_{,xx})_{,x}||^2_{\Gamma_i}\right)+\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}\right)\\
&+\sum_{r=1}^{N_c}\left(\frac{\varepsilon_1}{2}||(g^2u^h_{,xx})_{,x}||^2_{\Gamma_r}+\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}\right) \qquad (4.56)\\
&=\sum_{i=1}^{N_i}\frac{\varepsilon_1}{2}\left(||(g^2u^{h+}_{,xx})_{,x}||^2_{\Gamma_i}+||(g^2u^{h-}_{,xx})_{,x}||^2_{\Gamma_i}\right)+\sum_{r=1}^{N_c}\frac{\varepsilon_1}{2}||(g^2u^h_{,xx})_{,x}||^2_{\Gamma_r}\\
&+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}||(g^2u^h_{,xx})_{,x}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}.
\end{aligned}$$

Applying the trace inequality (A.39) and next the properties of Sobolev norms in (4.56), we conclude that

$$\begin{aligned}
&\sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}||(g^2u^h_{,xx})_{,x}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}C\left(h_e^{-1}|g^2u^h_{,xx}|^2_{1,\Omega_e}+h_e|g^2u^h_{,xx}|^2_{2,\Omega_e}\right)\\
&+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r} \qquad (4.57)\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}C\left(h_e^{-1}||g^2u^h_{,xx}||^2_{1,\Omega_e}+h_e||g^2u^h_{,xx}||^2_{2,\Omega_e}\right)\\
&+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u^h||^2_{\Gamma_r}.
\end{aligned}$$

Hence, making use of inverse estimate (A.37), (4.57) gives

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} ||g^2 u^h_{,xx}||^2_{1,\Omega_e} + h_e ||g^2 u^h_{,xx}||^2_{2,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1} ||u^h||^2_{\Gamma_r} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} C_I^2 h_e^{-2} ||g^2 u^h_{,xx}||^2_{\Omega_e} + h_e C_{II}^2 h_e^{-4} ||g^2 u^h_{,xx}||^2_{\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1} ||u^h||^2_{\Gamma_r} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C_1 h_e^{-3} ||g^2 u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1} ||u^h||^2_{\Gamma_r} \\
&= \sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 g^2}{2h_e^3} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1 \alpha} ||\alpha^{1/2} [\![u^h]\!]||^2_{\Gamma_i} \\
&+ \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1 \alpha_c} ||\alpha_c^{1/2} u^h||^2_{\Gamma_r},
\end{aligned}
\tag{4.58}
$$

where $C_1 = C \max\{C_I^2, C_{II}^2\}$. We denote by C_I, C_{II} the constants resulting from an inverse estinate.

Therefore, from (4.55) – (4.58), we reach the conclusion that the terms into the first bracket, on the right-hand side of (4.54), can be bounded as follows

$$
\begin{aligned}
&\langle (g^2 u^h_{,xx})_{,x} \rangle [\![u^h]\!]_{\tilde{\Gamma}} + (g^2 u^h_{,xx})_{,x} \cdot n u^h|_{\Gamma_c} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 g^2}{2h_e^3} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1 \alpha} ||\alpha^{1/2} [\![u^h]\!]||^2_{\Gamma_i} \\
&+ \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1 \alpha_c} ||\alpha_c^{1/2} u^h||^2_{\Gamma_r}.
\end{aligned}
\tag{4.59}
$$

In addition, we shall analogously estimate the terms enclosed into the second parenthesis on the right-hand side of (4.54). By applying the Cauchy-Schwarz inequality (A.12) and afterwards the Young inequality (A.17), we obtain

$$\begin{aligned}
&\left|\langle g^2 u^h_{,xx}\rangle [\![u^h_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|g^2 u^h_{,xx} u^h_{,x} \cdot n|_{\Gamma_q}\right| \\
&\leqslant ||\langle g^2 u^h_{,xx}\rangle||_{\tilde{\Gamma}} ||[\![u^h_{,x}]\!]||_{\tilde{\Gamma}} + ||g^2 u^h_{,xx}||_{\Gamma_q} ||u^h_{,x}||_{\Gamma_q} \\
&\leqslant \left(\frac{\varepsilon_2}{2}||\langle g^2 u^h_{,xx}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\tilde{\Gamma}}\right) + \left(\frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\Gamma_q} + \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_q}\right) \\
&= \sum_{i=1}^{N_i} \left(\frac{\varepsilon_2}{2}||\langle g^2 u^h_{,xx}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i}\right) \\
&+ \sum_{j=1}^{N_q} \left(\frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j}\right),
\end{aligned} \tag{4.60}$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.
The above terms can be bounded by using the mean value inequality (A.19) in (4.60), then we arrive at

$$\begin{aligned}
&\sum_{i=1}^{N_i} \left(\frac{\varepsilon_2}{2}||\langle g^2 u^h_{,xx}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} \left(\frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j}\right) \\
&\leqslant \sum_{i=1}^{N_i} \left(\frac{\varepsilon_2}{2}\left(||g^2 u^{h+}_{,xx}||^2_{\Gamma_i} + ||g^2 u^{h-}_{,xx}||^2_{\Gamma_i}\right) + \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i}\right) \\
&+ \sum_{j=1}^{N_q} \left(\frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j}\right) \\
&= \sum_{i=1}^{N_i} \frac{\varepsilon_2}{2}\left(||g^2 u^{h+}_{,xx}||^2_{\Gamma_i} + ||g^2 u^{h-}_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} \frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\Gamma_j} \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j}.
\end{aligned} \tag{4.61}$$

Recalling the trace inequality (A.38), followed by the properties of Sobolev norms in (4.61) , we deduce

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2}||g^2 u^h_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} C \left(h_e^{-1}||g^2 u^h_{,xx}||^2_{\Omega_e} + h_e||g^2 (u^h_{,xx})_{,x}||^2_{\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j} \qquad (4.62) \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} C \left(h_e^{-1}||g^2 u^h_{,xx}||^2_{\Omega_e} + h_e||g^2 u^h_{,xx}||^2_{1,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j}.
\end{aligned}
$$

As a consequence, making use of inverse estimate (A.37) in (4.62), we have

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} C \left(h_e^{-1}||g^2 u^h_{,xx}||^2_{\Omega_e} + h_e||g^2 u^h_{,xx}||^2_{1,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} C \left(h_e^{-1}||g^2 u^h_{,xx}||^2_{\Omega_e} + h_e C_I^2 h_e^{-2}||g^2 u^h_{,xx}||^2_{\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h||^2_{\Gamma_j} \qquad (4.63) \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} C_2 h_e^{-1}||g^2 u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u^h_{,x}||^2_{\Gamma_j} \\
&= \sum_{e=1}^{N_{el}} \frac{\varepsilon_2 C_2 g^2}{2h_e}||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2 \beta}||\beta^{1/2}[\![u^h_{,x}]\!]||^2_{\Gamma_i} \\
&+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2 \alpha_q}||\alpha_q^{1/2} u^h_{,x}||^2_{\Gamma_j},
\end{aligned}
$$

where $C_2 = C\max\{1, C_I^2\}$. We denote by C_I the constant resulting from an inverse estimate.

Ergo, from (4.60) – (4.63), we arrive to the conclusion that the terms into the second bracket, on the right-hand side of (4.54), can be estimated as follows

$$\begin{aligned} &\left|\langle g^2 u^h_{,xx}\rangle [\![u^h_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|g^2 u^h_{,xx} u^h_{,x}\cdot n|_{\Gamma_q}\right| \\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2 C_2 g^2}{2h_e} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2\beta} ||\beta^{1/2}[\![u^h_{,x}]\!]||^2_{\Gamma_i} \\ &+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2\beta_q} ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_j}. \end{aligned} \tag{4.64}$$

What is more, we shall make use of similar arguments to estimate the remaining terms, enclosed into the third parenthesis on the right-hand side of (4.54). By employing the Cauchy-Schwarz inequality (A.12) and the Young inequality (A.17) as well, we have

$$\begin{aligned} &\left|\langle u^h_{,x}\rangle [\![u^h]\!]_{\tilde{\Gamma}}\right| + \left|u^h_{,x}\cdot n u^h|_{\Gamma_c}\right| \\ &\leqslant ||\langle u^h_{,x}\rangle||_{\tilde{\Gamma}} ||[\![u^h]\!]||_{\tilde{\Gamma}} + ||u^h_{,x}||_{\Gamma_c} ||u^h||_{\Gamma_c} \\ &\leqslant \left(\frac{\varepsilon_3}{2}||\langle u^h_{,x}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\tilde{\Gamma}}\right) + \left(\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\Gamma_c} + \frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_c}\right) \\ &= \sum_{i=1}^{N_i}\left(\frac{\varepsilon_3}{2}||\langle u^h_{,x}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c}\left(\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\Gamma_r} + \frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\right), \end{aligned} \tag{4.65}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

The above terms can be bounded by recalling the mean value inequality

(A.19) in (4.65), thus we obtain

$$\begin{aligned}
&\sum_{i=1}^{N_i}\left(\frac{\varepsilon_3}{2}||\langle u^h_{,x}\rangle||^2_{\Gamma_i}+\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}\right)+\sum_{r=1}^{N_c}\left(\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\Gamma_r}+\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\right)\\
&\leqslant\sum_{i=1}^{N_i}\left(\frac{\varepsilon_3}{2}\left(||u^{h+}_{,x}||^2_{\Gamma_i}+||u^{h-}_{,x}||^2_{\Gamma_i}\right)+\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}\right)\\
&+\sum_{r=1}^{N_c}\left(\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\Gamma_r}+\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\right)\\
&=\sum_{i=1}^{N_i}\frac{\varepsilon_3}{2}\left(||u^{h+}_{,x}||^2_{\Gamma_i}+||u^{h-}_{,x}||^2_{\Gamma_i}\right)+\sum_{r=1}^{N_c}\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\Gamma_r}\\
&+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}.
\end{aligned}\tag{4.66}$$

Applying the trace inequality (A.38) and afterwards the properties of Sobolev norms in (4.66), we conclude that

$$\begin{aligned}
&\sum_{e=1}^{N_{el}}\frac{\varepsilon_3}{2}||u^h_{,x}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_3}{2}C\left(h_e^{-1}||u^h_{,x}||^2_{\Omega_e}+h_e||u^h_{,xx}||^2_{\Omega_e}\right)+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_3}{2}C\left(h_e^{-1}||u^h_{,x}||^2_{\Omega_e}+h_e||u^h_{,x}||^2_{1,\Omega_e}\right)+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u^h]\!]||^2_{\Gamma_i}+\sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u^h||^2_{\Gamma_r}.
\end{aligned}\tag{4.67}$$

By making use of inverse estimate (A.37) in (4.67), we consequently deduce

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_3}{2} C \left(h_e^{-1} ||u_{,x}^h||_{\Omega_e}^2 + h_e ||u_{,x}^h||_{1,\Omega_e}^2 \right) + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3} ||[\![u^h]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_3} ||u^h||_{\Gamma_r}^2 \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3}{2} C \left(h_e^{-1} ||u_{,x}^h||_{\Omega_e}^2 + h_e C_I^2 h_e^{-2} ||u_{,x}^h||_{\Omega_e}^2 \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3} ||[\![u^h]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_3} ||u^h||_{\Gamma_r}^2 \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3}{2} C_3 h_e^{-1} ||u_{,x}^h||_{\Omega_e}^2 + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3} ||[\![u^h]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_3} ||u^h||_{\Gamma_r}^2 \\
&= \sum_{e=1}^{N_{el}} \frac{\varepsilon_3 C_3}{2h_e} ||u_{,x}^h||_{\Omega_e}^2 + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3 \delta} ||\delta^{1/2} [\![u^h]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1 \delta_c} ||\delta_c^{1/2} u^h||_{\Gamma_r}^2,
\end{aligned} \tag{4.68}$$

where $C_3 = C \max\{1, C_I^2\}$. We denote by C_I the constant resulting from an inverse estinate.

Wherefore, from (4.65) – (4.68), we reach the conclusion that the terms into the third bracket, on the right-hand side of (4.54), can be bounded as follows

$$\begin{aligned}
&\left| \langle u_{,x}^h \rangle [\![u^h]\!]_{\tilde{\Gamma}} \right| + \left| u_{,x}^h \cdot n u^h |_{\Gamma_c} \right| \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3 C_3}{2h_e} ||u_{,x}^h||_{\Omega_e}^2 + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3 \delta} ||\delta^{1/2} [\![u^h]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_3 \delta_c} ||\delta_c^{1/2} u^h||_{\Gamma_r}^2 .
\end{aligned} \tag{4.69}$$

Further, inserting the inequalities (4.59), (4.64), as well as (4.69) on the right-hand side of (4.54), we have

$$
\begin{aligned}
B_{sb}(u^h, u^h) \geqslant\ & \sum_{e=1}^{N_{el}} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} + \sum_{e=1}^{N_{el}} ||u^h_{,x}||^2_{\Omega_e} \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 g^2}{h_e^3} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_1 \alpha} ||\alpha^{1/2} [\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{\varepsilon_1 \alpha_c} ||\alpha_c^{1/2} u^h||^2_{\Gamma_r} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_2 C_2 g^2}{h_e} ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_2 \beta} ||\beta^{1/2} [\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_2 \beta_q} ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_j} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_3 C_3}{h_e} ||u^h_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_3 \delta} ||\delta^{1/2} [\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{\varepsilon_3 \delta_c} ||\delta_c^{1/2} u^h||^2_{\Gamma_r} \Big) \\
& + \sum_{i=1}^{N_i} ||\alpha^{1/2} [\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\alpha_c^{1/2} u^h||^2_{\Gamma_r} \\
& + \sum_{i=1}^{N_i} ||\beta^{1/2} [\![u^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\delta^{1/2} [\![u^h]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\delta_c^{1/2} u^h||^2_{\Gamma_r}. \qquad (4.70)
\end{aligned}
$$

Also, with the aid of factorization on the right-hand side of (4.70), it follows that

$$\begin{aligned} B_{sb}(u^h, u^h) \geqslant & \sum_{e=1}^{N_{el}} \left(1 - \frac{\varepsilon_1 C_1 g^2}{h_e^3} - \frac{\varepsilon_2 C_2 g^2}{h_e}\right) ||(g^2)^{1/2} u^h_{,xx}||^2_{\Omega_e} \\ & + \sum_{e=1}^{N_{el}} \left(1 - \frac{\varepsilon_3 C_3}{h_e}\right) ||u^h_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_1 \alpha}\right) ||\alpha^{1/2} [\![u^h]\!]||^2_{\Gamma_i} \\ & + \sum_{r=1}^{N_c} \left(1 - \frac{1}{\varepsilon_1 \alpha_c}\right) ||\alpha_c^{1/2} u^h||^2_{\Gamma_r} + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_2 \beta}\right) ||\beta^{1/2} [\![u^h_{,x}]\!]||^2_{\Gamma_i} \\ & + \sum_{j=1}^{N_q} \left(1 - \frac{1}{\varepsilon_2 \beta_q}\right) ||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_j} + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_3 \delta}\right) ||\delta^{1/2} [\![u^h]\!]||^2_{\Gamma_i} \\ & + \sum_{r=1}^{N_c} \left(1 - \frac{1}{\varepsilon_3 \delta_c}\right) ||\delta_c^{1/2} u^h||^2_{\Gamma_r}. \end{aligned} \tag{4.71}$$

Then, by the use of definition of energy seminorm, (4.26), on the right-hand side of (4.71), we arrive at

$$B_{sb}(u^h, u^h) \geqslant m|||u^h|||^2_{sb},$$

which is the desired result. We denote by the constant m the minimum of the terms enclosed into the parentheses on the right-hand side of (4.71).

In particular, assuming that $\alpha = \alpha_c$, $\beta = \beta_q$ as well as $\delta = \delta_c$, we can prove (4.53) for $m = \frac{1}{2}$ if we choose

$$\varepsilon_1|_{\Omega_e} = \frac{h_e^3}{4C_1 g^2}, \quad \varepsilon_2|_{\Omega_e} = \frac{h_e}{4C_2 g^2} \quad \text{and} \quad \varepsilon_3|_{\Omega_e} = \frac{h_e}{2C_3},$$

in which case we obtain

$$\alpha = \alpha_c = \frac{8C_1 g^2}{h_e^3}, \quad \beta = \beta_q = \frac{8C_2 g^2}{h_e} \quad \text{and} \quad \delta = \delta_c = \frac{4C_3}{h_e},$$

as well. □

Let us now examine the coercivity of the bilinear form, $B_{sb}(\cdot, \cdot)$, for the *hp*-version SIPG finite element method.

Proposition 4.4.1.5. *The hp-version SIPG method* (4.49) *is stable in the energy seminorm* (4.26)*, that is, there exists a positive constant m such that*

$$B_{sb}(u,u) \geqslant m|||u|||^2_{sb} \quad \forall u \in \mathcal{V}^{hp}. \tag{4.72}$$

Proof. Similar to the series of steps of the previous proof, substituting u for w in in (4.23), for $\theta = -1$, and applying the triangle inequality, we obtain

$$\begin{aligned} B_{sb}(u,u) \geqslant\ & ||(g^2)^{1/2}u_{,xx}||^2_{\tilde{\Omega}} + ||u_{,x}||^2_{\tilde{\Omega}} \\ & + 2\Big(\langle (g^2 u_{,xx})_{,x}\rangle [\![u]\!]_{\tilde{\Gamma}} + (g^2 u_{,xx})_{,x} \cdot nu|_{\Gamma_c}\Big) \\ & - 2\Big(\big|\langle g^2 u_{,xx}\rangle [\![u_{,x}]\!]_{\tilde{\Gamma}}\big| + \big|g^2 u_{,xx} u_{,x} \cdot n|_{\Gamma_q}\big|\Big) \\ & - 2\Big(\big|\langle u_{,x}\rangle [\![u]\!]_{\tilde{\Gamma}}\big| + \big|u_{,x} \cdot nu|_{\Gamma_c}\big|\Big) \\ & + ||\alpha^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + ||\beta^{1/2}[\![u_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\delta^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} \\ & + ||\alpha_c^{1/2}u||^2_{\Gamma_c} + ||\beta_q^{1/2}u_{,x}||^2_{\Gamma_q} + ||\delta_c^{1/2}u||^2_{\Gamma_c}. \end{aligned} \tag{4.73}$$

To complete the proof, it only remains to estimate the terms enclosed into the parentheses on the right-hand side of (4.73).

As in h-version, we can write the terms into the first parenthesis, by using the Cauchy-Schwarz inequality (A.12), the Young inequality (A.17) as well as the mean value inequality (A.19)

$$\begin{aligned} & \langle (g^2 u_{,xx})_{,x}\rangle [\![u]\!]_{\tilde{\Gamma}} + (g^2 u_{,xx})_{,x} \cdot nu|_{\Gamma_c} \\ & \leqslant \sum_{i=1}^{N_i} \frac{\varepsilon_1}{2}\left(||(g^2 u^+_{,xx})_{,x}||^2_{\Gamma_i} + ||(g^2 u^-_{,xx})_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c} \frac{\varepsilon_1}{2}||(g^2 u_{,xx})_{,x}||^2_{\Gamma_r} \\ & + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1}||u||^2_{\Gamma_r} \\ & \leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e \subset \Omega)}^{N_{el}} \frac{\varepsilon_1}{2}\Big(||(g^2 u_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||(g^2 u_{,xx})_{,x}||^2_{\partial\Omega_e}\Big) \\ & + \sum_{e=1:(\partial\Omega_e \cap \Gamma_c):(\partial\Omega_e \subset \Gamma)}^{N_{el}} \frac{\varepsilon_1}{2}||(g^2 u_{,xx})_{,x}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_1}||u||^2_{\Gamma_r}, \end{aligned} \tag{4.74}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

The terms into the first two sums can be bounded by invoking the inverse inequality (A.21) in (4.74), then we deduce

$$\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_1}{2}\left(||(g^2u_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_1}{2}||(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u||^2_{\Gamma_r}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_1}{2}\left(c_1\frac{p^6_{e'}}{h^3_{e'}}||g^2u_{,xx}||^2_{\Omega_{e'}} + c_1\frac{p^6_e}{h^3_e}||g^2u_{,xx}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_1}{2}c_1\frac{p^6_e}{h^3_e}||g^2u_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u||^2_{\Gamma_r}\\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2}c_1\frac{p^6_e}{h^3_e}||g^2u_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1}||u||^2_{\Gamma_r}\\
&= \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2}c_1 g^2\frac{p^6_e}{h^3_e}||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1\alpha}||\alpha^{1/2}[\![u]\!]||^2_{\Gamma_i}\\
&+ \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1\alpha_c}||\alpha_c^{1/2}u||^2_{\Gamma_r},
\end{aligned} \tag{4.75}$$

where the constant c_1 is independent of h_e, p_e and u.

Therefore, from (4.74) – (4.75), we reach the conclusion that the terms, enclosed into the first bracket on the right-hand side of (4.73), can be estimated as follows

$$\begin{aligned}
&\langle(g^2u_{,xx})_{,x}\rangle[\![u]\!]_{\tilde\Gamma} + (g^2u_{,xx})_{,x}\cdot nu|_{\Gamma_c}\\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2}c_1 g^2\frac{p^6_e}{h^3_e}||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1\alpha}||\alpha^{1/2}[\![u]\!]||^2_{\Gamma_i}\\
&+ \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_1\alpha_c}||\alpha_c^{1/2}u||^2_{\Gamma_r}.
\end{aligned} \tag{4.76}$$

Moreover, we shall analogously estimate the terms appearing into the second parenthesis on the right-hand side of (4.73). By applying the Cauchy-Schwarz inequality (A.12), the Young inequality (A.17) and afterwards the mean value inequality (A.19), we conclude that

$$\begin{aligned}
&\left|\langle g^2 u_{,xx}\rangle [\![u_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|g^2 u_{,xx} u_{,x}\cdot n|_{\Gamma_q}\right| \\
&\leqslant \sum_{i=1}^{N_i} \frac{\varepsilon_2}{2}\left(||g^2 u^+_{,xx}||^2_{\Gamma_i} + ||g^2 u^-_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} \frac{\varepsilon_2}{2}||g^2 u_{,xx}||^2_{\Gamma_j} \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_2}{2}\left(||g^2 u_{,xx}||^2_{\partial\Omega_{e'}} + ||g^2 u_{,xx}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_2}{2}||g^2 u_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2}||[\![u_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_2}||u_{,x}||^2_{\Gamma_j},
\end{aligned} \tag{4.77}$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.

Recalling the inverse inequality (A.20) in (4.77), we arrive at

$$\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_2}{2}\left(||g^2u_{,xx}||^2_{\partial\Omega_{e'}}+||g^2u_{,xx}||^2_{\partial\Omega_e}\right)\\
&+\sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_2}{2}||g^2u_{,xx}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_2}||[\![u_{,x}]\!]||^2_{\Gamma_i}+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_2}||u_{,x}||^2_{\Gamma_j}\\
&\leqslant\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_2}{2}\left(c_2\frac{p_{e'}^2}{h_{e'}}||g^2u_{,xx}||^2_{\Omega_{e'}}+c_2\frac{p_e^2}{h_e}||g^2u_{,xx}||^2_{\Omega_e}\right)\\
&+\sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_2}{2}c_2\frac{p_e^2}{h_e}||g^2u_{,xx}||^2_{\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_2}||[\![u_{,x}]\!]||^2_{\Gamma_i}+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_2}||u_{,x}||^2_{\Gamma_j}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_2}{2}c_2\frac{p_e^2}{h_e}||g^2u_{,xx}||^2_{\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_2}||[\![u_{,x}]\!]||^2_{\Gamma_i}+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_2}||u_{,x}||^2_{\Gamma_j}\\
&=\sum_{e=1}^{N_{el}}\frac{\varepsilon_2}{2}c_2g^2\frac{p_e^2}{h_e}||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_2\beta}||\beta^{1/2}[\![u_{,x}]\!]||^2_{\Gamma_i}\\
&+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_2\beta_q}||\beta_q^{1/2}u_{,x}||^2_{\Gamma_j},
\end{aligned} \tag{4.78}$$

where the constant c_2 is independent of h_e, p_e and u.

Ergo, from (4.77) – (4.78), we arrive to the conclusion that the terms into the second bracket, on the right-hand side of (4.73), can be estimated as follows

$$\begin{aligned}
&\left|\langle g^2u_{,xx}\rangle[\![u_{,x}]\!]_{\tilde\Gamma}\right|+\left|g^2u_{,xx}u_{,x}\cdot n|_{\Gamma_q}\right|\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_2}{2}c_2g^2\frac{p_e^2}{h_e}||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_2\beta}||\beta^{1/2}[\![u_{,x}]\!]||^2_{\Gamma_i}\\
&+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_2\beta_q}||\beta_q^{1/2}u_{,x}||^2_{\Gamma_j}.
\end{aligned} \tag{4.79}$$

Furthermore, we shall use similar arguments to estimate the rest of the terms, which enter into the third parenthesis on the right-hand side of (4.73).

By employing the Cauchy-Schwarz inequality (A.12) the Young inequality (A.17) and next the mean value inequality (A.19), we have

$$
\begin{aligned}
&|\langle u_{,x}\rangle [\![u]\!]_{\tilde{\Gamma}}| + |u_{,x}\cdot nu|_{\Gamma_c}| \\
&\leqslant \sum_{i=1}^{N_i} \frac{\varepsilon_3}{2}\left(||u^+_{,x}||^2_{\Gamma_i} + ||u^-_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c}\frac{\varepsilon_3}{2}||u_{,x}||^2_{\Gamma_r} \\
&+ \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u||^2_{\Gamma_r} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_3}{2}\left(||u_{,x}||^2_{\partial\Omega_{e'}} + ||u_{,x}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_3}{2}||u_{,x}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u||^2_{\Gamma_r},
\end{aligned}
\tag{4.80}
$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.
We invoke the inverse inequality (A.20) in (4.80), as a consequence we obtain

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_3}{2}\left(||u_{,x}||^2_{\partial\Omega_{e'}} + ||u_{,x}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_3}{2}||u_{,x}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u||^2_{\Gamma_r} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_3}{2}\left(c_3\frac{p^2_{e'}}{h_{e'}}||u_{,x}||^2_{\Omega_{e'}} + c_3\frac{p^2_e}{h_e}||u_{,x}||^2_{\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_3}{2}c_3\frac{p^2_e}{h_e}||u_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3}||[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3}||u||^2_{\Gamma_r} \\
&\leqslant \sum_{e=1}^{N_{el}}\frac{\varepsilon_3}{2}c_3\frac{p^2_e}{h_e}||u_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_3\delta}||\delta^{1/2}[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c}\frac{1}{2\varepsilon_3\delta_c}||\delta_c^{1/2}u||^2_{\Gamma_r},
\end{aligned}
\tag{4.81}
$$

where the constant c_3 is independent of h_e, p_e and u.

Wherefore, from (4.80) – (4.81), we reach the conclusion that the terms, enclosed into the third bracket on the right-hand side of (4.73), can be bounded as follows

$$
\begin{aligned}
&|\langle u_{,x}\rangle [\![u]\!]_{\tilde{\Gamma}}| + |u_{,x} \cdot n u|_{\Gamma_c}| \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3}{2} c_3 \frac{p_e^2}{h_e} ||u_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3 \delta} ||\delta^{1/2} [\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{2\varepsilon_3 \delta_c} ||\delta_c^{1/2} u||^2_{\Gamma_r}.
\end{aligned}
\tag{4.82}
$$

After those series of steps, we gather the inequalities (4.76), (4.79) as well as (4.82) and insert them into the right-hand side of (4.73). As a result, we get

$$
\begin{aligned}
B_{sb}(u,u) \quad \geqslant \quad & \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e} + \sum_{e=1}^{N_{el}} ||u_{,x}||^2_{\Omega_e} \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_1 c_1 g^2 \frac{p_e^6}{h_e^3} ||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_1 \alpha} ||\alpha^{1/2}[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{\varepsilon_1 \alpha_c} ||\alpha_c^{1/2} u||^2_{\Gamma_r} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_2 c_2 g^2 \frac{p_e^2}{h_e} ||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_2 \beta} ||\beta^{1/2}[\![u_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_2 \beta_q} ||\beta_q^{1/2} u_{,x}||^2_{\Gamma_j} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_3 c_3 \frac{p_e^2}{h_e} ||u_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_3 \delta} ||\delta^{1/2}[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \frac{1}{\varepsilon_3 \delta_c} ||\delta_c^{1/2} u||^2_{\Gamma_r} \Big) \\
& + \sum_{i=1}^{N_i} ||\alpha^{1/2}[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\alpha_c^{1/2} u||^2_{\Gamma_r} \\
& + \sum_{i=1}^{N_i} ||\beta^{1/2}[\![u_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2} u_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\delta^{1/2}[\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\delta_c^{1/2} u||^2_{\Gamma_r}. \qquad (4.83)
\end{aligned}
$$

Now, with the aid of factorization on the right-hand side of (4.83), it is clear that

$$\begin{aligned}
B_{sb}(u,u) \geqslant & \sum_{e=1}^{N_{el}} \left(1 - \varepsilon_1 c_1 g^2 \frac{p_e^6}{h_e^3} - \varepsilon_2 c_2 g^2 \frac{p_e^2}{h_e}\right) ||(g^2)^{1/2} u_{,xx}||^2_{\Omega_e} \\
& + \sum_{e=1}^{N_{el}} \left(1 - \varepsilon_3 c_3 \frac{p_e^2}{h_e}\right) ||u_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_1 \alpha}\right) ||\alpha^{1/2} [\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \left(1 - \frac{1}{\varepsilon_1 \alpha_c}\right) ||\alpha_c^{1/2} u||^2_{\Gamma_r} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_2 \beta}\right) ||\beta^{1/2} [\![u_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \left(1 - \frac{1}{\varepsilon_2 \beta_q}\right) ||\beta_q^{1/2} u_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_3 \delta}\right) ||\delta^{1/2} [\![u]\!]||^2_{\Gamma_i} + \sum_{r=1}^{N_c} \left(1 - \frac{1}{\varepsilon_3 \delta_c}\right) ||\delta_c^{1/2} u||^2_{\Gamma_r}.
\end{aligned} \tag{4.84}$$

Then, by the use of definition of energy seminorm, (4.26), on the right-hand side of (4.84), we reach to

$$B_{sb}(u,u) \geqslant m|||u|||^2_{sb},$$

which is the desired result. We denote by the constant m the minimum of the terms enclosed into the parentheses on the right-hand side of (4.84).

In particular, assuming that $\alpha = \alpha_c$, $\beta = \beta_q$ as well as $\delta = \delta_c$, we can prove (4.72) for $m = \frac{1}{2}$ if we choose

$$\varepsilon_1|_{\Omega_e} = \frac{h_e^3}{4c_1 g^2 p_e^6}, \quad \varepsilon_2|_{\Omega_e} = \frac{h_e}{4c_2 g^2 p_e^2} \quad \text{and} \quad \varepsilon_3|_{\Omega_e} = \frac{h_e}{2c_3 p_e^2},$$

in which case we obtain

$$\alpha = \alpha_c = \frac{8c_1 g^2 p_e^6}{h_e^3}, \quad \beta = \beta_q = \frac{8c_2 g^2 p_e^2}{h_e} \quad \text{and} \quad \delta = \delta_c = \frac{4c_3 p_e^2}{h_e},$$

too. □

Wherefore, $B_s(\cdot,\cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{V}^{hp}$, and ergo the problem (4.49) has a unique solution.

4.4.2 Continuity of Bilinear Form

With the definition of the energy seminorm, (4.26), we have the following continuity result for the bilinear form (4.23), based on the Cauchy-Schwarz inequalities (A.12) and (A.13).

Proposition 4.4.2.1. *Let $B_{sb}(\cdot,\cdot)$ be the bilinear form defined in (4.23) with $\theta \in \{-1,1\}$ and $\alpha, \alpha_c, \beta, \beta_q, \delta, \delta_c \geqslant 0$. Then, there exists a constant $0 < C < \infty$, such that*

$$B_{sb}(u^h, w^h) \leqslant C|||u^h|||_{sb}|||w^h|||_{sb} \quad \forall u^h, w^h \in \mathcal{V}^h \tag{4.85}$$

where C is independent of h_e, for the h-version.

Proof. We can obtain (4.85) by applying at first the triangle inequality in the bilinear form

$$\begin{aligned} B_{sb}(u^h, w^h) &\leqslant |B_{sb}(u^h, w^h)| \\ &\leqslant |(g^2 u^h_{,xx}, w^h_{,xx})_{\tilde{\Omega}}| + |(u^h_{,x}, w^h_{,x})_{\tilde{\Omega}}| \\ &\quad + |\langle (g^2 u^h_{,xx})_{,x}\rangle [\![w^h]\!]_{\tilde{\Gamma}}| + |[\![u^h]\!]\langle (g^2 w^h_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| \\ &\quad + |\langle g^2 u^h_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}}| + |[\![u^h_{,x}]\!]\langle g^2 w^h_{,xx}\rangle_{\tilde{\Gamma}}| \\ &\quad + |\langle u^h_{,x}\rangle [\![w^h]\!]_{\tilde{\Gamma}}| + |[\![u^h]\!]\langle w^h_{,x}\rangle_{\tilde{\Gamma}}| \\ &\quad + |\alpha [\![u^h]\!][\![w^h]\!]_{\tilde{\Gamma}}| + |\beta [\![u^h_{,x}]\!][\![w^h_{,x}]\!]_{\tilde{\Gamma}}| + |\delta [\![u^h]\!][\![w^h]\!]_{\tilde{\Gamma}}| \\ &\quad + |(g^2 u^h_{,xx})_{,x} \cdot n w^h|_{\Gamma_c}| + |u^h (g^2 w^h_{,xx})_{,x} \cdot n|_{\Gamma_c}| \\ &\quad + |g^2 u^h_{,xx} w^h_{,x} \cdot n|_{\Gamma_q}| + |u^h_{,x} \cdot n g^2 w^h_{,xx}|_{\Gamma_q}| \\ &\quad + |u^h_{,x} \cdot n w^h|_{\Gamma_c}| + |u^h w^h_{,x} \cdot n|_{\Gamma_c}| \\ &\quad + |\alpha_c u^h w^h|_{\Gamma_c} + |\beta_q u^h_{,x} \cdot n w^h_{,x} \cdot n|_{\Gamma_q}| + |\delta_c u^h w^h|_{\Gamma_c}| \end{aligned} \tag{4.86}$$

and then the Cauchy-Schwarz inequality (A.12) on each term of the right-hand side of mathematical expression (4.86). As a consequence, we have

$$\begin{aligned}
B_{sb}(u^h,w^h) \leqslant\ & ||(g^2)^{1/2}u^h_{,xx}||_{\tilde{\Omega}}||(g^2)^{1/2}w^h_{,xx}||_{\tilde{\Omega}} + ||u^h_{,x}||_{\tilde{\Omega}}||w^h_{,x}||_{\tilde{\Omega}} \\
& + ||\alpha^{-1/2}\langle(g^2u^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}}||\alpha^{1/2}[\![w^h]\!]||_{\tilde{\Gamma}} \\
& + ||\alpha^{1/2}[\![u^h]\!]||_{\tilde{\Gamma}}||\alpha^{-1/2}\langle(g^2w^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}} \\
& + ||\beta^{-1/2}\langle g^2u^h_{,xx}\rangle||_{\tilde{\Gamma}}||\beta^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}} \\
& + ||\beta^{1/2}[\![u^h_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{-1/2}\langle g^2w^h_{,xx}\rangle||_{\tilde{\Gamma}} \\
& + ||\delta^{-1/2}\langle u^h_{,x}\rangle||_{\tilde{\Gamma}}||\delta^{1/2}[\![w^h]\!]||_{\tilde{\Gamma}} + ||\delta^{1/2}[\![u^h]\!]||_{\tilde{\Gamma}}||\delta^{-1/2}\langle w^h_{,x}\rangle||_{\tilde{\Gamma}} \\
& + ||\alpha^{1/2}[\![u^h]\!]||_{\tilde{\Gamma}}||\alpha^{1/2}[\![w^h]\!]||_{\tilde{\Gamma}} + ||\beta^{1/2}[\![u^h_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}} \\
& + ||\delta^{1/2}[\![u^h]\!]||_{\tilde{\Gamma}}||\delta^{1/2}[\![w^h]\!]||_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||_{\Gamma_c}||\alpha_c^{1/2}w^h||_{\Gamma_c} \\
& + ||\alpha_c^{1/2}u^h||_{\Gamma_c}||\alpha_c^{-1/2}(g^2w^h_{,xx})_{,x}||_{\Gamma_c} \\
& + ||\beta_q^{-1/2}g^2u^h_{,xx}||_{\Gamma_q}||\beta_q^{1/2}w^h_{,x}||_{\Gamma_q} \\
& + ||\beta_q^{1/2}u^h_{,x}||_{\Gamma_q}||\beta_q^{-1/2}g^2w^h_{,xx}||_{\Gamma_q} + ||\delta_c^{-1/2}u^h_{,x}||_{\Gamma_c}||\delta_c^{1/2}w^h||_{\Gamma_c} \\
& + ||\delta_c^{1/2}u^h||_{\Gamma_c}||\delta_c^{-1/2}w^h_{,x}||_{\Gamma_c} + ||\alpha_c^{1/2}u^h||_{\Gamma_c}||\alpha_c^{1/2}w^h||_{\Gamma_c} \\
& + ||\beta_q^{1/2}u^h_{,x}||_{\Gamma_q}||\beta_q^{1/2}w^h_{,x}||_{\Gamma_q} + ||\delta_c^{1/2}u^h||_{\Gamma_c}||\delta_c^{1/2}w^h||_{\Gamma_c}. \qquad (4.87)
\end{aligned}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (4.87), we get

$$\begin{aligned}
B_{sb}(u^h,w^h) \leqslant\ & \Big(||(g^2)^{1/2}u^h_{,xx}||^2_{\tilde{\Omega}} + ||u^h_{,x}||^2_{\tilde{\Omega}} + ||\alpha^{-1/2}\langle(g^2u^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} \\
& + ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\Gamma_c} + ||\beta^{-1/2}\langle g^2u^h_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2u^h_{,xx}||^2_{\Gamma_q} \\
& + ||\delta^{-1/2}\langle u^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_c} + 2||\alpha^{1/2}[\![u^h]\!]||^2_{\tilde{\Gamma}} \\
& + 2||\beta^{1/2}[\![u^h_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\delta^{1/2}[\![u^h]\!]||^2_{\tilde{\Gamma}} + 2||\alpha_c^{1/2}u^h||^2_{\Gamma_c} \\
& + 2||\beta_q^{1/2}u^h_{,x}||^2_{\Gamma_q} + 2||\delta_c^{1/2}u^h||^2_{\Gamma_c}\Big)^{1/2} \\
& \times \Big(||(g^2)^{1/2}w^h_{,xx}||^2_{\tilde{\Omega}} + ||w^h_{,x}||^2_{\tilde{\Omega}} + ||\alpha^{-1/2}\langle(g^2w^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} \\
& + ||\alpha_c^{-1/2}(g^2w^h_{,xx})_{,x}||^2_{\Gamma_c} + ||\beta^{-1/2}\langle g^2w^h_{,xx}\rangle||^2_{\tilde{\Gamma}} \\
& + ||\beta_q^{-1/2}g^2w^h_{,xx}||^2_{\Gamma_q} + ||\delta^{-1/2}\langle w^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}w^h_{,x}||^2_{\Gamma_c} \\
& + 2||\alpha^{1/2}[\![w^h]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\delta^{1/2}[\![w^h]\!]||^2_{\tilde{\Gamma}} \\
& + 2||\alpha_c^{1/2}w^h||^2_{\Gamma_c} + 2||\beta_q^{1/2}w^h_{,x}||^2_{\Gamma_q} + 2||\delta_c^{1/2}w^h||^2_{\Gamma_c}\Big)^{1/2}. \qquad (4.88)
\end{aligned}$$

Thus, to complete the proof, it only remains to estimate each of the mean value terms that enter into the parentheses on the right-hand side of (4.88).

Hence, by using the mean value inequality (A.19), we can write the first mean value terms, enclosed into the first parenthesis, as

$$\begin{aligned}
&||\alpha^{-1/2}\langle(g^2u^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\Gamma_c} \\
&= \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(g^2u^h_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\alpha^{-1/2}(g^2u^{h+}_{,xx})_{,x}||^2_{\Gamma_i} + ||\alpha^{-1/2}(g^2u^{h-}_{,xx})_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{e=1}^{N_{el}} ||\alpha^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\partial\Omega_e},
\end{aligned} \tag{4.89}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

Then, by applying the trace inequality (A.39) as well as the properties of Sobolev norms in (4.89), we conclude that

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} ||\alpha^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\partial\Omega_e} \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}|\alpha^{-1/2}g^2u^h_{,xx}|^2_{1,\Omega_e} + h_e|\alpha^{-1/2}g^2u^h_{,xx}|^2_{2,\Omega_e}\right) \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\alpha^{-1/2}g^2u^h_{,xx}||^2_{1,\Omega_e} + h_e||\alpha^{-1/2}g^2u^h_{,xx}||^2_{2,\Omega_e}\right).
\end{aligned} \tag{4.90}$$

So, making use of inverse estimate (A.37), (4.90) gives

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\alpha^{-1/2}g^2u^h_{,xx}||^2_{1,\Omega_e} + h_e||\alpha^{-1/2}g^2u^h_{,xx}||^2_{2,\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}C_I h_e^{-2}||\alpha^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e} + h_e C_{II} h_e^{-4}||\alpha^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C_1 h_e^{-3}||\alpha^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_1}{h_e^3}\alpha^{-1}g^2||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_1}{h_e^3}\frac{h_e^3}{C_\alpha g^2}g^2||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e},
\end{aligned} \tag{4.91}$$

with $C_1 = C\max\{C_I^2, C_{II}^2\}$ and by C_I, C_{II} the constants resulting from an inverse estinate. We denote by C_α the stabilization constant of the stabilization parameter $\alpha = \frac{C_\alpha g^2}{h_e^3}$ and we have picked out that $\frac{C_1}{C_\alpha} \leqslant 1$ without loss of generality.

Therefore, from (4.89) – (4.91), we reach the conclusion that the first mean value terms, on the right-hand side of (4.88), can be bounded as follows

$$||\alpha^{-1/2}\langle(g^2u^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2u^h_{,xx})_{,x}||^2_{\Gamma_c} \leqslant ||(g^2)^{1/2}u^h_{,xx}||^2_{\tilde{\Omega}}. \tag{4.92}$$

In addition, we shall analogously estimate the second mean value terms, enclosed into the first parenthesis on the right-hand side of (4.88). By recalling the mean value inequality (A.19), we deduce

$$\begin{aligned}
&||\beta^{-1/2}\langle g^2 u^h_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2} g^2 u^h_{,xx}||^2_{\Gamma_q} \\
&= \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle g^2 u^h_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2 u^h_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2} g^2 u^{h+}_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2} g^2 u^{h-}_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2 u^h_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} ||\beta^{-1/2} g^2 u^h_{,xx}||^2_{\partial\Omega_e},
\end{aligned} \tag{4.93}$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.
By invoking the trace inequality (A.38) and next the properties of Sobolev norms in (4.93), we have

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} ||\beta^{-1/2} g^2 u^h_{,xx}||^2_{\partial\Omega_e} \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2} g^2 u^h_{,xx}||^2_{\Omega_e} + h_e||\beta^{-1/2} g^2 (u^h_{,xx})_{,x}||^2_{\Omega_e}\right) \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2} g^2 u^h_{,xx}||^2_{\Omega_e} + h_e||\beta^{-1/2} g^2 u^h_{,xx}||^2_{1,\Omega_e}\right).
\end{aligned} \tag{4.94}$$

Afterwards, making use of inverse estimate (A.37), (4.94) yields

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e} + h_e||\beta^{-1/2}g^2u^h_{,xx}||^2_{1,\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e} + h_eC_I^2h_e^{-2}||\beta^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C_2h_e^{-1}||\beta^{-1/2}g^2u^h_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_2}{h_e}\beta^{-1}g^2||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_2}{h_e}\frac{h_e}{C_\beta g^2}g^2||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}u^h_{,xx}||^2_{\Omega_e},
\end{aligned}
\tag{4.95}
$$

with $C_2 = C\max\{1, C_I^2\}$ and by C_I the constant resulting from an inverse estimate. We denote by C_β the stabilization constant of the stabilization parameter $\beta = \frac{C_\beta g^2}{h_e}$ and we have selected that $\frac{C_2}{C_\beta} \leqslant 1$ without loss of generality.

Ergo, from (4.93) – (4.95), we arrive to the conclusion that the second mean value terms, on the right-hand side of (4.88), can be estimated as follows

$$
||\beta^{-1/2}\langle g^2u^h_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2u^h_{,xx}||^2_{\Gamma_q} \leqslant ||(g^2)^{1/2}u^h_{,xx}||^2_{\tilde{\Omega}}. \tag{4.96}
$$

What is more, we shall use similar arguments to bound the rest of the mean value terms, enclosed into the first parenthesis on the right-hand side of (4.88). By employing the mean value inequality (A.19), we obtain

$$
\begin{aligned}
&||\delta^{-1/2}\langle u^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_c} \\
&= \sum_{i=1}^{N_i} ||\delta^{-1/2}\langle u^h_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\delta^{-1/2}u^{h+}_{,x}||^2_{\Gamma_i} + ||\delta^{-1/2}u^{h-}_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{e=1}^{N_{el}} ||\delta^{-1/2}u^h_{,x}||^2_{\partial\Omega_e},
\end{aligned}
\tag{4.97}
$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.
Thus, using both the trace inequality (A.38) and the properties of Sobolev norms in (4.97), we get

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} ||\delta^{-1/2}u^h_{,x}||^2_{\partial\Omega_e} \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e} + h_e||\delta^{-1/2}u^h_{,xx}||^2_{\Omega_e}\right) \\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e} + h_e||\delta^{-1/2}u^h_{,x}||^2_{1,\Omega_e}\right).
\end{aligned}
\tag{4.98}
$$

By making use of inverse estimate (A.37) in (4.98), we subsequently deduce

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e} + h_e||\delta^{-1/2}u^h_{,x}||^2_{1,\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e} + h_e C_I^2 h_e^{-2}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C_3 h_e^{-1}||\delta^{-1/2}u^h_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_3}{h_e}\delta^{-1}||u^h_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_3}{h_e}\frac{h_e}{C_\delta}||u^h_{,x}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||u^h_{,x}||^2_{\Omega_e},
\end{aligned} \tag{4.99}$$

with $C_3 = C\max\{1, C_I^2\}$ and by C_I the constant resulting from an inverse estimate. We denote by C_δ the stabilization constant of the stabilization parameter $\delta = \frac{C_\delta}{h_e}$ and we have chosen that $\frac{C_3}{C_\delta} \leqslant 1$ without loss of generality.

Wherefore, from (4.97) – (4.99), we reach the conclusion that the remaining mean value terms, on the right-hand side of (4.88), can be bounded as follows

$$||\delta^{-1/2}\langle u^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_c} \leqslant ||u^h_{,x}||^2_{\tilde{\Omega}}. \tag{4.100}$$

We shall follow the above procedures in a similar manner to bound the mean value terms of w^h,enclosed into the second parenthesis on the right-hand side of (4.88). As a result, we have

$$\begin{aligned}
||\alpha^{-1/2}\langle(g^2 w^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2 w^h_{,xx})_{,x}||^2_{\Gamma_c} &\leqslant ||(g^2)^{1/2}w^h_{,xx}||^2_{\tilde{\Omega}},\\
||\beta^{-1/2}\langle g^2 w^h_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2 w^h_{,xx}||^2_{\Gamma_q} &\leqslant ||(g^2)^{1/2}w^h_{,xx}||^2_{\tilde{\Omega}},\\
||\delta^{-1/2}\langle w^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}w^h_{,x}||^2_{\Gamma_c} &\leqslant ||w^h_{,x}||^2_{\tilde{\Omega}}.
\end{aligned} \tag{4.101}$$

To boot, inserting the inequalities (4.92), (4.96), (4.100) as well as

(4.101) into the brackets in (4.88), we obtain

$$\begin{aligned} B_{sb}(u^h, w^h) \leqslant\ & \Big(3||(g^2)^{1/2} u^h_{,xx}||^2_{\tilde{\Omega}} + 2||u^h_{,x}||^2_{\tilde{\Omega}} + 2||\alpha^{1/2}[\![u^h]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![u^h_{,x}]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\delta^{1/2}[\![u^h]\!]||^2_{\tilde{\Gamma}} + 2||\alpha_c^{1/2} u^h||^2_{\Gamma_c} + 2||\beta_q^{1/2} u^h_{,x}||^2_{\Gamma_q} \\ & + 2||\delta_c^{1/2} u^h||^2_{\Gamma_c}\Big)^{1/2} \times \Big(3||(g^2)^{1/2} w^h_{,xx}||^2_{\tilde{\Omega}} + 2||w^h_{,x}||^2_{\tilde{\Omega}} \\ & + 2||\alpha^{1/2}[\![w^h]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\delta^{1/2}[\![w^h]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\alpha_c^{1/2} w^h||^2_{\Gamma_c} + 2||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_q} + 2||\delta_c^{1/2} w^h||^2_{\Gamma_c}\Big)^{1/2}. \end{aligned} \quad (4.102)$$

Also, by the use of definition of energy seminorm, (4.26), on the right-hand side of (4.102), we arrive at

$$B_{sb}(u^h, w^h) \leqslant C|||u^h|||_{sb}|||w^h|||_{sb},$$

where C is independent of h_e. □

Thereafter, let us examine the continuity of the bilinear form, $B_{sb}(\cdot,\cdot)$ for the *hp*-version interior penalty discontinuous Galerkin finite element method.

Proposition 4.4.2.2. *Let $B_{sb}(\cdot,\cdot)$ be the bilinear form defined in* (4.23) *with $\theta \in \{-1, 1\}$ and $\alpha, \alpha_c, \beta, \beta_q, \delta, \delta_c \geqslant 0$. Then, there exists a constant $0 < C < \infty$, such that*

$$B_{sb}(u, w) \leqslant C|||u|||_{sb}|||w|||_{sb} \quad \forall u, w \in \mathcal{V}^{hp}, \quad (4.103)$$

where C is independent of both h_e and p_e, for the hp-version.

Proof. Similar to the approach to the previous proof, we can obtain (4.103), by using at first the triangle inequality, then the Cauchy-Schwarz inequality (A.12) and finally the Cauchy-Schwarz discrete inequality (A.13).

As a consequence, we end up at the same result presented in mathematical expression (4.88). Up to this point

$$\begin{aligned} B_{sb}(u,w) \leqslant\ & \Big(||(g^2)^{1/2}u_{,xx}||^2_{\tilde{\Omega}} + ||u_{,x}||^2_{\tilde{\Omega}} + ||\alpha^{-1/2}\langle(g^2u_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} \\ & + ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\Gamma_c} + ||\beta^{-1/2}\langle g^2u_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2u_{,xx}||^2_{\Gamma_q} \\ & + ||\delta^{-1/2}\langle u_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u_{,x}||^2_{\Gamma_c} + 2||\alpha^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\beta^{1/2}[\![u_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\delta^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + 2||\alpha_c^{1/2}u||^2_{\Gamma_c} \\ & + 2||\beta_q^{1/2}u_{,x}||^2_{\Gamma_q} + 2||\delta_c^{1/2}u||^2_{\Gamma_c}\Big)^{1/2} \times \Big(||(g^2)^{1/2}w_{,xx}||^2_{\tilde{\Omega}} + ||w_{,x}||^2_{\tilde{\Omega}} \\ & + ||\alpha^{-1/2}\langle(g^2w_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2w_{,xx})_{,x}||^2_{\Gamma_c} \\ & + ||\beta^{-1/2}\langle g^2w_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2w_{,xx}||^2_{\Gamma_q} + ||\delta^{-1/2}\langle w_{,x}\rangle||^2_{\tilde{\Gamma}} \\ & + ||\delta_c^{-1/2}w_{,x}||^2_{\Gamma_c} + 2||\alpha^{1/2}[\![w]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\delta^{1/2}[\![w]\!]||^2_{\tilde{\Gamma}} + 2||\alpha_c^{1/2}w||^2_{\Gamma_c} + 2||\beta_q^{1/2}w_{,x}||^2_{\Gamma_q} \\ & + 2||\delta_c^{1/2}w||^2_{\Gamma_c}\Big)^{1/2}. \end{aligned} \tag{4.104}$$

Thereby, to complete the proof, it only remains to estimate each of the mean value terms appearing into the parentheses on the right-hand side of (4.104).

Hence, by applying the mean value inequality (A.19), we can write the first mean value terms, enclosed into the first parenthesis, as

$$\begin{aligned} & ||\alpha^{-1/2}\langle(g^2u_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\Gamma_c} \\ & = \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(g^2u_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\Gamma_r} \\ & \leqslant \sum_{i=1}^{N_i} \left(||\alpha^{-1/2}(g^2u^+_{,xx})_{,x}||^2_{\Gamma_i} + ||\alpha^{-1/2}(g^2u^-_{,xx})_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\Gamma_r} \\ & \leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\alpha^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||\alpha^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e}\right) \\ & + \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e}, \end{aligned} \tag{4.105}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

These terms can be bounded by invoking the inverse inequality (A.21) in (4.105), then we deduce

$$\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\alpha^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||\alpha^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\partial\Omega_e}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_1\frac{p_{e'}^6}{h_{e'}^3}||\alpha^{-1/2}g^2u_{,xx}||^2_{\Omega_{e'}} + c_1\frac{p_e^6}{h_e^3}||\alpha^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} c_1\frac{p_e^6}{h_e^3}||\alpha_c^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}||\alpha^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}\alpha^{-1}g^2||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}\frac{h_e^3}{C_\alpha g^2p_e^6}g^2||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e},
\end{aligned} \tag{4.106}$$

where the constant c_1 is independent of h_e, p_e and u. We denote by C_α the stabilization constant of the stabilization parameter $\alpha = \frac{C_\alpha g^2 p_e^6}{h_e^3}$ and we have picked out that $\frac{c_1}{C_\alpha} \leqslant 1$ without loss of generality.

Therefore, from (4.105) – (4.106), we reach the conclusion that the first mean value terms, enclosed into the first parenthesis on the right-hand side of (4.104), can be bounded as follows

$$||\alpha^{-1/2}\langle(g^2u_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2u_{,xx})_{,x}||^2_{\Gamma_c} \leqslant ||(g^2)^{1/2}u_{,xx}||^2_{\tilde{\Omega}}. \tag{4.107}$$

Moreover, we shall analogously estimate the second mean value terms on the right-hand side of (4.104). By recalling the mean value inequality

(A.19), we get

$$
\begin{aligned}
&||\beta^{-1/2}\langle g^2 u_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2} g^2 u_{,xx}||^2_{\Gamma_q} \\
&= \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle g^2 u_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2 u_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2} g^2 u^+_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2} g^2 u^-_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2 u_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \Big(||\beta^{-1/2} g^2 u_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2} g^2 u_{,xx}||^2_{\partial\Omega_e}\Big) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2} g^2 u_{,xx}||^2_{\partial\Omega_e},
\end{aligned}
\tag{4.108}
$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.

By employing the inverse inequality (A.20) in (4.108), we arrive at

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\beta^{-1/2}g^2u_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}g^2u_{,xx}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2}g^2u_{,xx}||^2_{\partial\Omega_e}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_2\frac{p_{e'}^2}{h_{e'}}||\beta^{-1/2}g^2u_{,xx}||^2_{\Omega_{e'}} + c_2\frac{p_e^2}{h_e}||\beta^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} c_2\frac{p_e^2}{h_e}||\beta_q^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} c_2\frac{p_e^2}{h_e}||\beta^{-1/2}g^2u_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_2\frac{p_e^2}{h_e}\beta^{-1}g^2||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_2\frac{p_e^2}{h_e}\frac{h_e}{C_\beta g^2p_e^2}g^2||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}u_{,xx}||^2_{\Omega_e},
\end{aligned}
\tag{4.109}
$$

where the constant c_2 is independent of h_e, p_e and u. We denote by C_β the stabilization constant of the stabilization parameter $\beta = \frac{C_\beta g^2 p_e^2}{h_e}$ and we have selected that $\frac{c_2}{C_\beta} \leqslant 1$ without loss of generality.

Ergo, from (4.108) – (4.109), we arrive to the conclusion that the second mean value terms, enclosed into the first parenthesis on the right-hand side of (4.104), can be estimated as follows

$$
||\beta^{-1/2}\langle g^2u_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2u_{,xx}||^2_{\Gamma_q} \leqslant ||(g^2)^{1/2}u_{,xx}||^2_{\tilde{\Omega}}. \tag{4.110}
$$

Furthermore, we shall use similar arguments to bound the remaining mean value terms appearing on the right-hand side of (4.104). By making use of the mean value inequality (A.19), we have

$$
\begin{aligned}
&||\delta^{-1/2}\langle u_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u_{,x}||^2_{\Gamma_c} \\
&= \sum_{i=1}^{N_i} ||\delta^{-1/2}\langle u_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}u_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\delta^{-1/2}u^+_{,x}||^2_{\Gamma_i} + ||\delta^{-1/2}u^-_{,x}||^2_{\Gamma_i}\right) + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}u_{,x}||^2_{\Gamma_r} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\delta^{-1/2}u_{,x}||^2_{\partial\Omega_{e'}} + ||\delta^{-1/2}u_{,x}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\delta_c^{-1/2}u_{,x}||^2_{\partial\Omega_e},
\end{aligned}
\tag{4.111}
$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

Thus, applying the inverse inequality (A.20), (4.111) yields

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\delta^{-1/2}u_{,x}||^2_{\partial\Omega_{e'}} + ||\delta^{-1/2}u_{,x}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c)/(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\delta_c^{-1/2}u_{,x}||^2_{\partial\Omega_e} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_3\frac{p_{e'}^2}{h_{e'}}||\delta^{-1/2}u_{,x}||^2_{\Omega_{e'}} + c_3\frac{p_e^2}{h_e}||\delta^{-1/2}u_{,x}||^2_{\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_c):(\partial\Omega_e\subset\Gamma)}^{N_{el}} c_3\frac{p_e^2}{h_e}||\delta_c^{-1/2}u_{,x}||^2_{\Omega_e} \\
&\leqslant \sum_{e=1}^{N_{el}} c_3\frac{p_e^2}{h_e}||\delta^{-1/2}u_{,x}||^2_{\Omega_e} \\
&= \sum_{e=1}^{N_{el}} c_3\frac{p_e^2}{h_e}\delta^{-1}||u_{,x}||^2_{\Omega_e} \\
&= \sum_{e=1}^{N_{el}} c_3\frac{p_e^2}{h_e}\frac{h_e}{C_\delta p_e^2}||u_{,x}||^2_{\Omega_e} \\
&\leqslant \sum_{e=1}^{N_{el}} ||u_{,x}||^2_{\Omega_e},
\end{aligned}
\tag{4.112}
$$

where the constant c_3 is independent of h_e, p_e and u. We denote by C_δ the stabilization constant of the stabilization parameter $\delta = \frac{C_\delta p_e^2}{h_e}$ and we have chosen $\frac{c_3}{C_\delta} \leqslant 1$ without loss of generality.

Wherefore, from (4.111) – (4.112), we reach the conclusion that the rest of the mean value terms, enclosed into the first parenthesis on the right-hand side of (4.104), can be bounded as follows

$$||\delta^{-1/2}\langle u^h_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2}u^h_{,x}||^2_{\Gamma_c} \leqslant ||u^h_{,x}||^2_{\tilde{\Omega}}. \tag{4.113}$$

We shall follow the above series of steps in a similar manner to estimate the mean value terms of w on the right-hand side of (4.104). As a result, we obtain

$$\begin{aligned} ||\alpha^{-1/2}\langle(g^2 w_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2 w_{,xx})_{,x}||^2_{\Gamma_c} &\leqslant ||(g^2)^{1/2} w_{,xx}||^2_{\tilde{\Omega}}, \\ ||\beta^{-1/2}\langle g^2 w_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2} g^2 w_{,xx}||^2_{\Gamma_q} &\leqslant ||(g^2)^{1/2} w_{,xx}||^2_{\tilde{\Omega}}, \\ ||\delta^{-1/2}\langle w_{,x}\rangle||^2_{\tilde{\Gamma}} + ||\delta_c^{-1/2} w_{,x}||^2_{\Gamma_c} &\leqslant ||w_{,x}||^2_{\tilde{\Omega}}. \end{aligned} \tag{4.114}$$

After that procedure, we gather the inequalities (4.107), (4.110), (4.113), and also (4.114) and insert them into the right-hand side of (4.104). In consequence, we conclude

$$\begin{aligned} B_{sb}(u,w) \;\leqslant\; & \Big(3||(g^2)^{1/2}u_{,xx}||^2_{\tilde{\Omega}} + 2||u_{,x}||^2_{\tilde{\Omega}} + 2||\alpha^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![u_{,x}]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\delta^{1/2}[\![u]\!]||^2_{\tilde{\Gamma}} + 2||\alpha_c^{1/2}u||^2_{\Gamma_c} + 2||\beta_q^{1/2}u_{,x}||^2_{\Gamma_q} \\ & + 2||\delta_c^{1/2}u||^2_{\Gamma_c}\Big)^{1/2} \times \Big(3||(g^2)^{1/2}w_{,xx}||^2_{\tilde{\Omega}} + 2||w_{,x}||^2_{\tilde{\Omega}} \\ & + 2||\alpha^{1/2}[\![w]\!]||^2_{\tilde{\Gamma}} + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\delta^{1/2}[\![w]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\alpha_c^{1/2}w||^2_{\Gamma_c} + 2||\beta_q^{1/2}w_{,x}||^2_{\Gamma_q} + 2||\delta_c^{1/2}w||^2_{\Gamma_c}\Big)^{1/2}. \end{aligned} \tag{4.115}$$

So, by the use of definition of energy seminorm, (4.26), on the right-hand side of (4.115), we reach to

$$B_{sb}(u,w) \leqslant C|||u|||_{sb}|||w|||_{sb},$$

where C is independent of both h_e and p_e. □

4.5 Error Analysis

In this section, we want to conduct an error analysis for interior penalty discontinuous Galerkin finite element methods (4.49). Specifically, our main concern is to prove both h- and hp-version a priori error estimates in the seminorm, $|||\cdot|||_{sb}$, for the methods introduced above. For this purpose, we have initially proved the consistency and we have showed stability of the methods in the preceding sections. With the results from consistency and stability, we can prove convergence of the methods. Let us assume for simplicity that g^2 is continuous on Ω.

4.5.1 Error Estimates in the Energy Seminorm

Convergence 4.5.1.1. *Let $\tilde{u}^h$ denote any interpolant of u from $H^{\mathbf{s}}(\Omega, \mathcal{P}(\Omega))$ onto the finite-dimensional space $\mathcal{V}^h$. Let us specify the interpolation error by $\eta = u - \tilde{u}^h$. Thereby, we can decompose the global error $u - u^h$ as follows*

$$u - u^h = (u - \tilde{u}^h) + (\tilde{u}^h - u^h) \equiv \eta + e^h. \tag{4.116}$$

So, using the triangle inequality, we have

$$|||u - u^h|||_{sb} \leqslant |||\eta|||_{sb} + |||e^h|||_{sb}, \tag{4.117}$$

where $e^h = \tilde{u}^h - u^h$ is the part of the error in the finite element space, i.e., $e^h \in \mathcal{V}^h$.

Our error analysis below will provide a bound on $|||e^h|||_{sb}$ in terms of suitable norms of η. As a consequence, we shall obtain a bound on $|||u-u^h|||_{sb}$ with respect to various norms of η. Hence, to complete the error analysis, we shall need to quantify norms of η in terms of the discretization parameter and Sobolev seminorms of the analytical solution u.

Theorem 4.5.1.2. *Assume that the consistency condition* (4.50) *and stability condition* (4.53) *(see Proposition 4.4.1.4) of the method hold. For each face, we define positive, real, piecewise constant functions α, α_c, β, β_q, δ and δ_c by*

$$\alpha = \alpha_c = \frac{C_\alpha g^2}{h_e^3}, \ \beta = \beta_q = \frac{C_\beta g^2}{h_e} \quad \text{and} \quad \delta = \delta_c = \frac{C_\delta}{h_e}.$$

Given that the conditions are satisfied for the interpolation estimates (A.30), (A.31) *and the trace inequalities* (A.38), (A.39) *hold, the error estimate for the symmetric interior penalty discontinuous Galerkin method* (4.49) *can be written as*

$$|||u - u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2, \tag{4.118}$$

where C is a constant dependent only on the space dimension and on k, and $|\cdot|_{k+1,\Omega_e}$ denotes the H^{k+1}-seminorm on Ω_e.

Proof. To begin with, we shall estimate e^h. For that purpose, we take advantage of the coercivity (4.53), the decomposition of the error (4.116) and the Galerkin orthogonality (4.50) yielding

$$
\begin{aligned}
m|||e^h|||_{sb}^2 &\leqslant B_{sb}(e^h, e^h) \\
&= B_{sb}(u - u^h - \eta, e^h) \\
&= B_{sb}(u - u^h, e^h) - B_{sb}(\eta, e^h) \\
&= -B_{sb}(\eta, e^h) \\
&\leqslant |B_{sb}(\eta, e^h)|.
\end{aligned} \tag{4.119}
$$

We continue by using the triangle inequality on the right-hand side of (4.119). Then, we obtain

$$
\begin{aligned}
m|||e^h|||_{sb}^2 \leqslant\ & |(g^2\eta_{,xx}, e^h_{,xx})_{\tilde{\Omega}}| + |(\eta_{,x}, e^h_{,x})_{\tilde{\Omega}}| \\
& + |\langle (g^2\eta_{,xx})_{,x}\rangle [\![e^h]\!]_{\tilde{\Gamma}}| + |[\![\eta]\!]\langle (g^2 e^h_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| \\
& + |\langle g^2\eta_{,xx}\rangle [\![e^h_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,x}]\!]\langle g^2 e^h_{,xx}\rangle_{\tilde{\Gamma}}| \\
& + |\langle \eta_{,x}\rangle [\![e^h]\!]_{\tilde{\Gamma}}| + |[\![\eta]\!]\langle e^h_{,x}\rangle_{\tilde{\Gamma}}| \\
& + |\alpha [\![\eta]\!][\![e^h]\!]_{\tilde{\Gamma}}| + |\beta [\![\eta_{,x}]\!][\![e^h_{,x}]\!]_{\tilde{\Gamma}}| + |\delta [\![\eta]\!][\![e^h]\!]_{\tilde{\Gamma}}| \\
& + |(g^2\eta_{,xx})_{,x} \cdot n e^h|_{\Gamma_c}| + |\eta (g^2 e^h_{,xx})_{,x} \cdot n|_{\Gamma_c}| \\
& + |g^2\eta_{,xx} e^h_{,x} \cdot n|_{\Gamma_q}| + |\eta_{,x} \cdot n g^2 e^h_{,xx}|_{\Gamma_q}| \\
& + |\eta_{,x} \cdot n e^h|_{\Gamma_c}| + |\eta e^h_{,x} \cdot n|_{\Gamma_c}| \\
& + |\alpha_c \eta e^h|_{\Gamma_c} + |\beta_q \eta_{,x} \cdot n e^h_{,x} \cdot n|_{\Gamma_q}| + |\delta_c \eta e^h|_{\Gamma_c}|.
\end{aligned} \tag{4.120}
$$

To bound the terms on the right-hand side of (4.120), we apply the

Cauchy-Schwarz inequality (A.12) giving

$$
\begin{aligned}
m|||e^h|||_{sb}^2 \leqslant\ & ||g^{1/2}\eta_{,xx}||_{\tilde{\Omega}}||g^{1/2}e^h_{,xx}||_{\tilde{\Omega}} + ||\eta_{,x}||_{\tilde{\Omega}}||e^h_{,x}||_{\tilde{\Omega}} \\
&+ ||\alpha^{-1/2}\langle(g^2\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}}||\alpha^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}} \\
&+ ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}||\alpha^{-1/2}\langle(g^2e^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}} \\
&+ ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\tilde{\Gamma}}||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}} \\
&+ ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{-1/2}\langle g^2e^h_{,xx}\rangle||_{\tilde{\Gamma}} \\
&+ ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\tilde{\Gamma}}||\delta^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}} + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}||\delta^{-1/2}\langle e^h_{,x}\rangle||_{\tilde{\Gamma}} \\
&+ ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}||\alpha^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}} + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}} \\
&+ ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}||\delta^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}} + ||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||_{\Gamma_c}||\alpha_c^{1/2}e^h||_{\Gamma_c} \\
&+ ||\alpha_c^{1/2}\eta||_{\Gamma_c}||\alpha_c^{-1/2}(g^2e^h_{,xx})_{,x}||_{\Gamma_c} \\
&+ ||\beta_q^{-1/2}g^2\eta_{,xx}||_{\Gamma_q}||\beta_q^{1/2}e^h_{,x}||_{\Gamma_q} \\
&+ ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}||\beta_q^{-1/2}g^2e^h_{,xx}||_{\Gamma_q} + ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_c}||\delta_c^{1/2}e^h||_{\Gamma_c} \\
&+ ||\delta_c^{1/2}\eta||_{\Gamma_c}||\delta_c^{-1/2}e^h_{,x}||_{\Gamma_c} + ||\alpha_c^{1/2}\eta||_{\Gamma_c}||\alpha_c^{1/2}e^h||_{\Gamma_c} \\
&+ ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}||\beta_q^{1/2}e^h_{,x}||_{\Gamma_q} + ||\delta_c^{1/2}\eta||_{\Gamma_c}||\delta_c^{1/2}e^h||_{\Gamma_c}. \qquad (4.121)
\end{aligned}
$$

As before in this chapter, we shall make use of Young inequality (A.17) on each term on the right-hand side of (4.121). For that reason, we deduce

$$
\begin{aligned}
m|||e^h|||_{sb}^2 \leqslant\ & \frac{1}{2\varepsilon}\Big(||(g^2)^{1/2}\eta_{,xx}||_{\tilde\Omega}^2 + ||\eta_{,x}||_{\tilde\Omega}^2 + ||\alpha^{-1/2}\langle(g^2\eta_{,xx})_{,x}\rangle||_{\tilde\Gamma}^2 \\
&+ ||\alpha^{1/2}[\![\eta]\!]||_{\tilde\Gamma}^2 + ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\tilde\Gamma}^2 + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde\Gamma}^2 \\
&+ ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\tilde\Gamma}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde\Gamma}^2 + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde\Gamma}^2 \\
&+ ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde\Gamma}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde\Gamma}^2 + ||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||_{\Gamma_c}^2 \\
&+ ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 + ||\beta_q^{-1/2}g^2\eta_{,xx}||_{\Gamma_q}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 \\
&+ ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_c}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 \\
&+ ||\delta_c^{1/2}\eta||_{\Gamma_c}^2\Big) \\
&+ \frac{\varepsilon}{2}\Big(||(g^2)^{1/2}e^h_{,xx}||_{\tilde\Omega}^2 + ||e^h_{,x}||_{\tilde\Omega}^2 + ||\alpha^{1/2}[\![e^h]\!]||_{\tilde\Gamma}^2 \\
&+ ||\alpha^{-1/2}\langle(g^2e^h_{,xx})_{,x}\rangle||_{\tilde\Gamma}^2 + ||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde\Gamma}^2 + ||\beta^{-1/2}\langle g^2e^h_{,xx}\rangle||_{\tilde\Gamma}^2 \\
&+ ||\delta^{1/2}[\![e^h]\!]||_{\tilde\Gamma}^2 + ||\delta^{-1/2}\langle e^h_{,x}\rangle||_{\tilde\Gamma}^2 + ||\alpha^{1/2}[\![e^h]\!]||_{\tilde\Gamma}^2 + ||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde\Gamma}^2 \\
&+ ||\delta^{1/2}[\![e^h]\!]||_{\tilde\Gamma}^2 + ||\alpha_c^{1/2}e^h||_{\Gamma_c}^2 + ||\alpha_c^{-1/2}(g^2e^h_{,xx})_{,x}||_{\Gamma_c}^2 \\
&+ ||\beta_q^{1/2}e^h_{,x}||_{\Gamma_q}^2 + ||\beta_q^{-1/2}g^2e^h_{,xx}||_{\Gamma_q}^2 + ||\delta_c^{1/2}e^h||_{\Gamma_c}^2 + ||\delta_c^{-1/2}e^h_{,x}||_{\Gamma_c}^2 \\
&+ ||\alpha_c^{1/2}e^h||_{\Gamma_c}^2 + ||\beta_q^{1/2}e^h_{,x}||_{\Gamma_q}^2 + ||\delta_c^{1/2}e^h||_{\Gamma_c}^2\Big).
\end{aligned} \tag{4.122}
$$

Thus, to proceed with the estimate of e^h, one of the steps remaining is to bound each of the mean value terms which appear into the second parenthesis, on the right-hand side of (4.122).

To achieve that, we shall exactly follow the same series of steps presented in mathematical formulas (4.89) – (4.100). As a result, we get

$$
\begin{aligned}
||\alpha^{-1/2}\langle(g^2e^h_{,xx})_{,x}\rangle||_{\tilde\Gamma}^2 + ||\alpha_c^{-1/2}(g^2e^h_{,xx})_{,x}||_{\Gamma_c}^2 &\leqslant ||(g^2)^{1/2}e^h_{,xx}||_{\tilde\Omega}^2, \\
||\beta^{-1/2}\langle g^2e^h_{,xx}\rangle||_{\tilde\Gamma}^2 + ||\beta_q^{-1/2}g^2e^h_{,xx}||_{\Gamma_q}^2 &\leqslant ||(g^2)^{1/2}e^h_{,xx}||_{\tilde\Omega}^2, \\
||\delta^{-1/2}\langle e^h_{,x}\rangle||_{\tilde\Gamma}^2 + ||\delta_c^{-1/2}e^h_{,x}||_{\Gamma_c}^2 &\leqslant ||e^h_{,x}||_{\tilde\Omega}^2.
\end{aligned} \tag{4.123}
$$

To boot, by inserting the inequalities, (4.123), into the second bracket on the right-hand side of (4.122), we have

$$
\begin{aligned}
m|||e^h|||_{sb}^2 \leqslant\ & \frac{1}{2\varepsilon}\Big(||(g^2)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}^2 + ||\eta_{,x}||_{\tilde{\Omega}}^2 + ||\alpha^{-1/2}\langle(g^2\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}}^2 \\
& + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\tilde{\Gamma}}^2 + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 \\
& + ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\tilde{\Gamma}}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 \\
& + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||_{\Gamma_c}^2 \\
& + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 + ||\beta_q^{-1/2}g^2\eta_{,xx}||_{\Gamma_q}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 \\
& + ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_c}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 \\
& + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2\Big) \\
& + \frac{\varepsilon}{2}\Big(3||(g^2)^{1/2}e^h_{,xx}||_{\tilde{\Omega}}^2 + 2||e^h_{,x}||_{\tilde{\Omega}}^2 \\
& + 2||\alpha^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}}^2 + 2||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}}^2 + 2||\delta^{1/2}[\![e^h]\!]||_{\tilde{\Gamma}}^2 \\
& + 2||\alpha_c^{1/2}e^h||_{\Gamma_c}^2 + ||\beta_q^{1/2}e^h_{,x}||_{\Gamma_q}^2 + 2||\delta_c^{1/2}e^h||_{\Gamma_c}^2\Big). \qquad (4.124)
\end{aligned}
$$

Now, by the use of the definition of energy seminorm, (4.26), in second parenthesis on the right-hand side of (4.124), it derives

$$
\begin{aligned}
m|||e^h|||_{sb}^2 \leqslant\ & \frac{1}{2\varepsilon}\Big(||(g^2)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}^2 + ||\eta_{,x}||_{\tilde{\Omega}}^2 + ||\alpha^{-1/2}\langle(g^2\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}}^2 \\
& + ||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||_{\Gamma_c}^2 + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 \\
& + ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\tilde{\Gamma}}^2 + ||\beta_q^{-1/2}g^2\eta_{,xx}||_{\Gamma_q}^2 + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 \\
& + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 + ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\tilde{\Gamma}}^2 + ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_c}^2 \\
& + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2 + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 \\
& + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2\Big) \\
& + \frac{3\varepsilon}{2}|||e^h|||_{sb}^2. \qquad (4.125)
\end{aligned}
$$

Afterwards, by choosing an appropriate value for ε in (4.125), it derives a bound on $|||e^h|||_{sb}$, in terms of suitable norms of η, being

$$
\begin{aligned}
\frac{m}{3}|||e^h|||^2_{sb} \leqslant\ & \frac{g^2}{m}||\eta_{,xx}||^2_{\tilde{\Omega}} + \frac{1}{m}||\eta_{,x}||^2_{\tilde{\Omega}} \\
&+ \left\{ \frac{(g^2)^2}{m}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}} + \frac{(g^2)^2}{m}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_c} \right\} \\
&+ \left\{ \frac{C_\alpha g^2}{m}||h_e^{-3/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{C_\alpha g^2}{m}||h_e^{-3/2}\eta||^2_{\Gamma_c} \right\} \\
&+ \left\{ \frac{(g^2)^2}{m}||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\tilde{\Gamma}} + \frac{(g^2)^2}{m}||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_q} \right\} \\
&+ \left\{ \frac{C_\beta g^2}{m}||h_e^{-1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + \frac{C_\beta g^2}{m}||h_e^{-1/2}\eta_{,x}||^2_{\Gamma_q} \right\} \\
&+ \left\{ \frac{1}{m}||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{m}||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_c} \right\} \\
&+ \left\{ \frac{C_\delta}{m}||h_e^{-1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{C_\delta}{m}||h_e^{-1/2}\eta||^2_{\Gamma_c} \right\} \\
&+ \left\{ \frac{1}{m}||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\alpha_c^{1/2}\eta||^2_{\Gamma_c} \right\} \\
&+ \left\{ \frac{1}{m}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} \right\} \\
&+ \left\{ \frac{1}{m}||\delta^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\delta_c^{1/2}\eta||^2_{\Gamma_c} \right\}. \qquad (4.126)
\end{aligned}
$$

We simultaneously note that the inverse estimates (see Theorem A.4.1 and Remarks A.4.2) do not hold for the interpolation error, since $\eta \notin \mathcal{V}^h$.

Therefore to complete the estimate of e^h, a subsequent step is to bound the terms enclosed into the brackets on the right-hand side of (4.126).

Hence, by invoking the mean value inequality (A.19), we can write the factors enclosed into the first bracket on the right-hand side of (4.126) as

$$\begin{aligned}
&\frac{(g^2)^2}{m}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma} + \frac{(g^2)^2}{m}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_c} \\
&= \frac{(g^2)^2}{m}\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \frac{(g^2)^2}{m}\sum_{r=1}^{N_c}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r} \\
&\leqslant \frac{(g^2)^2}{m}\sum_{i=1}^{N_i}\left(||\alpha^{-1/2}(\eta^+_{,xx})_{,x}||^2_{\Gamma_i} + ||\alpha^{-1/2}(\eta^-_{,xx})_{,x}||^2_{\Gamma_i}\right) \\
&+ \frac{(g^2)^2}{m}\sum_{r=1}^{N_c}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r} \\
&\leqslant \frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}}||\alpha^{-1/2}(\eta_{,xx})_{,x}||^2_{\partial\Omega_e},
\end{aligned} \tag{4.127}$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$.

Next, by applying the trace inequality (A.39) as well as the properties of Sobolev norms in (4.127), we conclude that

$$\begin{aligned}
&\frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}}||\alpha^{-1/2}(\eta_{,xx})_{,x}||^2_{\partial\Omega_e} \\
&\leqslant \frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}|\alpha^{-1/2}\eta_{,xx}|^2_{1,\Omega_e} + h_e|\alpha^{-1/2}\eta_{,xx}|^2_{2,\Omega_e}\right) \\
&\leqslant \frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\alpha^{-1/2}\eta_{,xx}||^2_{1,\Omega_e} + h_e||\alpha^{-1/2}\eta_{,xx}||^2_{2,\Omega_e}\right) \\
&= \frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}} C\alpha^{-1}\left(h_e^{-1}||\eta_{,xx}||^2_{1,\Omega_e} + h_e||\eta_{,xx}||^2_{2,\Omega_e}\right) \\
&= \frac{(g^2)^2}{m}\sum_{e=1}^{N_{el}} C\frac{h_e^3}{C_\alpha g^2}\left(h_e^{-1}||\eta_{,xx}||^2_{1,\Omega_e} + h_e||\eta_{,xx}||^2_{2,\Omega_e}\right) \\
&= C\sum_{e=1}^{N_{el}} h_e^3\left(h_e^{-1}||\eta_{,xx}||^2_{1,\Omega_e} + h_e||\eta_{,xx}||^2_{2,\Omega_e}\right).
\end{aligned} \tag{4.128}$$

In consequence, from (4.127) – (4.128), we reach the conclusion that the factors enclosed into the first bracket, on the right-hand side of (4.126), can be bounded as follows

$$\begin{aligned}&\frac{(g^2)^2}{m}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}}+\frac{(g^2)^2}{m}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_c}\\&\leqslant C\sum_{e=1}^{N_{el}}h_e^3\left(h_e^{-1}||\eta_{,xx}||^2_{1,\Omega_e}+h_e||\eta_{,xx}||^2_{2,\Omega_e}\right).\end{aligned}\tag{4.129}$$

Moreover, we shall follow the above procedure in a similar manner to bound the terms enclosed into the third and the fifth bracket respectively, on the right-hand side of (4.126). As a consequence, we get

$$\begin{aligned}&\frac{(g^2)^2}{m}||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\tilde{\Gamma}}+\frac{(g^2)^2}{m}||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_q}\\&\leqslant C\sum_{e=1}^{N_{el}}h_e\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e}+h_e||\eta_{,xxx}||^2_{\Omega_e}\right)\end{aligned}\tag{4.130}$$

and

$$\frac{1}{m}||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\tilde{\Gamma}}+\frac{1}{m}||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_c}\leqslant C\sum_{e=1}^{N_{el}}h_e\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e}+h_e||\eta_{,xx}||^2_{\Omega_e}\right).\tag{4.131}$$

Additionally, we shall analogously estimate the factors enclosed into the second bracket on the right-hand side of (4.126). By recalling the jump inequality (A.18), we obtain

$$\begin{aligned}&\frac{C_\alpha g^2}{m}||h_e^{-3/2}[\![\eta]\!]||^2_{\tilde{\Gamma}}+\frac{C_\alpha g^2}{m}||h_e^{-3/2}\eta||^2_{\Gamma_c}\\&=\frac{C_\alpha g^2}{m}\sum_{i=1}^{N_i}||h_e^{-3/2}[\![\eta]\!]||^2_{\Gamma_i}+\frac{C_\alpha g^2}{m}\sum_{r=1}^{N_c}||h_e^{-3/2}\eta||^2_{\Gamma_r}\\&\leqslant\frac{C_\alpha g^2}{m}\sum_{i=1}^{N_i}2\left(||h_e^{-3/2}\eta^+||^2_{\Gamma_i}+||h_e^{-3/2}\eta^-||^2_{\Gamma_i}\right)+\frac{C_\alpha g^2}{m}\sum_{r=1}^{N_c}||h_e^{-3/2}\eta||^2_{\Gamma_r}\\&\leqslant 2\frac{C_\alpha g^2}{m}\sum_{e=1}^{N_{el}}||h_e^{-3/2}\eta||^2_{\partial\Omega_e}.\end{aligned}\tag{4.132}$$

We employ the trace inequality (A.38) and next the properties of Sobolev norms in (4.132), so we deduce

$$\begin{aligned} &2\frac{C_\alpha g^2}{m}\sum_{e=1}^{N_{el}}||h_e^{-3/2}\eta||^2_{\partial\Omega_e} \\ &\leqslant 2\frac{C_\alpha g^2}{m}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||h_e^{-3/2}\eta||^2_{\Omega_e}+h_e||h_e^{-3/2}\eta_{,x}||^2_{\Omega_e}\right) \\ &= C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta||^2_{\Omega_e}+h_e||\eta_{,x}||^2_{\Omega_e}\right). \end{aligned} \tag{4.133}$$

Ergo, from (4.132) – (4.133), we arrive to the conclusion that the factors enclosed into the second bracket, on the right-hand side of (4.126), can be estimated as follows

$$\frac{C_\alpha g^2}{m}||h_e^{-3/2}[\![\eta]\!]||^2_{\tilde\Gamma}+\frac{C_\alpha g^2}{m}||h_e^{-3/2}\eta||^2_{\Gamma_c}\leqslant C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta||^2_{\Omega_e}+h_e||\eta_{,x}||^2_{\Omega_e}\right). \tag{4.134}$$

Furthermore, we shall follow the above series of steps in the same way to estimate the terms enclosed into the fourth and the sixth bracket respectively on the right-hand side of (4.126). For thar reason, we have

$$\begin{aligned} &\frac{C_\beta g^2}{m}||h_e^{-1/2}[\![\eta_{,x}]\!]||^2_{\tilde\Gamma}+\frac{C_\beta g^2}{m}||h_e^{-1/2}\eta_{,x}||^2_{\Gamma_q} \\ &\leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e}+h_e||\eta_{,xx}||^2_{\Omega_e}\right) \end{aligned} \tag{4.135}$$

and

$$\frac{C_\delta}{m}||h_e^{-1/2}[\![\eta]\!]||^2_{\tilde\Gamma}+\frac{C_\delta}{m}||h_e^{-1/2}\eta||^2_{\Gamma_c}\leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta||^2_{\Omega_e}+h_e||\eta_{,x}||^2_{\Omega_e}\right). \tag{4.136}$$

What is more, we shall use similar arguments to bound the factors that enclosed into the seventh bracket on the right-hand side of (4.126). By applying the jump inequality (A.18), we deduce

$$\begin{aligned}
&\frac{1}{m}||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\alpha_c^{1/2}\eta||^2_{\Gamma_c} \\
&= \frac{1}{m}\sum_{i=1}^{N_i}||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i} + \frac{1}{m}\sum_{r=1}^{N_c}||\alpha_c^{1/2}\eta||^2_{\Gamma_r} \\
&\leqslant \frac{1}{m}\sum_{i=1}^{N_i} 2\left(||\alpha^{1/2}\eta^+||^2_{\Gamma_i} + ||\alpha^{1/2}\eta^-||^2_{\Gamma_i}\right) + \frac{1}{m}\sum_{r=1}^{N_c}||\alpha_c^{1/2}\eta||^2_{\Gamma_r} \\
&\leqslant \frac{2}{m}\sum_{e=1}^{N_{el}}||\alpha^{1/2}\eta||^2_{\partial\Omega_e}.
\end{aligned} \tag{4.137}$$

Afterwards, in (4.137), we invoke the trace inequality (A.38) and the properties of Sobolev norms giving

$$\begin{aligned}
&\frac{2}{m}\sum_{e=1}^{N_{el}}||\alpha^{1/2}\eta||^2_{\partial\Omega_e} \\
&\leqslant \frac{2}{m}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\alpha^{1/2}\eta||^2_{\Omega_e} + h_e||\alpha^{1/2}\eta_{,x}||^2_{\Omega_e}\right) \\
&= \frac{2}{m}\sum_{e=1}^{N_{el}} C\alpha\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right) \\
&= \frac{2}{m}\sum_{e=1}^{N_{el}} C\frac{C_\alpha g^2}{h_e^3}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right) \\
&= C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right).
\end{aligned} \tag{4.138}$$

Wherefore, from (4.137) – (4.138), we reach the conclusion that the terms enclosed into the seventh bracket, on the right-hand side of (4.126), can be bounded as follows

$$\frac{1}{m}||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\alpha_c^{1/2}\eta||^2_{\Gamma_c} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right). \tag{4.139}$$

Also, by following the previous procedure step by step, we shall estimate the terms respectively enclosed into the eighth and the ninth bracket on the right-hand side of (4.126). Therefore, we arrive at

$$\frac{1}{m}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \tag{4.140}$$

and

$$\frac{1}{m}||\delta^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + \frac{1}{m}||\delta_c^{1/2}\eta||^2_{\Gamma_c} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right). \tag{4.141}$$

After that, gathering the inequalities (4.129) – (4.131), (4.134) – (4.136), (4.139) – (4.141) and inserting them on the right-hand side of (4.126), we obtain

$$\begin{aligned}\frac{m}{3}|||e^h|||^2_{sb} \;\leqslant\; & C\sum_{e=1}^{N_{el}}\Big\{||\eta_{,xx}||^2_{\Omega_e} + ||\eta_{,x}||^2_{\Omega_e} \\ & + h_e^3\left(h_e^{-1}||\eta_{,xx}||^2_{1,\Omega_e} + h_e||\eta_{,xx}||^2_{2,\Omega_e}\right) \\ & + h_e^{-3}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right) \\ & + h_e\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||\eta_{,xxx}||^2_{\Omega_e}\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \\ & + h_e\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta||^2_{\Omega_e} + h_e||\eta_{,x}||^2_{\Omega_e}\right)\Big\}.\end{aligned} \tag{4.142}$$

Application of interpolation estimates, (A.30), (A.31), yields for the terms on the right-hand side of (4.142)

$$||\eta_{,xx}||_{\Omega_e} \leqslant ||\eta_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{2,\Omega_e} \leqslant Ch_e^{k-1}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{4.143}$$

$$||\eta_{,x}||_{\Omega_e} \leqslant ||\eta||_{1,\Omega_e} \leqslant Ch_e^{k}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{4.144}$$

$$||\eta_{,xx}||_{1,\Omega_e} \leqslant ||\eta||_{3,\Omega_e} \leqslant Ch_e^{k-2}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{4.145}$$

$$||\eta_{,xx}||_{2,\Omega_e} \leqslant ||\eta||_{4,\Omega_e} \leqslant Ch_e^{k-3}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{4.146}$$

$$||\eta||_{\Omega_e} \leqslant Ch_e^{k+1}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e). \tag{4.147}$$

Substitution of (4.143) – (4.147) on the right-hand side of (4.142) leads to

$$\begin{aligned} \frac{m}{3}|||e^h|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}}\left(h_e^{2k}+h_e^{2(k-1)}\right)|u|_{k+1,\Omega_e}^2 \\ &\leqslant C\sum_{e=1}^{N_{el}}h_e^{2(k-1)}|u|_{k+1,\Omega_e}^2. \end{aligned}$$

Then, multiplying by $\frac{3}{m}$ both sides of the above inequality, we reach to the conclusion that e^h can be estimated as

$$|||e^h|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}}h_e^{2(k-1)}|u|_{k+1,\Omega_e}^2. \tag{4.148}$$

To go on, we shall estimate η by using similar arguments as in the case of e^h. By the definition of energy seminorm, (4.26), we get

$$\begin{aligned} |||\eta|||_{sb}^2 &= ||(g^2)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}^2 + ||\eta_{,x}||_{\tilde{\Omega}}^2 + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 \\ &\quad + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2}\eta||_{\Gamma_q}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2. \end{aligned}$$

We proceed by employing the inequalities (4.139) – (4.141), having ignored the coefficient $\frac{1}{m}$, we can bound the terms on the right-hand side of the seminorm as

$$\begin{aligned} |||\eta|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}}\Big\{||\eta_{,xx}||_{\Omega_e}^2 + ||\eta_{,x}||_{\Omega_e}^2 + h_e^{-3}\left(h_e^{-1}||\eta||_{\Omega_e}^2 + h_e||\eta_{,x}||_{\Omega_e}^2\right) \\ &\quad + h_e^{-1}\left(h_e^{-1}||\eta_{,x}||_{\Omega_e}^2 + h_e||\eta_{,xx}||_{\Omega_e}^2\right) \\ &\quad + h_e^{-1}\left(h_e^{-1}||\eta||_{\Omega_e}^2 + h_e||\eta_{,x}||_{\Omega_e}^2\right)\Big\}. \end{aligned} \tag{4.149}$$

Afterwards, insertion of the mathematical expressions (4.143), (4.144), (4.147) into the right-hand side of (4.149) yields

$$\begin{aligned} |||\eta|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}}\left(h_e^{2k}+h_e^{2(k-1)}\right)|u|_{k+1,\Omega_e}^2 \\ &\leqslant C\sum_{e=1}^{N_{el}}h_e^{2(k-1)}|u|_{k+1,\Omega_e}^2. \end{aligned}$$

As a result, we conclude that η can be bounded as

$$|||\eta|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2. \tag{4.150}$$

Now, combining (4.117) with the inequalities (4.148) and (4.150), we have

$$\begin{aligned} |||u-u^h|||_{sb}^2 &\leqslant \left(|||\eta|||_{sb} + |||e^h|||_{sb}\right)^2 \\ &\leqslant 2\left(|||\eta|||_{sb}^2 + |||e^h|||_{sb}^2\right) \\ &\leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2. \end{aligned}$$

Finally, it follows that

$$|||u-u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2,$$

which is the desired result. □

It is noteworthy that the resulting a priori error estimate is optimal in h. Let us return to the a priori error analysis of the $h-$NIPG method.

Theorem 4.5.1.3. *Assume that the consistency condition* (4.50) *and Propostion 4.4.1.2 of the method hold. For each face, we define positive, real, piecewise constant functions* α, α, β, β_q, δ *and* δ_c *by*

$$\alpha = \alpha_c = \frac{C_\alpha g^2}{h_e^3}, \; \beta = \beta_q = \frac{C_\beta g^2}{h_e} \quad \text{and} \quad \delta = \delta_c = \frac{C_\delta}{h_e}.$$

Given that the conditions are satisfied for the interpolation estimates (A.30), (A.31) *and the trace inequalities* (A.38), (A.39) *hold, the error estimate for the non-symmetric interior penalty discontinuous Galerkin method* (4.49) *can be written as*

$$|||u-u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2, \tag{4.151}$$

where C *is a constant dependent only on the space dimension and on* k, *and* $|\cdot|_{k+1,\Omega_e}$ *denotes the* H^{k+1}*-seminorm on* Ω_e.

Proof. To begin with, we shall estimate e^h. For that purpose, we take advantage of the coercivity (4.51), the decomposition of the error (4.116) and the Galerkin orthogonality (4.50) yielding

$$\begin{aligned} |||e^h|||_{sb}^2 &= B_{sb}(e^h, e^h) \\ &= B_{sb}(u - u^h - \eta, e^h) \\ &= B_{sb}(u - u^h, e^h) - B_{sb}(\eta, e^h) \\ &= -B_{sb}(\eta, e^h) \\ &\leqslant |B_{sb}(\eta, e^h)|. \end{aligned} \tag{4.152}$$

We shall make use of arguments being totally the same as in the proof of error estimate of the SIPG method. Hence, it derives a bound on $|||e^h|||_{sb}$, in terms of suitable norms of η, which is

$$\begin{aligned} \frac{1}{3}|||e^h|||_{sb}^2 \leqslant\; & g^2||\eta_{,xx}||_{\tilde{\Omega}}^2 + ||\eta_{,x}||_{\tilde{\Omega}}^2 \\ & + \left\{(g^2)^2||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}}^2 + (g^2)^2||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_c}^2\right\} \\ & + \left\{C_\alpha g^2||h_e^{-3/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + C_\alpha g^2||h_e^{-3/2}\eta||_{\Gamma_c}^2\right\} \\ & + \left\{(g^2)^2||\beta^{-1/2}\langle\eta_{,xx}\rangle||_{\tilde{\Gamma}}^2 + (g^2)^2||\beta_q^{-1/2}\eta_{,xx}||_{\Gamma_q}^2\right\} \\ & + \left\{C_\beta g^2||h_e^{-1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + C_\beta g^2||h_e^{-1/2}\eta_{,x}||_{\Gamma_q}^2\right\} \\ & + \left\{||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\tilde{\Gamma}}^2 + ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_c}^2\right\} \\ & + \left\{C_\delta||h_e^{-1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + C_\delta||h_e^{-1/2}\eta||_{\Gamma_c}^2\right\} \\ & + \left\{||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2\right\} \\ & + \left\{||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2\right\} \\ & + \left\{||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2\right\}. \end{aligned} \tag{4.153}$$

To complete the proof, the next step to be followed involves estimating the terms enclosed into the brackets on the right-hand side of (4.153). By following the similar series of steps as in the proof of error estimate of the SIPG method, we reach to conclusion that e^h can be bounded as

$$|||e^h|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}} h_e^{2(k-1)}|u|_{k+1,\Omega_e}^2. \tag{4.154}$$

By analogous procedure in keeping with the previous proof, we conclude that η can be estimated as

$$|||\eta|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2. \quad (4.155)$$

Now, employing (4.117) as well as (4.154) and (4.155), we deduce that

$$|||u - u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-1)} |u|_{k+1,\Omega_e}^2,$$

which is the desired result. □

It is worthy of notice that the resulting a priori error estimate of the NIPG is optimal in h.

Let us focus on a priori error estimate of the *hp*-version of the methods presented in this chapter.

Convergence 4.5.1.4. *Let $\Pi_{\mathbf{p}}$ denote any (linear) projection operator from $H^{\mathbf{s}}(\Omega, \mathcal{P}(\Omega))$ onto the finite element space $\mathcal{V}^{hp}$. We can then decompose the global error $u - u_{DG}$ as follows:*

$$u - u_{DG} = (u - \Pi_{\mathbf{p}} u) + (\Pi_{\mathbf{p}} u - u_{DG}) \equiv \eta + \xi. \quad (4.156)$$

So, using the triangle inequality, we have

$$|||u - u_{DG}|||_{sb} \leqslant |||\eta|||_{sb} + |||\xi|||_{sb}, \quad (4.157)$$

where $\xi = \Pi_{\mathbf{p}} u - u_{DG}$ is the part of the error in the finite element space, i.e., $\xi \in \mathcal{V}^{hp}$.

Our error analysis below will provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η. Thus, we shall obtain a bound on $|||u - u_{DG}|||_{sb}$ with respect to various norms of η. Ergo, to complete the error analysis, we shall need to quantify norms of η in terms of the discretization parameters and Sobolev seminorms of the analytical solution u.

Theorem 4.5.1.5. *Suppose that Ω is a bounded domain in $\Re$ and that $\mathcal{P}(\Omega)$ is a regular partition of Ω into elements Ω_e. Let $\mathbf{p} = (p_e : \Omega_e \in \mathcal{P}(\Omega), p_e \in \aleph, p_e \geqslant 3)$ be any polynomial degree vector of bounded local variation. For each* *face, we define positive, real, piecewise constant functions* α, α_c, β, β_q, δ *and* δ_c *by*

$$\alpha = \alpha_c = \frac{C_\alpha g^2 p_e^6}{h_e^3}, \;\; \beta = \beta_q = \frac{C_\beta g^2 p_e^2}{h_e} \quad \textit{and} \quad \delta = \delta_c = \frac{C_\delta p_e^2}{h_e}.$$

Let us also suppose that the stabilization constants C_α, C_β *and* C_δ *are such that the bilinear form* $B_{sb}(\cdot,\cdot)$ *is coercive (see Proposition 4.4.1.5). If the analytical solution* u *to the problem* (4.25) *belongs to the broken Sobolev space* $H^{\mathbf{t}}(\Omega,\mathcal{P}(\Omega))$, $\mathbf{t} = (t_e : \Omega_e \in \mathcal{P}(\Omega), t_e \geqslant 4)$, *then the solution* $u_{DG} \in \mathcal{V}^{hp}$ *of the problem* (4.49) *satisfies the following error bound*

$$|||u - u_{DG}|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}} ||u||_{t_e,\Omega_e}^2, \tag{4.158}$$

where $2 \leqslant s_e \leqslant \min(p_e+1, t_e)$, *and* C *is a constant dependent only on the space dimension and on* $t = \max_{\Omega_e \in \mathcal{P}(\Omega)} t_e$.

Proof. To begin with, we shall estimate ξ. For that purpose, we take advantage of the coercivity (4.72), the decomposition of the error (4.156) and the Galerkin orthogonality (4.50) yielding

$$\begin{aligned} m|||\xi|||_{sb}^2 &\leqslant B_{sb}(\xi,\xi) \\ &= B_{sb}(u - u_{DG} - \eta, \xi) \\ &= B_{sb}(u - u_{DG}, \xi) - B_{sb}(\eta,\xi) \\ &= -B_{sb}(\eta,\xi) \\ &\leqslant |B_{sb}(\eta,\xi)|. \end{aligned} \tag{4.159}$$

We continue by using the triangle inequality on the right-hand side of (4.159). Then, we obtain

$$\begin{aligned} m|||\xi|||_{sb}^2 \leqslant\; & |(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| + |(\eta_{,x}, \xi_{,x})_{\tilde{\Omega}}| \\ & + |\langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\tilde{\Gamma}}| + |[\![\eta]\!]\langle (g^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| \\ & + |\langle g^2\eta_{,xx}\rangle [\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\tilde{\Gamma}}| \\ & + |\langle \eta_{,x}\rangle [\![\xi]\!]_{\tilde{\Gamma}}| + |[\![\eta]\!]\langle \xi_{,x}\rangle_{\tilde{\Gamma}}| \\ & + |\alpha[\![\eta]\!][\![\xi]\!]_{\tilde{\Gamma}}| + |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |\delta[\![\eta]\!][\![\xi]\!]_{\tilde{\Gamma}}| \\ & + |(g^2\eta_{,xx})_{,x}\cdot n\xi|_{\Gamma_c}| + |\eta(g^2\xi_{,xx})_{,x}\cdot n|_{\Gamma_c}| \\ & + |g^2\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| + |\eta_{,x}\cdot n g^2\xi_{,xx}|_{\Gamma_q}| \\ & + |\eta_{,x}\cdot n\xi|_{\Gamma_c}| + |\eta\xi_{,x}\cdot n|_{\Gamma_c}| \\ & + |\alpha_c\eta\xi|_{\Gamma_c}| + |\beta_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}| + |\delta_c\eta\xi|_{\Gamma_c}|. \end{aligned} \tag{4.160}$$

Thereby, to provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η, it only remains to estimate the inner products on the right-hand side of (4.160).

With the aim of bounding the first inner product on the right-hand side of (4.160), we initially apply the triangle inequality yielding

$$\begin{aligned} |(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| &= \left| \sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e} \right| \\ &\leqslant \sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e}|. \end{aligned} \tag{4.161}$$

Then, by recalling the Cauchy-Schwarz inequality (A.12) and next the Cauchy-Schwarz discrete inequality (A.13) in (4.161), we have

$$\begin{aligned} &\sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e}| \\ &\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\eta_{,xx}||_{\Omega_e} ||(g^2)^{1/2}\xi_{,xx}||_{\Omega_e} \\ &\leqslant \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\eta_{,xx}||^2_{\Omega_e} \right)^{1/2} \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e} \right)^{1/2} \\ &= \left(||(g^2)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2}. \end{aligned} \tag{4.162}$$

By making use of the defintion of energy seminorm, (4.26), in (4.162), we get

$$\left(||(g^2)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{4.163}$$

Therefore, from (4.161) – (4.163), we reach the conclusion that the first inner product, on the right-hand side of (4.160), can be bounded as follows

$$|(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{4.164}$$

Also, the second inner product, on the right-hand side of (4.160), can analogously be bounded as

$$|(\eta_{,x}, \xi_{,x})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{4.165}$$

We shall additionally follow similar series of steps to estimate the stabilizing terms on the right-hand side of (4.160). Employing the triangle inequality, we deduce

$$\begin{aligned} |\alpha[\![\eta]\!][\![\xi]\!]_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i} \alpha[\![\eta]\!][\![\xi]\!]_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i} |\alpha[\![\eta]\!][\![\xi]\!]_{\Gamma_i}|. \end{aligned} \tag{4.166}$$

After that, by invoking the Cauchy-Schwarz inequality (A.12) and the Cauchy-Schwarz discrete inequality (A.13) in (4.166), we conclude

$$\begin{aligned} &\sum_{i=1}^{N_i} |\alpha[\![\eta]\!][\![\xi]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||_{\Gamma_i}||\alpha^{1/2}[\![\xi]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\xi]\!]||^2_{\Gamma_i}\right)^{1/2} \\ &= \left(||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||\alpha^{1/2}[\![\xi]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}. \end{aligned} \tag{4.167}$$

Using the defintion of energy seminorm, (4.26), in (4.167), it derives

$$\left(||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||\alpha^{1/2}[\![\xi]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{4.168}$$

Ergo, from (4.166) – (4.168), we arrive to the conclusion that the first stabilizing term, on right-hand side of (4.160), can be estimated as follows

$$|\alpha[\![\eta]\!][\![\xi]\!]_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{4.169}$$

Moreover, the rest of stabilizing terms, on the right hand side of (4.160), can correspondingly be bounded as

$$\begin{aligned} |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\delta[\![\eta]\!][\![\xi]\!]_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\alpha_c\eta\xi|_{\Gamma_c}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\beta_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\delta_c\eta\xi|_{\Gamma_c}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \end{aligned} \tag{4.170}$$

It's about time for us to estimate inner products, containing the mean value operator of η and the jump operator of ξ, on the right-hand side of (4.160). We use at first the triagle inequality, as a result we get

$$\begin{aligned} |\langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\tilde{\Gamma}}| &= \left| \sum_{i=1}^{N_i} \langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\Gamma_i} \right| \\ &\leqslant \sum_{i=1}^{N_i} |\langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\Gamma_i}|. \end{aligned} \tag{4.171}$$

Afterwards, applying the Cauchy-Schwarz inequality (A.12) and then the Cauchy-Schwarz discrete inequality (A.13) in (4.171), we have

$$\begin{aligned} &\sum_{i=1}^{N_i} |\langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||_{\Gamma_i} ||\alpha^{1/2}[\![\xi]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} \right)^{1/2} \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\xi]\!]||^2_{\Gamma_i} \right)^{1/2} \\ &= \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} \right)^{1/2} \left(||\alpha^{1/2}[\![\xi]\!]||^2_{\tilde{\Gamma}} \right)^{1/2}. \end{aligned} \tag{4.172}$$

Invoking the defintion of energy seminorm, (4.26), in (4.172), we obtain

$$\begin{aligned} &\left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} \right)^{1/2} \left(||\alpha^{1/2}[\![\xi]\!]||^2_{\tilde{\Gamma}} \right)^{1/2} \\ &\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} \right)^{1/2} |||\xi|||_{sb}. \end{aligned} \tag{4.173}$$

In consequence, from (4.171) – (4.173), we conclude that this type of inner product, on the right-hand side of (4.160), can subsequentlybe bounded as

$$|\langle (g^2\eta_{,xx})_{,x}\rangle [\![\xi]\!]_{\tilde{\Gamma}}| \leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} \right)^{1/2} |||\xi|||_{sb}. \tag{4.174}$$

Furthermore, we shall use similar arguments to estimate the remaining inner products of the corresponding form, on the right-hand side of (4.160). Thus, we deduce

$$
\begin{aligned}
|\langle g^2\eta_{,xx}\rangle [\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2} |||\xi|||_{sb},\\
|\langle \eta_{,x}\rangle [\![\xi]\!]_{\tilde{\Gamma}}| &\leqslant \left(\sum_{i=1}^{N_i} ||\delta^{-1/2}\langle \eta_{,x}\rangle||^2_{\Gamma_i}\right)^{1/2} |||\xi|||_{sb},\\
|(g^2\eta_{,xx})_{,x}\cdot n\xi|_{\Gamma_c}| &\leqslant \left(\sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||^2_{\Gamma_r}\right)^{1/2} |||\xi|||_{sb},\\
|g^2\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| &\leqslant \left(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}g^2\eta_{,xx}||^2_{\Gamma_j}\right)^{1/2} |||\xi|||_{sb},\\
|\eta_{,x}\cdot n\xi|_{\Gamma_c}| &\leqslant \left(\sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_r}\right)^{1/2} |||\xi|||_{sb},
\end{aligned}
\tag{4.175}
$$

where N_c denotes the number of exterior displacement boundary segments $\Gamma_r \subseteq \Gamma_c$ and N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$, as well.

A last step, for bounding $|||\xi|||_{sb}$ in terms of norms of η, is to estimate the rest of inner products, which contain the jump operator of η and the mean value operator of ξ, on the right-hand side of (4.160). As in the latter case, employing the triangle inequality, we get

$$
\begin{aligned}
|[\![\eta]\!]\langle (g^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}[\![\eta]\!]\langle (g^2\xi_{,xx})_{,x}\rangle_{\Gamma_i}\right|\\
&\leqslant \sum_{i=1}^{N_i} |[\![\eta]\!]\langle (g^2\xi_{,xx})_{,x}\rangle_{\Gamma_i}|.
\end{aligned}
\tag{4.176}
$$

Thereafter, by recalling the Cauchy-Schwarz inequality (A.12) and the Cauchy-

Schwarz discrete inequality (A.13) in (4.176), we conclude

$$\begin{aligned}
&\sum_{i=1}^{N_i} |[\![\eta]\!]\langle (g^2\xi_{,xx})_{,x}\rangle_{\Gamma_i}| \\
&\leqslant \sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||_{\Gamma_i} ||\alpha^{-1/2}\langle (g^2\xi_{,xx})_{,x}\rangle||_{\Gamma_i} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\xi_{,xx})_{,x}\rangle||^2_{\Gamma_i}\right)^{1/2} .
\end{aligned} \tag{4.177}$$

By invoking the mean value inequality (A.19) in (4.177), we now have

$$\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle (g^2\xi_{,xx})_{,x}\rangle||^2_{\Gamma_i}\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{i=1}^{N_i} (||\alpha^{-1/2}(g^2\xi^+_{,xx})_{,x}||^2_{\Gamma_i} + ||\alpha^{-1/2}(g^2\xi^-_{,xx})_{,x}||^2_{\Gamma_i})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (||\alpha^{-1/2}(g^2\xi_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||\alpha^{-1/2}(g^2\xi_{,xx})_{,x}||^2_{\partial\Omega_e})\right)^{1/2} .
\end{aligned} \tag{4.178}$$

Also, in (4.178), since $\xi \in \mathcal{V}^{hp}$ we can apply the inverse inequality (A.21), so we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (||\alpha^{-1/2}(g^2\xi_{,xx})_{,x}||^2_{\partial\Omega_{e'}} + ||\alpha^{-1/2}(g^2\xi_{,xx})_{,x}||^2_{\partial\Omega_e})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (c_1\frac{p^6_{e'}}{h^3_{e'}}||\alpha^{-1/2}g^2\xi_{,xx}||^2_{\Omega_{e'}} + c_1\frac{p^6_e}{h^3_e}||\alpha^{-1/2}g^2\xi_{,xx}||^2_{\Omega_e})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} c_1\frac{p^6_e}{h^3_e}||\alpha^{-1/2}g^2\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_1}{C_\alpha}||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2},
\end{aligned} \tag{4.179}$$

where the constant c_1 is independent of h_e, p_e and ξ. In (4.179), we choose $\frac{c_1}{C_\alpha} \leqslant 1$ without loss of generality. Thereby, we deduce

$$\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_1}{C_\alpha}||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}}\right)^{1/2}.
\end{aligned} \tag{4.180}$$

In (4.180), by making use of the definition of energy seminorm, (4.26), we conclude

$$\left(||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{4.181}$$

Wherefore, from (4.176) – (4.181), we arrive to the conclusion that this type of inner product, on the right-hand side of (4.160), can be bounded as follows

$$|[\![\eta]\!]\langle(g^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{4.182}$$

What is more, by following the above procedure in a similar manner, we shall achieve to estimate the rest of inner products of the corresponding form, on the right-hand side of (4.160). As a consequence, we have

$$\begin{aligned}
|[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb},\\
|[\![\eta]\!]\langle \xi_{,x}\rangle_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb},\\
|\eta(g^2\xi_{,xx})_{,x}\cdot n|_{\Gamma_c}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb},\\
|\eta_{,x}\cdot ng^2\xi_{,xx}|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb},\\
|\eta\xi_{,x}\cdot n|_{\Gamma_c}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}.
\end{aligned} \tag{4.183}$$

At this point, we gather the inequalities (4.164) – (4.165), (4.169) – (4.170), (4.174) – (4.175), (4.182) – (4.183) and insert them on the right-hand side of (4.160). So, it derives

$$\begin{aligned}
m|||\xi|||_{sb}^2 \;\leqslant\; & C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle(g^2\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2\Big)^{1/2}\\
&+\Big(\sum_{r=1}^{N_c}||\alpha_c^{-1/2}(g^2\eta_{,xx})_{,x}||_{\Gamma_r}^2\Big)^{1/2} + \Big(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\Gamma_i}^2\Big)^{1/2}\\
&+\Big(\sum_{j=1}^{N_q}||\beta_q^{-1/2}g^2\eta_{,xx}||_{\Gamma_j}^2\Big)^{1/2} + \Big(\sum_{i=1}^{N_i}||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\Gamma_i}^2\Big)^{1/2}\\
&+\Big(\sum_{r=1}^{N_c}||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_r}^2\Big)^{1/2}\Big\}|||\xi|||_{sb},
\end{aligned}$$

which implies that

$$
\begin{aligned}
|||\xi|||_{sb} \;\leqslant\; & C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\
& + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} + \Big(\sum_{i=1}^{N_i} ||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\
& + \Big(\sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r}\Big)^{1/2} + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2} \\
& + \Big(\sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_r}\Big)^{1/2}\Big\}. \qquad (4.184)
\end{aligned}
$$

By combining at once the mathematical expression (4.157) with (4.184), we get

$$
\begin{aligned}
|||u - u_{DG}|||_{sb} \;\leqslant\; & C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\
& + \Big(\sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r}\Big)^{1/2} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\
& + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2} + \Big(\sum_{i=1}^{N_i} ||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\
& + \Big(\sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_r}\Big)^{1/2}\Big\}
\end{aligned}
$$

or by successive use of (A.14), we have

$$
\begin{aligned}
|||u - u_{DG}|||^2_{sb} \;\leqslant\; & C\Big\{|||\eta|||^2_{sb} \\
& + \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r} \\
& + \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_r}\Big\}. \qquad (4.185)
\end{aligned}
$$

Therefore, we have obtained a bound on $|||u - u_{DG}|||_{sb}$ in terms of various norms of η. Thereby, to complete the proof, it only remains to estimate the terms appearing on the right-hand side of (4.185). We note that $\eta \notin \mathcal{V}^{hp}$.

To estimate the first term, we shall make use of the definition of energy seminorm, (4.26), yielding

$$\begin{aligned} |||\eta|||_{sb}^2 \;\; &\leqslant \;\; C\sum_{e=1}^{N_{el}} \left\{||\eta_{,xx}||_{\Omega_e}^2 + ||\eta_{,x}||_{\Omega_e}^2\right\} + ||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 \\ &\quad + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2}\eta_{,x}||_{\Gamma_q}^2 + ||\delta^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\delta_c^{1/2}\eta||_{\Gamma_c}^2. \end{aligned} \tag{4.186}$$

We shall additionally bound the factors on the right-hand side of (4.186). By recalling (A.32) for the first two norms, we obtain

$$||\eta_{,x}||_{\Omega_e} \leqslant ||\eta||_{1,\Omega_e} \leqslant C\frac{h_e^{s_e-1}}{p_e^{t_e-1}}||u||_{t_e,\Omega_e} \tag{4.187}$$

and

$$||\eta_{,xx}||_{\Omega_e} \leqslant ||\eta_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{2,\Omega_e} \leqslant C\frac{h_e^{s_e-2}}{p_e^{t_e-2}}||u||_{t_e,\Omega_e}. \tag{4.188}$$

Subsequently, we shall pay particular attention to estimate the terms, containing the stabilization parameters α and α_c, on the right-hand side of (4.186). By applying the jump inequality (A.18), we deduce that

$$\begin{aligned} &||\alpha^{1/2}[\![\eta]\!]||_{\tilde{\Gamma}}^2 + ||\alpha_c^{1/2}\eta||_{\Gamma_c}^2 \\ &= \sum_{i=1}^{N_i} ||\alpha^{1/2}[\![\eta]\!]||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} ||\alpha_c^{1/2}\eta||_{\Gamma_r}^2 \\ &\leqslant \sum_{i=1}^{N_i} 2\left(||\alpha^{1/2}\eta^+||_{\Gamma_i}^2 + ||\alpha^{1/2}\eta^-||_{\Gamma_i}^2\right) + \sum_{r=1}^{N_c} ||\alpha_c^{1/2}\eta||_{\Gamma_r}^2 \\ &\leqslant 2\sum_{e=1}^{N_{el}} ||\alpha^{1/2}\eta||_{\partial\Omega_e}^2. \end{aligned} \tag{4.189}$$

Afterwards, in (4.189), we get

$$2\sum_{e=1}^{N_{el}} ||\alpha^{1/2}\eta||_{\partial\Omega_e}^2 = C\sum_{e=1}^{N_{el}} \frac{p_e^6}{h_e^3}||\eta||_{\Omega_e}^2. \tag{4.190}$$

Now, employing (A.33) in (4.190), we have

$$\begin{aligned} & C\sum_{e=1}^{N_{el}}\frac{p_e^6}{h_e^3}||\eta||^2_{\Omega_e} \\ & \leqslant C\sum_{e=1}^{N_{el}}\frac{p_e^6}{h_e^3}\frac{h_e^{2s_e-1}}{p_e^{2t_e-1}}||u||^2_{t_e,\Omega_e} \\ & = C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{4.191}$$

Hence, from (4.189) – (4.191), we conclude that the factors, including the stabilization parameters α and α_c on the right hand side of (4.186), can be bounded as follows

$$||\alpha^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + ||\alpha_c^{1/2}\eta||^2_{\Gamma_c} \leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||^2_{t_e,\Omega_e}. \tag{4.192}$$

We analogously deduce that the remaining terms, containing the stabilization parameters β and β_q as well as δ and δ_q on the right hand side of (4.186), can be bounded as follows

$$\begin{aligned} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} &\leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}||u||^2_{t_e,\Omega_e}, \\ ||\delta^{1/2}[\![\eta]\!]||^2_{\tilde{\Gamma}} + ||\delta_c^{1/2}\eta||^2_{\Gamma_c} &\leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-2}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{4.193}$$

Thereafter, insertion of the mathematical inequalities (4.187) – (4.188) and (4.192) – (4.193) into the right-hand side of (4.186) yields

$$\begin{aligned}
|||\eta|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}}\left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-4}}+\frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}+\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}\right)||u||_{t_e,\Omega_e}^2 \\
&\quad + C\sum_{e=1}^{N_{el}}\left(\frac{h_e^{2s_e-2}}{p_e^{2t_e-2}}+\frac{h_e^{2s_e-2}}{p_e^{2t_e-3}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}}\left(\frac{h_e^{2s_e-2}}{p_e^{2t_e-3}}+\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||_{t_e,\Omega_e}^2.
\end{aligned}$$

As a result, we conclude that η can be bounded as

$$|||\eta|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||_{t_e,\Omega_e}^2. \tag{4.194}$$

Into the bargain, we shall estimate the remaining factors on the right-hand side of (4.185). By using the mean value inequality (A.19), we can write the terms including the stabilization parameters as

$$\begin{aligned}
&\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2+\sum_{r=1}^{N_c}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_r}^2 \\
&\leqslant \sum_{i=1}^{N_i}\left(||\alpha^{-1/2}(\eta_{,xx}^{+})_{,x}||_{\Gamma_i}^2+||\alpha^{-1/2}(\eta_{,xx}^{-})_{,x}||_{\Gamma_i}^2\right)+\sum_{r=1}^{N_c}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_r}^2 \\
&\leqslant \sum_{e=1}^{N_{el}}||\alpha^{-1/2}(\eta_{,xx})_{,x}||_{\partial\Omega_e}^2.
\end{aligned} \tag{4.195}$$

Next, in (4.195), we get

$$\sum_{e=1}^{N_{el}}||\alpha^{-1/2}(\eta_{,xx})_{,x}||_{\partial\Omega_e}^2 = C\sum_{e=1}^{N_{el}}\frac{h_e^3}{p_e^6}||(\eta_{,xx})_{,x}||_{\partial\Omega_e}^2. \tag{4.196}$$

Now, using (A.33) in (4.196), we have

$$\begin{aligned} & C\sum_{e=1}^{N_{el}} \frac{h_e^3}{p_e^6}||(\eta_{,xx})_{,x}||^2_{\partial\Omega_e} \\ & \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^3}{p_e^6}\frac{h_e^{2s_e-7}}{p_e^{2t_e-7}}||u||^2_{t_e,\Omega_e} \\ & = C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-1}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{4.197}$$

Ergo, from (4.195) – (4.197), we arrive to the conclusion that the terms, including the stabilization parameters α and α_c on the right-hand side of (4.185), can be bounded as follows

$$\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c}||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_r} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-1}}||u||^2_{t_e,\Omega_e}. \tag{4.198}$$

By following arguments in a same way, we deduce that the rest of the terms, containing the stabilization parameters β and β_q as well as δ and δ_c on the right of (4.185), can be estimated as

$$\begin{aligned} \sum_{i=1}^{N_i}||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q}||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}, \\ \sum_{i=1}^{N_i}||\delta^{-1/2}\langle\eta_{,x}\rangle||^2_{\Gamma_i} + \sum_{r=1}^{N_c}||\delta_c^{-1/2}\eta_{,x}||^2_{\Gamma_r} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-2}}{p_e^{2t_e-1}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{4.199}$$

Inserting the inequalities (4.194), (4.198) and (4.199), into the right-hand side of (4.185) and just by combining with each other, gives

$$\begin{aligned}
|||u-u_{DG}|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-2}}{p_e^{2t_e-1}}||u||_{t_e,\Omega_e}^2 \\
&\quad + C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-1}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-2}}{p_e^{2t_e-1}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||_{t_e,\Omega_e}^2.
\end{aligned}$$

So, we conclude that

$$|||u-u_{DG}|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||_{t_e,\Omega_e}^2,$$

which is the desired result. □

It is worth noting that the resulting a priori error estimate is optimal in h but is p-suboptimal by $\frac{3}{2}$ orders of p.

Let us return to the a priori error analysis of the hp-NIPG method.

Theorem 4.5.1.6. *Suppose that Ω is a bounded domain in $\Re$ and that $\mathcal{P}(\Omega)$ is a regular partition of Ω into elements Ω_e. Let $\mathbf{p} = (p_e : \Omega_e \in \mathcal{P}(\Omega), p_e \in \aleph$, $p_e \geqslant 3)$ be any polynomial degree vector of bounded local variation. To each face, we define positive, real, piecewise constant functions α, α, β, β_q, δ and δ_c by*

$$\alpha = \alpha_c = \frac{C_\alpha g^2 p_e^6}{h_e^3},\ \ \beta = \beta_q = \frac{C_\beta g^2 p_e^2}{h_e} \quad \textit{and} \quad \delta = \delta_c = \frac{C_\delta p_e^2}{h_e},$$

where the stabilization constants C_α, C_β and C_δ are arbitrary positive real numbers. Let us suppose that the analytical solution u to the problem (4.25) belongs to the broken Sobolev space $H^{\mathbf{t}}(\Omega, \mathcal{P}(\Omega))$, $\mathbf{t} = (t_e : \Omega_e \in \mathcal{P}(\Omega), t_e \geqslant 4)$. Then, the solution $u_{DG} \in \mathcal{V}^{hp}$ obtained from the NIPG method (4.49) satisfies the following error bound

$$|||u-u_{DG}|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}}||u||_{t_e,\Omega_e}^2, \tag{4.200}$$

where $2 \leqslant s_e \leqslant \min(p_e+1, t_e)$*, and* C *is a constant dependent only on the space dimension and on* $t = \max_{\Omega_e \in \mathcal{P}(\Omega)} t_e$.

Proof. To begin with, we shall estimate ξ. For that purpose, we take advantage of the coercivity (4.52), the decomposition of the error (4.156) and the Galerkin orthogonality (4.50) yielding

$$\begin{aligned} |||\xi|||_{sb}^2 &= B_{sb}(\xi,\xi) \\ &= B_{sb}(u - u_{DG} - \eta, \xi) \\ &= B_{sb}(u - u_{DG}, \xi) - B_{sb}(\eta,\xi) \\ &= -B_{sb}(\eta,\xi) \\ &\leqslant |B_{sb}(\eta,\xi)|. \end{aligned} \tag{4.201}$$

We shall employ arguments identical to the ones used in the proof of the error estimate of the SIPG method. Hence, it derives a bound on $|||\xi|||_{sb}$, in terms of suitable norms of η, being

$$\begin{aligned} |||\xi|||_{sb} &\leqslant C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2\Big)^{1/2} \\ &+ \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||_{\Gamma_i}^2\Big)^{1/2} + \Big(\sum_{i=1}^{N_i} ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\Gamma_i}^2\Big)^{1/2} \\ &+ \Big(\sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_r}^2\Big)^{1/2} + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||_{\Gamma_j}^2\Big)^{1/2} \\ &+ \Big(\sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_r}^2\Big)^{1/2}\Big\}. \end{aligned} \tag{4.202}$$

By combining at once the mathematical expression (4.157) with (4.202), we get

$$\begin{aligned} |||u - u_{DG}|||_{sb}^2 &\leqslant C\Big\{|||\eta|||_{sb}^2 + \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} ||\alpha_c^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_r}^2 \\ &+ \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||_{\Gamma_i}^2 + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||_{\Gamma_j}^2 \\ &+ \sum_{i=1}^{N_i} ||\delta^{-1/2}\langle\eta_{,x}\rangle||_{\Gamma_i}^2 + \sum_{r=1}^{N_c} ||\delta_c^{-1/2}\eta_{,x}||_{\Gamma_r}^2\Big\}. \end{aligned} \tag{4.203}$$

Therefore, we have achieved to provide a bound on $|||u - u_{DG}|||_{sb}$ in terms of various norms of η.

To complete the proof, we shall follow series of steps in a same way, as the above proof, in order to estimate the terms on the right-hand side of (4.203). We as well note that $\eta \notin \mathcal{V}^{hp}$.

In consequence, we can easily deduce that

$$|||u - u_{DG}|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-7}} ||u||_{t_e,\Omega_e}^2,$$

which is the desired result. □

It is significant to be mentioned that the resulting a priori error estimate is optimal in h but is p-suboptimal by $\frac{3}{2}$ orders of p.

4.6 Conclusions

The objective of this chapter is to establish a different approach for the one-dimensional Toupin-Mindlin strain gradient bar in tension. The interior penalty discontinuous Galerkin FEMs that we have introduced for this purpose exhibit the subsequent features:

1. The stabilizing terms have crucial importance for the convergence of the discontinuous Galerkin methods. Also, the choice of the stabilization constants is not critical for the convergence if their selected values are large enough.

2. Discontinuous piecewise quadratic up to 6 degree polynomials have been employed, leading to the straightforward implementations of the method.

3. The method is consistent, stable and convergent.

Chapter 5

CIPFEM for SGE in 1-D

5.1 Weak Formulation

We are ready to derive the weak formulation for the problem (4.9) – (4.10), which will lead to the continuous interior penalty finite element method. We shall suppose for the moment that the solution u of the problem is a sufficiently smooth function.

For each face $\Gamma_i \subseteq \tilde{\Gamma}$, let k and l be such indices that $k > l$ and the elements $\Omega_e := \Omega_e^k$ and $\Omega_{e'} := \Omega_e^l$ share the face Γ_i. Let us define the jump across Γ_i and the mean value on Γ_i of $u \in H^1(\Omega, \mathcal{P}(\Omega))$ by

$$[\![u]\!]_{\Gamma_i} := u|_{\partial\Omega_e \cap \Gamma_i} - u|_{\partial\Omega_{e'} \cap \Gamma_i} \quad \text{and} \quad \langle u \rangle_{\Gamma_i} := \frac{1}{2}\left(u|_{\partial\Omega_e \cap \Gamma_i} + u|_{\partial\Omega_{e'} \cap \Gamma_i}\right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and of the mean value to $\Gamma_j \subseteq \Gamma_q$ that belongs to the boundary Γ by letting:

$$[\![u]\!]_{\Gamma_j} = u|_{\Gamma_j} \quad \text{and} \quad \langle u \rangle_{\Gamma_j} = u|_{\Gamma_j}$$

In these definitions, the subscripts Γ_i and Γ_j will be supressed when no confusion is likely to occur. With each face $\Gamma_i \subseteq \tilde{\Gamma}$, we associate the unit normal vector $n = n_{\Omega_e^k}$, pointing from element Ω_e^k to Ω_e^l when $k > l$, and we choose $n = n_{\Omega_e}$ to be the unit outward normal when a node belongs to the boundary Γ.

Since the method will be non-conforming, we shall use the broken Sobolev space $H^4(\Omega, \mathcal{P}(\Omega))$ as trial space. We multiply the equation, (4.9), by a test function $w \in H^4(\Omega, \mathcal{P}(\Omega))$ and integrate over Ω

$$\int_\Omega (g^2 u_{,xx} - u)_{,xx} w dx = \int_\Omega f w dx.$$

Afterwards, we split the integrals

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} (g^2 u_{,xx} - u)_{,xx} w dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx,$$

and applying integration by parts on every elemental integral, so we get

$$\begin{aligned} &\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\ &- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} g^2 u_{,xx} w_{,x} \cdot n ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx, \end{aligned}$$

where n denotes the outward normal to each element boundary.

Now, we split the boundary terms as follows

$$\begin{aligned} &\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\ &+ \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot n w ds + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_P} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\ &- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot n ds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds \\ &- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_R} g^2 u_{,xx} w_{,x} \cdot n ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx, \end{aligned}$$

and hence we have

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\
&+ \int_{\Gamma_c} (g^2 u_{,xx} - u)_{,x} \cdot nwds + \int_{\Gamma_P} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot nds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds - \int_{\Gamma_R} g^2 u_{,xx} w_{,x} \cdot nds \\
&+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx.
\end{aligned}
\tag{5.1}
$$

We note that w vanishes on Γ_c. Next, using the natural boundary conditions, (4.10), on the fourth and on the seventh term respectively, on the left-hand side of (5.1) and moving it to the right-hand side, we obtain

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (g^2 u_{,xx} - u)_{,x} \cdot nwds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} g^2 u_{,xx} w_{,x} \cdot nds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot nds \\
&+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx + \int_{\Gamma_P} \frac{P}{AE} wds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot nds.
\end{aligned}
\tag{5.2}
$$

The second and the third term respectively on the left-hand side of (5.2) contain the boundary integrals over the interior element boundaries, i.e. the interior boundaries $\Gamma_i \subseteq \tilde{\Gamma}$. Consequently, in this sum of boundary integrals, we have two integrals over every interior boundary.

In order to evaluate the integrals on interior boundaries, we always use the interior trace of the test function w. Taking into account the Remark 4.2.0.1 and applying (4.13), we can see that the second and the third term respectively, on the left-hand side of (5.2), can subsequently be rewritten as

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Gamma}} \langle (g^2 u_{,xx} - u)_{,x} \rangle [\![w]\!] ds + \int_{\tilde{\Gamma}} [\![(g^2 u_{,xx} - u)_{,x}]\!] \langle w \rangle ds$$
$$- \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle ds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds$$
$$+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx + \int_{\Gamma_P} \frac{P}{AE} w ds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot n ds. \tag{5.3}$$

Since $w \in H^1(\Omega)$, the jump $[\![w]\!]$ vanishes on Ω and therefore on $\tilde{\Gamma}$. What's more, by noting that the fluxes $(g^2 u_{,xx} - u)_{,x} \cdot n$ and $g^2 u_{,xx}$ are continuous across the interelement boundaries Γ_i (e.g., when the exact solution $u \in H^4(\Omega)$), we have

$$\int_{\tilde{\Gamma}} [\![(g^2 u_{,xx} - u)_{,x}]\!] \langle w \rangle ds = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)),$$
$$\int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x} \rangle ds = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)).$$

Then, (5.3) reduces to

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx$$
$$- \int_{\tilde{\Gamma}} \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\Gamma_q} g^2 u_{,xx} w_{,x} \cdot n ds \tag{5.4}$$
$$= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx + \int_{\Gamma_P} \frac{P}{AE} w ds + \int_{\Gamma_R} \frac{R}{AE} w_{,x} \cdot n ds.$$

Moreover, from the boundary condition $u_{,x} \cdot n = q$, on Γ_q, upon multiplying by $-g^2 w_{,xx} + \beta_q w_{,x} \cdot n$ and integrating over Γ_q, we have

$$- \int_{\Gamma_q} u_{,x} \cdot n g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q u_{,x} \cdot n w_{,x} \cdot n ds = - \int_{\Gamma_q} q g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q q w_{,x} \cdot n ds. \tag{5.5}$$

The non-negative piecewise continuous function β_q, defined on Γ_q, is referred to as the stabilization parameter.

In addition, since we have an elliptic boundary value problem, elliptic regularity ensures us that $u_{,x}$ will be continuous on Ω. In that case the jump $[\![u_{,x}]\!]$ vanishes, i.e. $[\![u_{,x}]\!] = 0$. If we choose $-\langle g^2 w_{,xx}\rangle + \beta[\![w_{,x}]\!]$ as test function and integrate over $\tilde{\Gamma}$, it gives

$$-\int_{\tilde{\Gamma}} [\![u_{,x}]\!]\langle g^2 w_{,xx}\rangle ds + \int_{\tilde{\Gamma}} \beta[\![u_{,x}]\!][\![w_{,x}]\!] ds = 0, \tag{5.6}$$

where β is a non-negative continuous function, defined on $\tilde{\Gamma}$, which is referred to as the stabilization parameter.

Now adding (5.4) – (5.6), we get the continuous interior penalty weak formulation of the problem

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx \\
&- \int_{\tilde{\Gamma}} \langle g^2 u_{,xx}\rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}} [\![u_{,x}]\!]\langle g^2 w_{,xx}\rangle ds + \int_{\tilde{\Gamma}} \beta[\![u_{,x}]\!][\![w_{,x}]\!] ds \\
&- \int_{\Gamma_q} g^2 u_{,xx} w_{,x}\cdot n ds - \int_{\Gamma_q} u_{,x}\cdot n g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q u_{,x}\cdot n w_{,x}\cdot n ds \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx + \int_{\Gamma_P} \frac{P}{AE} w ds + \int_{\Gamma_R} \frac{R}{AE} w_{,x}\cdot n ds \\
&- \int_{\Gamma_q} q g^2 w_{,xx} ds + \int_{\Gamma_q} \beta_q q w_{,x}\cdot n ds.
\end{aligned} \tag{5.7}$$

The bilinear form is defined as

$$\begin{aligned}
B_{sb}(u,w) \;:=\; & (g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} + (u_{,x}, w_{,x})_{\tilde{\Omega}} \\
& - \langle g^2 u_{,xx}\rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} - [\![u_{,x}]\!]\langle g^2 w_{,xx}\rangle_{\tilde{\Gamma}} + \beta[\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} \\
& - g^2 u_{,xx} w_{,x}\cdot n|_{\Gamma_q} - u_{,x}\cdot n g^2 w_{,xx}|_{\Gamma_q} + \beta_q u_{,x}\cdot n w_{,x}\cdot n|_{\Gamma_q}.
\end{aligned} \tag{5.8}$$

We introduce the linear functional $L_{sb}(\cdot)$ on $H^4(\Omega, \mathcal{P}(\Omega))$

$$L_{sb}(w) \;:=\; (f,w)_{\tilde{\Omega}} + \frac{P}{AE} w|_{\Gamma_P} + \frac{R}{AE} w_{,x}\cdot n|_{\Gamma_R} - q g^2 w_{,xx}|_{\Gamma_q} + \beta_q q w_{,x}\cdot n|_{\Gamma_q}. \tag{5.9}$$

The stabilization parameters, β as well as β_q, depend on the discretization parameters h_e and p_e for the hp-method, in a manner that will be specified later in the text.

Then the broken weak formulation of the problem (4.9) – (4.10) reads as follows:

$$\text{Find } u \in bSs \text{ such that } B_{sb}(u,w) = L_{sb}(w) \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)), \tag{5.10}$$

where by bSs we denote the following function space

$$bSs = \{u \in H^4(\Omega, \mathcal{P}(\Omega)) : u_{,x} \cdot n, g^2 u_{,xx}, (g^2 u_{,xx} - u)_{,x} \cdot n \text{ are continuous across } \Gamma_i\}.$$

Note that the bilinear form $B_{sb}(\cdot,\cdot)$ is symmetric.

We shall associate with the bilinear form $B_{sb}(\cdot,\cdot)$ the energy seminorm, $|||\cdot|||_{sb}$, defined by

$$|||u|||_{sb} = \left(||(g^2)^{1/2} u_{,xx}||^2_{\tilde{\Omega}} + ||u_{,x}||^2_{\tilde{\Omega}} + ||\beta^{1/2} [\![u_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\beta_q^{1/2} u_{,x}||^2_{\Gamma_q} \right)^{1/2}, \quad u \in H^2(\Omega, \mathcal{P}(\Omega)). \tag{5.11}$$

We also notice that energy norm is mesh-dependent.

Proposition 5.1.0.1. *If β, $\beta_q > 0$, then $|||\cdot|||_{sb}$ is a seminorm on $H^2(\Omega, \mathcal{P}(\Omega))$.*

We note in passing that since $H^4(\Omega, \mathcal{P}(\Omega)) \subset H^2(\Omega, \mathcal{P}(\Omega))$, then $|||\cdot|||_{sb}$ is also a seminorm on $H^4(\Omega, \mathcal{P}(\Omega))$.

5.1.1 Consistency

We shall now show that a strong solution to the boundary value problem for the strain gradient bar in tension equation, which is smooth enough at the interelement boundaries, is the solution to the problem in the broken weak formulation. Let us start by demonstrating weak continuity of fluxes across the element faces Γ_i.

Lemma 5.1.1.1. *Suppose that $u \in H^4(\Omega)$; then, for any Γ_i, we have*

$$\int_{\Gamma_i} [\![u_{,x}]\!] w ds = \int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w ds = \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] w ds = 0 \quad \forall w \in L^2(\Gamma_i)$$

Proof. We follow the ideas of [181], where the first integral was shown to be equal to zero for all w in $L^2(\Gamma_i)$, when $u \in H^2(\Omega)$.

To establish the last equality, let Γ_i be an interior boundary and let $\Omega_{e'}$ and Ω_e be the elements sharing the face Γ_i. Let $\tilde{\Omega}_e = int(\overline{\Omega_{e'} \cup \Omega_e})$. Then, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, after integrating by parts, we have

$$
\begin{aligned}
\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\tilde{\Omega}_e} (g^2 u_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&= - \int_{\tilde{\Omega}_e} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx. \qquad (5.12)
\end{aligned}
$$

Then, we also split the left-hand side integral and apply the integration

by parts formula in each of $\Omega_{e'}$, Ω_e. As a result, we deduce

$$\begin{aligned}
\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\Omega_{e'}} (g^2 u_{,xx} - u)_{,xx} w dx \\
&\quad + \int_{\Omega_e} (g^2 u_{,xx} - u)_{,xx} w dx \\
&= \int_{\partial\Omega_{e'}} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\
&\quad - \int_{\Omega_{e'}} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\Omega_{e'}} u_{,x} w_{,x} dx \\
&\quad + \int_{\partial\Omega_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\
&\quad - \int_{\Omega_e} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\Omega_e} u_{,x} w_{,x} dx \\
&= -\int_{\Omega_{e'}} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\Omega_{e'}} u_{,x} w_{,x} dx \\
&\quad - \int_{\Omega_e} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\Omega_e} u_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds \\
&= -\int_{\tilde{\Omega}_e} (g^2 u_{,xx})_{,x} w_{,x} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds.
\end{aligned} \tag{5.13}$$

The identities (5.12) and (5.13), entail that

$$\int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{5.14}$$

Ergo,

$$\int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required.

Moreover, we shall use similar series of steps so as to establish the equality $\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w ds = 0$. Employing integration by parts formula twice, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, we get

$$\begin{aligned}
\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\tilde{\Omega}_e} g^2 (u_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&= \int_{\partial\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\partial\tilde{\Omega}_e} g^2 u_{,xx} w_{,x} \cdot n ds \\
&\quad + \int_{\tilde{\Omega}_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&= \int_{\tilde{\Omega}_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx. \qquad (5.15)
\end{aligned}$$

If we subsequently split the left-hand side integral and perform integration

by parts twice in each of $\Omega_{e'}$ and Ω_e, we conclude

$$\begin{aligned}
\int_{\tilde{\Omega}_e} (g^2 u_{,xx} - u)_{,xx} w dx &= \int_{\Omega_{e'}} (g^2 u_{,xx} - u)_{,xx} w dx \\
&\quad + \int_{\Omega_e} (g^2 u_{,xx} - u)_{,xx} w dx \\
&= \int_{\partial\Omega_{e'}} (g^2 u_{,xx} - u)_{,x} \cdot n w ds - \int_{\partial\Omega_{e'}} g^2 u_{,xx} w_{,x} \cdot n ds \\
&\quad + \int_{\Omega_{e'}} g^2 u_{,xx} w_{,xx} dx + \int_{\Omega_{e'}} u_{,x} w_{,x} dx \\
&\quad + \int_{\partial\Omega_e} (g^2 u_{,xx} - u)_{,x} \cdot n w ds \\
&\quad - \int_{\partial\Omega_e} g^2 u_{,xx} w_{,x} \cdot n ds + \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx \\
&\quad + \int_{\Omega_e} u_{,x} w_{,x} dx \\
&= \int_{\Omega_{e'}} g^2 u_{,xx} w_{,xx} dx + \int_{\Omega_{e'}} u_{,x} w_{,x} dx \\
&\quad + \int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx + \int_{\Omega_e} u_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds \\
&\quad - \int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds \\
&= \int_{\tilde{\Omega}_e} g^2 u_{,xx} w_{,xx} dx + \int_{\tilde{\Omega}_e} u_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds \\
&\quad - \int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds.
\end{aligned} \tag{5.16}$$

The identities (5.15), (5.16), entail that

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = \int_{\Gamma_i} [\![(g^2 u_{,xx} - u)_{,x}]\!] \cdot n w ds. \tag{5.17}$$

By substituting (5.14) into the equation (5.17), we reach the conclusion

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{5.18}$$

As a consequence,

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i} [\![g^2 u_{,xx}]\!] w_{,x} \cdot n ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required. □

Proposition 5.1.1.2. *The broken weak formulation* (5.10) *of the boundary value problem* (4.9) – (4.10) *is consistent in the space* $H^4(\Omega)$ *in the sense that any solution* u *to the boundary value problem, such that* $u \in H^4(\Omega)$, *solves* (5.10) *as well.*

Proof. To begin with, from (5.10) and the defining expressions for $B_{sb}(\cdot,\cdot)$, $L_{sb}(\cdot)$, for $u \in bSs$, we have

$$\begin{aligned} 0 &= B_{sb}(u,w) - L_{sb}(w) \\ &= (g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} + (u_{,x}, w_{,x})_{\tilde{\Omega}} - \langle g^2 u_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} - [\![u_{,x}]\!] \langle g^2 w_{,xx} \rangle_{\tilde{\Gamma}} \\ &\quad + \beta [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}} - g^2 u_{,xx} w_{,x} \cdot n|_{\Gamma_q} - u_{,x} \cdot n g^2 w_{,xx}|_{\Gamma_q} \\ &\quad + \beta_q u_{,x} \cdot n w_{,x} \cdot n|_{\Gamma_q} - (f,w)_{\tilde{\Omega}} - \frac{P}{AE} w|_{\Gamma_P} - \frac{R}{AE} w_{,x} \cdot n|_{\Gamma_R} \\ &\quad + q g^2 w_{,xx}|_{\Gamma_q} - \beta_q q w_{,x} \cdot n|_{\Gamma_q}. \end{aligned} \tag{5.19}$$

Next, performing integration by parts in $\int_{\tilde{\Omega}} u_{,x} w_{,x} dx$ and twice in $\int_{\tilde{\Omega}} g^2 u_{,xx} w_{,xx} dx$ respectively, we obtain

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,x} w_{,x} dx = \int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle w \rangle ds + \int_{\Gamma_P} u_{,x} \cdot n w ds - \sum_{e=1}^{N_{el}} \int_{\Omega_e} u_{,xx} w dx$$

or else

$$(u_{,x}, w_{,x})_{\tilde{\Omega}} = [\![u_{,x}]\!] \langle w \rangle_{\tilde{\Gamma}} + u_{,x} \cdot n w|_{\Gamma_P} - (u_{,xx}, w)_{\tilde{\Omega}}, \tag{5.20}$$

and

$$\begin{aligned}\sum_{e=1}^{N_{el}}\int_{\Omega_e} g^2 u_{,xx} w_{,xx} dx &= \int_{\tilde{\Gamma}} \langle g^2 u_{,xx}\rangle [\![w_{,x}]\!] ds + \int_{\tilde{\Gamma}} [\![g^2 u_{,xx}]\!] \langle w_{,x}\rangle ds \\ &+ \int_{\Gamma_q} g^2 u_{,xx} w_{,x}\cdot n ds + \int_{\Gamma_R} g^2 u_{,xx} w_{,x}\cdot n ds \\ &- \int_{\tilde{\Gamma}} [\![(g^2 u_{,xx})_{,x}]\!] \langle w\rangle ds - \int_{\Gamma_P} (g^2 u_{,xx})_{,x}\cdot n w ds \\ &+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} (g^2 u_{,xx})_{,xx} w dx \end{aligned} \tag{5.21}$$

or else

$$\begin{aligned}(g^2 u_{,xx}, w_{,xx})_{\tilde{\Omega}} &= \langle g^2 u_{,xx}\rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} + [\![g^2 u_{,xx}]\!] \langle w_{,x}\rangle_{\tilde{\Gamma}} + g^2 u_{,xx} w_{,x}\cdot n|_{\Gamma_q} \\ &+ g^2 u_{,xx} w_{,x}\cdot n|_{\Gamma_R} - [\![(g^2 u_{,xx})_{,x}]\!] \langle w\rangle_{\tilde{\Gamma}} \\ &- (g^2 u_{,xx})_{,x}\cdot n w|_{\Gamma_P} + ((g^2 u_{,xx})_{,xx}, w)_{\tilde{\Omega}}. \end{aligned} \tag{5.22}$$

Then, by substituting the mathematical formulas (5.20) and (5.22) into (5.19), we deduce that

$$\begin{aligned} 0 &= ((g^2 u_{,xx} - u)_{,xx} - f, w)_{\tilde{\Omega}} + [\![(u - g^2 u_{,xx})_{,x}]\!] \langle w\rangle_{\tilde{\Gamma}} + [\![g^2 u_{,xx}]\!] \langle w_{,x}\rangle_{\tilde{\Gamma}} \\ &- [\![u_{,x}]\!] \langle g^2 w_{,xx}\rangle_{\tilde{\Gamma}} - (u_{,x}\cdot n - q) g^2 w_{,xx}|_{\Gamma_q} \\ &+ \left(g^2 u_{,xx} - \frac{R}{AE}\right) w_{,x}\cdot n|_{\Gamma_R} + \left((u - g^2 u_{,xx})_{,x}\cdot n - \frac{P}{AE}\right) w|_{\Gamma_P} \\ &+ \beta [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}} + \beta_q (u_{,x}\cdot n - q) w_{,x}\cdot n|_{\Gamma_q}. \end{aligned} \tag{5.23}$$

Now, the mathematical equation, (5.23), is identical to zero for all w, when

$$[\![u_{,x}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{5.24}$$

$$[\![AEg^2 u_{,xx}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{5.25}$$

$$[\![AE(u - g^2 u_{,xx})_{,x}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{5.26}$$

and

$$AE(g^2 u_{,xx} - u)_{,xx} - \bar{f} = 0 \quad \text{in } \tilde{\Omega}, \tag{5.27}$$

$$u_{,x}\cdot n = q \quad \text{on } \Gamma_q, \tag{5.28}$$

$$AEg^2u_{,xx} = R \quad \text{on } \Gamma_R, \tag{5.29}$$

$$AE(u - g^2u_{,xx})_{,x} \cdot n = P \quad \text{on } \Gamma_P. \tag{5.30}$$

We note that (5.24) – (5.26) ensure the continuity (see Lemma (5.1.1.1)) of the displacement gradient, of the double force and of the (axial) force across interior boundaries. We also notice that (5.27) denotes the enforcement of the governing partial differential equation on element interiors and (5.28) – (5.30) account for the enforcement of the boundary condtions.

Wherefore, we conclude that any solution $u \in H^4(\Omega)$ to the boundary value problem (4.9) – (4.10) is a weak continuous interior penalty solution of (5.10). □

An immediate consequence of consistency is the Galerkin orthogonality property

$$B_{sb}(u - u^{wkhp}, w) = 0 \quad \forall w \in H^4(\Omega, \mathcal{P}(\Omega)), \tag{5.31}$$

where $u \in H^4(\Omega)$ is a strong solution to the boundary value problem (4.9) – (4.10) and $u^{wkhp} \in bSs$ is a solution to the broken weak formulation.

For the sake of simplicity, we shall suppose in what follows that the solution u to the boundary value problem (4.9) – (4.10) is sufficiently smooth, that is $u \in H^4(\Omega)$, and for that reason, the broken weak formulation (5.10) of the boundary value problem admits a (unique) solution.

5.2 Finite Element Spaces

In this section, we shall consider the finite-dimensional subspaces of the broken Sobolev space $H^4(\Omega, \mathcal{P}(\Omega))$ being used in the finite element approximation of the problem.

Thereby, for any element $\Omega_e \in \mathcal{P}(\Omega)$, we denote by $P_k(\Omega_e)$ the finite-dimensional space of all polynomials of degree less than or equal to k defined on Ω_e. Then, to each $\Omega_e \in \mathcal{P}(\Omega)$ we assign a non-negative integer p_e (the local polynomial index). We also remind that $h_e = diam(\Omega_e)$ is the element characteristic length.

We can now define the finite-dimensional trial solution and weighting function spaces as

$$\mathcal{U}^{hp} = \left\{ u^{hp} \in H^1(\Omega) |\ u^{hp}|_{\Omega_e} \in P_k(\Omega_e)\ \forall \Omega_e \in \mathcal{P}(\Omega),\ u^{hp}|_{\Gamma_c} = c \right\}, \tag{5.32}$$

$$\mathcal{W}^{hp} = \left\{ w^{hp} \in H^1(\Omega) |\ w^{hp}|_{\Omega_e} \in P_k(\Omega_e)\ \forall \Omega_e \in \mathcal{P}(\Omega),\ w^{hp}|_{\Gamma_c} = 0 \right\}, \tag{5.33}$$

where we have chosen approximations functions being continuous on the entire domain, but discontinuous in first and higher-order derivatives across interior bounaries.

5.3 CIP finite element method

We are ready to present the numerical method whose analysis we shall investigate in this chapter. Making use of the weak formulation derived in Section 5.1 and the finite element spaces constructed in the previous section, we state the continuous interior penalty finite element method for the problem (4.9) – (4.10):

$$\text{Find } u^{hp} \in \mathcal{U}^{hp} \text{ such that } B_{sb}(u^{hp}, w^{hp}) = L_{sb}(w^{hp}) \quad \forall w^{hp} \in \mathcal{W}^{hp}, \tag{5.34}$$

where the functions β, β_q, contained in $B_{sb}(\cdot,\cdot)$ and $L_{sb}(\cdot)$, will be defined in the coercivity property.

One can see from the definition of the bilinear form, (5.8), that the CIP method has non-local character. In addition, to element contributions we encounter terms on interior boundaries to the two elements adjacent to the respective interfaces.

By and large, the approximation $u^{hp} \in \mathcal{U}^{hp}$ to the solution will be continuous, but discontinuous in first and higher-order derivatives since there is no continuity requirement for the derivatives in the finite element space.

What's more, we shall suppose throughout that the strong solution u to the boundary value problem satisfies the smoothness assumption $u \in H^4(\Omega)$, so as to ensure that u is a solution to (5.10) and ergo to (5.34). Consequently, the Galerkin orthogonality property

$$B_{sb}(u - u^{hp}, w) = 0 \quad \forall w \in \mathcal{W}^{hp}, \tag{5.35}$$

where u is the analytical solution of the problem and u^{hp} is the continuous interior penalty approximation to u, defined by the method (5.34). Sufficient conditions for ensuring Galerkin orthogonality are: $u \in H^4(\Omega, \mathcal{P}(\Omega))$ and that $u_{,x}\cdot n$, $g^2 u_{,xx}$, $(g^2 u_{,xx} - u)_{,x}\cdot n$ are continuous across the element interfaces Γ_i. Note that the continuity $u_{,x}\cdot n$, $g^2 u_{,xx}$, $(g^2 u_{,xx} - u)_{,x}\cdot n$ in Ω is immediate if u is the weak solution of the problem with $f \in L^2(\Omega)$. Thus, no additional assumptions are posed for the Galerkin orthogonality to hold, because these are already subsumed in the definition of the space bSs.

Furthermore, we conclude that the advantages of stabilized DG methods may be counterbalanced by the disadvantage resulting from the introduction of additional unknowns. For fourth-order elliptic problems, however, we can envision formulations which are continuous and only exhibit discontinuities in first and higher-order derivatives. In many fourth-order elliptic problems, one is interested in solutions which are continuous in the variable and its derivatives, and by adopting a weak enforcement of the continuity of derivatives, while at the same time keeping interpolation functions C^0-continuous, one is able to overcome this disadvantage and retain the lower number of unknowns of continuous Galerkin methods [86].

Moreover, CIPFEMs have the following central features. They combine principles of the continuous Galerkin, discontinuous Galerkin and stabilized methods. Furthermore, the main feature of the CIP method is that it involves only the primary variable, eliminating first derivatives and Lagrange multipliers as unknowns. In addition, the approximation functions are C^0-continuous, a feature inherited from CG methods. Therefore, we will encounter discontinuities in first and higher-order derivatives, which leads to the adoption of concepts from DG methods. What's more, continuity of first and higher-order derivatives will be weakly enforced by adding weighted residual terms to the variational equation on interior boundaries, invoking stabilization techniques [86].

In addition, CIPFEMs have certain advantages over classical FEMs for fourth-order problems. First of all, they are much simpler than C^1-FEMs. Indeed, the lowest order CIP methods are as simple as classical non-conforming FEMs. But unlike classical non-conforming FEMs that only use low-order polynomials, CIP methods come in a natural hierarchy and higher-order CIP methods can capture smooth solutions efficiently [39]. Compared with mixed finite element methods, the stability of CIP methods can be established in a straightforward manner and the symmetric positive definiteness of the continuous problems is preserved by CIP methods. Note that in the literature most analyses of mixed methods for fourth-order problems focus on boundary conditions of the clamped plate [10]. The only results for other boundary conditions (that we know) were only obtained for smooth domains [32, 159]. Finally, we would also like to mention that naive mixed finite element methods that are equivalent to splitting the boundary value problem into two second-order boundary value problems produce wrong solutions if Ω is nonconvex [161].

5.3.1 Coercivity of Bilinear Form

Since the bilinear form $B_{sb}(\cdot,\cdot)$, (5.8), is symmetric, it yields the symmetric continuous interior penalty finite element method. The formulation is analogous to the one that was introduced by Baker [16] for the biharmonic problem, as well as is similar to that was introduced by Engel et al. [86] for the h-version continuous interior penalty finite element method for fourth-order elliptic problems.

Stability 5.3.1.1. *A method is stable when its bilinear form induces a norm which can be bounded from below.*

We showed earlier that $|||\cdot|||_{sb}$, (5.11), is a seminorm on the space $H^4(\Omega,\mathcal{P}(\Omega))$, thereby, since $\mathcal{W}^{hp}\subset H^4(\Omega,\mathcal{P}(\Omega))$, we have that $|||\cdot|||_{sb}$ is also a seminorm on $\mathcal{W}^{hp}$.

Let us now prove that the bilinear form $B_{sb}(\cdot,\cdot)$ of the method, presented in this chapter, is coercive on the finite-dimensional space $\mathcal{W}^{hp}$, and hence the problem (5.34) will have a unique solution in this space.

Proposition 5.3.1.2. *The hp-version continuous interior penalty finite element method* (5.34) *is stable in the energy seminorm* (5.11)*, that is, there exists a positive constant m such that*

$$B_{sb}(w,w)\geqslant m|||w|||_{sb}^2 \quad \forall w\in\mathcal{W}^{hp}. \tag{5.36}$$

Proof. Substituting w for u in the bilinear form, (5.8), and employing the triangle inequality, we obtain

$$\begin{aligned} B_{sb}(w,w) \;\geqslant\; & ||(g^2)^{1/2}w_{,xx}||_{\tilde\Omega}^2+||w_{,x}||_{\tilde\Omega}^2 \\ & -2\Big(\big|\langle g^2w_{,xx}\rangle[\![w_{,x}]\!]_{\tilde\Gamma}\big|+\big|g^2w_{,xx}w_{,x}\cdot n|_{\Gamma_q}\big|\Big) \\ & +||\beta^{1/2}[\![w_{,x}]\!]||_{\tilde\Gamma}^2+||\beta_q^{1/2}w_{,x}||_{\Gamma_q}^2. \end{aligned} \tag{5.37}$$

To thereby complete the proof, it only remains to estimate the terms appearing into the parenthesis on the right-hand side of (5.37).

So we can write the terms, enclosed into the parenthesis, by applying the Cauchy-Schwarz inequality (A.12) and afterwards the Young inequality (A.17)

$$
\begin{aligned}
&\left|\langle g^2 w_{,xx}\rangle [\![w_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|g^2 w_{,xx} w_{,x}\cdot n|_{\Gamma_q}\right| \\
&\leqslant ||\langle g^2 w_{,xx}\rangle||_{\tilde{\Gamma}} ||[\![w_{,x}]\!]||_{\tilde{\Gamma}} + ||g^2 w_{,xx}||_{\Gamma_q} ||w_{,x}||_{\Gamma_q} \\
&\leqslant \left(\frac{\varepsilon}{2}||\langle g^2 w_{,xx}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\tilde{\Gamma}}\right) + \left(\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\Gamma_q} + \frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_q}\right) \\
&= \sum_{i=1}^{N_i}\left(\frac{\varepsilon}{2}||\langle g^2 w_{,xx}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i}\right) \\
&+ \sum_{j=1}^{N_q}\left(\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\right),
\end{aligned}
\tag{5.38}
$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.
The above terms can be bounded by using the mean value inequality (A.19) in (5.38), then we arrive at

$$
\begin{aligned}
&\sum_{i=1}^{N_i}\left(\frac{\varepsilon}{2}||\langle g^2 w_{,xx}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q}\left(\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\right) \\
&\leqslant \sum_{i=1}^{N_i}\left(\frac{\varepsilon}{2}\left(||g^2 w^+_{,xx}||^2_{\Gamma_i} + ||g^2 w^-_{,xx}||^2_{\Gamma_i}\right) + \frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i}\right) \\
&+ \sum_{j=1}^{N_q}\left(\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\right) \\
&= \sum_{i=1}^{N_i}\frac{\varepsilon}{2}\left(||g^2 w^+_{,xx}||^2_{\Gamma_i} + ||g^2 w^-_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q}\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\Gamma_j} \\
&+ \sum_{i=1}^{N_i}\frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}}\frac{\varepsilon}{2}\left(||g^2 w_{,xx}||^2_{\partial\Omega_{e'}} + ||g^2 w_{,xx}||^2_{\partial\Omega_e}\right) \\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}}\frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}.
\end{aligned}
\tag{5.39}
$$

Recalling the inverse inequality (A.20) in (5.39), we deduce

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon}{2}\left(||g^2 w_{,xx}||^2_{\partial\Omega_{e'}} + ||g^2 w_{,xx}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon}{2}||g^2 w_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon}{2}\left(c_0\frac{p^2_{e'}}{h_{e'}}||g^2 w_{,xx}||^2_{\Omega_{e'}} + c_0\frac{p^2_e}{h_e}||g^2 w_{,xx}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon}{2}c_0\frac{p^2_e}{h_e}||g^2 w_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\\
&\leqslant \sum_{e=1}^{N_{el}}\frac{\varepsilon}{2}c_0\frac{p^2_e}{h_e}||g^2 w_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon}||w_{,x}||^2_{\Gamma_j}\\
&= \sum_{e=1}^{N_{el}}\frac{\varepsilon}{2}c_0 g^2\frac{p^2_e}{h_e}||(g^2)^{1/2} w_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon\beta}||\beta^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\
&+ \sum_{j=1}^{N_q}\frac{1}{2\varepsilon\beta_q}||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j},
\end{aligned}
\tag{5.40}
$$

where the constant c_0 is independent of h_e, p_e and w.

Ergo, from (5.38) – (5.40), we arrive to the conclusion that the terms, enclosed into the bracket on the right-hand side of (5.37), can subsequently be estimated as

$$
\begin{aligned}
&\left|\langle g^2 w_{,xx}\rangle[\![w_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|g^2 w_{,xx} w_{,x}\cdot n|_{\Gamma_q}\right|\\
&\leqslant \sum_{e=1}^{N_{el}}\frac{\varepsilon}{2}c_0 g^2\frac{p^2_e}{h_e}||(g^2)^{1/2} w_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon\beta}||\beta^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\
&+ \sum_{j=1}^{N_q}\frac{1}{2\varepsilon\beta_q}||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j}.
\end{aligned}
\tag{5.41}
$$

After that procedure, we insert the inequality (5.41) into the right-hand side of (5.37). As a result, we get

$$\begin{aligned} B_{sb}(w,w) \;\geqslant\; & \sum_{e=1}^{N_{el}} ||(g^2)^{1/2} w_{,xx}||^2_{\Omega_e} + \sum_{e=1}^{N_{el}} ||w_{,x}||^2_{\Omega_e} \\ & - \Big(\sum_{e=1}^{N_{el}} \varepsilon c_0 g^2 \frac{p_e^2}{h_e} ||(g^2)^{1/2} w_{,xx}||^2_{\Omega_e} \\ & + \sum_{i=1}^{N_i} \frac{1}{\varepsilon\beta} ||\beta^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon\beta_q} ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j} \Big) \\ & + \sum_{i=1}^{N_i} ||\beta^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j}. \end{aligned} \tag{5.42}$$

Now, with the aid of factorization on the right-hand side of (5.42), it is clear that

$$\begin{aligned} B_{sb}(w,w) \;\geqslant\; & \sum_{e=1}^{N_{el}} \left(1 - \varepsilon c_0 g^2 \frac{p_e^2}{h_e}\right) ||(g^2)^{1/2} w_{,xx}||^2_{\Omega_e} \\ & + \frac{1}{2} \sum_{e=1}^{N_{el}} ||w_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon\beta}\right) ||\beta^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} \\ & + \sum_{j=1}^{N_q} \left(1 - \frac{1}{\varepsilon\beta_q}\right) ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j}. \end{aligned} \tag{5.43}$$

Then, by the use of definition of energy seminorm, (5.11), on the right-hand side of (5.43), we reach to

$$B_{sb}(w,w) \geqslant m|||w|||^2_{sb},$$

which is the desired result. We denote by the constant m the minimum of $\frac{1}{2}$ and the terms enclosed into the parentheses on the right-hand side of (5.43).

In particular, assuming that $\beta = \beta_q$, we can prove (5.36) for $m = \frac{1}{2}$ if we choose

$$\varepsilon|_{\Omega_e} = \frac{h_e}{2 c_0 g^2 p_e^2},$$

in which case we obtain

$$\beta = \beta_q = \frac{4c_0 g^2 p_e^2}{h_e},$$

too. □

5.3.2 Continuity of Bilinear Form

With the definition of the energy seminorm, (5.11), we have the following continuity result for the bilinear form (4.23), based on the Cauchy-Schwarz inequalities (A.12) and (A.13).

Proposition 5.3.2.1. *Let $B_{sb}(\cdot,\cdot)$ be the bilinear form defined in (5.8) with $\beta, \beta_q \geqslant 0$. Then, there exists a constant $0 < C < \infty$, such that*

$$B_{sb}(\upsilon, w) \leqslant C|||\upsilon|||_{sb}|||w|||_{sb} \quad \forall \upsilon, w \in \mathcal{W}^{hp}, \tag{5.44}$$

where C is independent of h_e and p_e, for the hp-version.

Proof. We can obtain (5.44) by applying at first the triangle inequality in the bilinear form

$$\begin{aligned} B_{sb}(\upsilon, w) &\leqslant |B_{sb}(\upsilon, w)| \\ &\leqslant |(g^2\upsilon_{,xx}, w_{,xx})_{\tilde{\Omega}}| + |(\upsilon_{,x}, w_{,x})_{\tilde{\Omega}}| \\ &\quad + |\langle g^2\upsilon_{,xx}\rangle [\![w_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\upsilon_{,x}]\!] \langle g^2 w_{,xx}\rangle_{\tilde{\Gamma}}| + |\beta [\![\upsilon_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}}| \\ &\quad + |g^2\upsilon_{,xx} w_{,x}\cdot n|_{\Gamma_q}| + |\upsilon_{,x}\cdot n g^2 w_{,xx}|_{\Gamma_q}| + |\beta_q \upsilon_{,x}\cdot n w_{,x}\cdot n|_{\Gamma_q}|, \end{aligned} \tag{5.45}$$

and then the Cauchy-Schwarz inequality (A.12) on each term of the right-hand side of mathematical expression (5.45). As a consequence, we have

$$\begin{aligned} B_{sb}(\upsilon, w) \leqslant\ & ||(g^2)^{1/2}\upsilon_{,xx}||_{\tilde{\Omega}}||(g^2)^{1/2}w_{,xx}||_{\tilde{\Omega}} + ||\upsilon_{,x}||_{\tilde{\Omega}}||w_{,x}||_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle g^2\upsilon_{,xx}\rangle||_{\tilde{\Gamma}}||\beta^{1/2}[\![w_{,x}]\!]||_{\tilde{\Gamma}} \\ & + ||\beta^{1/2}[\![\upsilon_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{-1/2}\langle g^2 w_{,xx}\rangle||_{\tilde{\Gamma}} \\ & + ||\beta^{1/2}[\![\upsilon_{,x}]\!]||_{\tilde{\Gamma}}||\beta^{1/2}[\![w_{,x}]\!]||_{\tilde{\Gamma}} \\ & + ||\beta_q^{-1/2} g^2\upsilon_{,xx}||_{\Gamma_q}||\beta_q^{1/2} w_{,x}||_{\Gamma_q} \\ & + ||\beta_q^{1/2}\upsilon_{,x}||_{\Gamma_q}||\beta_q^{-1/2} g^2 w_{,xx}||_{\Gamma_q} \\ & + ||\beta_q^{1/2}\upsilon_{,x}||_{\Gamma_q}||\beta_q^{1/2} w_{,x}||_{\Gamma_q}. \end{aligned} \tag{5.46}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (5.46), we get

$$\begin{aligned} B_{sb}(\upsilon, w) \leqslant\ & \Big(||(g^2)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}} + ||\upsilon_{,x}||^2_{\tilde{\Omega}} + ||\beta^{-1/2}\langle g^2\upsilon_{,xx}\rangle||^2_{\tilde{\Gamma}} \\ & + ||\beta_q^{-1/2} g^2 \upsilon_{,xx}||^2_{\Gamma_q} + 2||\beta^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\beta_q^{1/2}\upsilon_{,x}||^2_{\Gamma_q}\Big)^{1/2} \\ & \times \Big(||(g^2)^{1/2} w_{,xx}||^2_{\tilde{\Omega}} + ||w_{,x}||^2_{\tilde{\Omega}} + ||\beta^{-1/2}\langle g^2 w_{,xx}\rangle||^2_{\tilde{\Gamma}} \\ & + ||\beta_q^{-1/2} g^2 w_{,xx}||^2_{\Gamma_q} + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_q}\Big)^{1/2}. \end{aligned} \tag{5.47}$$

Thus, to complete the proof, it only remains to estimate the mean value terms that enter into the parentheses on the right-hand side of (5.47).

Hence, by invoking the mean value inequality (A.19), we can write the mean value terms, enclosed into the first parenthesis, as

$$\begin{aligned} & ||\beta^{-1/2}\langle g^2\upsilon_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2} g^2\upsilon_{,xx}||^2_{\Gamma_q} \\ & = \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle g^2\upsilon_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2 \upsilon_{,xx}||^2_{\Gamma_j} \\ & \leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2} g^2 \upsilon^+_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2} g^2 \upsilon^-_{,xx}||^2_{\Gamma_i} \right) + \sum_{j=1}^{N_q} ||\beta_q^{-1/2} g^2\upsilon_{,xx}||^2_{\Gamma_j} \\ & \leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \Big(||\beta^{-1/2} g^2\upsilon_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2} g^2 \upsilon_{,xx}||^2_{\partial\Omega_e} \Big) \\ & + \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2} g^2 \upsilon_{,xx}||^2_{\partial\Omega_e}, \end{aligned} \tag{5.48}$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j \subseteq \Gamma_q$.

By employing the inverse inequality (A.20) in (5.48), we conclude

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\beta^{-1/2}g^2\upsilon_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}g^2\upsilon_{,xx}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2}g^2\upsilon_{,xx}||^2_{\partial\Omega_e}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_0\frac{p_{e'}^2}{h_{e'}}||\beta^{-1/2}g^2\upsilon_{,xx}||^2_{\Omega_{e'}} + c_0\frac{p_e^2}{h_e}||\beta^{-1/2}g^2\upsilon_{,xx}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} c_0\frac{p_e^2}{h_e}||\beta_q^{-1/2}g^2\upsilon_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} c_0\frac{p_e^2}{h_e}||\beta^{-1/2}g^2\upsilon_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_0\frac{p_e^2}{h_e}\beta^{-1}g^2||(g^2)^{1/2}\upsilon_{,xx}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_0\frac{p_e^2}{h_e}\frac{h_e}{C_\beta g^2 p_e^2}g^2||(g^2)^{1/2}\upsilon_{,xx}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\upsilon_{,xx}||^2_{\Omega_e},
\end{aligned}
\tag{5.49}
$$

where the constant c_0 is independent of h_e, p_e and υ. We denote by C_β the stabilization constant of the stabilization parameter $\beta = \frac{C_\beta g^2 p_e^2}{h_e}$ and we have chosen that $\frac{c_0}{C_\beta} \leqslant 1$ without loss of generality.

In consequence, from (5.48) – (5.49), we reach the conclusion that the mean value terms, enclosed into the first parenthesis on the right-hand side of (5.47), can be bounded as follows

$$||\beta^{-1/2}\langle g^2\upsilon_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2\upsilon_{,xx}||^2_{\Gamma_q} \leqslant ||(g^2)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}}. \tag{5.50}$$

We shall follow the above series of steps in a similar manner to estimate the mean value terms of w on the right-hand side of (5.47). As a result, we obtain

$$||\beta^{-1/2}\langle g^2 w_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}g^2 w_{,xx}||^2_{\Gamma_q} \leqslant ||(g^2)^{1/2}w_{,xx}||^2_{\tilde{\Omega}}. \tag{5.51}$$

After that procedure, we insert the inequalities (5.50) and (5.51) on the right-hand side of (5.47). Therefore, we deduce

$$\begin{aligned} B_{sb}(\upsilon, w) \leqslant\ & \Big(2||(g^2)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}} + ||u_{,x}||^2_{\tilde{\Omega}} + 2||\beta^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}} \\ & + 2||\beta_q^{1/2}\upsilon_{,x}||^2_{\Gamma_q}\Big)^{1/2} \times \Big(2||(g^2)^{1/2}w_{,xx}||^2_{\tilde{\Omega}} + ||w_{,x}||^2_{\tilde{\Omega}} \\ & + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}} + 2||\beta_q^{1/2}w_{,x}||^2_{\Gamma_q}\Big)^{1/2}. \end{aligned} \tag{5.52}$$

So, by the use of definition of energy seminorm, (5.11), on the right-hand side of (5.52), we arrive to

$$B_{sb}(\upsilon, w) \leqslant C|||\upsilon|||_{sb}|||w|||_{sb},$$

where C is independent of both h_e and p_e. □

5.4 Error Analysis

In this section, our concern is to conduct an error analysis for continuous interior penalty finite element method (5.34). Specifically, overall our research endeavor focuses on the proof of hp-version a priori error estimates in the seminorm, $|||\cdot|||_{sb}$, for the method introduced above. For this purpose, we have initially proved the consistency and we have showed stability of the method in the preceding sections. With the results from both consistency and stability, we can prove convergence of the methods. Let us assume for simplicity that g^2 is continuous on Ω.

5.4.1 Error Estimate in the Energy Seminorm

Convergence 5.4.1.1. *Let $\Pi_{\mathbf{p}}$ denote any (linear) projection operator from $H^{\mathbf{s}}(\Omega, \mathcal{P}(\Omega))$ onto the finite element space $\mathcal{U}^{hp}$. We can then decompose the global error $u - u^{hp}$ as follows:*

$$u - u^{hp} = (u - \Pi_{\mathbf{p}}u) + (\Pi_{\mathbf{p}}u - u^{hp}) \equiv \eta + \xi. \tag{5.53}$$

Then, using the triangle inequality, we have

$$|||u - u^{hp}|||_{sb} \leqslant |||\eta|||_{sb} + |||\xi|||_{sb}, \tag{5.54}$$

where $\xi = \Pi_{\mathbf{p}}u - u^{hp}$ is the part of the error in the finite element space, i.e., $\xi \in \mathcal{W}^{hp}$.

Our error analysis below will provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η. Thereby, we shall obtain a bound on $|||u - u^{hp}|||_{sb}$ with respect to various norms of η. Hence, to complete the error analysis, we shall need to quantify norms of η in terms of the discretization parameters and Sobolev seminorms of the analytical solution u.

Theorem 5.4.1.2. *Suppose that Ω is a bounded domain in $\Re$ and that $\mathcal{P}(\Omega)$ is a regular partition of Ω into N_{el} elements Ω_e. Let $\mathbf{p} = (p_e : \Omega_e \in \mathcal{P}(\Omega), p_e \in \aleph, p_e \geqslant 3)$ be any polynomial degree vector of bounded local variation. For each face, we define positive, real, piecewise constant functions β and β_q by*

$$\beta = \beta_q = \frac{C_\beta g^2 p_e^2}{h_e},$$

where the stabilization constant C_β is arbitrary positive real number. If the analytical solution u to the problem (5.10) belongs to the broken Sobolev space $H^{\mathbf{t}}(\Omega, \mathcal{P}(\Omega))$, $\mathbf{t} = (t_e : \Omega_e \in \mathcal{P}(\Omega), t_e \geqslant 4)$, then the solution $u^{hp} \in \mathcal{U}^{hp}$ of the problem (5.34) satisfies the following error bound

$$|||u - u^{hp}|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} ||u||_{t_e,\Omega_e}^2, \tag{5.55}$$

where $2 \leqslant s_e \leqslant \min(p_e + 1, t_e)$, and C is a constant dependent only on the space dimension and on $t = \max_{\Omega_e \in \mathcal{P}(\Omega)} t_e$.

Proof. To begin with, we shall estimate ξ. For that purpose, we take advantage of the coercivity (5.36), the decomposition of the error (5.53) and the Galerkin orthogonality (5.35) yielding

$$\begin{aligned} m|||\xi|||_{sb}^2 &\leqslant B_{sb}(\xi,\xi) \\ &= B_{sb}(u - u^{hp} - \eta, \xi) \\ &= B_{sb}(u - u^{hp}, \xi) - B_{sb}(\eta, \xi) \\ &= -B_{sb}(\eta,\xi) \\ &\leqslant |B_{sb}(\eta,\xi)|. \end{aligned} \tag{5.56}$$

We continue by using the triangle inequality on the right-hand side of (5.56). Then, we obtain

$$\begin{aligned} m|||\xi|||_{sb}^2 \leqslant\; & |(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| + |(\eta_{,x}, \xi_{,x})_{\tilde{\Omega}}| + |\langle g^2\eta_{,xx}\rangle [\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,x}]\!] \langle g^2 \xi_{,xx}\rangle_{\tilde{\Gamma}}| \\ & + |\beta [\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |g^2\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| + |\eta_{,x}\cdot n g^2 \xi_{,xx}|_{\Gamma_q}| \\ & + |\beta_q \eta_{,x}\cdot n \xi_{,x}\cdot n|_{\Gamma_q}|. \end{aligned} \tag{5.57}$$

Thereby, to provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η, it only remains to estimate the inner products on the right-hand side of (5.57).

With the purpose of bounding the first inner product on the right-hand side of (5.57), we initially apply the triangle inequality yielding

$$\begin{aligned} |(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| &= \left| \sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e} \right| \\ &\leqslant \sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e}|. \end{aligned} \tag{5.58}$$

So, by recalling the Cauchy-Schwarz inequality (A.12) and next the Cauchy-Schwarz discrete inequality (A.13) in (5.58), we have

$$\begin{aligned} &\sum_{e=1}^{N_{el}} |(g^2\eta_{,xx}, \xi_{,xx})_{\Omega_e}| \\ &\leqslant \sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\eta_{,xx}||_{\Omega_e} ||(g^2)^{1/2}\xi_{,xx}||_{\Omega_e} \\ &\leqslant \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\eta_{,xx}||^2_{\Omega_e} \right)^{1/2} \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e} \right)^{1/2} \\ &= \left(||(g^2)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2}. \end{aligned} \tag{5.59}$$

By making use of the defintion of energy seminorm, (5.11), in (5.59), we get

$$\left(||(g^2)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}} \right)^{1/2} \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{5.60}$$

Therefore, from (5.58) – (5.60), we reach to conclusion that the first inner product, on the right-hand side of (5.57), can be bounded as follows

$$|(g^2\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{5.61}$$

Also, the second inner product, on the right-hand side of (5.57), can analogously be bounded as

$$|(\eta_{,x}, \xi_{,x})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb} |||\xi|||_{sb}. \tag{5.62}$$

We shall additionally follow similar series of steps to estimate the stabilizing terms on the right-hand side of (5.57). Employing the triangle inequality, we deduce

$$\begin{aligned} |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i} \beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i} |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}|. \end{aligned} \tag{5.63}$$

After that, by invoking the Cauchy-Schwarz inequality (A.12) and the Cauchy-Schwarz discrete inequality (A.13) in (5.63), we conclude

$$\begin{aligned} &\sum_{i=1}^{N_i} |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\Gamma_i} ||\beta^{1/2}[\![\xi_{,x}]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\ &= \left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||\alpha^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}. \end{aligned} \tag{5.64}$$

Using the definition of energy seminorm, (5.11) in (5.64), derives

$$\left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.65}$$

Ergo, from (5.63) – (5.65), we arrive to the conclusion that the first stabilizing term, on right-hand side of (5.57), can be estimated as follows

$$|\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.66}$$

Moreover, the rest of stabilizing terms on the right hand side of (5.57) can correspondingly be bounded as

$$|\beta_q \eta_{,x} \cdot n \xi_{,x} \cdot n|_{\Gamma_q}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.67}$$

It's about time for us to estimate inner products, containing the mean value operator of η and the jump operator of ξ, on the right-hand side of

(5.57). We use at first the triagle inequality, as a result we get

$$\begin{aligned} |\langle g^2\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}\langle g^2\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i}|\langle g^2\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}|. \end{aligned} \tag{5.68}$$

Afterwards, applying the Cauchy-Schwarz inequality (A.12) and then the Cauchy-Schwarz discrete inequality (A.13) in (5.68), we have

$$\begin{aligned} &\sum_{i=1}^{N_i}|\langle g^2\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||_{\Gamma_i}||\beta^{1/2}[\![\xi_{,x}]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\ &= \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}. \end{aligned} \tag{5.69}$$

Invoking the definition of energy seminorm, (5.11), in (5.69), we obtain

$$\begin{aligned} &\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \\ &\leqslant \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}|||\xi|||_{sb}. \end{aligned} \tag{5.70}$$

In consequence, from (5.68) – (5.70), we conclude that this type of inner product, on the right-hand side of (5.57), can subsequently be bounded as

$$|\langle g^2\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}| \leqslant \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}|||\xi|||_{sb}. \tag{5.71}$$

Furthermore, we shall use similar arguments to estimate the remaining inner product of the corresponding form, on the right-hand side of (5.57). Thus, we deduce

$$|g^2\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| \leqslant \left(\sum_{j=1}^{N_q}||\beta_q^{-1/2}g^2\eta_{,xx}||^2_{\Gamma_j}\right)^{1/2}|||\xi|||_{sb}, \tag{5.72}$$

where N_q denotes the number of exterior displacement gradient boundary segments $\Gamma_j\subseteq\Gamma_q$, as well.

A last step, for bounding $|||\xi|||_{sb}$ in terms of norms of η, is to estimate the rest of inner products which contain the jump operator of η and the mean value operator of ξ, on the right-hand side of (5.57). As in the latter case, employing the triangle inequality, we get

$$\begin{aligned}|[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\Gamma_i}\right|\\ &\leqslant \sum_{i=1}^{N_i}|[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\Gamma_i}|.\end{aligned} \tag{5.73}$$

Thereafter, by recalling the Cauchy-Schwarz inequality (A.12) and the Cauchy-Schwarz discrete inequality (A.13) in (5.73), we conclude

$$\begin{aligned}&\sum_{i=1}^{N_i}|[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\Gamma_i}|\\ &\leqslant\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||_{\Gamma_i}||\beta^{-1/2}\langle g^2\xi_{,xx}\rangle||_{\Gamma_i}\\ &\leqslant\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\xi_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}.\end{aligned} \tag{5.74}$$

By invoking the mean value inequality (A.19) in (5.74), we now have

$$\begin{aligned}
&\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle g^2\xi_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\\
&\leqslant\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2}\\
&\times\left(\sum_{i=1}^{N_i}(||\beta^{-1/2}g^2\xi^+_{,xx}||^2_{\Gamma_i}+||\beta^{-1/2}g^2\xi^-_{,xx}||^2_{\Gamma_i})\right)^{1/2}\\
&\leqslant\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2}\\
&\times\left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}}(||\beta^{-1/2}g^2\xi_{,xx}||^2_{\partial\Omega_{e'}}+||\beta^{-1/2}g^2\xi_{,xx}||^2_{\partial\Omega_e})\right)^{1/2}.
\end{aligned}\tag{5.75}$$

Also, since $\xi\in\mathcal{W}^{hp}$, we can apply the inverse inequality (A.20) in (5.75), so we obtain

$$
\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (||\beta^{-1/2}g^2\xi_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}g^2\xi_{,xx}||^2_{\partial\Omega_e})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (c_0\frac{p_{e'}^2}{h_{e'}}||\beta^{-1/2}g^2\xi_{,xx}||^2_{\Omega_{e'}} + c_0\frac{p_e^2}{h_e}||\beta^{-1/2}g^2\xi_{,xx}||^2_{\Omega_e})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} c_0\frac{p_e^2}{h_e}||\beta^{-1/2}g^2\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_0}{C_\beta}||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2},
\end{aligned}
\tag{5.76}
$$

where the constant c_0 is independent of h_e, p_e and ξ. In (5.76), we choose $\frac{c_0}{C_\beta} \leqslant 1$ without loss of generality. Thereby, we deduce

$$
\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_0}{C_\beta}||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} ||(g^2)^{1/2}\xi_{,xx}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}}\right)^{1/2}.
\end{aligned}
\tag{5.77}
$$

In (5.77), by making use of the definition of energy seminorm, (5.11), we conclude

$$
\left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(g^2)^{1/2}\xi_{,xx}||^2_{\tilde{\Omega}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.78}
$$

Wherefore, from (5.73) – (5.78), we arrive to the conclusion that this type of inner product, on the right-hand side of (5.57), can be bounded as follows

$$|[\![\eta_{,x}]\!]\langle g^2\xi_{,xx}\rangle_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.79}$$

What is more, by following the above procedure in a similar manner, we shall achieve to estimate the rest of inner products of the corresponding form, on the right-hand side of (5.57). As a consequence, we have

$$|\eta_{,x} \cdot ng^2\xi_{,xx}|_{\Gamma_q}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{5.80}$$

At this point, we gather the inequalities (5.61) – (5.62), (5.66) – (5.67), (5.71) – (5.72), (5.79) – (5.80) and insert them on the right-hand side of (5.57). So, it derives

$$\begin{aligned} m|||\xi|||^2_{sb} \;\; &\leqslant \;\; C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle g^2\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ &\quad + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}g^2\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2}\Big\}|||\xi|||_{sb}, \end{aligned}$$

which implies that

$$|||\xi|||_{sb} \leqslant C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2}\Big\}. \tag{5.81}$$

By combining at once the mathematical expression (5.54) with (5.81), we get

$$\begin{aligned} |||u - u^{hp}|||_{sb} \;\; &\leqslant \;\; C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ &\quad + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2}\Big\} \end{aligned}$$

or by successive use of (A.14), we have

$$|||u - u^{hp}|||^2_{sb} \leqslant C\Big\{|||\eta|||^2_{sb} + \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big\}. \tag{5.82}$$

Therefore, we have obtained a bound on $|||u - u^{hp}|||_{sb}$ in terms of various norms of η. Thereby, to complete the proof, it only remains to estimate the terms appearing on the right-hand side of (5.82). We note that $\eta \notin \mathcal{W}^{hp}$.

To estimate the first term, we shall make use of the definition of energy seminorm, (5.11), yielding

$$|||\eta|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} \left\{ ||\eta_{,xx}||_{\Omega_e}^2 + ||\eta_{,x}||_{\Omega_e}^2 \right\} + ||\beta^{1/2} [\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2} \eta_{,x}||_{\Gamma_q}^2. \quad (5.83)$$

We shall additionally bound the factors on the right-hand side of (5.83). By recalling (A.32) for the first two norms, we obtain

$$||\eta_{,x}||_{\Omega_e} \leqslant ||\eta||_{1,\Omega_e} \leqslant C \frac{h_e^{s_e - 1}}{p_e^{t_e - 1}} ||u||_{t_e,\Omega_e} \quad (5.84)$$

and

$$||\eta_{,xx}||_{\Omega_e} \leqslant ||\eta_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{2,\Omega_e} \leqslant C \frac{h_e^{s_e - 2}}{p_e^{t_e - 2}} ||u||_{t_e,\Omega_e}. \quad (5.85)$$

Subsequently, we shall pay particular attention to estimate the terms, containing the stabilization parameters β and β_q, on the right-hand side of (5.83). By applying the jump inequality (A.18), we deduce that

$$\begin{aligned} & ||\beta^{1/2} [\![\eta_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\beta_q^{1/2} \eta_{,x}||_{\Gamma_q}^2 \\ & = \sum_{i=1}^{N_i} ||\beta^{1/2} [\![\eta_{,x}]\!]||_{\Gamma_i}^2 + \sum_{j=1}^{N_q} ||\beta_q^{1/2} \eta_{,x}||_{\Gamma_j}^2 \\ & \leqslant \sum_{i=1}^{N_i} 2 \left(||\beta^{1/2} \eta_{,x}^+||_{\Gamma_i}^2 + ||\beta^{1/2} \eta_{,x}^-||_{\Gamma_i}^2 \right) + \sum_{j=1}^{N_q} ||\beta_q^{1/2} \eta_{,x}||_{\Gamma_j}^2 \\ & \leqslant 2 \sum_{e=1}^{N_{el}} ||\beta^{1/2} \eta_{,x}||_{\partial\Omega_e}^2. \end{aligned} \quad (5.86)$$

Afterwards, in (5.86), we get

$$2 \sum_{e=1}^{N_{el}} ||\beta^{1/2} \eta_{,x}||_{\partial\Omega_e}^2 = C \sum_{e=1}^{N_{el}} \frac{p_e^2}{h_e} ||\eta_{,x}||_{\partial\Omega_e}^2. \quad (5.87)$$

Now, employing (A.33) in (5.87), we have

$$\begin{aligned} & C\sum_{e=1}^{N_{el}} \frac{p_e^2}{h_e} ||\eta_{,x}||^2_{\partial\Omega_e} \\ & \leqslant C\sum_{e=1}^{N_{el}} \frac{p_e^2}{h_e} \frac{h_e^{2s_e-3}}{p_e^{2t_e-3}} ||u||^2_{t_e,\Omega_e} \\ & = C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} ||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{5.88}$$

Hence, from (5.86) – (5.88), we conclude that the factors, including the stabilization parameters β and β_q on the right hand side of (5.83), can be bounded as follows

$$||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} ||u||^2_{t_e,\Omega_e}. \tag{5.89}$$

Thereafter, insertion of the mathematical inequalities (5.84) – (5.85) and (5.89) into the right-hand side of (5.83) yields

$$\begin{aligned} |||\eta|||_s^2 \; & \leqslant \; C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-4}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} \right) ||u||^2_{t_e,\Omega_e} \\ & \quad + C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-2}}{p_e^{2t_e-2}} ||u||^2_{t_e,\Omega_e} \\ & \leqslant \; C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-2}}{p_e^{2t_e-2}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} \right) ||u||^2_{t_e,\Omega_e} \\ & \leqslant \; C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} ||u||^2_{t_e,\Omega_e}. \end{aligned}$$

As a result, we conclude that η can be bounded as

$$|||\eta|||_s^2 \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} ||u||^2_{t_e,\Omega_e}. \tag{5.90}$$

Into the bargain, we shall estimate the remaining factors on the right-hand side of (5.82). By using the mean value inequality (A.19), we can write the terms including the stabilization parameters as

$$\begin{aligned}
&\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2}\eta^+_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}\eta^-_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} ||\beta^{-1/2}\eta_{,xx}||^2_{\partial\Omega_e}.
\end{aligned} \tag{5.91}$$

Next, in (5.91), we get

$$\sum_{e=1}^{N_{el}} ||\beta^{-1/2}\eta_{,xx}||^2_{\partial\Omega_e} = C\sum_{e=1}^{N_{el}} \frac{h_e}{p_e^2}||\eta_{,xx}||^2_{\partial\Omega_e}. \tag{5.92}$$

Now, using (A.33) in (5.92), we have

$$\begin{aligned}
&C\sum_{e=1}^{N_{el}} \frac{h_e}{p_e^2}||\eta_{,xx}||^2_{\partial\Omega_e} \\
&\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e}{p_e^2}\frac{h_e^{2s_e-5}}{p_e^{2t_e-5}}||u||^2_{t_e,\Omega_e} \\
&= C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}.
\end{aligned} \tag{5.93}$$

Ergo, from (5.91) – (5.93), we arrive to the conclusion that the terms, including the stabilization parameters β and β_q on the right-hand side of (5.82), can be bounded as follows

$$\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}. \tag{5.94}$$

Inserting the inequalities (5.90), (5.94), into the right-hand side of (5.82) and just by combining with each other, it gives

$$\begin{aligned}|||u - u^{hp}|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}}\left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-3}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}\right)||u||_{t_e,\Omega_e}^2\\ &\leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}||u||_{t_e,\Omega_e}^2.\end{aligned}$$

So, we conclude that

$$|||u - u^{hp}|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}}\frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}||u||_{t_e,\Omega_e}^2, \tag{5.95}$$

which is the desired result. □

It is worth noting that the resulting a priori error estimate is optimal in h but is p-suboptimal by $\frac{1}{2}$ orders of p.

5.5 Conclusions

The objective of this chapter is to establish an alternative approach for the one-dimensional Toupin-Mindlin strain gradient bar in tension. The continuous interior penalty finite element method that we have introduced for this purpose exhibits the subsequent features:

1. It is formulated only in the primary variable.
2. Only piecewise continuous polynomials are employed.
3. Continuity requirements, for the derivatives, are satisfied weakly.
4. The method is consistent, stable and convergent.

Chapter 6

CIPFEM for a 6th-order Equation of SGE

6.1 Model Problem

Toupin and Mindlin included higher-order stresses and strains in the theory of linear elasticity, which serves today as the foundation of more advanced strain gradient elasticity and plasticity formulation [192, 157, 102], respectively. Let us introduce a one-dimensional model problem following their concepts.

Let $\Omega \subset \Re$ be an open, bounded domain and Γ its boundary. Let Γ_c, Γ_q, Γ_r, Γ_m, Γ_M and Γ_V denote the transverse displacement, slope, curvature, double moment, bending moment and shear force boundaries, respectively [167, 116].

We consider the equation:

$$EI(-g^2 u_{,xx} + u)_{,xxxx} = f \quad \text{in } \Omega. \tag{6.1}$$

We supplement the equation with the following boundary conditions

$$\begin{aligned} u &= c \quad \text{on } \Gamma_c, \\ u_{,x} \cdot n &= q \quad \text{on } \Gamma_q, \\ u_{,xx} &= r \quad \text{on } \Gamma_r, \\ (EIg^2 u_{,xx})_{,x} \cdot n &= m \quad \text{on } \Gamma_m, \\ EI(u - g^2 u_{,xx})_{,xx} &= M \quad \text{on } \Gamma_M, \\ EI(u - g^2 u_{,xx})_{,xxx} \cdot n &= V \quad \text{on } \Gamma_V, \end{aligned} \tag{6.2}$$

where n is the unit normal vector to the boundary exterior to Ω and $f \in L^2(\Omega)$. In the above, u denotes the transverse displacement, EI is a bending stiffness, f is a given distributed load and c, q, r, m, M and V denote the prescribed boundary transverse displacement, slope, curvature, double moment, bending moment and shear force, respectively.
Note that we have the relationships

$$
\begin{aligned}
\overline{\Gamma_c \cup \Gamma_V} &= \Gamma, &(6.3)\\
\Gamma_c \cap \Gamma_V &= \emptyset, &(6.4)\\
\overline{\Gamma_q \cup \Gamma_M} &= \Gamma, &(6.5)\\
\Gamma_q \cap \Gamma_M &= \emptyset, &(6.6)\\
\overline{\Gamma_r \cup \Gamma_m} &= \Gamma, &(6.7)\\
\Gamma_r \cap \Gamma_m &= \emptyset, &(6.8)
\end{aligned}
$$

between the different parts of the boundary. The constitutive equations for the stress σ and the higher-order $\bar{\sigma}$ can be expressed as

$$
\begin{aligned}
\sigma &= E u_{,x}, &(6.9)\\
\bar{\sigma} &= E g^2 u_{,xx}, &(6.10)
\end{aligned}
$$

where E is a material parameter (the modulus of elasticity) and g a length scale (which represents material length related to the volumetric elastic strain energy).

What is more, we mention that the first three boundary conditions are called essential and the three remaining are called natural, respectively. Specifically, the last two are called Robin boundary conditions, as well.

Under suitable conditions on Ω and on the data f, c, q, r, m, M and V, the boundary value problem (6.1) – (6.2), possesses a unique solution $u \in H^6(\Omega)$ that depends continuously on the data of the problem.

6.2 Weak Formulation

We are ready to derive the weak formulation for the problem (6.1) – (6.2), which will lead to the continuous interior penalty finite element method. We shall suppose for the moment that the solution u of the problem is a sufficiently smooth function.

For each face $\Gamma_i \subseteq \tilde{\Gamma}$, let k and l be such indices that $k > l$ and the elements $\Omega_e := \Omega_e^k$ and $\Omega_{e'} := \Omega_e^l$ share the face Γ_i. Let us define the jump across Γ_i and the mean value on Γ_i of $u \in H^1(\Omega, \mathcal{P}(\Omega))$ by

$$[\![u]\!]_{\Gamma_i} := u|_{\partial\Omega_e \cap \Gamma_i} - u|_{\partial\Omega_{e'} \cap \Gamma_i} \quad \text{and} \quad \langle u \rangle_{\Gamma_i} := \frac{1}{2}\left(u|_{\partial\Omega_e \cap \Gamma_i} + u|_{\partial\Omega_{e'} \cap \Gamma_i}\right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and of the mean value to $\Gamma_j \subseteq \Gamma_q$, $\Gamma_s \subseteq \Gamma_r$ that belong to the boundary Γ by letting:

$$\begin{aligned} [\![u]\!]_{\Gamma_j} &= u|_{\Gamma_j} \quad \text{and} \quad \langle u \rangle_{\Gamma_j} = u|_{\Gamma_j}, \\ [\![u]\!]_{\Gamma_s} &= u|_{\Gamma_s} \quad \text{and} \quad \langle u \rangle_{\Gamma_s} = u|_{\Gamma_s}. \end{aligned}$$

In these definitions, the subscripts Γ_i and $\Gamma_{j:s}$ will be supressed when no confusion is likely to occur. With each face $\Gamma_i \subseteq \tilde{\Gamma}$, we associate the unit normal vector $n = n_{\Omega_e^k}$, pointing from element Ω_e^k to Ω_e^l when $k > l$, and we choose $n = n_{\Omega_e}$ to be the unit outward normal when a node belongs to the boundary Γ.

The method will be non-conforming. So, we shall use the broken Sobolev space $H^6(\Omega, \mathcal{P}(\Omega))$ as trial space. Then, we multiply the equation, (6.1), by a test function $w \in H^6(\Omega, \mathcal{P}(\Omega))$ and integrate over Ω

$$\int_\Omega EI(-g^2 u_{,xx} + u)_{,xxxx} w dx = \int_\Omega f w dx.$$

Afterwards, we split the integrals

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx,$$

and applying integration by parts on every elemental integral, so we get

$$\begin{aligned} &\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n w ds \\ &- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot n ds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e} (EIg^2 u_{,xx})_{,x} \cdot n w_{,xx} ds \\ &+ \sum_{e=1}^{N_{el}} \int_{\Omega_e} EI u_{,xx} w_{,xx} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx, \end{aligned}$$

where n denotes the outward normal to each element boundary.

Now, we split the boundary terms as follows

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds \\
&+ \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_c} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds \\
&+ \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_V} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_q} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_M} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(g^2 u_{,xx})_{,x} \cdot nw_{,xx} ds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_r} EI(g^2 u_{,xx})_{,x} \cdot nw_{,xx} ds \\
&- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \Gamma_m} EI(g^2 u_{,xx})_{,x} \cdot nw_{,xx} ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EIu_{,xx} w_{,xx} dx \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx,
\end{aligned}
$$

and hence we have

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds$$
$$+ \int_{\Gamma_c} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds + \int_{\Gamma_V} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds$$
$$- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds$$
$$- \int_{\Gamma_q} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds - \int_{\Gamma_M} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds$$
$$- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (EIg^2 u_{,xx})_{,x} \cdot nw_{,xx} ds - \int_{\Gamma_r} (EIg^2 u_{,xx})_{,x} \cdot nw_{,xx} ds$$
$$- \int_{\Gamma_m} (EIg^2 u_{,xx})_{,x} \cdot nw_{,xx} ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EIu_{,xx} w_{,xx} dx = \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx. \tag{6.11}$$

We note that w vanishes on Γ_c. Next, using the natural boundary conditions, (6.2), on the fourth, on the seventh and on the tenth term respectively, on the left-hand side of (6.11), and moving it to the right-hand side, we obtain

$$\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot nwds$$
$$- \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds$$
$$- \int_{\Gamma_q} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot nds - \sum_{e=1}^{N_{el}} \int_{\partial\Omega_e \cap \tilde{\Gamma}} (EIg^2 u_{,xx})_{,x} \cdot nw_{,xx} ds$$
$$- \int_{\Gamma_r} (EIg^2 u_{,xx})_{,x} \cdot nw_{,xx} ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EIu_{,xx} w_{,xx} dx$$
$$= \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx - \int_{\Gamma_V} Vwds + \int_{\Gamma_M} Mw_{,x} \cdot nds + \int_{\Gamma_m} mw_{,xx} ds. \tag{6.12}$$

The second, the third and the fifth term respectively, on the left-hand side of (6.12), contain the boundary integrals over the interior element boundaries, i.e. the interior boundaries $\Gamma_i \subseteq \tilde{\Gamma}$. Consequently, in this sum of boundary integrals, we have two integrals over every interior boundary.

In order to evaluate the integrals on interior boundaries, we always use the interior trace of the test function w. Taking into account the Remark 4.2.0.1 and applying (4.13), we can see that the second, the third and the fifth term respectively, on the left-hand side of (6.12), can be reconfigured as follows

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \int_{\tilde{\Gamma}} \langle EI(-g^2 u_{,xx} + u)_{,xxx} \rangle [\![w]\!] ds \\
&+ \int_{\tilde{\Gamma}} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \langle w \rangle ds - \int_{\tilde{\Gamma}} \langle EI(-g^2 u_{,xx} + u)_{,xx} \rangle [\![w_{,x}]\!] ds \\
&- \int_{\tilde{\Gamma}} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] \langle w_{,x} \rangle ds - \int_{\Gamma_q} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot n ds \\
&- \int_{\tilde{\Gamma}} \langle EI(g^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!] ds - \int_{\tilde{\Gamma}} [\![(EIg^2 u_{,xx})_{,x}]\!] \langle w_{,xx} \rangle ds \\
&- \int_{\Gamma_r} (EIg^2 u_{,xx})_{,x} \cdot n w_{,xx} ds + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EI u_{,xx} w_{,xx} dx \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx - \int_{\Gamma_V} V w ds + \int_{\Gamma_M} M w_{,x} \cdot n ds + \int_{\Gamma_m} m w_{,xx} ds.
\end{aligned} \tag{6.13}$$

Since $w \in H^1(\Omega)$, the jump $[\![w]\!]$ vanishes on Ω and therefore on $\tilde{\Gamma}$. What's more, by noting that the fluxes $EI(-g^2 u_{,xx}+u)_{,xxx} \cdot n$, $EI(-g^2 u_{,xx}+u)_{,xx}$ and $(EIg^2 u_{,xx})_{,x} \cdot n$ are continuous across the interelement boundaries Γ_i (e.g., when the exact solution $u \in H^6(\Omega)$), we have

$$\begin{aligned}
\int_{\tilde{\Gamma}} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \langle w \rangle ds = 0 \quad \forall w \in H^6(\Omega, \mathcal{P}(\Omega)), \\
\int_{\tilde{\Gamma}} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] \langle w_{,x} \rangle ds = 0 \quad \forall w \in H^6(\Omega, \mathcal{P}(\Omega)), \\
\int_{\tilde{\Gamma}} [\![(EIg^2 u_{,xx})_{,x}]\!] \langle w_{,xx} \rangle ds = 0 \quad \forall w \in H^6(\Omega, \mathcal{P}(\Omega)).
\end{aligned}$$

Then, (6.13) reduces to

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EI u_{,xx} w_{,xx} dx \\
&+ \int_{\tilde{\Gamma}} \langle (EIg^2 u_{,xx})_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}} \langle EI u_{,xx} \rangle [\![w_{,x}]\!] ds \\
&+ \int_{\Gamma_q} (EIg^2 u_{,xx})_{,xx} w_{,x} \cdot n ds - \int_{\Gamma_q} EI u_{,xx} w_{,x} \cdot n ds \\
&- \int_{\tilde{\Gamma}} \langle (EIg^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!] ds - \int_{\Gamma_r} (EIg^2 u_{,xx})_{,x} \cdot n w_{,xx} ds \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx - \int_{\Gamma_V} V w ds + \int_{\Gamma_M} M w_{,x} \cdot n ds + \int_{\Gamma_m} m w_{,xx} ds.
\end{aligned} \tag{6.14}$$

Moreover, we multiply the boundary condition $u_{,x} \cdot n = q$, on Γ_q, by $(EIg^2 w_{,xx})_{,xx} + \beta_q w_{,x} \cdot n$ and by $-EI w_{,xx} + \alpha_q w_{,x} \cdot n$. Subsequently, integrating over Γ_q, we obtain

$$\begin{aligned}
&\int_{\Gamma_q} u_{,x} \cdot n (EIg^2 w_{,xx})_{,xx} ds + \int_{\Gamma_q} \beta_q u_{,x} \cdot n w_{,x} \cdot n ds \\
&= \int_{\Gamma_q} q (EIg^2 w_{,xx})_{,xx} ds + \int_{\Gamma_q} \beta_q q w_{,x} \cdot n ds,
\end{aligned} \tag{6.15}$$

and

$$-\int_{\Gamma_q} u_{,x} \cdot n EI w_{,xx} ds + \int_{\Gamma_q} \alpha_q u_{,x} \cdot n w_{,x} \cdot n ds = -\int_{\Gamma_q} q EI w_{,xx} ds + \int_{\Gamma_q} \alpha_q q w_{,x} \cdot n ds. \tag{6.16}$$

The non-negative piecewise continuous functions β_q and α_q, defined on Γ_q, are referred to as the stabilization parameters.

In addition, $u_{,x}$ is continuous on Ω, in that case the jump $[\![u_{,x}]\!]$ vanishes on each Γ_i, i.e. $[\![u_{,x}]\!] = 0$. If we choose $\langle (EIg^2 w_{,xx})_{,xx} \rangle + \beta [\![w_{,x}]\!]$ as well as $-\langle EI w_{,xx} \rangle + \alpha [\![w_{,x}]\!]$ as test functions and integrate over $\tilde{\Gamma}$, it derives

$$\int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle (EIg^2 w_{,xx})_{,xx} \rangle ds + \int_{\tilde{\Gamma}} \beta [\![u_{,x}]\!] [\![w_{,x}]\!] ds = 0, \tag{6.17}$$

and

$$-\int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle EI w_{,xx} \rangle ds + \int_{\tilde{\Gamma}} \alpha [\![u_{,x}]\!] [\![w_{,x}]\!] ds = 0, \tag{6.18}$$

where β and α are non-negative continuous functions, defined on $\tilde{\Gamma}$, which are mentioned as the stabilization parameters.

Furthermore, from the boundary condition $u_{,xx} = r$, on Γ_r, upon multiplying by $-(EIg^2 w_{,xx})_{,x} \cdot n + \gamma_r w_{,xx}$ and integrating over Γ_r, we have

$$\begin{aligned} &-\int_{\Gamma_r} u_{,xx}(EIg^2 w_{,xx})_{,x} \cdot n ds + \int_{\Gamma_r} \gamma_r u_{,xx} w_{,xx} ds \\ &= -\int_{\Gamma_r} r(EIg^2 w_{,xx})_{,x} \cdot n ds + \int_{\Gamma_r} \gamma_r r w_{,xx} ds. \end{aligned} \tag{6.19}$$

The non-negative piecewise continuous function γ_r, defined on Γ_r, is referred to as the stabilization parameter.

To boot, $u_{,xx}$ is continuous on Ω, then it follows that the jump $[\![u_{,xx}]\!]$ vanishes on each Γ_i, i.e. $[\![u_{,x}]\!] = 0$. If we choose $-\langle (EIg^2 w_{,xx})_{,x}\rangle + \gamma [\![w_{,xx}]\!]$ as test function and integrate over $\tilde{\Gamma}$, it yields

$$-\int_{\tilde{\Gamma}} [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x}\rangle ds + \int_{\tilde{\Gamma}} \gamma [\![u_{,xx}]\!] [\![w_{,xx}]\!] ds = 0, \tag{6.20}$$

where γ is a non-negative continuous function, defined on $\tilde{\Gamma}$, which is mentioned as the stabilization parameter.

By adding at this point (6.15) – (6.20), we get the continuous interior penalty weak formulation of the problem

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EI u_{,xx} w_{,xx} dx \\
&+ \int_{\tilde{\Gamma}} \langle (EIg^2 u_{,xx})_{,xx} \rangle [\![w_{,x}]\!] ds + \int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle (EIg^2 w_{,xx})_{,xx} \rangle ds \\
&- \int_{\tilde{\Gamma}} \langle (EIg^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!] ds - \int_{\tilde{\Gamma}} [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x} \rangle ds \\
&- \int_{\tilde{\Gamma}} \langle EI u_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}} [\![u_{,x}]\!] \langle EI w_{,xx} \rangle ds \\
&+ \int_{\tilde{\Gamma}} \beta [\![u_{,x}]\!] [\![w_{,x}]\!] + \int_{\tilde{\Gamma}} \gamma [\![u_{,xx}]\!] [\![w_{,xx}]\!] ds + \int_{\tilde{\Gamma}} \alpha [\![u_{,x}]\!] [\![w_{,x}]\!] ds \\
&+ \int_{\Gamma_q} (EIg^2 u_{,xx})_{,xx} w_{,x} \cdot n ds + \int_{\Gamma_q} u_{,x} \cdot n (EIg^2 w_{,xx})_{,xx} ds \\
&- \int_{\Gamma_r} (EIg^2 u_{,xx})_{,x} \cdot n w_{,xx} ds - \int_{\Gamma_r} u_{,xx} (EIg^2 w_{,xx})_{,x} \cdot n ds \\
&- \int_{\Gamma_q} EI u_{,xx} w_{,x} \cdot n ds - \int_{\Gamma_q} u_{,x} \cdot n EI w_{,xx} ds \\
&+ \int_{\Gamma_q} \beta_q u_{,x} \cdot n w_{,x} \cdot n ds + \int_{\Gamma_r} \gamma_r u_{,xx} w_{,xx} ds + \int_{\Gamma_q} \alpha_q u_{,x} \cdot n w_{,x} \cdot n ds \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} f w dx - \int_{\Gamma_V} V w ds + \int_{\Gamma_M} M w_{,x} \cdot n ds + \int_{\Gamma_m} m w_{,xx} ds \\
&+ \int_{\Gamma_q} q (EIg^2 w_{,xx})_{,xx} ds - \int_{\Gamma_r} r (EIg^2 w_{,xx})_{,x} \cdot n ds - \int_{\Gamma_q} q EI w_{,xx} ds \\
&+ \int_{\Gamma_q} \beta_q q w_{,x} \cdot n ds + \int_{\Gamma_r} \gamma_r r w_{,xx} ds + \int_{\Gamma_q} \alpha_q q w_{,x} \cdot n ds.
\end{aligned}
\tag{6.21}
$$

Using the inner products (3.7) and (3.8), (6.21) can alternatively be rewritten in a more compressed form as

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} \int_{\Omega_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \sum_{e=1}^{N_{el}} \int_{\Omega_e} EIu_{,xx} w_{,xx} dx \\
&+ \int_{\tilde{\Gamma}_1} \langle (EIg^2 u_{,xx})_{,xx} \rangle [\![w_{,x}]\!] ds + \int_{\tilde{\Gamma}_1} [\![u_{,x}]\!] \langle (EIg^2 w_{,xx})_{,xx} \rangle ds \\
&- \int_{\tilde{\Gamma}_2} \langle (EIg^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!] ds - \int_{\tilde{\Gamma}_2} [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x} \rangle ds \\
&- \int_{\tilde{\Gamma}_1} \langle EIu_{,xx} \rangle [\![w_{,x}]\!] ds - \int_{\tilde{\Gamma}_1} [\![u_{,x}]\!] \langle EIw_{,xx} \rangle ds \\
&+ \int_{\tilde{\Gamma}_1} \beta [\![u_{,x}]\!] [\![w_{,x}]\!] + \int_{\tilde{\Gamma}_2} \gamma [\![u_{,xx}]\!] [\![w_{,xx}]\!] ds + \int_{\tilde{\Gamma}_1} \alpha [\![u_{,x}]\!] [\![w_{,x}]\!] ds \\
&= \sum_{e=1}^{N_{el}} \int_{\Omega_e} fwdx - \int_{\Gamma_V} Vwds + \int_{\Gamma_M} Mw_{,x} \cdot nds + \int_{\Gamma_m} mw_{,xx} ds \\
&+ \int_{\Gamma_q} q(EIg^2 w_{,xx})_{,xx} ds - \int_{\Gamma_r} r(EIg^2 w_{,xx})_{,x} \cdot nds - \int_{\Gamma_q} qEIw_{,xx} ds \\
&+ \int_{\Gamma_q} \beta_q q w_{,x} \cdot nds + \int_{\Gamma_r} \gamma_r r w_{,xx} ds + \int_{\Gamma_q} \alpha_q q w_{,x} \cdot nds.
\end{aligned} \tag{6.22}
$$

The bilinear form $B_{sb}(\cdot,\cdot)$ is defined as

$$
\begin{aligned}
B_{sb}(u,w) \;:=\; & ((EIg^2 u_{,xx})_{,x}, (w_{,xx})_{,x})_{\tilde{\Omega}} + (EIu_{,xx}, w_{,xx})_{\tilde{\Omega}} \\
& + \langle (EIg^2 u_{,xx})_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}_1} + [\![u_{,x}]\!] \langle (EIg^2 w_{,xx})_{,xx} \rangle_{\tilde{\Gamma}_1} \\
& - \langle (EIg^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!]_{\tilde{\Gamma}_2} - [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x} \rangle_{\tilde{\Gamma}_2} \\
& - \langle EIu_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}_1} - [\![u_{,x}]\!] \langle EIw_{,xx} \rangle_{\tilde{\Gamma}_1} \\
& + \beta [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}_1} + \gamma [\![u_{,xx}]\!] [\![w_{,xx}]\!]_{\tilde{\Gamma}_2} + \alpha [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}_1}.
\end{aligned} \tag{6.23}
$$

We introduce the linear functional $L_{sb}(\cdot)$ on $H^6(\Omega, \mathcal{P}(\Omega))$

$$
\begin{aligned}
L_{sb}(w) \;:=\; & (f,w)_{\tilde{\Omega}} - Vw|_{\Gamma_V} + Mw_{,x} \cdot n|_{\Gamma_M} + mw_{,xx}|_{\Gamma_m} \\
& + q(EIg^2 w_{,xx})_{,xx}|_{\Gamma_q} - r(EIg^2 w_{,xx})_{,x} \cdot n|_{\Gamma_r} - qEIw_{,xx}|_{\Gamma_q} \\
& + \beta_q q w_{,x} \cdot n|_{\Gamma_q} + \gamma_r r w_{,xx}|_{\Gamma_r} + \alpha_q q w_{,x} \cdot n|_{\Gamma_q}.
\end{aligned} \tag{6.24}
$$

The stabilization parameters, $\beta, \beta_q, \gamma, \gamma_r, \alpha, \alpha_q$, depend on the discretization parameter h_e for the h-method, and on the discretization parameters h_e, p_e

for the hp-method respectively, in a manner that will be specified later in the text.

Then the broken weak formulation of the problem (6.1) – (6.2) reads as follows:

$$\text{Find } u \in bSs \text{ such that } B_{sb}(u,w) = L_{sb}(w) \quad \forall w \in H^6(\Omega, \mathcal{P}(\Omega)), \tag{6.25}$$

where by bSs we denote the following function space

$$\begin{aligned} bSs \ = \ & \{u \in H^6(\Omega, \mathcal{P}(\Omega)) : u_{,x} \cdot n, u_{,xx}, (EIg^2 u_{,xx})_{,x} \cdot n, \\ & EI(-g^2 u_{,xx} + u)_{,xx}, EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n \text{ are continuous across } \Gamma_i\}. \end{aligned}$$

Note that the bilinear form $B_{sb}(\cdot,\cdot)$ is symmetric.

Since our main goal is to present the continuous interior penalty finite element method, we shall associate with the bilinear form, $B_{sb}(\cdot,\cdot)$, the energy seminorm, $|||\cdot|||_{sb}$, defined by

$$\begin{aligned} |||u|||_{sb} \ = \ & \Big(||(EIg^2)^{1/2}(u_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2} u_{,xx}||^2_{\tilde{\Omega}} + ||\beta^{1/2} [\![u_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\ & + ||\gamma^{1/2}[\![u_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + ||\alpha^{1/2}[\![u_{,x}]\!]||^2_{\tilde{\Gamma}_1} \Big)^{1/2}, \\ & u \in H^3(\Omega, \mathcal{P}(\Omega)). \end{aligned} \tag{6.26}$$

We also notice that energy seminorm is mesh-dependent.

Proposition 6.2.0.1. *If β, β_q, γ, γ_r, α, $\alpha_c > 0$, then $|||\cdot|||_{sb}$ is a seminorm on $H^3(\Omega, \mathcal{P}(\Omega))$.*

We note in passing that since $H^6(\Omega, \mathcal{P}(\Omega)) \subset H^3(\Omega, \mathcal{P}(\Omega))$, then $|||\cdot|||_{sb}$ is also a seminorm on $H^6(\Omega, \mathcal{P}(\Omega))$.

6.2.1 Consistency

We shall now show that a strong solution to the boundary value problem for the strain gradient beam in bending equation , which is smooth enough at the interelement boundaries, is the solution to the problem in the broken weak formulation. Let us start by demonstrating weak continuity of fluxes across the element faces Γ_i.

Lemma 6.2.1.1. *Suppose that $u \in H^6(\Omega)$; then, for any Γ_i, we have*

$$\begin{aligned}\int_{\Gamma_i} [\![u_{,x}]\!] w ds &= \int_{\Gamma_i} [\![u_{,xx}]\!] w ds = \int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] w ds \\ &= \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] w ds \\ &= \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] w ds = 0 \quad \forall w \in L^2(\Gamma_i).\end{aligned}$$

Proof. We follow the ideas of [181], where the first two integrals were shown to be equal to zero for all w in $L^2(\Gamma_i)$, when $u \in H^2(\Omega)$.

To establish the last equality, let Γ_i be an interior boundary and let $\Omega_{e'}$ and Ω_e be the elements sharing the face Γ_i. Let $\tilde{\Omega}_e = int(\overline{\Omega_{e'} \cup \Omega_e})$. Then, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, after integrating by parts, we have

$$\begin{aligned}\int_{\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx &= \int_{\partial\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n w ds \\ &\quad + \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\ &\quad - \int_{\tilde{\Omega}_e} (EIu_{,xx})_{,x} w_{,x} dx \\ &= + \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\ &\quad - \int_{\tilde{\Omega}_e} (EIu_{,xx})_{,x} w_{,x} dx. \end{aligned} \tag{6.27}$$

Then, we also split the left-hand side integral and apply the integration

by parts formula in each of $\Omega_{e'}$, Ω_e. As a result, we deduce

$$\begin{aligned}
\int_{\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx &= \int_{\Omega_{e'}} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx \\
&\quad + \int_{\Omega_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx \\
&= \int_{\partial\Omega_{e'}} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n w ds \\
&\quad + \int_{\Omega_{e'}} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\
&\quad - \int_{\Omega_{e'}} (EIu_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\partial\Omega_e} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n w ds \\
&\quad + \int_{\Omega_e} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\
&\quad - \int_{\Omega_e} (EIu_{,xx})_{,x} w_{,x} dx \\
&= + \int_{\Omega_{e'}} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\
&\quad - \int_{\Omega_{e'}} (EIu_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\Omega_e} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\
&\quad - \int_{\Omega_e} (EIu_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds \\
&= + \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,xxx} w_{,x} dx \\
&\quad - \int_{\tilde{\Omega}_e} (EIu_{,xx})_{,x} w_{,x} dx \\
&\quad + \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds.
\end{aligned} \tag{6.28}$$

At this point, the identities (6.27) and (6.28), entail that

$$\int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{6.29}$$

Ergo,

$$\int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required.

Moreover, we shall use similar series of steps so as to establish the equality $\int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] w ds = 0$. Employing integration by parts formula twice, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, we get

$$\begin{aligned} \int_{\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx &= \int_{\partial\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n w ds \\ &\quad - \int_{\partial\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xx} w_{,x} \cdot n ds \\ &\quad - \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,xx} w_{,xx} dx \\ &\quad + \int_{\tilde{\Omega}_e} EI u_{,xx} w_{,xx} dx \\ &= - \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,xx} w_{,xx} dx \\ &\quad + \int_{\tilde{\Omega}_e} EI u_{,xx} w_{,xx} dx. \end{aligned} \tag{6.30}$$

If we subsequently split the left-hand side integral and perform integration

by parts twice in each of $\Omega_{e'}$ and Ω_e, we conclude

$$\begin{aligned}\int_{\tilde{\Omega}_e} EI(-g^2u_{,xx}+u)_{,xxxx}w dx &= \int_{\Omega_{e'}} EI(-g^2u_{,xx}+u)_{,xxxx}w dx \\ &\quad + \int_{\Omega_e} EI(-g^2u_{,xx}+u)_{,xxxx}w dx \\ &= -\int_{\tilde{\Omega}_e} (EIg^2u_{,xx})_{,xx}w_{,xx} dx \\ &\quad + \int_{\tilde{\Omega}_e} EIu_{,xx}w_{,xx} dx \\ &\quad + \int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xxx}]\!]\cdot nw ds \\ &\quad - \int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xx}]\!] w_{,x}\cdot n ds.\end{aligned} \tag{6.31}$$

The identities (6.30), (6.31), entail that

$$\int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xx}]\!] w_{,x}\cdot n ds = \int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xxx}]\!]\cdot nw ds. \tag{6.32}$$

By substituting (6.29) into the equation (6.32), we reach to conclusion

$$\int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xx}]\!] w_{,x}\cdot n ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{6.33}$$

As a consequence,

$$\int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xx}]\!] w_{,x}\cdot n ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it implies that

$$\int_{\Gamma_i} [\![EI(-g^2u_{,xx}+u)_{,xx}]\!] w_{,x}\cdot n ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required.

In addition, we shall follow similar arguments to establish the equality $\int_{\Gamma_i} [\![(EIg^2u_{,xx})_{,x}]\!] w ds = 0$. Applying integration by parts formula three times, for any $w \in \mathcal{D}(\tilde{\Omega}_e) = C_0^\infty(\tilde{\Omega}_e)$, we obtain

$$\int_{\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx = + \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx + \int_{\tilde{\Omega}_e} EI u_{,xx} w_{,xx} dx. \tag{6.34}$$

Thereafter, we additionally split the left-hand side integral and make use of the integration by parts formula three times in each of $\Omega_{e'}$, Ω_e. In consequence, it yields

$$\begin{aligned} \int_{\tilde{\Omega}_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx &= \int_{\Omega_{e'}} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx \\ &\quad + \int_{\Omega_e} EI(-g^2 u_{,xx} + u)_{,xxxx} w dx \\ &= + \int_{\tilde{\Omega}_e} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx \\ &\quad + \int_{\tilde{\Omega}_e} EI u_{,xx} w_{,xx} dx \\ &\quad + \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds \\ &\quad - \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] w_{,x} \cdot n ds \\ &\quad - \int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] \cdot n w_{,xx} ds. \end{aligned} \tag{6.35}$$

The identities (6.34), (6.35), yield that

$$\begin{aligned} \int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] \cdot n w_{,xx} ds &= \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xxx}]\!] \cdot n w ds \\ &\quad - \int_{\Gamma_i} [\![EI(-g^2 u_{,xx} + u)_{,xx}]\!] w_{,x} \cdot n ds. \end{aligned} \tag{6.36}$$

By inserting (6.29) and (6.33) into the equation (6.36), we reach to the conclusion

$$\int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] \cdot n w_{,xx} ds = 0 \quad \forall w \in \mathcal{D}(\tilde{\Omega}_e). \tag{6.37}$$

Hence,

$$\int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] \cdot n w_{,xx} ds = 0 \quad \forall w \in \mathcal{D}(\Gamma_i).$$

As $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, it means that

$$\int_{\Gamma_i} [\![(EIg^2 u_{,xx})_{,x}]\!] \cdot n w_{,xx} ds = 0 \quad \forall w \in L^2(\Gamma_i),$$

as required. □

Proposition 6.2.1.2. *The broken weak formulation* (6.25) *of the boundary value problem* (6.1) – (6.2) *is consistent in the space $H^6(\Omega)$ in the sense that any solution u to the boundary value problem, such that $u \in H^6(\Omega)$, solves* (6.25) *as well.*

Proof. To begin with, from (6.25) and the defining expressions for $B_{sb}(\cdot,\cdot)$, $L_{sb}(\cdot)$, for $u \in bSs$, we have

$$\begin{aligned} 0 &= B_{sb}(u,w) - L_{sb}(w) \\ &= ((EIg^2 u_{,xx})_{,x}, (w_{,xx})_{,x})_{\tilde{\Omega}} + (EIu_{,xx}, w_{,xx})_{\tilde{\Omega}} \\ &\quad + \langle (EIg^2 u_{,xx})_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}_1} + [\![u_{,x}]\!] \langle (EIg^2 w_{,xx})_{,xx} \rangle_{\tilde{\Gamma}_1} \\ &\quad - \langle (EIg^2 u_{,xx})_{,x} \rangle [\![w_{,xx}]\!]_{\tilde{\Gamma}_2} - [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x} \rangle_{\tilde{\Gamma}_2} \\ &\quad - \langle EIu_{,xx} \rangle [\![w_{,x}]\!]_{\tilde{\Gamma}_1} - [\![u_{,x}]\!] \langle EIw_{,xx} \rangle_{\tilde{\Gamma}_1} \\ &\quad + \beta [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}_1} + \gamma [\![u_{,xx}]\!] [\![w_{,xx}]\!]_{\tilde{\Gamma}_2} + \alpha [\![u_{,x}]\!] [\![w_{,x}]\!]_{\tilde{\Gamma}_1} \\ &\quad - (f,w)_{\tilde{\Omega}} + Vw|_{\Gamma_V} - Mw_{,x} \cdot n|_{\Gamma_M} - mw_{,xx}|_{\Gamma_m} \\ &\quad - q(EIg^2 w_{,xx})_{,xx}|_{\Gamma_q} + r(EIg^2 w_{,xx})_{,x} \cdot n|_{\Gamma_r} + qEIw_{,xx}|_{\Gamma_q} \\ &\quad - \beta_q q w_{,x} \cdot n|_{\Gamma_q} - \gamma_r r w_{,xx}|_{\Gamma_r} - \alpha_q q w_{,x} \cdot n|_{\Gamma_q}. \end{aligned} \tag{6.38}$$

Next, performing integration by parts twice in $\int_{\tilde{\Omega}} Eu_{,x} w_{,x} dx$ and three times in $\int_{\tilde{\Omega}} (EIg^2 u_{,xx})_{,x} (w_{,xx})_{,x} dx$ respectively, we deduce

$$\begin{aligned} 0 \;=\; & (EI(-g^2 u_{,xx} + u)_{,xxxx} - f, w)_{\tilde{\Omega}} - [\![u_{,x}]\!] \langle EI(w - g^2 w_{,xx})_{,xx} \rangle_{\tilde{\Gamma}} \\ & - [\![u_{,xx}]\!] \langle (EIg^2 w_{,xx})_{,x} \rangle_{\tilde{\Gamma}} + [\![(EIg^2 u_{,xx})_{,x}]\!] \langle w_{,xx} \rangle_{\tilde{\Gamma}} \\ & + [\![EI(u - g^2 u_{,xx})_{,xx}]\!] \langle w_{,x} \rangle_{\tilde{\Gamma}} - [\![EI(u - g^2 u_{,xx})_{,xxx}]\!] \langle w \rangle_{\tilde{\Gamma}} \\ & - (u_{,x} \cdot n - q) EI(w - g^2 w_{,xx})_{,xx}|_{\Gamma_q} - (u_{,xx} - r)(EIg^2 w_{,xx})_{,x} \cdot n|_{\Gamma_r} \\ & + \big((EIg^2 u_{,xx})_{,x} \cdot n - m\big)\, w_{,xx}|_{\Gamma_m} \\ & + \big(EI(u - g^2 u_{,xx})_{,xx} - M\big)\, w_{,x} \cdot n|_{\Gamma_M} \\ & - \big(EI(u - g^2 u_{,xx})_{,xxx} \cdot n - V\big)\, w|_{\Gamma_V} \\ & + \beta [\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} + \gamma [\![u_{,xx}]\!][\![w_{,xx}]\!]_{\tilde{\Gamma}} + \alpha [\![u_{,x}]\!][\![w_{,x}]\!]_{\tilde{\Gamma}} \\ & + \beta_q (u_{,x} \cdot n - q) w_{,x} \cdot n|_{\Gamma_q} + \gamma_r (u_{,xx} - r) w_{,xx}|_{\Gamma_r} \\ & + \alpha_q (u_{,x} \cdot n - q) w_{,x} \cdot n|_{\Gamma_q}. \end{aligned} \tag{6.39}$$

Now, the mathematical equation, (6.39), is identical to zero for all w, when

$$[\![u_{,x}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{6.40}$$

$$[\![u_{,xx}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{6.41}$$

$$[\![(EIg^2 u_{,xx})_{,x}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{6.42}$$

$$[\![EI(u - g^2 u_{,xx})_{,xx}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{6.43}$$

$$[\![EI(u - g^2 u_{,xx})_{,xxx}]\!] = 0 \quad \text{on } \tilde{\Gamma}, \tag{6.44}$$

and

$$EI(-g^2 u_{,xx} + u)_{,xxxx} - f = 0 \quad \text{in } \tilde{\Omega}, \tag{6.45}$$

$$u_{,x} \cdot n = q \quad \text{on } \Gamma_q, \tag{6.46}$$

$$u_{,xx} = r \quad \text{on } \Gamma_r, \tag{6.47}$$

$$(EIg^2 u_{,xx})_{,x} \cdot n = m \quad \text{on } \Gamma_m, \tag{6.48}$$

$$EI(u - g^2 u_{,xx})_{,xx} = M \quad \text{on } \Gamma_M, \tag{6.49}$$

$$EI(u - g^2 u_{,xx})_{,xxx} \cdot n = V \quad \text{on } \Gamma_V. \tag{6.50}$$

We note that (6.40) – (6.44) ensure the continuity (see Lemma (6.2.1.1)) of the slope, of the curvature, of the double moment, of the bending moment and of the shear force across interior boundaries. We also notice that (6.45) denotes the enforcement of the governing partial differential equation on element interiors and (6.46) – (6.50) account for the enforcement of the boundary condtions.

Wherefore, we conclude that any solution $u \in H^6(\Omega)$ to the boundary value problem (6.1) – (6.2) is a weak continuous interior penalty solution of (6.25). □

An immediate consequence of consistency is the Galerkin orthogonality property

$$B_s(u - u^{wkh}, w) = 0 \quad \forall w \in H^6(\Omega, \mathcal{P}(\Omega)), \tag{6.51}$$

where $u \in H^6(\Omega)$ is a strong solution to the boundary value problem (6.1) – (6.2) and $u^{wkh} \in bSs$ is a solution to the broken weak formulation.

For the sake of simplicity, we shall suppose in what follows that the solution u to the boundary value problem (6.1) – (6.2) is sufficiently smooth, that is $u \in H^6(\Omega)$, and for that reason, the broken weak formulation (6.25) of the boundary value problem admits a (unique) solution.

6.3 Finite Element Spaces

In this section, we shall consider the finite-dimensional subspaces of the broken Sobolev space $H^6(\Omega, \mathcal{P}(\Omega))$ being used in the finite element approximation of the problem.

Thereby, for any element $\Omega_e \in \mathcal{P}(\Omega)$, we denote by $P_k(\Omega_e)$ the finite-dimensional space of all polynomials of degree less than or equal to k defined on Ω_e. Then, to each $\Omega_e \in \mathcal{P}(\Omega)$ we assign a non-negative integer p_e (the local polynomial index). We also remind that $h_e = diam(\Omega_e)$ is the element characteristic length.

We can now define the finite-dimensional trial solution and weighting function spaces (for the h-version) as

$$\mathcal{U}^h = \left\{ u^h \in H^1(\Omega) |\; u^h|_{\Omega_e} \in P_k(\Omega_e)\; \forall \Omega_e \in \mathcal{P}(\Omega),\; u^h|_{\Gamma_c} = c \right\}, \tag{6.52}$$

$$\mathcal{W}^h = \left\{ w^h \in H^1(\Omega) |\; w^h|_{\Omega_e} \in P_k(\Omega_e)\; \forall \Omega_e \in \mathcal{P}(\Omega),\; w^h|_{\Gamma_c} = 0 \right\}, \tag{6.53}$$

where we have chosen approximation functions being continuous on the entire domain, but discontinuous in first and higher-order derivatives across interior boundaries.

For the hp-version continuous interior penalty finite element method, we denote the finite-dimensional trial solution and weighting function spaces by $\mathcal{U}^{hp}$ and $\mathcal{W}^{hp}$, respectively.

6.4 CIP finite element method

We are ready to present the numerical method whose analysis we shall investigate in this chapter. Making use of the weak formulation derived in Section 6.2 and the finite element spaces constructed in the previous section, we state the continuous interior penalty finite element method for the problem (6.1) – (6.2):

$$\text{Find } u^h \in \mathcal{U}^h \text{ such that } B_{sb}(u^h, w^h) = L_{sb}(w^h) \quad \forall w^h \in \mathcal{W}^h, \tag{6.54}$$

where the functions β, β_q, γ, γ_r, α, α_q contained in $B_{sb}(\cdot,\cdot)$ and $L_{sb}(\cdot)$, will be defined in the coercivity property.

One can see from the definition of the bilinear form, (6.23), that the CIP method has non-local character. In addition, to element contributions, we encounter terms on interior boundaries to the two elements adjacent to the respective interfaces.

Generally speaking, the approximation $u^h \in \mathcal{U}^h$ to the solution will be continuous, but discontinuous in first and higher-order derivatives since there is no continuity requirement for the derivatives in the finite element space.

What is more, we shall suppose throughout that the strong solution u to the boundary value problem satisfies the smoothness assumption $u \in H^6(\Omega)$, so as to ensure that u is a solution to (6.25) and ergo to (6.54). Consequently, the Galerkin orthogonality property

$$B_s(u - u^h, w) = 0 \quad \forall w \in \mathcal{W}^h, \tag{6.55}$$

where u is the analytical solution of the problem and u^h is the continuous interior penalty approximation to u, defined by the method (6.54). Sufficient conditions for ensuring Galerkin orthogonality are: $u \in H^6(\Omega, \mathcal{P}(\Omega))$ and that $u_{,x} \cdot n$, $u_{,xx}$, $(EIg^2 u_{,xx})_{,x} \cdot n$, $EI(-g^2 u_{,xx} + u)_{,xx}$, $EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n$ are continuous across the element interfaces Γ_i. Note that the continuity of $u_{,x} \cdot n$, $u_{,xx}$, $(EIg^2 u_{,xx})_{,x} \cdot n$, $EI(-g^2 u_{,xx} + u)_{,xx}$, $EI(-g^2 u_{,xx} + u)_{,xxx} \cdot n$ in Ω is immediate if u is the weak solution of the problem with $f \in L^2(\Omega)$. Thus, no additional assumptions are posed for the Galerkin orthogonality to hold, because these are already subsumed in the definition of the space bSs.

We conclude that the advantages of stabilized DG methods may be counterbalanced by the disadvantage resulting from the introduction of additional unknowns. For elliptic problems, however, we can envision formulations which are continuous and only exhibit discontinuities in first and higher-order derivatives. In many elliptic problems, one is interested in solutions which are continuous in the variable and its derivatives, and by adopting a weak enforcement of the continuity of derivatives, while at the same time keeping interpolation functions C^0-continuous, one is able to overcome this disadvantage and retain the lower number of unknowns of continuous Galerkin methods [86].

The CIPFEMs have the following central features. They combine principles of the CG, DG and stabilized methods. Furthermore, the main feature of the CIP method is that it involves only the primary variable, eliminating first derivatives and Lagrange multipliers as unknowns. In addition, the approximation functions are C^0-continuous, a feature inherited from CG methods. Therefore, we will encounter discontinuities in first and higher-order derivatives, which leads to the adoption of concepts from DG methods. What is more, continuity of first and higher-order derivatives will be weakly enforced by adding weighted residual terms to the variational equation on interior boundaries, invoking stabilization techniques [86].

6.4.1 Coercivity of Bilinear Form

Since the bilinear form $B_{sb}(\cdot,\cdot)$, (6.23), is symmetric, it yields the symmetric continuous interior penalty finite element method. The formulation is analogous to the one that was introduced by Baker [16] for the biharmonic problem and by Engel et al. [86] for fourth-order elliptic problems.

Stability 6.4.1.1. *A method is stable when its bilinear form induces a norm which can be bounded from below.*

We showed earlier that $|||\cdot|||_{sb}$, (6.26), is a seminorm on the space $H^6(\Omega,\mathcal{P}(\Omega))$, thus, since $\mathcal{W}^h \subset H^6(\Omega,\mathcal{P}(\Omega))$, we have that $|||\cdot|||_{sb}$ is also a seminorm on $\mathcal{W}^h$.

Let us now prove that the bilinear form $B_{sb}(\cdot,\cdot)$ of the method, presented in this chapter, is coercive on the finite-dimensional space $\mathcal{W}^h$, and hence the problem (6.54) will have a unique solution in this space.

Proposition 6.4.1.2. *The h-version continuous interior penalty finite element method* (6.54) *is stable in the energy seminorm* (6.26)*, that is, there exists a positive constant θ such that*

$$B_{sb}(w^h,w^h) \geqslant \theta |||w^h|||_{sb}^2 \quad \forall w^h \in \mathcal{W}^h. \tag{6.56}$$

Proof. Substituting w^h for u^h in the bilinear form, (6.23), employing the inner products (3.7) and (3.8) as well as the triangle inequality, we obtain

$$\begin{aligned}
B_s(w^h, w^h) \geqslant\ & ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2}w^h_{,xx}||^2_{\tilde{\Omega}} \\
& + 2\Big(\langle (EIg^2 w^h_{,xx})_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2 w^h_{,xx})_{,xx} w^h_{,x} \cdot n|_{\Gamma_q}\Big) \\
& - 2\Big(\big|\langle (EIg^2 w^h_{,xx})_{,x}\rangle [\![w^h_{,xx}]\!]_{\tilde{\Gamma}}\big| + \big|(EIg^2 w^h_{,xx})_{,x} \cdot n w^h_{,xx}|_{\Gamma_r}\big|\Big) \\
& - 2\Big(\big|\langle EI w^h_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}}\big| + \big|EI w^h_{,xx} w^h_{,x} \cdot n|_{\Gamma_q}\big|\Big) \\
& + ||\beta^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\gamma^{1/2}[\![w^h_{,xx}]\!]||^2_{\tilde{\Gamma}} + ||\alpha^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}} \\
& + ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_q} + ||\gamma_r^{1/2} w^h_{,xx}||^2_{\Gamma_r} + ||\alpha_q^{1/2} w^h_{,x}||^2_{\Gamma_q}.
\end{aligned} \tag{6.57}$$

Thus, to complete the proof, it only remains to estimate each of the terms appearing into the parentheses on the right-hand side of (6.57).

So we can write the terms, enclosed into the first parenthesis, by using the Cauchy-Schwarz inequality (A.12), as well as the Young inequality (A.17)

$$\begin{aligned}
& \langle (EIg^2 w^h_{,xx})_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2 w^h_{,xx})_{,xx} w^h_{,x} \cdot n|_{\Gamma_q} \\
& \leqslant ||\langle (EIg^2 w^h_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}} ||[\![w^h_{,x}]\!]||_{\tilde{\Gamma}} + ||(EIg^2 w^h_{,xx})_{,xx}||_{\Gamma_q} ||w^h_{,x}||_{\Gamma_q} \\
& \leqslant \left(\frac{\varepsilon_1}{2}||\langle (EIg^2 w^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}} + \frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}}\right) \\
& + \left(\frac{\varepsilon_1}{2}||(EIg^2 w^h_{,xx})_{,xx}||^2_{\Gamma_q} + \frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_q}\right) \\
& = \sum_{i=1}^{N_i}\left(\frac{\varepsilon_1}{2}||\langle (EIg^2 u^h_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\Gamma_i}\right) \\
& + \sum_{j=1}^{N_q}\left(\frac{\varepsilon_1}{2}||(EIg^2 w^h_{,xx})_{,xx}||^2_{\Gamma_j} + \frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_j}\right),
\end{aligned} \tag{6.58}$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$. The above terms can be bounded by invoking the mean value inequality

(A.19) in (6.58), then we deduce

$$\begin{aligned}
&\sum_{i=1}^{N_i}\left(\frac{\varepsilon_1}{2}||\langle(EIg^2w^h_{,xx})_{,xx}\rangle||^2_{\Gamma_i}+\frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\Gamma_i}\right)\\
&+\sum_{j=1}^{N_q}\left(\frac{\varepsilon_1}{2}||(EIg^2w^h_{,xx})_{,xx}||^2_{\Gamma_j}+\frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_j}\right)\\
&\leqslant\sum_{i=1}^{N_i}\left(\frac{\varepsilon_1}{2}\left(||(EIg^2w^{h+}_{,xx})_{,xx}||^2_{\Gamma_i}+||(EIg^2w^{h-}_{,xx})_{,xx}||^2_{\Gamma_i}\right)+\frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\Gamma_i}\right)\\
&+\sum_{j=1}^{N_q}\left(\frac{\varepsilon_1}{2}||(EIg^2w^h_{,xx})_{,xx}||^2_{\Gamma_j}+\frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_j}\right)\\
&=\sum_{i=1}^{N_i}\frac{\varepsilon_1}{2}\left(||(EIg^2w^{h+}_{,xx})_{,xx}||^2_{\Gamma_i}+||(EIg^2w^{h-}_{,xx})_{,xx}||^2_{\Gamma_i}\right)\\
&+\sum_{j=1}^{N_q}\frac{\varepsilon_1}{2}||(EIg^2w^h_{,xx})_{,xx}||^2_{\Gamma_j}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\Gamma_i}+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_j}\\
&\leqslant\sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}||(EIg^2w^h_{,xx})_{,xx}||^2_{\partial\Omega_e}+\sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![w^h_{,x}]\!]||^2_{\Gamma_i}+\sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1}||w^h_{,x}||^2_{\Gamma_j}.
\end{aligned} \tag{6.59}$$

Applying the trace inequality (A.39), followed by the properties of Sobolev norms in (6.59), we conclude that

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} ||(EIg^2 w^h_{,xx})_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} |(EIg^2 w^h_{,xx})_{,x}|^2_{1,\Omega_e} + h_e |(EIg^2 w^h_{,xx})_{,x}|^2_{2,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} ||(EIg^2 w^h_{,xx})_{,x}||^2_{1,\Omega_e} + h_e ||(EIg^2 w^h_{,xx})_{,x}||^2_{2,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w^h_{,x}||^2_{\Gamma_j}.
\end{aligned} \tag{6.60}$$

Hence, making use of inverse estimate (A.37), (6.60) gives

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} ||(EIg^2 w^h_{,xx})_{,x}||^2_{1,\Omega_e} + h_e ||(EIg^2 w^h_{,xx})_{,x}||^2_{2,\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} C \left(h_e^{-1} C_I^2 h_e^{-2} ||(EIg^2 w^h_{,xx})_{,x}||^2_{\Omega_e} + h_e C_{II}^2 h_e^{-4} ||(EIg^2 w^h_{,xx})_{,x}||^2_{\Omega_e} \right) \\
&+ \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w^h_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w^h_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 EIg^2}{2h_e^3} ||(EIg^2)^{1/2} (w^h_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1 \beta} ||\beta^{1/2} [\![w^h_{,x}]\!]||^2_{\Gamma_i} \\
&+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1 \beta_q} ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_j},
\end{aligned} \tag{6.61}$$

where $C_1 = C\max\{C_I^2, C_{II}^2\}$. We denote by C_I, C_{II} the constants resulting from an inverse estinate.

Therefore, from (6.58) – (6.61), we reach the conclusion that the terms into the first bracket, on the right-hand side of (6.57), can be bounded as follows

$$\begin{aligned}&\langle (EIg^2 w^h_{,xx})_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2 u^h_{,xx})_{,xx} w^h_{,x}\cdot n|_{\Gamma_q}\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 EIg^2}{2h_e^3} ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1 \beta} ||\beta^{1/2} [\![w^h_{,x}]\!]||^2_{\Gamma_i}\\ &+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1 \beta_q} ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_j}.\end{aligned} \tag{6.62}$$

Moreover, we shall follow the above procedure in a similar manner to estimate the terms respectively enclosed into the second and the third parenthesis on the right-hand side of (6.57).

As a consequence, we deduce

$$\begin{aligned}&\left|\langle (EIg^2 w^h_{,xx})_{,x}\rangle [\![w^h_{,xx}]\!]_{\tilde{\Gamma}}\right| + \left|(EIg^2 w^h_{,xx})_{,x}\cdot n w^h_{,xx}|_{\Gamma_r}\right|\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2 C_2 EIg^2}{2h_e} ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2 \gamma} ||\gamma^{1/2} [\![w^h_{,xx}]\!]||^2_{\Gamma_i}\\ &+ \sum_{s=1}^{N_r} \frac{1}{2\varepsilon_2 \gamma_r} ||\gamma_r^{1/2} w^h_{,xx}||^2_{\Gamma_s},\end{aligned} \tag{6.63}$$

and

$$\begin{aligned}&\left|\langle EI w^h_{,xx}\rangle [\![w^h_{,x}]\!]_{\tilde{\Gamma}}\right| + \left|EI w^h_{,xx} w^h_{,x}\cdot n|_{\Gamma_q}\right|\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3 C_3 EI}{2h_e} ||(EI)^{1/2} w^h_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3 \alpha} ||\alpha^{1/2} [\![w^h_{,x}]\!]||^2_{\Gamma_i}\\ &+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_3 \alpha_q} ||\alpha_q^{1/2} w^h_{,x}||^2_{\Gamma_j}.\end{aligned} \tag{6.64}$$

We denote by N_r the number of exterior curvature boundary segments $\Gamma_s \subseteq \Gamma_r$.

Thereafter, inserting the inequalities (6.62) – (6.64) on the right-hand side of (6.57), we have

$$
\begin{aligned}
B_{sb}(w^h, w^h) \ \geqslant \ & \sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} + \sum_{e=1}^{N_{el}} ||(EI)^{1/2} w^h_{,xx}||^2_{\Omega_e} \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_1 C_1 EIg^2}{h_e^3} ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_1 \beta} ||\beta^{1/2} [\![w^h_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_1 \beta_q} ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_j} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_2 C_2 EIg^2}{h_e} ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_2 \gamma} ||\gamma^{1/2} [\![w^h_{,xx}]\!] ||^2_{\Gamma_i} + \sum_{s=1}^{N_r} \frac{1}{\varepsilon_2 \gamma_r} ||\gamma_r^{1/2} w^h_{,xx}||^2_{\Gamma_s} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \frac{\varepsilon_3 C_3 EI}{h_e} ||(EI)^{1/2} w^h_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_3 \alpha} ||\alpha^{1/2} [\![w^h_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_3 \alpha_q} ||\alpha_q^{1/2} w^h_{,x}||^2_{\Gamma_j} \Big) \\
& + \sum_{i=1}^{N_i} ||\beta^{1/2} [\![w^h_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\gamma^{1/2} [\![w^h_{,xx}]\!] ||^2_{\Gamma_i} + \sum_{s=1}^{N_r} ||\gamma_r^{1/2} w^h_{,xx}||^2_{\Gamma_s} \\
& + \sum_{i=1}^{N_i} ||\alpha^{1/2} [\![w^h_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\alpha_q^{1/2} w^h_{,x}||^2_{\Gamma_j}. \qquad (6.65)
\end{aligned}
$$

Also, with the aid of factorization on the right-hand side of (6.65), it follows that

$$\begin{aligned}
B_{sb}(w^h, w^h) \geqslant & \sum_{e=1}^{N_{el}} \left(1 - \frac{\varepsilon_1 C_1 EIg^2}{h_e^3} - \frac{\varepsilon_2 C_2 EIg^2}{h_e}\right) ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\Omega_e} \\
& + \sum_{e=1}^{N_{el}} \left(1 - \frac{\varepsilon_3 C_3 EI}{h_e}\right) ||(EI)^{1/2} w^h_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_1 \beta}\right) ||\beta^{1/2} [\![w^h_{,x}]\!]||^2_{\Gamma_i} \\
& + \sum_{j=1}^{N_q} \left(1 - \frac{1}{\varepsilon_1 \beta_q}\right) ||\beta_q^{1/2} w^h_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_2 \gamma}\right) ||\gamma^{1/2} [\![w^h_{,xx}]\!]||^2_{\Gamma_i} \\
& + \sum_{s=1}^{N_r} \left(1 - \frac{1}{\varepsilon_2 \gamma_r}\right) ||\gamma_r^{1/2} w^h_{,xx}||^2_{\Gamma_s} \\
& + \sum_{i=1}^{N_i} \left(1 - \frac{1}{\varepsilon_3 \alpha}\right) ||\alpha^{1/2} [\![w^h_{,x}]\!]||^2_{\Gamma_i} \\
& + \sum_{j=1}^{N_q} \left(1 - \frac{1}{\varepsilon_3 \alpha_q}\right) ||\alpha_q^{1/2} w^h_{,x}||^2_{\Gamma_j}.
\end{aligned} \tag{6.66}$$

Then, by the use of definition of energy seminorm, (6.26), on the right-hand side of (6.66), we arrive at

$$B_{sb}(w^h, w^h) \geqslant \theta |||w^h|||^2_{sb},$$

which is the desired result. We denote by the constant θ the minimum of the terms enclosed into the parentheses on the right-hand side of (6.66).

In particular, assuming that $\beta = \beta_q$, $\gamma = \gamma_r$ as well as $\alpha = \alpha_q$, we can prove (6.56) for $\theta = \frac{1}{2}$ if we choose

$$\varepsilon_1|_{\Omega_e} = \frac{h_e^3}{4C_1 EIg^2}, \quad \varepsilon_2|_{\Omega_e} = \frac{h_e}{4C_2 EIg^2} \quad \text{and} \quad \varepsilon_3|_{\Omega_e} = \frac{h_e}{2C_3 EI},$$

in which case we obtain

$$\beta = \beta_q = \frac{8C_1 EIg^2}{h_e^3}, \quad \gamma = \gamma_r = \frac{8C_2 EIg^2}{h_e} \quad \text{and} \quad \alpha = \alpha_q = \frac{4C_3 EI}{h_e},$$

as well. □

Let us now examine the coercivity of the bilinear form, $B_{sb}(\cdot,\cdot)$, for the hp-version continuous interior penalty finite element method, on the finite-dimensional space $\mathcal{W}^{hp}$.

Proposition 6.4.1.3. *The hp-version continuous interior penalty finite element method* (6.54) *is stable in the energy seminorm* (6.26)*, that is, there exists a positive constant θ such that*

$$B_{sb}(w,w) \geqslant \theta|||w|||_{sb}^2 \quad \forall w \in \mathcal{W}^{hp}. \tag{6.67}$$

Proof. Similar to the series of steps of the previous proof, substituting w for u in (6.23), applying the inner products (3.7) and (3.8) as well as the triangle inequality, we obtain

$$\begin{aligned} B_{sb}(w,w) \;\geqslant\; & ||(EIg^2)^{1/2}(w_{,xx})_{,x}||_{\tilde{\Omega}}^2 + ||(EI)^{1/2}w_{,xx}||_{\tilde{\Omega}}^2 \\ & + 2\Big(\langle(EIg^2w_{,xx})_{,xx}\rangle[\![w_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2w_{,xx})_{,xx}w_{,x}\cdot n|_{\Gamma_q}\Big) \\ & - 2\Big(\big|\langle(EIg^2w_{,xx})_{,x}\rangle[\![w_{,xx}]\!]_{\tilde{\Gamma}}\big| + \big|(EIg^2w_{,xx})_{,x}\cdot nw_{,xx}|_{\Gamma_r}\big|\Big) \\ & - 2\Big(\big|\langle EIw_{,xx}\rangle[\![w_{,x}]\!]_{\tilde{\Gamma}}\big| + \big|EIw_{,xx}w_{,x}\cdot n|_{\Gamma_q}\big|\Big) \\ & + ||\beta^{1/2}[\![w_{,x}]\!]||_{\tilde{\Gamma}}^2 + ||\gamma^{1/2}[\![w_{,xx}]\!]||_{\tilde{\Gamma}}^2 + ||\alpha^{1/2}[\![w_{,x}]\!]||_{\tilde{\Gamma}}^2 \\ & + ||\beta_q^{1/2}w_{,x}||_{\Gamma_q}^2 + ||\gamma_r^{1/2}w_{,xx}||_{\Gamma_r}^2 + ||\alpha_q^{1/2}w_{,x}||_{\Gamma_q}^2. \end{aligned} \tag{6.68}$$

To complete the proof, it only remains to estimate the terms enclosed into the parentheses on the right-hand side of (6.68).

As in h-version, we can write the terms into the first parenthesis, by using the Cauchy-Schwarz inequality (A.12), the Young inequality (A.17) as well as the mean value inequality (A.19)

$$
\begin{aligned}
&\langle (EIg^2 w_{,xx})_{,xx}\rangle [\![w_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2 w_{,xx})_{,xx} w_{,x} \cdot n|_{\Gamma_q} \\
&\leqslant \sum_{i=1}^{N_i} \frac{\varepsilon_1}{2} \left(||(EIg^2 w^+_{,xx})_{,xx}||^2_{\Gamma_i} + ||(EIg^2 w^-_{,xx})_{,xx}||^2_{\Gamma_i} \right) \\
&+ \sum_{j=1}^{N_q} \frac{\varepsilon_1}{2} ||(EIg^2 w_{,xx})_{,xx}||^2_{\Gamma_j} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e \subset \Omega)}^{N_{el}} \frac{\varepsilon_1}{2} \left(||(EIg^2 w_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||(EIg^2 w_{,xx})_{,xx}||^2_{\partial\Omega_e} \right) \qquad (6.69) \\
&+ \sum_{e=1:(\partial\Omega_e \cap \Gamma_q):(\partial\Omega_e \subset \Gamma)}^{N_{el}} \frac{\varepsilon_1}{2} ||(EIg^2 w_{,xx})_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1} ||[\![w_{,x}]\!]||^2_{\Gamma_i} \\
&+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1} ||w_{,x}||^2_{\Gamma_j},
\end{aligned}
$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$. The terms into the first two sums can be bounded by invoking the inverse inequality (A.21) in (6.69), then we deduce

$$
\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_1}{2}\left(||(EIg^2w_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||(EIg^2w_{,xx})_{,xx}||^2_{\partial\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_1}{2}||(EIg^2w_{,xx})_{,xx}||^2_{\partial\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![w_{,x}]\!]||^2_{\Gamma_i}\\
&+ \sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1}||w_{,x}||^2_{\Gamma_j}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \frac{\varepsilon_1}{2}\left(c_1\frac{p_{e'}^6}{h_{e'}^3}||(EIg^2w_{,xx})_{,x}||^2_{\Omega_{e'}} + c_1\frac{p_e^6}{h_e^3}||(EIg^2w_{,xx})_{,x}||^2_{\Omega_e}\right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} \frac{\varepsilon_1}{2}c_1\frac{p_e^6}{h_e^3}||(EIg^2w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![w_{,x}]\!]||^2_{\Gamma_i}\\
&+ \sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1}||w_{,x}||^2_{\Gamma_j}\\
&\leqslant \sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}c_1\frac{p_e^6}{h_e^3}||(EIg^2w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1}||[\![w_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1}||w_{,x}||^2_{\Gamma_j}\\
&= \sum_{e=1}^{N_{el}}\frac{\varepsilon_1}{2}c_1EIg^2\frac{p_e^6}{h_e^3}||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i}\frac{1}{2\varepsilon_1\beta}||\beta^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\
&+ \sum_{j=1}^{N_q}\frac{1}{2\varepsilon_1\beta_q}||\beta_q^{1/2}w_{,x}||^2_{\Gamma_j},
\end{aligned}
\tag{6.70}
$$

where the constant c_1 is independent of h_e, p_e and w.

In consequence, from (6.69) – (6.70), we arrive to the conclusion that the terms into the first bracket, on the right-hand side of (6.68), can be estimated as follows

$$\begin{aligned}&\langle(EIg^2 w_{,xx})_{,xx}\rangle[\![w_{,x}]\!]_{\tilde{\Gamma}} + (EIg^2 w_{,xx})_{,xx} w_{,x}\cdot n|_{\Gamma_q}\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_1}{2} c_1 EIg^2 \frac{p_e^6}{h_e^3} ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_1 \beta} ||\beta^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\ &+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_1 \beta_q} ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j}.\end{aligned} \tag{6.71}$$

Furthermore, we shall analogously estimate the terms appearing into the second and the third parenthesis on the right-hand side of (6.68). As a result, we arrive at

$$\begin{aligned}&\left|\langle(EIg^2 w_{,xx})_{,x}\rangle[\![w_{,xx}]\!]_{\tilde{\Gamma}}\right| + \left|(EIg^2 w_{,xx})_{,x}\cdot n w_{,xx}|_{\Gamma_r}\right|\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_2}{2} c_2 EIg^2 \frac{p_e^2}{h_e} ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_2 \gamma} ||\gamma^{1/2}[\![w_{,xx}]\!]||^2_{\Gamma_i}\\ &+ \sum_{s=1}^{N_r} \frac{1}{2\varepsilon_2 \gamma_r} ||\gamma_r^{1/2} w_{,xx}||^2_{\Gamma_s},\end{aligned} \tag{6.72}$$

and

$$\begin{aligned}&|\langle EIw_{,xx}\rangle[\![w_{,x}]\!]_{\tilde{\Gamma}}| + \left|EIw_{,xx} w\cdot n|_{\Gamma_q}\right|\\ &\leqslant \sum_{e=1}^{N_{el}} \frac{\varepsilon_3}{2} c_3 EI \frac{p_e^2}{h_e} ||(EI)^{1/2} w_{,xx}||^2_{\Omega_e} + \sum_{i=1}^{N_i} \frac{1}{2\varepsilon_3 \alpha} ||\alpha^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\ &+ \sum_{j=1}^{N_q} \frac{1}{2\varepsilon_3 \alpha_q} ||\alpha_q^{1/2} w_{,x}||^2_{\Gamma_j}.\end{aligned} \tag{6.73}$$

We denote by N_r the number of exterior curvature boundary segments $\Gamma_s \subseteq \Gamma_r$.

After those series of steps, we gather the inequalities (6.71) – (6.73) and insert them into the right-hand side of (6.68). Hence, we get

$$
\begin{aligned}
B_{sb}(w,w) \ \geqslant\ & \sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} + \sum_{e=1}^{N_{el}} ||(EI)^{1/2} w_{,xx}||^2_{\Omega_e} \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_1 c_1 EIg^2 \frac{p_e^6}{h_e^3} ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_1 \beta} ||\beta^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_1 \beta_q} ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_2 c_2 EIg^2 \frac{p_e^2}{h_e} ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_2 \gamma} ||\gamma^{1/2} [\![w_{,xx}]\!] ||^2_{\Gamma_i} + \sum_{s=1}^{N_r} \frac{1}{\varepsilon_2 \gamma_r} ||\gamma_r^{1/2} w_{,xx}||^2_{\Gamma_s} \Big) \\
& - \Big(\sum_{e=1}^{N_{el}} \varepsilon_3 c_3 EI \frac{p_e^2}{h_e} ||(EI)^{1/2} w_{,xx}||^2_{\Omega_e} \\
& + \sum_{i=1}^{N_i} \frac{1}{\varepsilon_3 \alpha} ||\alpha^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} \frac{1}{\varepsilon_3 \alpha_q} ||\alpha_q^{1/2} w_{,x}||^2_{\Gamma_r} \Big) \\
& + \sum_{i=1}^{N_i} ||\beta^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2} w_{,x}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\gamma^{1/2} [\![w_{,xx}]\!] ||^2_{\Gamma_i} + \sum_{s=1}^{N_r} ||\gamma_r^{1/2} w_{,xx}||^2_{\Gamma_j} \\
& + \sum_{i=1}^{N_i} ||\alpha^{1/2} [\![w_{,x}]\!] ||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\alpha_q^{1/2} w_{,x}||^2_{\Gamma_j}.
\end{aligned}
\tag{6.74}
$$

Now, with the aid of factorization on the right-hand side of (6.74), it is clear that

$$\begin{aligned}B_{sb}(w,w) \geqslant & \sum_{e=1}^{N_{el}}\left(1-\varepsilon_1 c_1 EIg^2\frac{p_e^6}{h_e^3}-\varepsilon_2 c_2 EIg^2\frac{p_e^2}{h_e}\right)||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\Omega_e}\\ &+\sum_{e=1}^{N_{el}}\left(1-\varepsilon_3 c_3 EI\frac{p_e^2}{h_e}\right)||(EI)^{1/2}w_{,xx}||^2_{\Omega_e}\\ &+\sum_{i=1}^{N_i}\left(1-\frac{1}{\varepsilon_1\beta}\right)||\beta^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\ &+\sum_{j=1}^{N_q}\left(1-\frac{1}{\varepsilon_1\beta_q}\right)||\beta_q^{1/2}w_{,x}||^2_{\Gamma_j}\\ &+\sum_{i=1}^{N_i}\left(1-\frac{1}{\varepsilon_2\gamma}\right)||\gamma^{1/2}[\![w_{,xx}]\!]||^2_{\Gamma_i}\\ &+\sum_{s=1}^{N_r}\left(1-\frac{1}{\varepsilon_2\gamma_r}\right)||\gamma_r^{1/2}w_{,xx}||^2_{\Gamma_s}\\ &+\sum_{i=1}^{N_i}\left(1-\frac{1}{\varepsilon_3\alpha}\right)||\alpha^{1/2}[\![w_{,x}]\!]||^2_{\Gamma_i}\\ &+\sum_{j=1}^{N_q}\left(1-\frac{1}{\varepsilon_3\alpha_q}\right)||\alpha_q^{1/2}w_{,x}||^2_{\Gamma_j}. \end{aligned} \tag{6.75}$$

So, by the use of definition of energy seminorm, (6.26), on the right-hand side of (6.75), we reach to

$$B_{sb}(w,w) \geqslant \theta|||w|||^2_{sb},$$

which is the desired result. We denote by the constant θ the minimum of the terms enclosed into the parentheses on the right-hand side of (6.75).

In particular, assuming that $\beta = \beta_q$, $\gamma = \gamma_r$ as well as $\alpha = \alpha_q$, we can prove (6.67) for $\theta = \frac{1}{2}$ if we choose

$$\varepsilon_1|_{\Omega_e} = \frac{h_e^3}{4c_1 EIg^2p_e^6}, \quad \varepsilon_2|_{\Omega_e} = \frac{h_e}{4c_2 EIg^2p_e^2} \quad \text{and} \quad \varepsilon_3|_{\Omega_e} = \frac{h_e}{2c_3 EIp_e^2},$$

in which case we obtain

$$\beta = \beta_q = \frac{8c_1 EIg^2p_e^6}{h_e^3}, \quad \gamma = \gamma_r = \frac{8c_2 EIg^2p_e^2}{h_e} \quad \text{and} \quad \alpha = \alpha_q = \frac{4c_3 EIp_e^2}{h_e},$$

too. $\square$

Wherefore, $B_{sb}(\cdot,\cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{W}^{hp}$, and ergo the problem (6.54) (respectively for the finite-dimensional spaces $\mathcal{U}^{hp}$ and $\mathcal{W}^{hp}$) has a unique solution.

6.4.2 Continuity of Bilinear Form

With the definition of the energy seminorm, (6.26), we have the following continuity result for the bilinear form (6.23), based on the Cauchy-Schwarz inequalities (A.12) and (A.13).

Proposition 6.4.2.1. *Let $B_{sb}(\cdot,\cdot)$ be the bilinear form defined in* (6.23) *with $\beta, \beta_q, \gamma, \gamma_r, \alpha, \alpha_q \geqslant 0$. Then, there exists a constant $0 < C < \infty$, such that*

$$B_{sb}(\upsilon^h, w^h) \leqslant C|||\upsilon^h|||_{sb}|||w^h|||_{sb} \quad \forall \upsilon^h, w^h \in \mathcal{W}^h, \tag{6.76}$$

where C is independent of h_e.

Proof. We can obtain (6.76) by applying at first the triangle inequality in the bilinear form

$$\begin{aligned} B_{sb}(\upsilon^h, w^h) \leqslant\ & |B_{sb}(\upsilon^h, w^h)| \\ \leqslant\ & |((EIg^2\upsilon^h_{,xx})_{,x}, (w^h_{,xx})_{,x})_{\tilde{\Omega}}| + |(EI\upsilon^h_{,xx}, w^h_{,xx})_{\tilde{\Omega}}| \\ & + |\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle[\![w^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\upsilon^h_{,x}]\!]\langle(EIg^2w^h_{,xx})_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ & + |\langle(EIg^2\upsilon^h_{,xx})_{,x}\rangle[\![w^h_{,xx}]\!]_{\tilde{\Gamma}_2}| + |[\![\upsilon^h_{,xx}]\!]\langle(EIg^2w^h_{,xx})_{,x}\rangle_{\tilde{\Gamma}_2}| \\ & + |\langle EI\upsilon^h_{,xx}\rangle[\![w^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\upsilon^h_{,x}]\!]\langle EIw^h_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ & + |\beta[\![\upsilon^h_{,x}]\!][\![w^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |\gamma[\![\upsilon^h_{,xx}]\!][\![w^h_{,xx}]\!]_{\tilde{\Gamma}_2}| + |\alpha[\![\upsilon^h_{,x}]\!][\![w^h_{,x}]\!]_{\tilde{\Gamma}_1}|, \end{aligned} \tag{6.77}$$

and then the Cauchy-Schwarz inequality (A.12) on the right-hand side of

(6.77). As a consequence, we get

$$\begin{aligned}B_{sb}(\upsilon^h, w^h) \;\leqslant\; & ||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||_{\tilde{\Omega}}||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||_{\tilde{\Omega}} \\ & + ||(EI)^{1/2}\upsilon^h_{,xx}||_{\tilde{\Omega}}||(EI)^{1/2}w^h_{,xx}||_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1}||\beta^{1/2}[\![w^h_x]\!]||_{\tilde{\Gamma}_1} \\ & + ||\beta^{1/2}[\![\upsilon^h_{,x}]\!]||_{\tilde{\Gamma}_1}||\beta^{-1/2}\langle(EIg^2w^h_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1} \\ & + ||\gamma^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2}||\gamma^{1/2}[\![w^h_{,xx}]\!]||_{\tilde{\Gamma}_2} \\ & + ||\gamma^{1/2}[\![\upsilon^h_{,xx}]\!]||_{\tilde{\Gamma}_2}||\gamma^{-1/2}\langle(EIg^2w^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2} \\ & + ||\alpha^{-1/2}\langle EI\upsilon^h_{,xx}\rangle||_{\tilde{\Gamma}_1}||\alpha^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}_1} \\ & + ||\alpha^{1/2}[\![\upsilon^h_{,x}]\!]||_{\tilde{\Gamma}_1}||\alpha^{-1/2}\langle EIw^h_{,xx}\rangle||_{\tilde{\Gamma}_1} \\ & + ||\beta^{1/2}[\![\upsilon^h_{,x}]\!]||_{\tilde{\Gamma}_1}||\beta^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}_1} \\ & + ||\gamma^{1/2}[\![\upsilon^h_{,xx}]\!]||_{\tilde{\Gamma}_2}||\gamma^{1/2}[\![w^h_{,xx}]\!]||_{\tilde{\Gamma}_2} \\ & + ||\alpha^{1/2}[\![\upsilon^h_{,x}]\!]||_{\tilde{\Gamma}_1}||\alpha^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}_1}.\end{aligned} \tag{6.78}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (6.78), we have

$$\begin{aligned}B_{sb}(\upsilon^h, w^h) \;\leqslant\; & \Big(||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2}\upsilon^h_{,xx}||^2_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + ||\gamma^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \\ & + ||\alpha^{-1/2}\langle EI\upsilon^h_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\beta^{1/2}[\![\upsilon^h_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![\upsilon^h_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \\ & + 2||\alpha^{1/2}[\![\upsilon^h_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big)^{1/2} \times \Big(||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\tilde{\Omega}} \\ & + ||(EI)^{1/2}w^h_{,xx}||^2_{\tilde{\Omega}} + ||\beta^{-1/2}\langle(EIg^2w^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\ & + ||\gamma^{-1/2}\langle(EIg^2w^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} + ||\alpha^{-1/2}\langle EIw^h_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\ & + 2||\beta^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![w^h_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + ||\alpha^{1/2}[\![w^h_{,x}]\!]||_{\tilde{\Gamma}_1}\Big)^{1/2}.\end{aligned} \tag{6.79}$$

Thus, to complete the proof, it only remains to estimate each of the mean value terms that enter into the parentheses on the right-hand side of (6.79).

Hence, by using the mean value inequality (A.19), we can write the first mean value term, appearing into the first parenthesis, as

$$
\begin{aligned}
&||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1}\\
&= ||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}(EIg^2\upsilon^h_{,xx})_{,xx}||^2_{\Gamma_q}\\
&= \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(EIg^2\upsilon^h_{,xx})_{,xx}||^2_{\Gamma_j}\\
&\leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2}(EIg^2\upsilon^{h+}_{,xx})_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}(EIg^2\upsilon^{h-}_{,xx})_{,xx}||^2_{\Gamma_i}\right)\\
&+ \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(EIg^2\upsilon^h_{,xx})_{,xx}||^2_{\Gamma_j}\\
&\leqslant \sum_{e=1}^{N_{el}} ||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,xx}||^2_{\partial\Omega_e},
\end{aligned}
\tag{6.80}
$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$. Then, by applying the trace inequality (A.39) as well as the properties of Sobolev norms in (6.80), we conclude that

$$
\begin{aligned}
&\sum_{e=1}^{N_{el}} ||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,xx}||^2_{\partial\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}|\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}|^2_{1,\Omega_e} + h_e|\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}|^2_{2,\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{2,\Omega_e}\right).
\end{aligned}
\tag{6.81}
$$

So, making use of inverse estimate (A.37), (6.81) gives

$$\begin{aligned}
&\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{2,\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C\left(h_e^{-1}C_I h_e^{-2}||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{\Omega_e} + h_e C_{II} h_e^{-4}||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{\Omega_e}\right)\\
&\leqslant \sum_{e=1}^{N_{el}} C_1 h_e^{-3}||\beta^{-1/2}(EIg^2\upsilon^h_{,xx})_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_1}{h_e^3}\beta^{-1}EIg^2||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} \frac{C_1}{h_e^3}\frac{h_e^3}{C_\beta EIg^2}EIg^2||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\Omega_e},
\end{aligned} \tag{6.82}$$

with $C_1 = C\max\{C_I^2, C_{II}^2\}$ and by C_I, C_{II} the constants resulting from an inverse estinate. We denote by C_β the stabilization constant of the stabilization parameter $\beta = \frac{C_\beta EIg^2}{h_e^3}$ and we have chosen that $\frac{C_1}{C_\beta} \leqslant 1$ without loss of generality.

Wherefore, from (6.80) – (6.82), we reach the conclusion that the first mean value term, enclosed into the first bracket on the right-hand side of (6.79), can be bounded as follows

$$||\beta^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant ||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\tilde{\Omega}}. \tag{6.83}$$

In addition, we shall follow the above series of steps in the same way to bound the remaining mean value terms, enclosed into the first parenthesis on the right-hand side of (6.79).

Ergo, we arrive to the conclusion that these factors can subsequently be bounded as

$$||\gamma^{-1/2}\langle(EIg^2\upsilon^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \leqslant ||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\tilde{\Omega}}, \tag{6.84}$$

and

$$||\alpha^{-1/2}\langle EI\upsilon^h_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant ||(EI)^{1/2}\upsilon^h_{,xx}||^2_{\tilde{\Omega}}. \tag{6.85}$$

What is more, we shall use similar arguments to bound the mean value terms of w^h, enclosed into the second parenthesis on the right-hand side of (6.79). In consequence, we deduce

$$\begin{aligned} ||\beta^{-1/2}\langle(EIg^2 w^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\gamma^{-1/2}\langle(EIg^2 w^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} &\leqslant ||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\alpha^{-1/2}\langle EI w^h_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EI)^{1/2} w^h_{,xx}||^2_{\tilde{\Omega}}. \end{aligned} \tag{6.86}$$

To boot, inserting the inequalities (6.83) – (6.85), as well as (6.86) into the brackets on the right-hand side of (6.79), it yields

$$\begin{aligned} B_{sb}(\upsilon^h, w^h) \;\leqslant\; & \Big(3||(EIg^2)^{1/2}(\upsilon^h_{,xx})_{,x}||^2_{\tilde{\Omega}} + 2||(EI)^{1/2}\upsilon^h_{,xx}||^2_{\tilde{\Omega}} + 2||\beta^{1/2}[\![\upsilon^h_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\ & + 2||\gamma^{1/2}[\![\upsilon^h_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + 2||\alpha^{1/2}[\![\upsilon^h_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big)^{1/2} \\ & \times \Big(3||(EIg^2)^{1/2}(w^h_{,xx})_{,x}||^2_{\tilde{\Omega}} + 2||(EI)^{1/2} w^h_{,xx}||^2_{\tilde{\Omega}} \\ & + 2||\beta^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![w^h_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + 2||\alpha^{1/2}[\![w^h_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big)^{1/2}. \end{aligned} \tag{6.87}$$

Also, by the use of definition of energy seminorm, (6.26), on the right-hand side of (6.87), we arrive at

$$B_{sb}(\upsilon^h, w^h) \leqslant C|||\upsilon^h|||_{sb}|||w^h|||_{sb},$$

where C is independent of h_e. □

Thereafter, let us examine the continuity of the bilinear form, $B_{sb}(\cdot,\cdot)$ for the *hp*-version continuous interior penalty finite element method.

Proposition 6.4.2.2. *Let $B_{sb}(\cdot,\cdot)$ be the bilinear form defined in* (6.23) *with $\beta, \beta_q, \gamma, \gamma_r, \alpha, \alpha_q \geqslant 0$. Then, there exists a constant $0 < C < \infty$, such that*

$$B_{sb}(\upsilon, w) \leqslant C|||\upsilon|||_{sb}|||w|||_{sb} \quad \forall \upsilon, w \in \mathcal{W}^{hp}, \tag{6.88}$$

where C is independent of both h_e and p_e, for the hp-version.

Proof. Similar to the approach to the previous proof, we can obtain (6.88), by using at first the triangle inequality, then the Cauchy-Schwarz inequality (A.12) and next the Cauchy-Schwarz discrete inequality (A.13).

As a consequence, we end up at the same result presented in mathematical expression (6.79). Up to this point

$$\begin{aligned} B_{sb}(\upsilon, w) \leqslant\; & \Big(||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle (EIg^2\upsilon_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + ||\gamma^{-1/2}\langle (EIg^2\upsilon_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \\ & + ||\alpha^{-1/2}\langle EI\upsilon_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\beta^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![\upsilon_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \\ & + 2||\alpha^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}_1} \Big)^{1/2} \times \Big(||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\tilde{\Omega}} \\ & + ||(EI)^{1/2}w_{,xx}||^2_{\tilde{\Omega}} + ||\beta^{-1/2}\langle (EIg^2 w_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\ & + ||\gamma^{-1/2}\langle (EIg^2 w_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} + ||\alpha^{-1/2}\langle EIw_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\ & + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![w_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + ||\alpha^{1/2}[\![w_{,x}]\!]||_{\tilde{\Gamma}_1} \Big)^{1/2}. \end{aligned} \tag{6.89}$$

Thereby, to complete the proof, it only remains to estimate the mean value terms appearing into the parentheses on the right-hand side of (6.89).

Hence, by applying the mean value inequality (A.19), we can write the first mean value term, enclosed into the first parenthesis, as

$$\begin{aligned} & ||\beta^{-1/2}\langle (EIg^2\upsilon_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\ & = ||\beta^{-1/2}\langle (EIg^2\upsilon_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}} + ||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\Gamma_q} \\ & = \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle (EIg^2\upsilon_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\Gamma_j} \\ & \leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2}(EIg^2\upsilon^+_{,xx})_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}(EIg^2\upsilon^-_{,xx})_{,xx}||^2_{\Gamma_i} \right) \\ & + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\Gamma_j} \\ & \leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \Big(||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_e} \Big) \\ & + \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_e}, \end{aligned} \tag{6.90}$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$. These terms can be bounded by invoking the inverse inequality (A.21) in (6.90), then we deduce

$$\begin{aligned}
&\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_e} \right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} ||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,xx}||^2_{\partial\Omega_e}\\
&\leqslant \sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_1\frac{p_{e'}^6}{h_{e'}^3}||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,x}||^2_{\Omega_{e'}} + c_1\frac{p_e^6}{h_e^3}||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,x}||^2_{\Omega_e} \right)\\
&+ \sum_{e=1:(\partial\Omega_e\cap\Gamma_q):(\partial\Omega_e\subset\Gamma)}^{N_{el}} c_1\frac{p_e^6}{h_e^3}||\beta_q^{-1/2}(EIg^2\upsilon_{,xx})_{,x}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}||\beta^{-1/2}(EIg^2\upsilon_{,xx})_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}\beta^{-1}EIg^2||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\Omega_e}\\
&= \sum_{e=1}^{N_{el}} c_1\frac{p_e^6}{h_e^3}\frac{h_e^3}{C_\beta EIg^2p_e^6}EIg^2||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\Omega_e}\\
&\leqslant \sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\Omega_e},
\end{aligned} \tag{6.91}$$

where the constant c_1 is independent of h_e, p_e and υ. We denote by C_β the stabilization constant of the stabilization parameter $\beta = \frac{C_\beta EIg^2p_e^6}{h_e^3}$ and we have chosen that $\frac{c_1}{C_\beta} \leqslant 1$ without loss of generality.

Therefore, from (6.90) – (6.91), we easily conclude that the first mean value term, enclosed into the first parenthesis on the right-hand side of (6.89), can subsequently be bounded as

$$||\beta^{-1/2}\langle(EIg^2\upsilon_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant \sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\Omega_e}. \tag{6.92}$$

Furthermore, by following the above procedure step by step, we shall estimate the rest of the mean value terms, enclosed into the first parenthesis on the right-hand side of (6.89).

Accordingly, we reach the conclusion that the corresponding mean value terms can be bounded as follows

$$||\gamma^{-1/2}\langle(EIg^2\upsilon_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \leqslant ||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\tilde{\Omega}}, \tag{6.93}$$

and

$$||\alpha^{-1/2}\langle EI\upsilon_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant ||(EI)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}}. \tag{6.94}$$

Moreover, we shall use the same arguments to bound the mean value terms of w, enclosed into the second parenthesis on the right-hand side of (6.89). As a result, we obtain

$$\begin{aligned} ||\beta^{-1/2}\langle(EIg^2w_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\gamma^{-1/2}\langle(EIg^2w_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} &\leqslant ||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\alpha^{-1/2}\langle EIw_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EI)^{1/2}w_{,xx}||^2_{\tilde{\Omega}}. \end{aligned} \tag{6.95}$$

After that procedure, we gather the inequalities (6.92) – (6.95) and insert them into the brackets on the right-hand side of (6.89). That produces

$$\begin{aligned} B_{sb}(\upsilon, w) \leqslant\ & \Big(3||(EIg^2)^{1/2}(\upsilon_{,xx})_{,x}||^2_{\tilde{\Omega}} + 2||(EI)^{1/2}\upsilon_{,xx}||^2_{\tilde{\Omega}} + 2||\beta^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\ & + 2||\gamma^{1/2}[\![\upsilon_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + 2||\alpha^{1/2}[\![\upsilon_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big)^{1/2} \\ & \times \Big(3||(EIg^2)^{1/2}(w_{,xx})_{,x}||^2_{\tilde{\Omega}} + 2||(EI)^{1/2}w_{,xx}||^2_{\tilde{\Omega}} \\ & + 2||\beta^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![w_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + 2||\alpha^{1/2}[\![w_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big)^{1/2}. \end{aligned} \tag{6.96}$$

So, by the use of definition of energy seminorm, (6.26), on the right-hand side of (6.96), we reach to

$$B_{sb}(\upsilon, w) \leqslant C|||\upsilon|||_{sb}|||w|||_{sb},$$

where C is independent of both h_e and p_e. □

6.5 Error Analysis

In this section, we aim to conduct an error analysis for continuous interior penalty finite element method (6.54). Specifically, our objective is to prove h and hp-version, as well, a priori error estimates in the seminorm, $|||\cdot|||_{sb}$, for the method introduced above. For this purpose, we have covered the usual ground of consistency and stability of the method in the preceding sections. With the results from both consistency and stability, we can prove convergence of the method. Let us assume for simplicity that both EIg^2 and EI are continuous on Ω.

6.5.1 Error Estimates in the Energy Seminorm

Convergence 6.5.1.1. *Let $\tilde{u}^h$ denote any interpolant of u from $H^{\mathbf{s}}(\Omega, \mathcal{P}(\Omega))$ onto the finite-dimensional space $\mathcal{U}^h$. Let us specify the interpolation error by $\eta = u - \tilde{u}^h$. Thereby, we can decompose the global error $u - u^h$ as follows*

$$u - u^h = (u - \tilde{u}^h) + (\tilde{u}^h - u^h) \equiv \eta + e^h. \tag{6.97}$$

So, using the triangle inequality, we have

$$|||u - u^h|||_{sb} \leqslant |||\eta|||_{sb} + |||e^h|||_{sb}, \tag{6.98}$$

where $e^h = \tilde{u}^h - u^h$ is the part of the error in the finite element space, i.e., $e^h \in \mathcal{W}^h$.

Our error analysis below will provide a bound on $|||e^h|||_{sb}$ in terms of suitable norms of η. As a consequence, we shall obtain a bound on $|||u-u^h|||_{sb}$ with respect to various norms of η. Hence, to complete the error analysis, we shall need to quantify norms of η in terms of the discretization parameter and Sobolev seminorms of the analytical solution u.

Theorem 6.5.1.2. *Assume that the consistency condition (6.55) and stability condition (6.56) (see Proposition 6.4.1.2) of the method hold. For each face, we define positive, real, piecewise constant functions β, β_q, γ, γ_r, α and α_q by*

$$\beta = \beta_q = \frac{C_\beta EIg^2}{h_e^3}, \ \gamma = \gamma_r = \frac{C_\gamma EIg^2}{h_e} \quad \text{and} \quad \alpha = \alpha_q = \frac{C_\delta EI}{h_e}.$$

Given that the conditions are satisfied for the interpolation estimates (A.30), (A.31) *and the trace inequalities* (A.38), (A.39) *hold, the error estimate for the continuous interior penalty method* (6.54) *can be written as*

$$|||u - u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2, \tag{6.99}$$

where C *is a constant dependent only on the space dimension and on* k, *and* $|\cdot|_{k+1,\Omega_e}$ *denotes the* H^{k+1}*-seminorm on* Ω_e.

Proof. To begin with, we shall estimate e^h. For that purpose, we take advantage of the coercivity (6.56), the decomposition of the error (6.97) and the Galerkin orthogonality (6.55) yielding

$$\begin{aligned} \theta|||e^h|||_{sb}^2 &\leqslant B_{sb}(e^h, e^h) \\ &= B_{sb}(u - u^h - \eta, e^h) \\ &= B_{sb}(u - u^h, e^h) - B_s(\eta, e^h) \\ &= -B_{sb}(\eta, e^h) \\ &\leqslant |B_{sb}(\eta, e^h)|. \end{aligned} \tag{6.100}$$

We continue by using the triangle inequality on the right-hand side of (6.100). Then, we obtain

$$\begin{aligned} \theta|||e^h|||_{sb}^2 \leqslant\ & |((EIg^2\eta_{,xx})_{,x}, (e^h_{,xx})_{,x})_{\tilde{\Omega}}| + |(EI\eta_{,xx}, e^h_{,xx})_{\tilde{\Omega}}| \\ &+ |\langle (EIg^2\eta_{,xx})_{,xx}\rangle [\![e^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\eta_{,x}]\!]\langle (EIg^2 e^h_{,xx})_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ &+ |\langle (EIg^2\eta_{,xx})_{,x}\rangle [\![e^h_{,xx}]\!]_{\tilde{\Gamma}_2}| + |[\![\eta_{,xx}]\!]\langle (EIg^2 e^h_{,xx})_{,x}\rangle_{\tilde{\Gamma}_2}| \\ &+ |\langle EI\eta_{,xx}\rangle [\![e^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\eta_{,x}]\!]\langle EI e^h_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ &+ |\beta[\![\eta_{,x}]\!][\![e^h_{,x}]\!]_{\tilde{\Gamma}_1}| + |\gamma[\![\eta_{,xx}]\!][\![e^h_{,xx}]\!]_{\tilde{\Gamma}_2}| + |\alpha[\![\eta_{,x}]\!][\![e^h_{,x}]\!]_{\tilde{\Gamma}_1}|. \end{aligned} \tag{6.101}$$

To bound the terms on the right-hand side of (6.101), we apply the

Cauchy-Schwarz inequality (A.12) giving

$$\begin{aligned}
\theta|||e^h|||_{sb}^2 \leqslant\ & ||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\tilde{\Omega}}||(EIg^2)^{1/2}(e^h_{,xx})_{,x}||_{\tilde{\Omega}} \\
& + ||(EI)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}||(EI)^{1/2}e^h_{,xx}||_{\tilde{\Omega}} \\
& + ||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1}||\beta^{1/2}[\![e^h_x]\!]||_{\tilde{\Gamma}_1} \\
& + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}||\beta^{-1/2}\langle(EIg^2e^h_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1} \\
& + ||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2}||\gamma^{1/2}[\![e^h_{,xx}]\!]||_{\tilde{\Gamma}_2} \\
& + ||\gamma^{1/2}[\![\eta_{,xx}]\!]||_{\tilde{\Gamma}_2}||\gamma^{-1/2}\langle(EIg^2e^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2} \\
& + ||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||_{\tilde{\Gamma}_1}||\alpha^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}_1} \\
& + ||\alpha^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}||\alpha^{-1/2}\langle EIe^h_{,xx}\rangle||_{\tilde{\Gamma}_1} \\
& + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}_1} \\
& + ||\gamma^{1/2}[\![\eta_{,xx}]\!]||_{\tilde{\Gamma}_2}||\gamma^{1/2}[\![e^h_{,xx}]\!]||_{\tilde{\Gamma}_2} \\
& + ||\alpha^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}||\alpha^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}_1}. \qquad (6.102)
\end{aligned}$$

As before in this chapter, we shall make use of Young inequality (A.17) on each term on the right-hand side of (6.102). For that reason, we deduce

$$\begin{aligned}
\theta|||e^h|||_{sb}^2 \leqslant\ & \frac{1}{2\varepsilon}\Big(||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\tilde{\Omega}}^2 + ||(EI)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}^2 \\
& + ||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1}^2 + 2||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}^2 \\
& + ||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2}^2 + 2||\gamma^{1/2}[\![\eta_{,xx}]\!]||_{\tilde{\Gamma}_2}^2 \\
& + ||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||_{\tilde{\Gamma}_1}^2 + 2||\alpha^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}^2\Big) \\
& + \frac{\varepsilon}{2}\Big(||(EIg^2)^{1/2}(e^h_{,xx})_{,x}||_{\tilde{\Omega}}^2 + ||(EI)^{1/2}e^h_{,xx}||_{\tilde{\Omega}}^2 \\
& + 2||\beta^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}_1}^2 + ||\beta^{-1/2}\langle(EIg^2e^h_{,xx})_{,xx}\rangle||_{\tilde{\Gamma}_1}^2 \\
& + 2||\gamma^{1/2}[\![e^h_{,xx}]\!]||_{\tilde{\Gamma}_2}^2 + ||\gamma^{-1/2}\langle(EIg^2e^h_{,xx})_{,x}\rangle||_{\tilde{\Gamma}_2}^2 \\
& + 2||\alpha^{1/2}[\![e^h_{,x}]\!]||_{\tilde{\Gamma}_1}^2 + ||\alpha^{-1/2}\langle EIe^h_{,xx}\rangle||_{\tilde{\Gamma}_1}^2\Big). \qquad (6.103)
\end{aligned}$$

Thus, to proceed with the estimate of e^h, one of the steps remaining is to bound each of the mean value terms which are enclosed into the second parenthesis, on the right-hand side of (6.103).

To achieve that, we shall follow exactly the same series of steps presented in mathematical formulas (6.80) – (6.85). As a result, we get

$$\begin{aligned} ||\beta^{-1/2}\langle(EIg^2 e^h_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EIg^2)^{1/2}(e^h_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\gamma^{-1/2}\langle(EIg^2 e^h_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} &\leqslant ||(EIg^2)^{1/2}(e^h_{,xx})_{,x}||^2_{\tilde{\Omega}}, \\ ||\alpha^{-1/2}\langle EIe^h_{,xx}\rangle||^2_{\tilde{\Gamma}_1} &\leqslant ||(EI)^{1/2}e^h_{,xx}||^2_{\tilde{\Omega}}. \end{aligned} \tag{6.104}$$

To boot, by inserting the inequalities, (6.104), into the second bracket on the right-hand side of (6.103), we have

$$\begin{aligned} \theta|||e^h|||^2_{sb} \leqslant\ & \frac{1}{2\varepsilon}\Big(||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\ & + ||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} + 2||\gamma^{1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \\ & + ||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\alpha^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big) \\ & + \frac{\varepsilon}{2}\Big(3||(EIg^2)^{1/2}(e^h_{,xx})_{,x}||^2_{\tilde{\Omega}} + 2||(EI)^{1/2}e^h_{,xx}||^2_{\tilde{\Omega}} \\ & + 2||\beta^{1/2}[\![e^h_{,x}]\!]||^2_{\tilde{\Gamma}_1} + 2||\gamma^{1/2}[\![e^h_{,xx}]\!]||^2_{\tilde{\Gamma}_2} + 2||\alpha^{1/2}[\![e^h_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big). \end{aligned} \tag{6.105}$$

Now, by the use of the definition of energy seminorm, (6.26), in second parenthesis on the right-hand side of (6.105), derives

$$\begin{aligned} \theta|||e^h|||^2_{sb} \leqslant\ & \frac{1}{2\varepsilon}\Big(||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||^2_{\tilde{\Omega}} + ||(EI)^{1/2}\eta_{,xx}||^2_{\tilde{\Omega}} \\ & + ||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\ & + ||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} + 2||\gamma^{1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \\ & + ||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||^2_{\tilde{\Gamma}_1} + 2||\alpha^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1}\Big) \\ & + \frac{3\varepsilon}{2}|||e^h|||^2_{sb}. \end{aligned} \tag{6.106}$$

Afterwards, by choosing an appropriate value for ε in (6.106) derives a bound on $|||e^h|||_{sb}$, in terms of suitable norms of η, being

$$
\begin{aligned}
\frac{\theta}{3}|||e^h|||^2_{sb} \leqslant\ & \frac{EIg^2}{\theta}||(\eta_{,xx})_{,x}||^2_{\tilde{\Omega}} + \frac{EI}{\theta}||\eta_{,xx}||^2_{\tilde{\Omega}} \\
& + \left\{ \frac{(EIg^2)^2}{\theta}||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \right\} \\
& + \left\{ \frac{C_\beta EIg^2}{\theta}||h_e^{-3/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \right\} \\
& + \left\{ \frac{(EIg^2)^2}{\theta}||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \right\} \\
& + \left\{ \frac{C_\gamma EIg^2}{\theta}||h_e^{-1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \right\} \\
& + \left\{ \frac{(EI)^2}{\theta}||\alpha^{-1/2}\langle\eta_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \right\} \\
& + \left\{ \frac{C_\alpha EI}{\theta}||h_e^{-1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \right\} \\
& + \left\{ \frac{1}{\theta}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \right\} \\
& + \left\{ \frac{1}{\theta}||\gamma^{1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \right\} \\
& + \left\{ \frac{1}{\theta}||\alpha^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \right\}.
\end{aligned}
\tag{6.107}
$$

We simultaneously note that the inverse estimates (see Theorem A.4.1 and Remarks A.4.2) do not hold for the interpolation error, since $\eta \notin \mathcal{W}^h$.

Therefore to complete the estimate of e^h, a subsequent step is to bound the terms enclosed into the brackets on the right-hand side of (6.107).

Hence, by invoking the mean value inequality (A.19), we can write the factors enclosed into the first bracket on the right-hand side of (6.107) as

$$\begin{aligned}
&\frac{(EIg^2)^2}{\theta}||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \\
&= \frac{(EIg^2)^2}{\theta}||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}} + \frac{(EIg^2)^2}{\theta}||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_q} \\
&= \frac{(EIg^2)^2}{\theta}\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \frac{(EIg^2)^2}{\theta}\sum_{j=1}^{N_q}||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j} \\
&\leqslant \frac{(EIg^2)^2}{\theta}\sum_{i=1}^{N_i}\left(||\beta^{-1/2}(\eta^+_{,xx})_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}(\eta^-_{,xx})_{,xx}||^2_{\Gamma_i}\right) \\
&+ \frac{(EIg^2)^2}{\theta}\sum_{j=1}^{N_q}||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j} \\
&\leqslant \frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}||\beta^{-1/2}(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e},
\end{aligned} \tag{6.108}$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$. Next, by applying the trace inequality (A.39) as well as the properties of Sobolev norms in (6.108), we conclude that

$$\begin{aligned}
&\frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}||\beta^{-1/2}(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e} \\
&\leqslant \frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}C\left(h_e^{-1}|\beta^{-1/2}(\eta_{,xx})_{,x}|^2_{1,\Omega_e} + h_e|\beta^{-1/2}(\eta_{,xx})_{,x}|^2_{2,\Omega_e}\right) \\
&\leqslant \frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}C\left(h_e^{-1}||\beta^{-1/2}(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||\beta^{-1/2}(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right) \\
&= \frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}C\beta^{-1}\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right) \\
&= \frac{(EIg^2)^2}{\theta}\sum_{e=1}^{N_{el}}C\frac{h_e^3}{C_\beta EIg^2}\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right) \\
&= C\sum_{e=1}^{N_{el}}h_e^3\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right).
\end{aligned} \tag{6.109}$$

In consequence, from (6.108) – (6.109), we reach the conclusion that the factors enclosed into the first bracket, on the right-hand side of (6.107), can be bounded as follows

$$\frac{(EIg^2)^2}{\theta}||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} h_e^3\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right). \tag{6.110}$$

Moreover, we shall follow the above procedure in a similar manner to bound the terms enclosed into the third and the fifth bracket respectively, on the right-hand side of (6.107). As a consequence, we get

$$\frac{(EIg^2)^2}{\theta}||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\tilde{\Gamma}_2} \leqslant C\sum_{e=1}^{N_{el}} h_e\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,xx}||^2_{\Omega_e}\right), \tag{6.111}$$

and

$$\frac{(EI)^2}{\theta}||\alpha^{-1/2}\langle\eta_{,xx}\rangle||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} h_e\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{\Omega_e}\right). \tag{6.112}$$

Additionally, we shall analogously estimate the factors enclosed into the second bracket on the right-hand side of (6.107). By recalling the jump inequality (A.18), we obtain

$$\begin{aligned}
&\frac{C_\beta EIg^2}{\theta}||h_e^{-3/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1}\\
&= \frac{C_\beta EIg^2}{\theta}||h_e^{-3/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + \frac{C_\beta EIg^2}{\theta}||h_e^{-3/2}\eta_{,x}||^2_{\Gamma_q}\\
&= \frac{C_\beta EIg^2}{\theta}\sum_{i=1}^{N_i}||h_e^{-3/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i} + \frac{C_\beta EIg^2}{\theta}\sum_{j=1}^{N_q}||h_e^{-3/2}\eta_{,x}||^2_{\Gamma_j}\\
&\leqslant \frac{C_\beta EIg^2}{\theta}\sum_{i=1}^{N_i}2\left(||h_e^{-3/2}\eta^+_{,x}||^2_{\Gamma_i} + ||h_e^{-3/2}\eta^-_{,x}||^2_{\Gamma_i}\right) + \frac{C_\beta EIg^2}{\theta}\sum_{j=1}^{N_q}||h_e^{-3/2}\eta_{,x}||^2_{\Gamma_j}\\
&\leqslant 2\frac{C_\beta EIg^2}{\theta}\sum_{e=1}^{N_{el}}||h_e^{-3/2}\eta_{,x}||^2_{\partial\Omega_e}.
\end{aligned} \tag{6.113}$$

We employ the trace inequality (A.38) and next the properties of Sobolev norms in (6.113), so we deduce

$$
\begin{aligned}
&2\frac{C_\beta EIg^2}{\theta}\sum_{e=1}^{N_{el}}||h_e^{-3/2}\eta_{,x}||^2_{\partial\Omega_e} \\
&\leqslant 2\frac{C_\beta EIg^2}{\theta}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||h_e^{-3/2}\eta_{,x}||^2_{\Omega_e} + h_e||h_e^{-3/2}\eta_{,xx}||^2_{\Omega_e}\right) \\
&= C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right).
\end{aligned}
\tag{6.114}
$$

Ergo, from (6.113) – (6.114), we arrive to the conclusion that the factors enclosed into the second bracket, on the right-hand side of (6.107), can be estimated as follows

$$
\frac{C_\beta EIg^2}{\theta}||h_e^{-3/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right). \tag{6.115}
$$

Furthermore, we shall follow the above series of steps in the same way to estimate the terms enclosed into the fourth and the sixth bracket respectively on the right-hand side of (6.107). For thar reason, we have

$$
\frac{C_\gamma EIg^2}{\theta}||h_e^{-1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{\Omega_e}\right), \tag{6.116}
$$

and

$$
\frac{C_\alpha EI}{\theta}||h_e^{-1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right). \tag{6.117}
$$

What is more, we shall use similar arguments to bound the factors that enclosed into the seventh bracket on the right-hand side of (6.107). By applying the jump inequality (A.18), we deduce

$$\begin{aligned}
&\frac{1}{\theta}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_1} \\
&= \frac{1}{\theta}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + \frac{1}{\theta}||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} \\
&= \frac{1}{\theta}\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i} + \frac{1}{\theta}\sum_{j=1}^{N_q}||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_j} \\
&\leqslant \frac{1}{\theta}\sum_{i=1}^{N_i} 2\left(||\beta^{1/2}\eta^+_{,x}||^2_{\Gamma_i} + ||\beta^{1/2}\eta^-_{,x}||^2_{\Gamma_i}\right) + \frac{1}{\theta}\sum_{j=1}^{N_q}||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_j} \\
&\leqslant \frac{2}{\theta}\sum_{e=1}^{N_{el}}||\beta^{1/2}\eta_{,x}||^2_{\partial\Omega_e}.
\end{aligned} \tag{6.118}$$

Afterwards, in (6.118), we invoke the trace inequality (A.38) and the properties of Sobolev norms giving

$$\begin{aligned}
&\frac{2}{\theta}\sum_{e=1}^{N_{el}}||\beta^{1/2}\eta_{,x}||^2_{\partial\Omega_e} \\
&\leqslant \frac{2}{\theta}\sum_{e=1}^{N_{el}} C\left(h_e^{-1}||\beta^{1/2}\eta_{,x}||^2_{\Omega_e} + h_e||\beta^{1/2}\eta_{,xx}||^2_{\Omega_e}\right) \\
&= \frac{2}{\theta}\sum_{e=1}^{N_{el}} C\beta\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \\
&= \frac{2}{\theta}\sum_{e=1}^{N_{el}} C\frac{C_\beta EIg^2}{h_e^3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \\
&= C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right).
\end{aligned} \tag{6.119}$$

Wherefore, from (6.118) – (6.119), we reach the conclusion that the terms enclosed into the seventh bracket, on the right-hand side of (6.107), can be bounded as follows

$$\frac{1}{\theta}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_1} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right). \tag{6.120}$$

Also, by following the previous procedure step by step, we shall estimate the terms enclosed into the eighth and the ninth bracket respectively on the right-hand side of (6.107). Therefore, we arrive at

$$\frac{1}{\theta}||\gamma^{1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{\Omega_e}\right), \tag{6.121}$$

and

$$\frac{1}{\theta}||\alpha^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right). \tag{6.122}$$

After that, gathering the inequalities (6.110) – (6.112), (6.115) – (6.117), (6.120) – (6.122) and inserting them on the right-hand side of (6.107), we obtain

$$\begin{aligned}\frac{\theta}{3}|||e^h|||^2_{sb} \;\leqslant\; & C\sum_{e=1}^{N_{el}}\big\{||(\eta_{,xx})_{,x}||^2_{\Omega_e} + ||\eta_{,xx}||^2_{\Omega_e} \\ & + h_e^3\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{1,\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{2,\Omega_e}\right) \\ & + h_e^{-3}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right) \\ & + h_e\left(h_e^{-1}||(\eta_{,xx})_{,x}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,xx}||^2_{\Omega_e}\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{\Omega_e}\right) \\ & + h_e\left(h_e^{-1}||\eta_{,xx}||^2_{\Omega_e} + h_e||(\eta_{,xx})_{,x}||^2_{\Omega_e}\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta_{,x}||^2_{\Omega_e} + h_e||\eta_{,xx}||^2_{\Omega_e}\right)\big\}.\end{aligned} \tag{6.123}$$

Application of interpolation estimates, (A.30), (A.31), yields for the terms on the right-hand side of (6.123)

$$||(\eta_{,xx})_{,x}||_{\Omega_e} \leqslant ||\eta_{,xx}||_{1,\Omega_e} \leqslant ||\eta||_{3,\Omega_e} \leqslant Ch_e^{k-2}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{6.124}$$

$$||\eta_{,xx}||_{\Omega_e} \leqslant ||\eta_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{2,\Omega_e} \leqslant Ch_e^{k-1}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{6.125}$$

$$||(\eta_{,xx})_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{4,\Omega_e} \leqslant Ch_e^{k-3}|u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{6.126}$$

$$||(\eta_{,xx})_{,x}||_{2,\Omega_e} \leqslant ||\eta||_{5,\Omega_e} \leqslant C h_e^{k-4} |u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e), \tag{6.127}$$

$$||\eta_{,x}||_{\Omega_e} \leqslant ||\eta||_{1,\Omega_e} \leqslant C h_e^{k} |u|_{k+1,\Omega_e} \quad \forall u \in H^{k+1}(\Omega_e). \tag{6.128}$$

Substitution of (6.124) – (6.128) on the right-hand side of (6.123) leads to

$$\begin{aligned} \frac{\theta}{3}|||e^h|||_{sb}^2 &\leqslant C \sum_{e=1}^{N_{el}} \left(h_e^{2(k-1)} + h_e^{2(k-2)}\right) |u|_{k+1,\Omega_e}^2 \\ &\leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2 . \end{aligned}$$

Then, multiplying by $\frac{3}{\theta}$ both sides of the above inequality, we reach to the conclusion that e^h can be estimated as

$$|||e^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2 . \tag{6.129}$$

To go on, we shall estimate η by using similar arguments as in the case of e^h. By the definition of energy seminorm, (6.26), we get

$$\begin{aligned} |||\eta|||_{sb}^2 &= ||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\tilde{\Omega}}^2 + ||(EI)^{1/2}\eta_{,xx}||_{\tilde{\Omega}}^2 + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}^2 \\ &\quad + ||\gamma^{1/2}[\![\eta_{,xx}]\!]||_{\tilde{\Gamma}_2}^2 + ||\alpha^{1/2}[\![\eta_{x}]\!]||_{\tilde{\Gamma}_1}^2 . \end{aligned}$$

Next, by employing the inequalities (6.120) – (6.122), having ignored the coefficient $\frac{1}{\theta}$, we can bound the terms on the right-hand side of the seminorm as

$$\begin{aligned} |||\eta|||_{sb}^2 \leqslant\ & C \sum_{e=1}^{N_{el}} \Big\{ ||(\eta_{,xx})_{,x}||_{\Omega_e}^2 + ||\eta_{,xx}||_{\Omega_e}^2 + h_e^{-3}\left(h_e^{-1}||\eta_{,x}||_{\Omega_e}^2 + h_e||\eta_{,xx}||_{\Omega_e}^2\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta_{,xx}||_{\Omega_e}^2 + h_e||(\eta_{,xx})_{,x}||_{\Omega_e}^2\right) \\ & + h_e^{-1}\left(h_e^{-1}||\eta_{,x}||_{\Omega_e}^2 + h_e||\eta_{,xx}||_{\Omega_e}^2\right) \Big\}. \end{aligned} \tag{6.130}$$

Afterwards, insertion of the mathematical expressions (6.124), (6.125), (6.128) into the right-hand side of (6.130) yields

$$\begin{aligned} |||\eta|||_{sb}^2 &\leqslant C \sum_{e=1}^{N_{el}} \left(h_e^{2(k-1)} + h_e^{2(k-2)}\right) |u|_{k+1,\Omega_e}^2 \\ &\leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2 . \end{aligned}$$

As a result, we conclude that η can be bounded as

$$|||\eta|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2. \tag{6.131}$$

Now, combining (6.98) with the inequalities (6.129) and (6.131), we have

$$\begin{aligned} |||u - u^h|||_{sb}^2 &\leqslant \left(|||\eta|||_{sb} + |||e^h|||_{sb}\right)^2 \\ &\leqslant 2\left(|||\eta|||_s^2 + |||e^h|||_s^2\right) \\ &\leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2. \end{aligned}$$

Finally, it follows that

$$|||u - u^h|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} h_e^{2(k-2)} |u|_{k+1,\Omega_e}^2,$$

which is the desired result. □

It is noteworthy that the resulting a priori error estimate is optimal in h.

Now, it is imperative that we pay our attention to a priori error estimate of the *hp*-version of the method presented in this chapter.

Convergence 6.5.1.3. *Let $\Pi_{\mathbf{p}}$ denote any (linear) projection operator from $H^{\mathbf{s}}(\Omega, \mathcal{P}(\Omega))$ onto the finite element space $\mathcal{U}^{hp}$. We can then decompose the global error $u - u^{hp}$ as follows:*

$$u - u^{hp} = (u - \Pi_{\mathbf{p}} u) + (\Pi_{\mathbf{p}} u - u^{hp}) \equiv \eta + \xi. \tag{6.132}$$

So, using the triangle inequality, we get

$$|||u - u^{hp}|||_{sb} \leqslant |||\eta|||_{sb} + |||\xi|||_{sb}, \tag{6.133}$$

where $\xi = \Pi_{\mathbf{p}} u - u^{hp}$ is the part of the error in the finite element space, i.e., $\xi \in \mathcal{W}^{hp}$.

Our error analysis below will provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η. Thus, we shall obtain a bound on $|||u - u^{hp}|||_{sb}$ with respect to various norms of η. Ergo, to complete the error analysis, we shall need to quantify norms of η in terms of the discretization parameters and Sobolev seminorms of the analytical solution u.

Theorem 6.5.1.4. *Suppose that* Ω *is a bounded domain in* $\Re$ *and that* $\mathcal{P}(\Omega)$ *is a regular partition of* Ω *into elements* Ω_e*. Let* $\mathbf{p} = (p_e : \Omega_e \in \mathcal{P}(\Omega), p_e \in \aleph, p_e \geqslant 4)$ *be any polynomial degree vector of bounded local variation. For each face, we define positive, real, piecewise constant functions* β, β_q, γ, γ_r, α *and* α_q *by*

$$\beta = \beta_q = \frac{C_\beta EIg^2 p_e^6}{h_e^3}, \; \gamma = \gamma_r = \frac{C_\gamma EIg^2 p_e^2}{h_e} \quad \text{and} \quad \alpha = \alpha_q = \frac{C_\alpha EI p_e^2}{h_e},$$

where the stabilization constants C_β, C_γ *and* C_α *are arbitrary positive real numbers. If the analytical solution* u *to the problem (6.25) belongs to the broken Sobolev space* $H^{\mathbf{t}}(\Omega, \mathcal{P}(\Omega))$, $\mathbf{t} = (t_e : \Omega_e \in \mathcal{P}(\Omega), t_e \geqslant 6)$*, then the solution* $u^{hp} \in \mathcal{U}^{hp}$ *of the problem (6.54) satisfies the following error bound*

$$|||u - u^{hp}|||_{sb}^2 \leqslant C \sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}} ||u||_{t_e,\Omega_e}^2, \tag{6.134}$$

where $3 \leqslant s_e \leqslant \min(p_e+1, t_e)$*, and* C *is a constant dependent only on the space dimension and on* $t = \max_{\Omega_e \in \mathcal{P}(\Omega)} t_e$*.*

Proof. To begin with, we shall estimate ξ. For that purpose, we take advantage of the coercivity (6.67), the decomposition of the error (6.132) and the Galerkin orthogonality (6.55) yielding

$$\begin{aligned} \theta|||\xi|||_{sb}^2 &\leqslant B_{sb}(\xi,\xi) \\ &= B_{sb}(u - u^{hp} - \eta, \xi) \\ &= B_{sb}(u - u^{hp}, \xi) - B_{sb}(\eta,\xi) \\ &= -B_{sb}(\eta,\xi) \\ &\leqslant |B_{sb}(\eta,\xi)|. \end{aligned} \tag{6.135}$$

We continue by using the triangle inequality on the right-hand side of (6.135). Then, we obtain

$$\begin{aligned} \theta|||\xi|||_{sb}^2 \leqslant\; & |((EIg^2\eta_{,xx})_{,x}, (\xi_{,xx})_{,x})_{\tilde{\Omega}}| + |(EI\eta_{,xx}, \xi_{,xx})_{\tilde{\Omega}}| \\ & + |\langle (EIg^2\eta_{,xx})_{,xx}\rangle [\![\xi_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\eta_{,x}]\!]\langle (EIg^2\xi_{,xx})_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ & + |\langle (EIg^2\eta_{,xx})_{,x}\rangle [\![\xi_{,xx}]\!]_{\tilde{\Gamma}_2}| + |[\![\eta_{,xx}]\!]\langle (EIg^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}_2}| \\ & + |\langle EI\eta_{,xx}\rangle [\![\xi_{,x}]\!]_{\tilde{\Gamma}_1}| + |[\![\eta_{,x}]\!]\langle EI\xi_{,xx}\rangle_{\tilde{\Gamma}_1}| \\ & + |\beta [\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}_1}| + |\gamma [\![\eta_{,xx}]\!][\![\xi_{,xx}]\!]_{\tilde{\Gamma}_2}| + |\alpha [\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}_1}| \end{aligned}$$

or by applying inner products (3.7), (3.8)

$$\begin{aligned}\theta|||\xi|||_{sb}^2 \leqslant\ & |((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\tilde{\Omega}}| + |(EI\eta_{,xx},\xi_{,xx})_{\tilde{\Omega}}| \\ & + |\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\tilde{\Gamma}}| \\ & + |\langle(EIg^2\eta_{,xx})_{,x}\rangle[\![\xi_{,xx}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,xx}]\!]\langle(EIg^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| \\ & + |\langle EI\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |[\![\eta_{,x}]\!]\langle EI\xi_{,xx}\rangle_{\tilde{\Gamma}}| \\ & + |\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| + |\gamma[\![\eta_{,xx}]\!][\![\xi_{,xx}]\!]_{\tilde{\Gamma}}| + |\alpha[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| \\ & + |(EIg^2\eta_{,xx})_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| + |\eta_{,x}\cdot n(EIg^2\xi_{,xx})_{,xx}|_{\Gamma_q}| \\ & + |(EIg^2\eta_{,xx})_{,x}\cdot n\xi_{,xx}|_{\Gamma_r}| + |\eta_{,xx}(EIg^2\xi_{,xx})_{,x}\cdot n|_{\Gamma_r}| \\ & + |EI\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}| + |\eta_{,x}\cdot nEI\xi_{,xx}|_{\Gamma_q}| \\ & + |\beta_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}| + |\gamma_r\eta_{,xx}\xi_{,xx}|_{\Gamma_r}| + |\alpha_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}|.\end{aligned} \tag{6.136}$$

Thereby, to provide a bound on $|||\xi|||_{sb}$ in terms of suitable norms of η, it only remains to estimate the inner products on the right-hand side of (6.136).

With the aim of bounding the first inner product on the right-hand side of (6.136), we initially apply the triangle inequality yielding

$$\begin{aligned}|((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\tilde{\Omega}}| &= \left|\sum_{e=1}^{N_{el}}|((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\Omega_e}\right| \\ &\leqslant \sum_{e=1}^{N_{el}}|((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\Omega_e}|.\end{aligned} \tag{6.137}$$

Then, by recalling the Cauchy-Schwarz inequality (A.12) and next the Cauchy-Schwarz discrete inequality (A.13) in (6.137), we have

$$\begin{aligned}&\sum_{e=1}^{N_{el}}|((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\Omega_e}| \\ &\leqslant \sum_{e=1}^{N_{el}}||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\Omega_e}||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||_{\Omega_e} \\ &\leqslant \left(\sum_{e=1}^{N_{el}}||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\Omega_e}^2\right)^{1/2}\left(\sum_{e=1}^{N_{el}}||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||_{\Omega_e}^2\right)^{1/2} \\ &= \left(||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||_{\tilde{\Omega}}^2\right)^{1/2}\left(||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||_{\tilde{\Omega}}^2\right)^{1/2}.\end{aligned} \tag{6.138}$$

By making use of the definition of energy seminorm, (6.26), in (6.138), we get

$$\left(||(EIg^2)^{1/2}(\eta_{,xx})_{,x}||^2_{\tilde{\Omega}}\right)^{1/2}\left(||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\tilde{\Omega}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.139}$$

Therefore, from (6.137) – (6.139), we reach the conclusion that the first inner product, on the right-hand side of (6.136), can be bounded as follows

$$|((EIg^2\eta_{,xx})_{,x},(\xi_{,xx})_{,x})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.140}$$

Also, the second inner product, on the right-hand side of (6.136), can analogously be bounded as

$$|(EI\eta_{,xx},\xi_{,xx})_{\tilde{\Omega}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.141}$$

We shall additionally follow similar series of steps to estimate the stabilizing terms on the right-hand side of (6.136). Employing the triangle inequality, we deduce

$$\begin{aligned}|\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i}|\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}|.\end{aligned} \tag{6.142}$$

After that, by invoking the Cauchy-Schwarz inequality (A.12) and the Cauchy-Schwarz discrete inequality (A.13) in (6.142), we conclude

$$\begin{aligned}&\sum_{i=1}^{N_i}|\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||_{\Gamma_i}||\beta^{1/2}[\![\xi_{,x}]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\ &= \left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}\left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}.\end{aligned} \tag{6.143}$$

Using the defintion of energy seminorm, (6.26), in (6.143), it derives

$$\left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}\left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.144}$$

Ergo, from (6.142) – (6.144), we arrive to the conclusion that the first stabilizing term, on right-hand side of (6.136), can be estimated as follows

$$|\beta[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.145}$$

Moreover, the rest of stabilizing terms on the right hand side of (6.136) can correspondingly be bounded as follows

$$\begin{aligned} |\gamma[\![\eta_{,xx}]\!][\![\xi_{,xx}]\!]_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\alpha[\![\eta_{,x}]\!][\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\beta_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\gamma_r\eta_{,xx}\xi_{,xx}|_{\Gamma_r}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\alpha_q\eta_{,x}\cdot n\xi_{,x}\cdot n|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \end{aligned} \tag{6.146}$$

It's about time for us to estimate inner products, containing the mean value operator of η and the jump operator of ξ, on the right-hand side of (6.136). We use at first the triagle inequality and as a result we get

$$\begin{aligned} |\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i}|\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}|. \end{aligned} \tag{6.147}$$

Afterwards, applying the Cauchy-Schwarz inequality (A.12) and then the Cauchy-Schwarz discrete inequality (A.13) in (6.147), we have

$$\begin{aligned} &\sum_{i=1}^{N_i}|\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||_{\Gamma_i}||\beta^{1/2}[\![\xi_{,x}]\!]||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\ &= \left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\left(||\beta^{1/2}[\![\xi_x]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}. \end{aligned} \tag{6.148}$$

Invoking the definition of energy seminorm, (6.26), in (6.148), we obtain

$$\begin{aligned}&\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\quad\left(||\beta^{1/2}[\![\xi_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2}\\&\leqslant\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\quad|||\xi|||_{sb}.\end{aligned}\tag{6.149}$$

In consequence, from (6.147) – (6.149), we conclude that this type of inner product, on the right-hand side of (6.136), can be bounded as follows

$$|\langle(EIg^2\eta_{,xx})_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}|\leqslant\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\quad|||\xi|||_{sb}.\tag{6.150}$$

Furthermore, we shall use similar arguments to estimate the remaining inner products of the corresponding form, on the right-hand side of (6.136). Thus, we deduce

$$\begin{aligned}|\langle(EIg^2\eta_{,xx})_{,x}\rangle[\![\xi_{,xx}]\!]_{\tilde{\Gamma}}|&\leqslant\left(\sum_{i=1}^{N_i}||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i}\right)^{1/2}\quad|||\xi|||_{sb},\\|\langle EI\eta_{,xx}\rangle[\![\xi_{,x}]\!]_{\tilde{\Gamma}}|&\leqslant\left(\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2}\quad|||\xi|||_{sB},\\|(EIg^2\eta_{,xx})_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}|&\leqslant\left(\sum_{j=1}^{N_q}||\beta_q^{-1/2}(EIg^2\eta_{,xx})_{,xx}||^2_{\Gamma_j}\right)^{1/2}\quad|||\xi|||_{sb},\\|(EIg^2\eta_{,xx})_{,x}\cdot n\xi_{,xx}|_{\Gamma_r}|&\leqslant\left(\sum_{s=1}^{N_r}||\gamma_r^{-1/2}(EIg^2\eta_{,xx})_{,x}||^2_{\Gamma_s}\right)^{1/2}\quad|||\xi|||_{sb},\\|EI\eta_{,xx}\xi_{,x}\cdot n|_{\Gamma_q}|&\leqslant\left(\sum_{j=1}^{N_q}||\beta_q^{-1/2}EI\eta_{,xx}||^2_{\Gamma_j}\right)^{1/2}\quad|||\xi|||_{sb},\end{aligned}\tag{6.151}$$

where N_q denotes the number of exterior slope boundary segments $\Gamma_j \subseteq \Gamma_q$ and N_r denotes the number of exterior curvature boundary segments $\Gamma_s \subseteq \Gamma_r$, as well.

A last step, for bounding $|||\xi|||_{sb}$ in terms of norms of η, is to estimate the rest of inner products, which contain the jump operator of η and the mean value operator of ξ, in (6.136). As in the latter case, employing the triangle inequality produces

$$\begin{aligned} |[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\tilde{\Gamma}}| &= \left|\sum_{i=1}^{N_i}[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\Gamma_i}\right| \\ &\leqslant \sum_{i=1}^{N_i}|[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\Gamma_i}|. \end{aligned} \tag{6.152}$$

Thereafter, by recalling the Cauchy-Schwarz inequality (A.12) and the Cauchy-Schwarz discrete inequality (A.13) in (6.152), we conclude

$$\begin{aligned} &\sum_{i=1}^{N_i}|[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\Gamma_i}| \\ &\leqslant \sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||_{\Gamma_i}||\beta^{-1/2}\langle(EIg^2\xi_{,xx})_{,xx}\rangle||_{\Gamma_i} \\ &\leqslant \left(\sum_{i=1}^{N_i}||\beta^{1/2}[\![\eta_{,x}]\!]||_{\Gamma_i}^2\right)^{1/2}\left(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\xi_{,xx})_{,xx}\rangle||_{\Gamma_i}^2\right)^{1/2}. \end{aligned} \tag{6.153}$$

By invoking the mean value inequality (A.19) in (6.153), we now have

$$
\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(EIg^2\xi_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{i=1}^{N_i}(||\beta^{-1/2}(EIg^2\xi^+_{,xx})_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}(EIg^2\xi^-_{,xx})_{,xx}||^2_{\Gamma_i})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}}(||\beta^{-1/2}(EIg^2\xi_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}(EIg^2\xi_{,xx})_{,xx}||^2_{\partial\Omega_e})\right)^{1/2}.
\end{aligned}
\tag{6.154}
$$

Also, in (6.154), since $\xi \in \mathcal{W}^{hp}$ we can apply the inverse inequality (A.21), so we obtain

$$
\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left(\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} (||\beta^{-1/2}(EIg^2\xi_{,xx})_{,xx}||^2_{\partial\Omega_{e'}} + ||\beta^{-1/2}(EIg^2\xi_{,xx})_{,xx}||^2_{\partial\Omega_e})\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \\
&\times \left\{\sum_{e',e=1:(\partial\Omega_{e'},\partial\Omega_e\subset\Omega)}^{N_{el}} \left(c_1\frac{p^6_{e'}}{h^3_{e'}}||\beta^{-1/2}(EIg^2\xi_{,xx})_{,x}||^2_{\Omega_{e'}}\right.\right. \\
&\left.\left.+c_1\frac{p^6_e}{h^3_e}||\beta^{-1/2}(EIg^2\xi_{,xx})_{,x}||^2_{\Omega_e}\right)\right\}^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} c_1\frac{p^6_e}{h^3_e}||\beta^{-1/2}(EIg^2\xi_{,xx})_{,x}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_1}{C_\beta}||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\Omega_e}\right)^{1/2},
\end{aligned}
\tag{6.155}
$$

where the constant c_1 is independent of h_e, p_e and ξ. In (6.155), we choose $\frac{c_1}{C_\beta} \leqslant 1$ without loss of generality. Thereby, we deduce

$$
\begin{aligned}
&\left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} \frac{c_1}{C_\beta}||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\Omega_e}\right)^{1/2} \\
&\leqslant \left(\sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i}\right)^{1/2} \left(\sum_{e=1}^{N_{el}} ||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\Omega_e}\right)^{1/2} \\
&= \left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\tilde{\Omega}}\right)^{1/2}.
\end{aligned}
\tag{6.156}
$$

In (6.156), by making use of the definition of energy seminorm, (6.26), we conclude

$$
\left(||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}}\right)^{1/2} \left(||(EIg^2)^{1/2}(\xi_{,xx})_{,x}||^2_{\tilde{\Omega}}\right)^{1/2} \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.157}
$$

Wherefore, from (6.152) – (6.157), we arrive to the conclusion that this type of inner product, on the right-hand side of (6.136), can be bounded as follows

$$|[\![\eta_{,x}]\!]\langle(EIg^2\xi_{,xx})_{,xx}\rangle_{\tilde{\Gamma}}| \leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \tag{6.158}$$

What is more, by following the above procedure in a similar manner, we shall achieve to estimate the rest of inner products of the corresponding form, on the right-hand side of (6.136). As a consequence, we have

$$\begin{aligned} |[\![\eta_{,xx}]\!]\langle(EIg^2\xi_{,xx})_{,x}\rangle_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |[\![\eta_{,x}]\!]\langle EI\xi_{,xx}\rangle_{\tilde{\Gamma}}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\eta_{,x}\cdot n(EIg^2\xi_{,xx})_{,xx}|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\eta_{,xx}(EIg^2\xi_{,xx})_{,x}\cdot n|_{\Gamma_r}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}, \\ |\eta_{,x}\cdot nEI\xi_{,xx}|_{\Gamma_q}| &\leqslant |||\eta|||_{sb}|||\xi|||_{sb}. \end{aligned} \tag{6.159}$$

At this point, we gather the inequalities (6.140) – (6.141), (6.145) – (6.146), (6.150) – (6.151), (6.158) – (6.159) and insert them on the right-hand side of (6.136). So, it derives

$$\begin{aligned} \theta|||\xi|||_{sb}^2 \;\leqslant\; C\Big\{ & |||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i}||\beta^{-1/2}\langle(EIg^2\eta_{,xx})_{,xx}\rangle||_{\Gamma_i}^2\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q}||\beta_q^{-1/2}(EIg^2\eta_{,xx})_{,xx}||_{\Gamma_j}^2\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{N_i}||\gamma^{-1/2}\langle(EIg^2\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2\Big)^{1/2} \\ & + \Big(\sum_{s=1}^{N_r}||\gamma_r^{-1/2}(EIg^2\eta_{,xx})_{,x}||_{\Gamma_s}^2\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{N_i}||\alpha^{-1/2}\langle EI\eta_{,xx}\rangle||_{\Gamma_i}^2\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q}||\alpha_q^{-1/2}EI\eta_{,xx}||_{\Gamma_j}^2\Big)^{1/2}\Big\}|||\xi|||_{sb}, \end{aligned}$$

which implies that

$$\begin{aligned} |||\xi|||_{sb} \;\leqslant\; & C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{N_i} ||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} + \Big(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j}\Big)^{1/2} + \Big(\sum_{s=1}^{N_r} ||\gamma_r^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_s}\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q} ||\alpha_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2}\Big\}. \end{aligned} \tag{6.160}$$

By combining at once the mathematical expression (6.133) with (6.160), we get

$$\begin{aligned} |||u-u^{hp}|||_{sb} \;\leqslant\; & C\Big\{|||\eta|||_{sb} + \Big(\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j}\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{N_i} ||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ & + \Big(\sum_{s=1}^{N_r} ||\gamma_r^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_s}\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i}\Big)^{1/2} \\ & + \Big(\sum_{j=1}^{N_q} ||\alpha_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j}\Big)^{1/2}\Big\} \end{aligned}$$

or by successive use of (A.14), we have

$$\begin{aligned} |||u-u^{hp}|||_{sb}^2 \;\leqslant\; & C\Big\{|||\eta|||_{sb}^2 \\ & + \sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||_{\Gamma_i}^2 + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||_{\Gamma_j}^2 \\ & + \sum_{i=1}^{N_i} ||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||_{\Gamma_i}^2 + \sum_{s=1}^{N_r} ||\gamma_r^{-1/2}(\eta_{,xx})_{,x}||_{\Gamma_s}^2 \\ & + \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle\eta_{,xx}\rangle||_{\Gamma_i}^2 + \sum_{j=1}^{N_q} ||\alpha_q^{-1/2}\eta_{,xx}||_{\Gamma_j}^2\Big\}. \quad (6.161) \end{aligned}$$

Therefore, we have obtained a bound on $|||u-u^{hp}|||_{sb}$ in terms of various norms of η. Thereby, to complete the proof, it only remains to estimate the terms appearing on the right-hand side of (6.161). We note that $\eta \notin \mathcal{W}^{hp}$.

To estimate the first term, we shall make use of the definition of energy seminorm, (6.26), yielding

$$\begin{aligned} |||\eta|||_{sb}^2 \;\leqslant\; & C\sum_{e=1}^{N_{el}} \big\{||(\eta_{,xx})_{,x}||_{\Omega_e}^2 + ||\eta_{,xx}||_{\Omega_e}^2\big\} + ||\beta^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}^2 \\ & + ||\gamma^{1/2}[\![\eta_{,xx}]\!]||_{\tilde{\Gamma}_2}^2 + ||\alpha^{1/2}[\![\eta_{,x}]\!]||_{\tilde{\Gamma}_1}^2. \quad (6.162) \end{aligned}$$

We shall additionally bound the factors on the right-hand side of (6.162). By recalling (A.32) for the first two norms, we obtain

$$||(\eta_{,xx})_{,x}||_{\Omega_e} \leqslant ||\eta_{,xx}||_{1,\Omega_e} \leqslant ||\eta||_{3,\Omega_e} \leqslant C\frac{h_e^{s_e-3}}{p_e^{t_e-3}}||u||_{t_e,\Omega_e} \quad (6.163)$$

and

$$||\eta_{,xx}||_{\Omega_e} \leqslant ||\eta_{,x}||_{1,\Omega_e} \leqslant ||\eta||_{2,\Omega_e} \leqslant C\frac{h_e^{s_e-2}}{p_e^{t_e-2}}||u||_{t_e,\Omega_e}. \quad (6.164)$$

Subsequently, we shall pay particular attention to estimate the term, containing the stabilization parameter β, on the right-hand side of (6.162).

By applying the jump inequality (A.18), we deduce that

$$\begin{aligned}
&||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \\
&= ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}} + ||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_q} \\
&= \sum_{i=1}^{N_i} ||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_j} \\
&\leqslant \sum_{i=1}^{N_i} 2\left(||\beta^{1/2}\eta^+_{,x}||^2_{\Gamma_i} + ||\beta^{1/2}\eta^-_{,x}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} ||\beta_q^{1/2}\eta_{,x}||^2_{\Gamma_j} \\
&\leqslant 2\sum_{e=1}^{N_{el}} ||\beta^{1/2}\eta_{,x}||^2_{\partial\Omega_e}.
\end{aligned} \tag{6.165}$$

Afterwards, in (6.165), we get

$$2\sum_{e=1}^{N_{el}} ||\beta^{1/2}\eta_{,x}||^2_{\partial\Omega_e} = C\sum_{e=1}^{N_{el}} \frac{p_e^6}{h_e^3}||\eta_{,x}||^2_{\partial\Omega_e}. \tag{6.166}$$

Now, by invoking (A.33) in (6.166), we have

$$\begin{aligned}
&C\sum_{e=1}^{N_{el}} \frac{p_e^6}{h_e^3}||\eta_{,x}||^2_{\partial\Omega_e} \\
&\leqslant C\sum_{e=1}^{N_{el}} \frac{p_e^6}{h_e^3}\frac{h_e^{2s_e-3}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e} \\
&= C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||^2_{t_e,\Omega_e}.
\end{aligned} \tag{6.167}$$

Hence, from (6.165) – (6.167), we conclude that the factor, including the stabilization parameter β on the right hand side of (6.162), can be bounded as follows

$$||\beta^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||^2_{t_e,\Omega_e}. \tag{6.168}$$

We analogously deduce that the remaining terms, containing the stabilization parameters γ and α on the right hand side of (6.162), can be bounded as follows

$$
\begin{aligned}
||\gamma^{1/2}[\![\eta_{,xx}]\!]||^2_{\tilde{\Gamma}_2} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-7}}||u||^2_{t_e,\Omega_e}, \\
||\alpha^{1/2}[\![\eta_{,x}]\!]||^2_{\tilde{\Gamma}_1} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}||u||^2_{t_e,\Omega_e}.
\end{aligned}
\tag{6.169}
$$

Thereafter, insertion of the mathematical inequalities (6.163) – (6.164) and (6.168) – (6.169) into the right-hand side of (6.162) yields

$$
\begin{aligned}
|||\eta|||^2_{sb} &\leqslant C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-6}}{p_e^{2t_e-6}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-7}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}\right)||u||^2_{t_e,\Omega_e} \\
&\quad + C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-4}} + \frac{h_e^{2s_e-4}}{p_e^{2t_e-5}}\right)||u||^2_{t_e,\Omega_e} \\
&\leqslant C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-5}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}\right)||u||^2_{t_e,\Omega_e} \\
&\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||^2_{t_e,\Omega_e}.
\end{aligned}
$$

As a result, we conclude that η can be bounded as

$$
|||\eta|||^2_{sb} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||^2_{t_e,\Omega_e}. \tag{6.170}
$$

Into the bargain, we shall estimate the remaining factors on the right-hand side of (6.161). By using the mean value inequality (A.19), we can configure the terms including the stabilization parameters as

$$
\begin{aligned}
&\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{i=1}^{N_i} \left(||\beta^{-1/2}(\eta^+_{,xx})_{,xx}||^2_{\Gamma_i} + ||\beta^{-1/2}(\eta^-_{,xx})_{,xx}||^2_{\Gamma_i}\right) + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j} \\
&\leqslant \sum_{e=1}^{N_{el}} ||\beta^{-1/2}(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e}.
\end{aligned}
\tag{6.171}
$$

Next, in (6.171), we get

$$\sum_{e=1}^{N_{el}} ||\beta^{-1/2}(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e} = C\sum_{e=1}^{N_{el}} \frac{h_e^3}{p_e^6}||(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e}. \tag{6.172}$$

Now, using (A.33) in (6.172), we have

$$\begin{aligned} &C\sum_{e=1}^{N_{el}} \frac{h_e^3}{p_e^6}||(\eta_{,xx})_{,xx}||^2_{\partial\Omega_e} \\ &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^3}{p_e^6}\frac{h_e^{2s_e-9}}{p_e^{2t_e-9}}||u||^2_{t_e,\Omega_e} \\ &= C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{6.173}$$

Ergo, from (6.171) – (6.173), we arrive to the conclusion that the terms, including the stabilization parameters β and β_q on the right-hand side of (6.161), can be bounded as follows

$$\sum_{i=1}^{N_i} ||\beta^{-1/2}\langle(\eta_{,xx})_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\beta_q^{-1/2}(\eta_{,xx})_{,xx}||^2_{\Gamma_j} \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}. \tag{6.174}$$

By following arguments in a same way, we deduce that the rest of the terms, containing the stabilization parameters γ and γ_r as well as α and α_q on the right of (6.161), can be estimated as

$$\begin{aligned} \sum_{i=1}^{N_i} ||\gamma^{-1/2}\langle(\eta_{,xx})_{,x}\rangle||^2_{\Gamma_i} + \sum_{s=1}^{N_r} ||\gamma_r^{-1/2}(\eta_{,xx})_{,x}||^2_{\Gamma_s} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-5}}||u||^2_{t_e,\Omega_e}, \\ \sum_{i=1}^{N_i} ||\alpha^{-1/2}\langle\eta_{,xx}\rangle||^2_{\Gamma_i} + \sum_{j=1}^{N_q} ||\alpha_q^{-1/2}\eta_{,xx}||^2_{\Gamma_j} &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}}||u||^2_{t_e,\Omega_e}. \end{aligned} \tag{6.175}$$

Inserting the inequalities (6.170), (6.174) and (6.175), into the right-hand side of (6.161) and just by combining with each other, gives

$$
\begin{aligned}
|||u-u^{hp}|||_{sb}^2 &\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-4}}{p_e^{2t_e-3}}||u||_{t_e,\Omega_e}^2 \\
&\quad + C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-6}}{p_e^{2t_e-3}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-5}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}} \left(\frac{h_e^{2s_e-4}}{p_e^{2t_e-3}} + \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}\right)||u||_{t_e,\Omega_e}^2 \\
&\leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||_{t_e,\Omega_e}^2.
\end{aligned}
$$

So, we conclude that

$$
|||u-u^{hp}|||_{sb}^2 \leqslant C\sum_{e=1}^{N_{el}} \frac{h_e^{2s_e-6}}{p_e^{2t_e-9}}||u||_{t_e,\Omega_e}^2,
$$

which is the desired result. □

It is worth noting that the resulting a priori error estimate is optimal in h but is p-suboptimal by $\frac{3}{2}$ orders of p.

6.6 Conclusions

The objective of this chapter is to establish an alternative approach for the one-dimensional Toupin-Mindlin strain gradient beam in bending. The continuous interior penalty finite element method that we have introduced for this purpose exhibits the subsequent features:

1. It is formulated only in the primary variable.
2. Only piecewise continuous polynomials are employed.
3. Continuity requirements, for the derivatives, are satisfied weakly.
4. The method is consistent, stable and convergent.

Part II

Higher Dimensional Problems

Chapter 7

IPDGFEMs for a Bending Plate Model

7.1 Preliminaries

Suppose that Ω is a bounded, open, convex domain in $\Re^2$ with boundary Γ_{bd}. Let $\mathcal{T}$ be a subdivision of Ω into disjoint open convex elements domains $K = K_j$ such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K},$$

$$K_i \cap K_j = \emptyset \quad \text{for } i \neq j$$

and the intersection $\bar{K}_i \cap \bar{K}_j$ is either empty, a vertex or an edge. We define a piecewise constant mesh function $h_{\mathcal{T}}$ by

$$h_{\mathcal{T}}(x) = h_K = diam(K), \quad x \in K, \quad K \in \mathcal{T}$$

and put

$$h = \max_{K \in \mathcal{T}} h_K.$$

Let $\hat{K}$ be a fixed reference element in $\Re^2$. We shall further assume that each $K \in \mathcal{T}$ is an affine image of the reference element $\hat{K}$

$$K = F_K(\hat{K}), \quad K \in \mathcal{T}.$$

Let $\mathcal{E}$ be the set of all open one-dimensional element faces, associated with the subdivision $\mathcal{T}$. We also define a piecewise constant face-function on $\mathcal{E}$

$$h_{\mathcal{E}}(x) = h_e = diam(e), \quad x \in e, \ e \in \mathcal{E}.$$

Let us assume that the subdivision $\mathcal{T}$ is shape-regular (see either p. 124 in [55] or Remark 2.2, p. 114 in [31] or Definition A.1.7). We note that for a shape-regular family there exists a positive constant c (the shape-regularity constant), independent of h, such that

$$ch_K \leqslant h_e \leqslant h_K, \quad \forall K \in \mathcal{T},\ \forall e \in \partial K,$$

hence, for any element $K \in \mathcal{T}$, h_K and h_e are equal to within a constant.

To each $K \in \mathcal{T}$ we assign a non-negative integer p_K (the local polynomial degree) and a non-negative integer s_K (the local Sobolev space index). Then, we collect the p_K, s_K and F_K in the vectors

$$\mathbf{p} = (p_K : K \in \mathcal{T}),\ \mathbf{s} = (s_K : K \in \mathcal{T}) \text{ and } \mathbf{F} = (F_K : K \in \mathcal{T}).$$

We now return to the set $\mathcal{E}$. We also assume that $\mathcal{E}$ is decomposed into two subsets, namely $\mathcal{E}_{\text{int}}$ and $\mathcal{E}_\partial$, which contain the set of all elements of $\mathcal{E}$ that are not subsets of Γ_{bd}, i.e.,

$$\mathcal{E}_{\text{int}} = \{e \in \mathcal{E} : e \subset \Omega\}$$

and the set of all elements of $\mathcal{E}$ that are subsets of Γ_{bd}, i.e.,

$$\mathcal{E}_\partial = \{e \in \mathcal{E} : e \subset \Gamma_{bd}\}.$$

The subset $\mathcal{E}_\partial$ is further decomposed either into $\mathcal{E}_c$ and $\mathcal{E}_Q$ or into $\mathcal{E}_q$ and $\mathcal{E}_M$, i.e.,

$$\mathcal{E}_\partial = \mathcal{E}_c \cup \mathcal{E}_Q \quad \text{or} \quad \mathcal{E}_\partial = \mathcal{E}_q \cup \mathcal{E}_M \equiv \mathcal{E}_{\partial'}.$$

For an integer m we define

$$\langle p^m \rangle_{\mathcal{E}}(x) = \langle p^m \rangle_e = \frac{p_K^m + p_{K'}^m}{2}, \quad x \in e,\ e \in \mathcal{E}_{\text{int}},$$

where the elements K and K' share the face e, as well as

$$\langle p^m \rangle_{\mathcal{E}}(x) = \langle p^m \rangle_e = p_K^m, \quad x \in e,\ e \in \mathcal{E}_\partial,$$

where $e \subset \partial K$.

What is more, we define the set Γ as

$$\Gamma := \bigcup_{e \in \mathcal{E}} e$$

and the sets $\Gamma_{\rm int}$, Γ_c, Γ_q, Γ_M together with Γ_Q as

$$\Gamma_{\rm int} := \bigcup_{e\in\mathcal{E}_{\rm int}} e, \quad \Gamma_c := \bigcup_{e\in\mathcal{E}_c} e, \quad \Gamma_q := \bigcup_{e\in\mathcal{E}_q} e, \quad \Gamma_M := \bigcup_{e\in\mathcal{E}_M} e, \quad \Gamma_Q := \bigcup_{e\in\mathcal{E}_Q} e,$$

all with the obvious meanings respectively. We note that either

$$\Gamma = \Gamma_{\rm int} \cup \Gamma_{bd} = \Gamma_{\rm int} \cup \Gamma_c \cup \Gamma_Q,$$

or

$$\Gamma = \Gamma_{\rm int} \cup \Gamma_{bd} = \Gamma_{\rm int} \cup \Gamma_q \cup \Gamma_M \equiv \Gamma'.$$

Let $\Gamma_0 = \Gamma_{\rm int} \cup \Gamma_c$ and $\Gamma_1 = \Gamma_{\rm int} \cup \Gamma_q$. We define for $u, w \in L^2(\Gamma_0)$ and for $u, w \in L^2(\Gamma_1)$, the inner products

$$\int_{\Gamma_0} uwdr = \int_{\Gamma_{\rm int}} uwdr + \int_{\Gamma_c} uwdr \tag{7.1}$$

$$\int_{\Gamma_1} uwdr = \int_{\Gamma_{\rm int}} uwdr + \int_{\Gamma_q} uwdr \tag{7.2}$$

with associated norms $||\cdot||_{\Gamma_0}$ and $||\cdot||_{\Gamma_1}$. So, it will hold as well

$$||u||^2_{\Gamma_0} = ||u||^2_{\Gamma_{\rm int}} + ||u||^2_{\Gamma_c} \tag{7.3}$$

or

$$\sum_{e\in\mathcal{E}_0} ||u||^2_e = \sum_{e\in\mathcal{E}_{\rm int}} ||u||^2_e + \sum_{e\in\mathcal{E}_c} ||u||^2_e \tag{7.4}$$

and

$$||u||^2_{\Gamma_1} = ||u||^2_{\Gamma_{\rm int}} + ||u||^2_{\Gamma_q}. \tag{7.5}$$

or

$$\sum_{e\in\mathcal{E}_1} ||u||^2_e = \sum_{e\in\mathcal{E}_{\rm int}} ||u||^2_e + \sum_{e\in\mathcal{E}_q} ||u||^2_e. \tag{7.6}$$

7.2 Kirchhoff-Love Plate Model Problem

Let us consider a thin plate, the medium surface of which is denoted by Ω, the boundary by Γ_{bd} and the thickness being 2ε. The mechanical framework that we consider is linear elasticity. The material constituting the structure is assumed to be homogeneous and isotropic (this is not a restriction, but just a simplification), see [80].

A transverse loading is applied, the force density of which is represented by the function f_3. In addition, the lateral boundary is clamped on a part γ_0, simply supported on another one, say γ_1 and free on a last one, denoted γ_2. Then, the Kirchhoff-Love plate model consists in finding an element u_3 which represents the deflection of the plate.

We consider the equation:

$$\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = f_3 \quad \text{on } \Omega, \tag{7.7}$$

where $f_3 \in L^2(\Omega)$. We denote by E and by ν the Young modulus and the Poisson ratio, respectively. Let us recall that $E > 0$ and $0 < \nu < \frac{1}{2}$.

We supplement the equation with the following boundary conditions

$$\begin{aligned} u_3 &= 0 \quad \text{on } \gamma_0 \cup \gamma_1, \\ \frac{\partial u_3}{\partial b} &= 0 \quad \text{on } \gamma_0, \\ m_{\alpha\beta} b_\alpha b_\beta &= 0 \quad \text{on } \gamma_1 \cup \gamma_2, \\ \partial_s \left(m_{\alpha\beta} a_\alpha b_\beta \right) + \partial_\alpha m_{\alpha\beta} b_\beta &= 0 \quad \text{on } \gamma_2, \end{aligned} \tag{7.8}$$

where $\{b_\alpha\}$ are the components of the unit outwards normal along the boundary of Ω and $\{a_\alpha\}$ are the components of the unit tangent to the boundary of Ω. We denote by $m_{\alpha\beta}$ the bending moments being

$$m_{\alpha\beta} = -\frac{2E\varepsilon}{1-\nu^2}\left\{(1-\nu)\partial_{\alpha\beta} u_3 + \nu \Delta u_3 \delta_{\alpha\beta}\right\}. \tag{7.9}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.
Let us recall once for all that $\frac{\partial u_3}{\partial b} = \partial_\alpha u_3 b_\alpha$ is the normal derivative and $\partial_s(\cdot) = \partial_\alpha(\cdot) a_\alpha = \frac{\partial \cdot}{\partial s}$ is the derivative along the boundary. Moreover, for the local basis $(\{a_\alpha\}, \{b_\alpha\})$, it holds

$$b_1 = -a_2 \quad \text{and} \quad b_2 = a_1.$$

We can rewrite the boundary conditions (7.8) with (7.9), so the boundary value problem is formulated as:

$$\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = f_3 \quad \text{on } \Omega, \tag{7.10}$$

$$
\begin{aligned}
u_3 &= 0 \quad \text{on } \Gamma_c, \\
\frac{\partial u_3}{\partial b} &= 0 \quad \text{on } \Gamma_q, \\
\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} &= 0 \quad \text{on } \Gamma_M \\
\frac{\partial}{\partial b}(\Delta u_3) + (1-\nu)\frac{\partial^3 u_3}{\partial s^2 \partial b} &= 0 \quad \text{on } \Gamma_Q,
\end{aligned}
\tag{7.11}
$$

where $\Gamma_c = (\gamma_0 \cup \gamma_1)$, $\Gamma_q = \gamma_0$, $\Gamma_M = (\gamma_1 \cup \gamma_2)$ and $\Gamma_Q = \gamma_2$. In the above, R denotes the radius of curvature of the boundary Γ_{bd} of Ω (counted positively along the unit outwards normal). Hence $R < 0$ if the domain is locally concave and $R > 0$ if it is locally convex.

We mention that the first two boundary conditions are called essential and the other two are called natural, respectively.

Furthermore, the third boundary condition derives from the combination of mathematical expression

$$
\Delta u_3 = \frac{\partial^2 u_3}{\partial b^2} + \frac{\partial^2 u_3}{\partial s^2} + \frac{1}{R}\frac{\partial u_3}{\partial b}. \tag{7.12}
$$

with (7.9).

What is more, note that we have the relationships

$$
\begin{aligned}
\overline{\Gamma_c \cup \Gamma_Q} &= \Gamma_{bd}, \\
\Gamma_c \cap \Gamma_Q &= \emptyset, \\
\overline{\Gamma_q \cup \Gamma_M} &= \Gamma_{bd}, \\
\Gamma_q \cap \Gamma_M &= \emptyset,
\end{aligned}
\tag{7.13}
$$

between the different parts of the boundary. Let Γ_∂ signify the union of one-dimensional open edges of Ω. Also notice that by construction Γ_{bd} differs from Γ_∂ on a set of one-dimensional measure zero which contains the vertices of the (polygonal) boundary of Ω.

7.3 Weak Formulation

We are ready to derive the weak formulation for the problem (7.10) – (7.11), which will lead to the discontinuous Galerkin finite element method. We shall assume for the moment that the solution u_3 of the problem is a sufficiently smooth function.

For each face $e \in \mathcal{E}_{\text{int}}$, let i and j be such indices that $i > j$ and the elements $K := K_i$ and $K' := K_j$ share the face e. Let us define the jump across e and the mean value on e of $u_3 \in H^1(\Omega, \mathcal{T})$ by

$$[\![u_3]\!]_e := u_3|_{\partial K \cap e} - u_3|_{\partial K' \cap e} \quad \text{and} \quad \langle u_3 \rangle_e := \frac{1}{2}\left(u_3|_{\partial K \cap e} + u_3|_{\partial K' \cap e}\right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and of the mean value to faces $e \in \mathcal{E}_\partial$ by letting:

$$[\![u_3]\!]_e = u_3|_e \quad \text{and} \quad \langle u_3 \rangle_e = u_3|_e.$$

In the above definitions, the subscript e will be supressed when no confusion is likely to occur. With each face $e \in \mathcal{E}_{\text{int}}$ we associate the unit normal vector $b = b_{K_i}$ to e, pointing from element K_i to K_j when $i > j$, and with each $e \in \mathcal{E}_\partial$ we associate the external unit normal vector $b = b_K$, where $e \subset \partial K$.

Since the method will be non-conforming, we shall use the broken Sobolev space $H^4(\Omega, \mathcal{T})$ as trial space. We multiply the equation, (7.10), by a test function $w_3 \in H^4(\Omega, \mathcal{T})$ and integrate over Ω

$$\int_\Omega \frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 w_3 dv = \int_\Omega f_3 w_3 dv.$$

Afterwards, we split the integrals

$$\sum_{K \in \mathcal{T}} \int_K \frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 w_3 dv = \sum_{K \in \mathcal{T}} \int_K f_3 w_3 dv. \tag{7.14}$$

In addition, it holds

$$\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 = -\frac{\varepsilon^2}{3} \partial_{\alpha\beta} m_{\alpha\beta},$$

so by applying the "double" Stokes formula for plates (B.5) on every elemental integral in (7.14) together with the aid of (7.9) and (7.12), we get

$$\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3dv+\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3dv$$
$$+\sum_{K\in\mathcal{T}}\int_{\partial K} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr$$
$$-\sum_{K\in\mathcal{T}}\int_{\partial K} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr$$
$$=\sum_{K\in\mathcal{T}}\int_K f_3w_3dv,$$

where b denotes the outward normal to each element edge.

Now, we split the boundary terms as follows

$$\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3dv+\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3dv$$
$$+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr$$
$$+\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr$$
$$+\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_Q} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr$$
$$-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr$$
$$-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_q} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr$$
$$-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_M} \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr$$
$$=\sum_{K\in\mathcal{T}}\int_K f_3w_3dv,$$

and hence we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3dv + \sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3dv \\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr \\
&+\int_{\Gamma_c}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr \\
&+\int_{\Gamma_Q}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3dr \\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr \\
&-\int_{\Gamma_q}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr \\
&-\int_{\Gamma_M}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr \\
&=\sum_{K\in\mathcal{T}}\int_K f_3w_3dv.
\end{aligned} \tag{7.15}$$

Using the natural boundary conditions, (7.11), on the fifth and on the eighth term respectively, on the left-hand side of (7.15) and moving it to the right-hand side, we obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3 dv+\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3 dv\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3 dr\\
&+\int_{\Gamma_c}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)w_3 dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr\\
&-\int_{\Gamma_q}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr\\
&=\sum_{K\in\mathcal{T}}\int_K f_3 w_3 dv.
\end{aligned}
\tag{7.16}
$$

The third and the fifth term respectively on the left-hand side of (7.16) contain the boundary integrals over the interior element edges, i.e. the edges $e\in\Gamma_{\text{int}}$. Consequently, in this sum of boundary integrals, we have two integrals over every interior edge.

Remark 7.3.0.1. *Let us note that, for a given face $e\in\mathcal{E}_{int}$ shared by two adjacent elements K_i and K_j $(i>j)$, we can write*

$$
\frac{\partial u_3|_{K_i}}{\partial b_{K_i}}w_3|_{K_i}+\frac{\partial u_3|_{K_j}}{\partial b_{K_j}}w_3|_{K_j}=\frac{\partial u_3|_{K_i}}{\partial b}w_3|_{K_i}-\frac{\partial u_3|_{K_j}}{\partial b}w_3|_{K_j}.
$$

Hence, by analogy with the formula

$$
ac-bd=\frac{1}{2}(a+b)(c-d)+\frac{1}{2}(a-b)(c+d)\quad\forall a,b,c,d\in\Re,
$$

we get

$$
\frac{\partial u_3|_{K_i}}{\partial b_{K_i}}w_3|_{K_i}+\frac{\partial u_3|_{K_j}}{\partial b_{K_j}}w_3|_{K_j}=\left\langle\frac{\partial u_3}{\partial b}\right\rangle [\![w_3]\!]+\left[\frac{\partial u_3}{\partial b}\right]\langle w_3\rangle\ \forall u_3,w_3\in H^1(\Omega,\mathcal{T}).
\tag{7.17}
$$

In order to evaluate these integrals, we always use the interior trace of the test function w_3. Taking into account the Remark 7.3.0.1 (together with the orientation convention that we have adopted) and applying (7.17), we can see that the third and the fifth term respectively, on the left-hand side of (7.16), can be rewritten as follows

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3 dv + \sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3 dv\\
&+\int_{\Gamma_{\text{int}}}\left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)\right\rangle [\![w_3]\!] dr\\
&+\int_{\Gamma_{\text{int}}}\left[\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)\right]\langle w_3\rangle dr\\
&+\int_{\Gamma_c}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right) w_3 dr\\
&-\int_{\Gamma_{\text{int}}}\left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\right\rangle \left[\frac{\partial w_3}{\partial b}\right] dr\\
&-\int_{\Gamma_{\text{int}}}\left[\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\right]\left\langle \frac{\partial w_3}{\partial b}\right\rangle dr\\
&-\int_{\Gamma_q}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b} dr\\
&=\sum_{K\in\mathcal{T}}\int_K f_3 w_3 dv.
\end{aligned} \tag{7.18}$$

By noting that the fluxes

$$\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)$$

and

$$\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2 u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)$$

are continuous across the element faces $e\in\mathcal{E}_{\text{int}}$ (e.g., when the exact solution $u\in H^4(\Omega)$), we have

$$\int_{\Gamma_{\text{int}}}\left[\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3 u_3}{\partial s^2\partial b}\right)\right]\langle w_3\rangle dr = 0,$$

$$\int_{\Gamma_{\text{int}}} \left[\frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} \right) \right] \left\langle \frac{\partial w_3}{\partial b} \right\rangle dr = 0, \quad \forall w \in H^4(\Omega, \mathcal{T}).$$

Then, (7.18) reduces to

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \int_K \frac{2E\varepsilon^3}{3(1+\nu)} \partial_{\alpha\beta} u_3 \partial_{\alpha\beta} w_3 dv + \sum_{K\in\mathcal{T}} \int_K \frac{2E\varepsilon^3 \nu}{3(1-\nu^2)} \Delta u_3 \Delta w_3 dv \\
&+ \int_{\Gamma_{\text{int}}} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta u_3) + (1-\nu)\frac{\partial^3 u_3}{\partial s^2 \partial b} \right) \right\rangle [\![w_3]\!] dr \\
&+ \int_{\Gamma_c} \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta u_3) + (1-\nu)\frac{\partial^3 u_3}{\partial s^2 \partial b} \right) w_3 dr \\
&- \int_{\Gamma_{\text{int}}} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} \right) \right\rangle \left[\frac{\partial w_3}{\partial b} \right] dr \\
&- \int_{\Gamma_q} \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} \right) \frac{\partial w_3}{\partial b} dr \\
&= \sum_{K\in\mathcal{T}} \int_K f_3 w_3 dv.
\end{aligned} \tag{7.19}$$

Next, we multiply the boundary condition $u_3 = 0$, on Γ_c, by

$$-\theta \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta w_3) + (1-\nu)\frac{\partial^3 w_3}{\partial s^2 \partial b} \right) + \gamma_c w_3.$$

Then, integrating over Γ_c, we get

$$-\int_{\Gamma_c} \theta \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta w_3) + (1-\nu)\frac{\partial^3 w_3}{\partial s^2 \partial b} \right) u_3 dr + \int_{\Gamma_c} \gamma_c u_3 w_3 dr = 0, \tag{7.20}$$

where θ is the symmetrization parameter. We restrict ourselves to the case $\theta \in \{-1, 1\}$. The non-negative piecewise continuous function γ_c, defined on Γ_c, is referred to as the stabilization parameter.

Furthermore, u_3 is continuous on Ω. So, the jump $[\![u_3]\!]$ vanishes, i.e. $[\![u_3]\!] = 0$. If we choose

$$-\theta \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta w_3) + (1-\nu)\frac{\partial^3 w_3}{\partial s^2 \partial b} \right) \right\rangle + \gamma [\![w_3]\!]$$

as test function and integrate over Γ_{int}, we shall deduce

$$\begin{aligned}&-\int_{\Gamma_{\text{int}}}\theta\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta w_3)+(1-\nu)\frac{\partial^3 w_3}{\partial s^2\partial b}\right)\right\rangle [\![u_3]\!]dr\\&+\int_{\Gamma_{\text{int}}}\gamma[\![u_3]\!][\![w_3]\!]dr=0,\end{aligned}\tag{7.21}$$

where γ is a non-negative piecewise continuous function, defined on Γ_{int}, which is referred to as stabilization parameter.

Moreover, from the boundary condition $\frac{\partial u_3}{\partial b}=0$, on Γ_q, upon multiplying by

$$\theta\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2 w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)+\zeta_q\frac{\partial w_3}{\partial b}$$

and integrating over Γ_q, we have

$$\begin{aligned}&\int_{\Gamma_q}\theta\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2 w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)\frac{\partial u_3}{\partial b}dr\\&+\int_{\Gamma_q}\zeta_q\frac{\partial u_3}{\partial b}\frac{\partial w_3}{\partial b}dr=0.\end{aligned}\tag{7.22}$$

The non-negative piecewise continuous function ζ_q, defined on Γ_q, is referred to as the stabilization parameter.

In addition, $\frac{\partial u_3}{\partial b}$ is continuous on Ω. In that case the jump $\left[\frac{\partial u_3}{\partial b}\right]$ vanishes, i.e. $\left[\frac{\partial u_3}{\partial b}\right]=0$. If we choose

$$\theta\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2 w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)\right\rangle+\zeta\left[\frac{\partial w_3}{\partial b}\right]$$

as test function and integrate over Γ_{int}, it will yield

$$\begin{aligned}&\int_{\Gamma_{\text{int}}}\theta\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2 w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)\right\rangle\left[\frac{\partial u_3}{\partial b}\right]dr\\&+\int_{\Gamma_{\text{int}}}\zeta\left[\frac{\partial u_3}{\partial b}\right]\left[\frac{\partial w_3}{\partial b}\right]dr=0,\end{aligned}\tag{7.23}$$

where ζ is a non-negative continuous function, defined on Γ_{int}, which is referred to as the stabilization parameter.

At this point, adding (7.19) – (7.23), we get the discontinuous Galerkin weak formulation of the problem

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K \frac{2E\varepsilon^3}{3(1+\nu)}\partial_{\alpha\beta}u_3\partial_{\alpha\beta}w_3dv+\sum_{K\in\mathcal{T}}\int_K\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\Delta u_3\Delta w_3dv\\
&+\int_{\Gamma_{\rm int}}\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3u_3}{\partial s^2\partial b}\right)\right\rangle [\![w_3]\!]dr\\
&-\int_{\Gamma_{\rm int}}\theta\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta w_3)+(1-\nu)\frac{\partial^3w_3}{\partial s^2\partial b}\right)\right\rangle [\![u_3]\!]dr\\
&-\int_{\Gamma_{\rm int}}\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\right\rangle\left[\frac{\partial w_3}{\partial b}\right]dr\\
&+\int_{\Gamma_{\rm int}}\theta\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)\right\rangle\left[\frac{\partial u_3}{\partial b}\right]dr\\
&+\int_{\Gamma_{\rm int}}\gamma[\![u_3]\!][\![w_3]\!]dr+\int_{\Gamma_{\rm int}}\zeta\left[\frac{\partial u_3}{\partial b}\right]\left[\frac{\partial w_3}{\partial b}\right]dr\\
&+\int_{\Gamma_c}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta u_3)+(1-\nu)\frac{\partial^3u_3}{\partial s^2\partial b}\right)w_3dr\\
&-\int_{\Gamma_c}\theta\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\frac{\partial}{\partial b}(\Delta w_3)+(1-\nu)\frac{\partial^3w_3}{\partial s^2\partial b}\right)u_3dr\\
&-\int_{\Gamma_q}\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta u_3-(1-\nu)\frac{\partial^2u_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b}\right)\frac{\partial w_3}{\partial b}dr\\
&+\int_{\Gamma_q}\theta\frac{2E\varepsilon^3}{3(1-\nu^2)}\left(\Delta w_3-(1-\nu)\frac{\partial^2w_3}{\partial s^2}-\frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b}\right)\frac{\partial u_3}{\partial b}dr\\
&+\int_{\Gamma_c}\gamma_cu_3w_3dr+\int_{\Gamma_q}\zeta_q\frac{\partial u_3}{\partial b}\frac{\partial w_3}{\partial b}dr=\sum_{K\in\mathcal{T}}\int_K f_3w_3dv.
\end{aligned}
\tag{7.24}
$$

Using the inner products (7.1) and (7.2), (7.24) can alternatively be rewritten in a more compressed form as

$$
\begin{aligned}
&\int_\Omega \frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 : D_h^2 w_3 dv + \int_\Omega \frac{2E\varepsilon^3\nu}{3(1-\nu^2)} \Delta_h u_3 \Delta_h w_3 dv \\
&+ \int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta u_3) + (1-\nu)\frac{\partial^3 u_3}{\partial s^2 \partial b} \right) \right\rangle [\![w_3]\!] dr \\
&- \int_{\Gamma_0} \theta \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta w_3) + (1-\nu)\frac{\partial^3 w_3}{\partial s^2 \partial b} \right) \right\rangle [\![u_3]\!] dr \\
&- \int_{\Gamma_1} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} \right) \right\rangle \left[\frac{\partial w_3}{\partial b} \right] dr \\
&+ \int_{\Gamma_1} \theta \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta w_3 - (1-\nu)\frac{\partial^2 w_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b} \right) \right\rangle \left[\frac{\partial u_3}{\partial b} \right] dr \\
&+ \int_{\Gamma_0} \gamma [\![u_3]\!] [\![w_3]\!] dr + \int_{\Gamma_1} \zeta \left[\frac{\partial u_3}{\partial b} \right] \left[\frac{\partial w_3}{\partial b} \right] dr = \int_\Omega f_3 w_3 dv,
\end{aligned}
\tag{7.25}
$$

where D_h^2 defines the broken Hessian matrix and Δ_h defines the broken Laplacian with respect to the subdivision $\mathcal{T}$, respectively.

The bilinear form $B_{pl}(\cdot,\cdot)$ is defined as

$$
\begin{aligned}
&B_{pl}(u_3, w_3) := \int_\Omega \frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 : D_h^2 w_3 dv + \int_\Omega \frac{2E\varepsilon^3\nu}{3(1-\nu^2)} \Delta_h u_3 \Delta_h w_3 dv \\
&+ \int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta u_3) + (1-\nu)\frac{\partial^3 u_3}{\partial s^2 \partial b} \right) \right\rangle [\![w_3]\!] dr \\
&- \int_{\Gamma_0} \theta \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b}(\Delta w_3) + (1-\nu)\frac{\partial^3 w_3}{\partial s^2 \partial b} \right) \right\rangle [\![u_3]\!] dr \\
&- \int_{\Gamma_1} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu)\frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial u_3}{\partial b} \right) \right\rangle \left[\frac{\partial w_3}{\partial b} \right] dr \\
&+ \int_{\Gamma_1} \theta \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta w_3 - (1-\nu)\frac{\partial^2 w_3}{\partial s^2} - \frac{(1-\nu)}{R}\frac{\partial w_3}{\partial b} \right) \right\rangle \left[\frac{\partial u_3}{\partial b} \right] dr \\
&+ \int_{\Gamma_0} \gamma [\![u_3]\!] [\![w_3]\!] dr + \int_{\Gamma_1} \zeta \left[\frac{\partial u_3}{\partial b} \right] \left[\frac{\partial w_3}{\partial b} \right] dr.
\end{aligned}
\tag{7.26}
$$

We introduce the linear functional $L_{pl}(\cdot)$ on $H^4(\Omega,\mathcal{T})$

$$
L_{pl}(w_3) := \int_\Omega f_3 w_3 dv. \tag{7.27}
$$

The stabilization parameters, γ, ζ, depend on the discretization parameters h and p for the hp-method, in a manner that will be specified later in the text.

Then the broken weak formulation of the problem (7.10) – (7.11) reads as follows:

$$\text{Find } u_3 \in bSs \text{ such that } B_{pl}(u_3, w_3) = L_{pl}(w_3) \quad \forall w_3 \in H^4(\Omega, \mathcal{T}), \tag{7.28}$$

where by bSs we denote the following function space

$$\begin{aligned} bSs \;=\; & \Big\{ u \in H^4(\Omega, \mathcal{T}) : u,\ \frac{\partial u_3}{\partial b}, \\ & \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\Delta u_3 - (1-\nu) \frac{\partial^2 u_3}{\partial s^2} - \frac{(1-\nu)}{R} \frac{\partial u_3}{\partial b} \right), \\ & \frac{2E\varepsilon^3}{3(1-\nu^2)} \left(\frac{\partial}{\partial b} (\Delta u_3) + (1-\nu) \frac{\partial^3 u_3}{\partial s^2 \partial b} \right) \\ & \text{are continuous across } e \in \mathcal{E}_{\text{int}} \Big\}. \end{aligned}$$

Note that for $\theta = -1$ the bilinear form $B_{pl}(\cdot, \cdot)$ is symmetric, whereas for $\theta = 1$ it is not symmetric.

We shall associate with the bilinear form $B_{pl}(\cdot, \cdot)$ the energy seminorm, $|||\cdot|||_{pl}$, defined by

$$\begin{aligned} |||u_3|||_{pl} \;=\; & \left(\left\| \left(\frac{2E\varepsilon^3}{3(1+\nu)} \right)^{1/2} D_h^2 u_3 \right\|_\Omega^2 + \left\| \left(\frac{2E\varepsilon^3 \nu}{3(1-\nu^2)} \right)^{1/2} \Delta_h u_3 \right\|_\Omega^2 \right. \\ & \left. + \, ||\gamma^{1/2} [\![u_3]\!]||_{\Gamma_0}^2 + \left\| \zeta^{1/2} \left[\frac{\partial u_3}{\partial b} \right] \right\|_{\Gamma_1}^2 \right)^{1/2}, \quad u \in H^2(\Omega, \mathcal{T}). \end{aligned} \tag{7.29}$$

Proposition 7.3.0.2. *If $\gamma, \zeta > 0$, then $|||\cdot|||_{pl}$ is a seminorm on $H^2(\Omega, \mathcal{T})$.*

We note in passing that since $H^4(\Omega, \mathcal{T}) \subset H^2(\Omega, \mathcal{T})$, then $|||\cdot|||_{pl}$ is also a seminorm on $H^4(\Omega, \mathcal{T})$.

7.4 Finite Element Spaces

In this section, we will consider the finite-dimensional subspace of the broken Sobolev space $H^4(\Omega, \mathcal{T})$ which is used in the finite element approximation of the problem.

For a non-negative integer p, we denote by $\mathcal{Q}_p(\hat{K})$ the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. We collect the h_K and p_K into the element-wise constant functions

$$\mathbf{h}, \mathbf{p} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K \text{ and } \mathbf{p}|_K = p_K, \ K \in \mathcal{T},$$

respectively. We consider the finite element space

$$\mathcal{S}_h^p \equiv \mathcal{S}^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}) := \left\{ v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{Q}_{p_K}(\hat{K}), \ K \in \mathcal{T} \right\}. \quad (7.30)$$

We shall assume throughout that the mesh size function $\mathbf{h}$ and polynomial degree function $\mathbf{p}$, with $p_K \geqslant 2$ for each $K \in \mathcal{T}$, have bounded local variation (see Remark A.3.5). What's more, we will refer to the functions in $\mathcal{S}_h^p$ as test functions. We note that the test functions are discontinuous along the edges of the mesh.

7.5 DG Finite Element Method

We are ready to present the numerical method whose analysis we shall investigate in this chapter. Making use of the weak formulation derived in Section 7.3 and the finite element spaces constructed in the previous section, we state the discontinuous Galerkin finite element method for the problem (7.10) – (7.11):

$$\text{Find } u_{3:DG} \in \mathcal{S}_h^p \text{ such that } \ B_{pl}(u_{3:DG}, w_3) = L_{pl}(w_3) \quad \forall w_3 \in \mathcal{S}_h^p, \quad (7.31)$$

where the functions γ, ζ, contained in $B_{pl}(\cdot, \cdot)$, will be defined in the coercivity property. We shall allude to the discontinuous Galerkin finite element method with $\theta = -1$ as the symmetric interior penalty Galerkin (SIPG), whereas for $\theta = 1$ the discontinuous Galerkin finite element method will be referred to as the non-symmetric interior penalty Galerkin (NIPG).

One can see from the definition of the bilinear form, (7.26), that the DG method has non-local character. In addition to element contributions, we encounter terms on interior boundaries to the two elements adjacent to the respective interfaces.

The approximation $u_{3:DG} \in \mathcal{S}_h^p$ to the solution will be generally discontinuous since there is no continuity requirement in the finite element space.

Clearly, the number of degrees of freedom of $\mathcal{S}_h^p$ is greater than that of the corresponding finite element space for a conforming *hp*-finite element method, as continuity is imposed weakly by the method and not through the choice of shared inter-element degrees of freedom as in a continuous finite element space. Moreover, since typically all basis functions used in discontinuous Galerkin finite element method have non zero-trace on the element interfaces, no static condensation of degrees of freedom can be performed to reduce degrees of freedom.

On the other hand, the weak imposition of inter-element continuity may give rise to sparser linear systems, being easier to solve. Furthermore, discontinuous Galerkin finite element methods allow greater flexibility in the choice of polynomial degree p on every element. Indeed, as no continuity requirements are imposed across the element interfaces, in practice polynomial degree may vary almost arbitrarily across adjacent elements (cf. also the bounded local variation condition in Remark A.3.5). Thereby *hp*-discontinuous Galerkin finite element method is a very attractive contender in the context of *hp*-adaptivity [15, 188, 130, 112, 190, 114]. Considering also that *hp*-adaptation is superior to *h*-adaptive mesh refinement techniques, particularly when the approximating solutions admit high local regularity, discontinuous Galerkin finite element method offers a very suitable framework for adaptivity.

It is well established that in problems where steep gradients are present in the analytical solution (for instance, the presence of boundary or interior layers, etc.), standard conforming finite element methods produce oscillatory approximations, when the degrees of freedom are insufficient to resolve the rapid variation in the solution. In such instances, stabilization methods (stramline-diffusion stabilization, bubble stabilization) are often employed to counteract the undesirable oscillatory effects. However, it appears that such stabilizations are unnecessary for the *hp*-discontinuous Galerkin method [130], as numerical dissipation introduced by the discontinuities in the numerical solution stabilises the numerical solution and reduces the oscillations. This fact was indicated theoretically in [188, 130] as it was shown therein that it is not necessary to include streamline-diffusion stabilization terms to prove meaningful error bounds.

7.5.1 Coercivity of Bilinear Form

Stability 7.5.1.1. *A method is stable when its bilinear form induces a norm which can be bounded from below.*

The choice $\theta = 1$ gives rise to the non-symmetric interior penalty Galerkin (NIPG) formulation. It is straightforward to show that the corresponding bilinear form is coercive.

Proposition 7.5.1.2. *Let $\theta = 1$, $\gamma > 0$, $\zeta > 0$, then the hp-version NIPG method has a unique solution $u_{3:DG} \in \mathcal{S}_h^p$.*

Proof. As it is easy to see from the bilinear form (7.26), by substituting u_3 for w_3 and for $\theta = 1$, we have

$$B_{pl}(u_3, u_3) = |||u_3|||_{pl}^2 \quad \forall u_3 \in \mathcal{S}_h^p. \tag{7.32}$$

We showed earlier that $|||\cdot|||_{pl}$ is a seminorm on the space $H^4(\Omega, \mathcal{T})$, thus, since $\mathcal{S}_h^p \subset H^4(\Omega, \mathcal{T})$, we get that $|||\cdot|||_{pl}$ is also a seminorm on $\mathcal{S}_h^p$.

Wherefore, $B_{pl}(\cdot, \cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{S}_h^p$, and as a result the problem (7.31) has a unique solution in this space. □

Setting $\theta = -1$ yields the symmetric interior penalty Galerkin (SIPG) formulation with a symmetric bilinear form. Unfortunately, this bilinear form is non-coercive unless the stabilization parameters are chosen sufficiently large.

Let us now examine the coercivity of the bilinear form, $B_{sb}(\cdot, \cdot)$, for the hp-version SIPG finite element method.

Proposition 7.5.1.3. *The hp-version SIPG method* (7.31) *is stable in the energy seminorm* (7.29)*, that is, there exists a positive constant m such that*

$$B_{pl}(u_3, u_3) \geqslant m|||u_3|||_{pl}^2 \quad \forall u_3 \in \mathcal{S}_h^p. \tag{7.33}$$

Proof. Substituting u_3 for w_3 in the bilinear form (7.26), for $\theta = -1$, and applying the triangle inequality, we obtain

$$
\begin{aligned}
B_{pl}(u_3, u_3) \;\geqslant\; & \left|\left|\left(\frac{2E\varepsilon^3}{3(1+\nu)}\right)^{1/2} D_h^2 u_3\right|\right|_{\Omega}^2 + \left|\left|\left(\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right)^{1/2} \Delta_h u_3\right|\right|_{\Omega}^2 \\
& + 2\int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle [\![u_3]\!] dr \\
& + 2\int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1+\nu)} \frac{\partial^3 u_3}{\partial s^2 \partial b} \right\rangle [\![u_3]\!] dr \\
& - 2\int_{\Gamma_1} \left| \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta u_3 \right\rangle \left[\!\left[\frac{\partial u_3}{\partial b}\right]\!\right] \right| dr \\
& + 2\int_{\Gamma_1} \left\langle \frac{2E\varepsilon^3}{3(1+\nu)} \left(\frac{\partial^2 u_3}{\partial s^2} + \frac{1}{R}\frac{\partial u_3}{\partial b}\right) \right\rangle \left[\!\left[\frac{\partial u_3}{\partial b}\right]\!\right] dr \\
& + ||\gamma^{1/2} [\![u_3]\!]||_{\Gamma_0}^2 + \left|\left|\zeta^{1/2} \left[\!\left[\frac{\partial u_3}{\partial b}\right]\!\right]\right|\right|_{\Gamma_1}^2 . \qquad (7.34)
\end{aligned}
$$

Thus, to complete the proof, it only remains to estimate each of the integrals apprearing on the right hand side of (7.34).

So we can write the first integral, by using the the Cauchy-Schwarz inequality (A.12), as well as the Young inequality (A.17)

$$
\begin{aligned}
& \int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle [\![u_3]\!] dr \\
& \leqslant \left|\left| \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle \right|\right|_{\Gamma_0} ||[\![u_3]\!]||_{\Gamma_0} \\
& \leqslant \frac{\varepsilon_1}{2} \left|\left| \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle \right|\right|_{\Gamma_0}^2 + \frac{1}{2\varepsilon_1} ||[\![u_3]\!]||_{\Gamma_0}^2 \qquad (7.35) \\
& = \sum_{e\in\mathcal{E}_0} \left(\frac{\varepsilon_1}{2} \left|\left| \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle \right|\right|_{e}^2 + \frac{1}{2\varepsilon_1} ||[\![u_3]\!]||_{e}^2 \right).
\end{aligned}
$$

By employing the mean value inequality (A.19) in (7.35), we deduce

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_0}\left(\frac{\varepsilon_1}{2}\left|\left|\left\langle\frac{2E\varepsilon^3}{3(1-\nu^2)}\frac{\partial}{\partial b}(\Delta u_3)\right\rangle\right|\right|_e^2+\frac{1}{2\varepsilon_1}||[\![u_3]\!]||_e^2\right)\\
&\leqslant\sum_{e\in\mathcal{E}_{\text{int}}}\frac{\varepsilon_1}{2}\left(\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla(\Delta u_3)^+\right|\right|_e^2+\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla(\Delta u_3)^-\right|\right|_e^2\right)\\
&+\sum_{e\in\mathcal{E}_c}\frac{\varepsilon_1}{2}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_e^2+\sum_{e\in\mathcal{E}_0}\frac{1}{2\varepsilon_1}||[\![u_3]\!]||_e^2\\
&\leqslant\sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial}\frac{\varepsilon_1}{2}\left(\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K'}^2+\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K}^2\right)\\
&+\sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c}\frac{\varepsilon_1}{2}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K}^2+\sum_{e\in\mathcal{E}_0}\frac{1}{2\varepsilon_1}||[\![u_3]\!]||_e^2.
\end{aligned}\tag{7.36}$$

The sums of ∂K can be bounded by recalling the inverse inequality (A.21) in (7.36), then we conclude

$$\begin{aligned}
&\sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial}\frac{\varepsilon_1}{2}\left(\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K'}^2+\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K}^2\right)\\
&+\sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c}\frac{\varepsilon_1}{2}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\nabla\Delta u_3\right|\right|_{\partial K}^2+\sum_{e\in\mathcal{E}_0}\frac{1}{2\varepsilon_1}||[\![u_3]\!]||_e^2\\
&\leqslant\sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial}\frac{\varepsilon_1}{2}\left(c_1\frac{p_{K'}^6}{h_{K'}^3}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta u_3\right|\right|_{K'}^2\right.\\
&\left.+c_1\frac{p_K^6}{h_K^3}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta u_3\right|\right|_K^2\right)\\
&+\sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c}\frac{\varepsilon_1}{2}c_1\frac{p_K^6}{h_K^3}\left|\left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta u_3\right|\right|_K^2+\sum_{e\in\mathcal{E}_0}\frac{1}{2\varepsilon_1}||[\![u_3]\!]||_e^2\\
&\leqslant\sum_{K\in\mathcal{T}}\frac{\varepsilon_1}{2}c_1\frac{2E\varepsilon^3p_K^6}{3\nu(1-\nu^2)h_K^3}\left|\left|\left(\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right)^{1/2}\Delta u_3\right|\right|_K^2+\sum_{e\in\mathcal{E}_0}\frac{1}{2\varepsilon_1\gamma}||\gamma^{1/2}[\![u_3]\!]||_e^2.
\end{aligned}\tag{7.37}$$

In consequence, from (7.35) – (7.37), we arrive to the conclusion that the first integral, on the right-hand side of (7.34), can be estimated as follows

$$\int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \frac{\partial}{\partial b}(\Delta u_3) \right\rangle [\![u_3]\!] dr$$
$$\leqslant \sum_{K\in\mathcal{T}} \frac{\varepsilon_1}{2} c_1 \frac{2E\varepsilon^3 p_K^6}{3\nu(1-\nu^2)h_K^3} \left|\left| \left(\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right)^{1/2} \Delta u_3 \right|\right|_K^2 + \sum_{e\in\mathcal{E}_0} \frac{1}{2\varepsilon_1\gamma} ||\gamma^{1/2}[\![u_3]\!]||_e^2. \tag{7.38}$$

Furthermore, we shall analogously estimate the integrals entering on the right-hand side of (7.34). As a result, we arrive at

$$\int_{\Gamma_0} \left\langle \frac{2E\varepsilon^3}{3(1+\nu)} \frac{\partial^3 u_3}{\partial s^2 \partial b} \right\rangle [\![u_3]\!] dr$$
$$\leqslant \sum_{K\in\mathcal{T}} \frac{\varepsilon_2}{2} c_1 \frac{2E\varepsilon^3 p_K^6}{3(1+\nu)h_K^3} \left|\left| \left(\frac{2E\varepsilon^3}{3(1+\nu)}\right)^{1/2} D^2 u_3 \right|\right|_K^2 + \sum_{e\in\mathcal{E}_0} \frac{1}{2\varepsilon_2\gamma} ||\gamma^{1/2}[\![u_3]\!]||_e^2, \tag{7.39}$$

$$\int_{\Gamma_1} \left| \left\langle \frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta u_3 \right\rangle \left[\frac{\partial u_3}{\partial b}\right] \right| dr$$
$$\leqslant \sum_{K\in\mathcal{T}} \frac{\varepsilon_3}{2} c_0 \frac{2E\varepsilon^3 p_K^2}{3\nu(1-\nu^2)h_K} \left|\left| \left(\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right)^{1/2} \Delta u_3 \right|\right|_K^2 \tag{7.40}$$
$$+ \sum_{e\in\mathcal{E}_1} \frac{1}{2\varepsilon_3\zeta} \left|\left| \zeta^{1/2} \left[\frac{\partial u_3}{\partial b}\right] \right|\right|_e^2$$

and

$$\int_{\Gamma_1} \left\langle \frac{2E\varepsilon^3}{3(1+\nu)} \left(\frac{\partial^2 u_3}{\partial s^2} + \frac{1}{R}\frac{\partial u_3}{\partial b} \right) \right\rangle \left[\frac{\partial u_3}{\partial b}\right] dr$$
$$\leqslant \sum_{K\in\mathcal{T}} \frac{\varepsilon_4}{2} c_0 \frac{2E\varepsilon^3 p_K^2}{3(1+\nu)h_K} \left|\left| \left(\frac{2E\varepsilon^3}{3(1+\nu)}\right)^{1/2} D^2 u_3 \right|\right|_K^2 \tag{7.41}$$
$$+ \sum_{e\in\mathcal{E}_1} \frac{1}{2\varepsilon_4\zeta} \left|\left| \zeta^{1/2} \left[\frac{\partial u_3}{\partial b}\right] \right|\right|_e^2.$$

After those series of steps, we gather the inequalities (7.38) – (7.41) and insert them into the right-hand side of (7.34). Hence, we get

$$
\begin{aligned}
&B_{pl}(u_3, u_3) \\
&\geqslant \sum_{K\in\mathcal{T}} \left(1 - \varepsilon_2 c_1 \frac{2E\varepsilon^3 p_K^6}{3(1+\nu)h_K^3} - \varepsilon_4 c_0 \frac{2E\varepsilon^3 p_K^2}{3(1+\nu)h_K}\right) \left\| \left(\frac{2E\varepsilon^3}{3(1+\nu)}\right)^{1/2} D^2 u_3 \right\|_K^2 \\
&+ \sum_{K\in\mathcal{T}} \left(1 - \varepsilon_1 c_1 \frac{2E\varepsilon^3 p_K^6}{3\nu(1-\nu^2)h_K^3} - \varepsilon_3 c_0 \frac{2E\varepsilon^3 p_K^2}{3\nu(1-\nu^2)h_K}\right) \left\| \left(\frac{2E\varepsilon^3 \nu}{3(1-\nu^2)}\right)^{1/2} \Delta u_3 \right\|_K^2 \\
&+ \sum_{e\in\mathcal{E}_0} \left(1 - \frac{1}{\varepsilon_1\gamma} - \frac{1}{\varepsilon_2\gamma}\right) ||\gamma^{1/2} [\![u_3]\!]||_e^2 \\
&+ \sum_{e\in\mathcal{E}_1} \left(1 - \frac{1}{\varepsilon_3\zeta} - \frac{1}{\varepsilon_4\zeta}\right) \left\| \zeta^{1/2} \left[\frac{\partial u_3}{\partial b}\right] \right\|_e^2 .
\end{aligned}
\tag{7.42}
$$

So, by the use of definition of energy seminorm, (7.29), on the right-hand side of (7.42), we reach to

$$
B_{pl}(u_3, u_3) \geqslant m|||u_3|||_{pl}^2, \tag{7.43}
$$

which is the desired result. We denote by the constant m the minimum of the terms enclosed into the parentheses on the right-hand side of (7.42).

In particular, we can prove (7.33) for $m = \frac{1}{2}$ if we choose

$$
\varepsilon_1|_K = \frac{3\nu(1-\nu^2)h_K^3}{8c_1 E\varepsilon^3 p_K^6}, \quad \varepsilon_2|_K = \frac{3(1+\nu)h_K^3}{8c_1 E\varepsilon^3 p_K^6},
$$

$$
\varepsilon_3|_K = \frac{3\nu(1-\nu^2)h_K}{8c_0 E\varepsilon^3 p_K^2}, \quad \text{and} \quad \varepsilon_4|_K = \frac{3(1+\nu)h_K}{8c_0 E\varepsilon^3 p_K^2},
$$

in which case we obtain

$$
\gamma = \frac{16c_1 E\varepsilon^3(-\nu^2+\nu+1)p_K^6}{3\nu(1-\nu^2)h_K^3} \quad \text{and} \quad \zeta = \frac{16c_0 E\varepsilon^3(-\nu^2+\nu+1)p_K^2}{3\nu(1-\nu^2)h_K},
$$

as well. □

Wherefore, $B_{pl}(\cdot,\cdot)$ is a coercive bilinear form on the finite-dimensional space $\mathcal{S}_h^p$, and ergo the problem (7.31) has a unique solution.

By knowing the form of the stabilization parameters γ and ζ, on Γ_0 and Γ_1 respectively, we can define the discontinuous Galerkin finite element method by using lifting operators as well as deduce the stability of these operators.

7.6 DGFEM with Lifting Operators

We would like to present the interior penalty discontinuous Galerkin method by using appropriate lifting operators for the problem (7.10) – (7.11). We shall employ the weak formulation which derives in Section 7.3 and the finite element space $\mathcal{S}_h^p$ constructed in the Section 7.4.

Let us first introduce the following functional space

$$V_3^0 = \left\{ \upsilon | \upsilon \in H^2(\Omega) : \upsilon = 0 \text{ on } \Gamma_c,\ \frac{\partial \upsilon}{\partial b} = 0 \text{ on } \Gamma_q \right\}, \tag{7.44}$$

which is equipped with the norm induced by the Sobolev space $H^2(\Omega)$, see [80].

Next, we introduce the lifting operators $\mathcal{L}_i : \mathcal{S} := \mathcal{S}_h^p + V_3^0 \to \mathcal{S}_h^p, i = 1, 2$ by

$$\int_\Omega \mathcal{L}_1(u) w dv = \int_{\Gamma_0} [\![u]\!] \langle \nabla w \rangle dr - \int_{\Gamma_1} \langle w \rangle [\![\nabla u]\!] dr \quad \forall w \in \mathcal{S}_h^p, \tag{7.45}$$

$$\int_\Omega \mathcal{L}_2(u) w dv = \int_{\Gamma_0} [\![u]\!] \langle (w)_t \rangle dr + \int_{\Gamma_1} \langle w \rangle [\![\nabla u]\!] dr \quad \forall w \in \mathcal{S}_h^p, \tag{7.46}$$

where $(\cdot)_t$ denotes the tangential derivative along the edge e.

Now, we can rewrite the discontinuous Galerkin weak formulation, (7.25), of the problem (7.10) – (7.11), by employing the lifting operators $\mathcal{L}_i$, as

$$\begin{aligned}
&\int_\Omega \frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 : D_h^2 w_3 dv + \int_\Omega \frac{2E\varepsilon^3 \nu}{3(1-\nu^2)} \Delta_h u_3 \Delta_h w_3 dv \\
&+ \int_\Omega \left(\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta_h u_3 \mathcal{L}_1(w_3) - \theta \frac{2E\varepsilon^3}{3(1-\nu^2)} \mathcal{L}_1(u_3) \Delta_h w_3 \right) dv \\
&+ \int_\Omega \left(\frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 \mathcal{L}_2(w_3) - \theta \frac{2E\varepsilon^3}{3(1+\nu)} \mathcal{L}_2(u_3) D_h^2 w_3 \right) dv \\
&+ \int_{\Gamma_0} \gamma [\![u_3]\!][\![w_3]\!] dr + \int_{\Gamma_1} \zeta [\![\nabla u_3]\!][\![\nabla w_3]\!] dr = \int_\Omega f_3 w_3 dv.
\end{aligned} \tag{7.47}$$

The bilinear form $B_{pl} : \mathcal{S} \times \mathcal{S} \to \Re$ is defined as

$$\begin{aligned} B_{pl}(u_3, w_3) \quad := & \int_\Omega \frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 : D_h^2 w_3 dv + \int_\Omega \frac{2E\varepsilon^3 \nu}{3(1-\nu^2)} \Delta_h u_3 \Delta_h w_3 dv \\ & + \int_\Omega \left(\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta_h u_3 \mathcal{L}_1(w_3) - \theta \frac{2E\varepsilon^3}{3(1-\nu^2)} \mathcal{L}_1(u_3) \Delta_h w_3 \right) dv \\ & + \int_\Omega \left(\frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 \mathcal{L}_2(w_3) - \theta \frac{2E\varepsilon^3}{3(1+\nu)} \mathcal{L}_2(u_3) D_h^2 w_3 \right) dv \\ & + \int_{\Gamma_0} \gamma [\![u_3]\!][\![w_3]\!] dr + \int_{\Gamma_1} \zeta [\![\nabla u_3]\!][\![\nabla w_3]\!] dr, \end{aligned} \tag{7.48}$$

for any $u_3, w_3 \in \mathcal{S}$.

The linear form $L_{pl} : \mathcal{S} \to \Re$ is given by

$$L_{pl}(w_3) := \int_\Omega f_3 w_3 dv, \tag{7.49}$$

for any $w_3 \in \mathcal{S}$.

Then, the interior penalty discontinuous Galerkin method of the problem (7.10), (7.11), reads as folllows:

$$\text{Find } u_{3:DG} \in \mathcal{S}_h^p \text{ such that } \; B_{pl}(u_{3:DG}, w_3) = L_{pl}(w_3) \quad \forall w_3 \in \mathcal{S}_h^p. \tag{7.50}$$

We notice that this formulation is incosistent for trial and test functions belonging either to the solution space $\mathcal{S}$ or to the solution space V_3^0.

In practice, the right-hand side is approximated by the L^2 projection of the source of the function f onto the finite element space $\mathcal{S}_h^p$. We denote the L^2 projection of f onto $\mathcal{S}_h^p$ by Πf.

We shall associate with the bilinear form $B_{pl}(\cdot,\cdot)$, (7.48), the energy seminorm, $|||\cdot|||_{pl}$, defined by

$$\begin{aligned} |||u_3|||_{pl} \quad = & \left(\left|\left| \left\{ \frac{2E\varepsilon^3}{3(1+\nu)} \right\}^{1/2} D_h^2 u_3 \right|\right|_\Omega^2 + \left|\left| \left\{ \frac{2E\varepsilon^3 \nu}{3(1-\nu^2)} \right\}^{1/2} \Delta_h u_3 \right|\right|_\Omega^2 \right. \\ & \left. + ||\gamma^{1/2} [\![u_3]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2} [\![\nabla u_3]\!]||_{\Gamma_1}^2 \right)^{1/2}, \quad u \in H^2(\Omega, \mathcal{T}) \end{aligned} \tag{7.51}$$

Proposition 7.6.0.1. *If $\gamma, \zeta > 0$, then $|||\cdot|||_{pl}$ is a seminorm on $H^2(\Omega, \mathcal{T})$.*

We note in passing that since $H^4(\Omega, \mathcal{T}) \subset H^2(\Omega, \mathcal{T})$, then $|||\cdot|||_{pl}$ is also a seminorm on $H^4(\Omega, \mathcal{T})$.

7.6.1 Stability Bounds of Lifting Operators

In this section, we aim to derive the stability of the trace liftings $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 7.6.1.1. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be the trace liftings defined in* (7.45) *and in* (7.46)*, respectively. Then, for $u \in \mathcal{S}$, the following bounds hold:*

$$||\mathcal{L}_1(u)||_\Omega^2 \leqslant C(\varepsilon, E, \nu)\left(||\gamma^{1/2}[\![u]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla u]\!]||_{\Gamma_1}^2\right), \tag{7.52}$$

$$||\mathcal{L}_2(u)||_\Omega^2 \leqslant C(\varepsilon, E, \nu)\left(||\gamma^{1/2}[\![u]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla u]\!]||_{\Gamma_1}^2\right), \tag{7.53}$$

where

$$C(\varepsilon, E, \nu) = \frac{\nu(1-\nu^2)}{E\varepsilon^3(-\nu^2+\nu+1)} \tag{7.54}$$

is a positive constant, that is independent of u and of discretization parameters. We denote by $\gamma : \Gamma_0 \to \Re$ and $\zeta : \Gamma_1 \to \Re$ piecewise constant functions, defined by

$$\gamma = C_\gamma E\varepsilon^3 \frac{(-\nu^2+\nu+1)}{\nu(1-\nu^2)}\left\langle \frac{\mathbf{p^6}}{\mathbf{h^3}} \right\rangle$$

and

$$\zeta = C_\zeta E\varepsilon^3 \frac{(-\nu^2+\nu+1)}{\nu(1-\nu^2)}\left\langle \frac{\mathbf{p^2}}{\mathbf{h}} \right\rangle,$$

with C_γ as well as C_ζ sufficiently large positive constants depending only on the mesh parameters.

Proof. We denote by $\Pi : L^2(\Omega) \to S_h^p$ the (orthogonal) L^2-projection operator onto the finite element S_h^p. By invoking the definition of the L^2-norm, the orthogonality of the L^2-projection operator and the definition of the trace lifting $\mathcal{L}_1$, we get

$$\begin{aligned} ||\mathcal{L}_1(u)||_\Omega &= \sup_{z\in L^2(\Omega)} \frac{\int_\Omega \mathcal{L}_1(u) z dv}{||z||_\Omega} \\ &= \sup_{z\in L^2(\Omega)} \frac{\int_\Omega \mathcal{L}_1(u) \Pi z dv}{||z||_\Omega} \\ &= \sup_{z\in L^2(\Omega)} \frac{\int_{\Gamma_0} [\![u]\!]\langle\nabla(\Pi z)\rangle dr - \int_{\Gamma_1}\langle \Pi z\rangle [\![\nabla u]\!] dr}{||z||_\Omega}. \end{aligned} \tag{7.55}$$

By recalling the Cauchy-Schwarz inequality (A.12) and then the Cauchy-Schwarz discrete inequality (A.13) in (7.55), we obtain

$$\begin{aligned}
&\sup_{z\in L^2(\Omega)} \frac{\int_{\Gamma_0}[\![u]\!]\langle\nabla(\Pi z)\rangle dr - \int_{\Gamma_1}\langle\Pi z\rangle[\![\nabla u]\!]dr}{||z||_\Omega}\\
&\leqslant \sup_{z\in L^2(\Omega)} \frac{||\gamma^{1/2}[\![u]\!]||_{\Gamma_0}||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||_{\Gamma_0} + ||\zeta^{-1/2}\langle\Pi z\rangle||_{\Gamma_1}||\zeta^{1/2}[\![\nabla u]\!]||_{\Gamma_1}}{||z||_\Omega}\\
&\leqslant \sup_{z\in L^2(\Omega)} \frac{\left(||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||^2_{\Gamma_0} + ||\zeta^{-1/2}\langle\Pi z\rangle||^2_{\Gamma_1}\right)^{\frac{1}{2}}}{||z||_\Omega}\\
&\times\left(||\gamma^{1/2}[\![u]\!]||^2_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla u]\!]||^2_{\Gamma_1}\right)^{\frac{1}{2}}.
\end{aligned} \tag{7.56}$$

As a consequence, from (7.55) – (7.56), we deduce

$$\begin{aligned}
||\mathcal{L}_1(u)||_\Omega \quad \leqslant \quad & \sup_{z\in L^2(\Omega)} \frac{\left(||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||^2_{\Gamma_0} + ||\zeta^{-1/2}\langle\Pi z\rangle||^2_{\Gamma_1}\right)^{\frac{1}{2}}}{||z||_\Omega}\\
&\times\left(||\gamma^{1/2}[\![u]\!]||^2_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla u]\!]||^2_{\Gamma_1}\right)^{\frac{1}{2}}.
\end{aligned} \tag{7.57}$$

Thereby, to complete the proof, it only remains to estimate each of the mean value terms appearing on the right-hand side of (7.57).

Hence, by applying the mean value inequality (A.19), we can write the first mean value term as

$$\begin{aligned}
&||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||^2_{\Gamma_0}\\
&= \sum_{e\in\mathcal{E}_0} ||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||^2_e\\
&\leqslant \sum_{e\in\mathcal{E}_{\rm int}} \left(||\gamma^{-1/2}\nabla(\Pi z)^+||^2_e + ||\gamma^{-1/2}\nabla(\Pi z)^-||^2_e\right) + \sum_{e\in\mathcal{E}_c} ||\gamma^{-1/2}\nabla(\Pi z)||^2_e\\
&\leqslant \sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial} \left(||\gamma^{-1/2}\nabla(\Pi z)||^2_{\partial K'} + ||\gamma^{-1/2}\nabla(\Pi z)||^2_{\partial K}\right)\\
&+ \sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c} ||\gamma^{-1/2}\nabla(\Pi z)||^2_{\partial K}\\
&\leqslant \sum_{K\in\mathcal{T}} ||\gamma^{-1/2}\nabla(\Pi z)||^2_{\partial K}.
\end{aligned} \tag{7.58}$$

Afterwards, by using the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions, on the finite element space S_h^p, as well as the inverse inequality (A.21) in (7.58), we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} ||\gamma^{-1/2}\nabla(\Pi z)||^2_{\partial K} \\
&\leqslant \frac{\nu(1-\nu^2)}{\tilde{C}C_\gamma E\varepsilon^3(-\nu^2+\nu+1)} \sum_{K\in\mathcal{T}} \frac{h_K^3}{p_K^6} ||\nabla(\Pi z)||^2_{\partial K} \\
&\leqslant \frac{c_1\nu(1-\nu^2)}{\tilde{C}C_\gamma E\varepsilon^3(-\nu^2+\nu+1)} \sum_{K\in\mathcal{T}} ||z||^2_K \\
&\leqslant \frac{\nu(1-\nu^2)}{2E\varepsilon^3(-\nu^2+\nu+1)} ||z||^2_\Omega,
\end{aligned} \tag{7.59}$$

where $\tilde{C} = \tilde{C}(\eta, \rho)$ is a positive constant and $C_\gamma \geqslant \frac{2c_1}{\tilde{C}}$.

Therefore, from (7.58) – (7.59), we reach the conclusion that the first mean value term, on the right-hand side of (7.57), can be bounded as

$$||\gamma^{-1/2}\langle\nabla(\Pi z)\rangle||^2_{\Gamma_0} \leqslant \frac{\nu(1-\nu^2)}{2E\varepsilon^3(-\nu^2+\nu+1)} ||z||^2_\Omega. \tag{7.60}$$

In addition, we shall follow the above series of steps in the same way to estimate the remaining mean value term, on the right-hand side of (7.57). By employing the mean value inequality (A.19), we conclude

$$\begin{aligned}
&||\zeta^{-1/2}\langle\Pi z\rangle||^2_{\Gamma_1} \\
&= \sum_{e\in\mathcal{E}_1} ||\zeta^{-1/2}\langle\Pi z\rangle||^2_e \\
&\leqslant \sum_{e\in\mathcal{E}_{\rm int}} \left(||\zeta^{-1/2}(\Pi z)^+||^2_e + ||\zeta^{-1/2}(\Pi z)^-||^2_e\right) + \sum_{e\in\mathcal{E}_q} ||\zeta^{-1/2}\Pi z||^2_e \\
&\leqslant \sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial} \left(||\zeta^{-1/2}\Pi z||^2_{\partial K'} + ||\zeta^{-1/2}\Pi z||^2_{\partial K}\right) \\
&+ \sum_{K\in\mathcal{T}:\partial K\cap\Gamma_q} ||\zeta^{-1/2}\Pi z||^2_{\partial K} \\
&\leqslant \sum_{K\in\mathcal{T}} ||\zeta^{-1/2}\Pi z||^2_{\partial K}.
\end{aligned} \tag{7.61}$$

Next, by invoking the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions, on the finite element space S_h^p, as well as the inverse inequality (A.20) in (7.61), we get

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}} ||\zeta^{-1/2}\Pi z||^2_{\partial K} \\
&\leqslant \frac{\nu(1-\nu^2)}{CC_\zeta E\varepsilon^3(-\nu^2+\nu+1)} \sum_{K\in\mathcal{T}} \frac{h_K}{p_K^2}||\Pi z||^2_{\partial K} \\
&\leqslant \frac{c_0\nu(1-\nu^2)}{CC_\zeta E\varepsilon^3(-\nu^2+\nu+1)} \sum_{K\in\mathcal{T}} ||z||^2_K \\
&\leqslant \frac{\nu(1-\nu^2)}{2E\varepsilon^3(-\nu^2+\nu+1)}||z||^2_\Omega,
\end{aligned}
\tag{7.62}
$$

where $C = C(\eta, \rho)$ is a positive constant and $C_\zeta \geqslant \frac{2c_0}{C}$.

Ergo, from (7.61) – (7.62), we arrive to the conclusion that the second mean value term, on the right-hand side of (7.57), can subsequently be estimated as

$$
||\zeta^{-1/2}\langle\Pi z\rangle||^2_{\Gamma_1} \leqslant \frac{\nu(1-\nu^2)}{2E\varepsilon^3(-\nu^2+\nu+1)}||z||^2_\Omega. \tag{7.63}
$$

Inserting the inequalities (7.60) – (7.63) on the right-hand side of (7.57) yields to boot

$$
||\mathcal{L}_1(u)||^2_\Omega \leqslant \frac{\nu(1-\nu^2)}{E\varepsilon^3(-\nu^2+\nu+1)}\left(||\gamma^{1/2}[\![u]\!]||^2_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla u]\!]||^2_{\Gamma_1}\right),
$$

which is one of the desired results.

What is more, by following the above procedure step by step, we shall bound the trace lifitng $\mathcal{L}_2$ as

$$
||\mathcal{L}_2(u)||^2_\Omega \leqslant \frac{\nu(1-\nu^2)}{E\varepsilon^3(-\nu^2+\nu+1)}\left(||\gamma^{1/2}[\![u]\!]||^2_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla u]\!]||^2_{\Gamma_1}\right),
$$

which is the other desired result. □

In the following sections, we are going to prove the coercivity and the continuity property of the bilinear form for the symmetric interior penalty discontinuous Galerkin method.

7.6.2 Coercivity of Bilinear Form

In this section, our concern is to examine the coercivity of the bilinear form $B_{pl}(\cdot,\cdot)$ for the symmetric interior penalty discontinuous Galerkin finite element method.

We showed earlier that $|||\cdot|||_{pl}$, (7.51), is a seminorm on the space $H^4(\Omega,\mathcal{T})$, thereby, since $S_h^p \subset H^4(\Omega,\mathcal{T})$, we get that $|||\cdot|||_{pl}$ is also a seminorm on S_h^p.

Proposition 7.6.2.1. *Let $\gamma:\Gamma_0\to\Re$ and $\zeta:\Gamma_1\to\Re$ be piecewise constant functions. Then, the bilinear form $B_{pl}(\cdot,\cdot)$, defined in (7.48), is coercive in the sense that*

$$B_{pl}(u_3,u_3) \geqslant m|||u_3|||_{pl}^2 \quad \forall u_3 \in \mathcal{S}_h^p, \tag{7.64}$$

where m is a positive constant depending only on the mesh parameters.

Proof. Substituting u_3 for w_3 in the bilinear form, (7.48), and for $\theta=-1$, we obtain

$$\begin{aligned} B_{pl}(u_3,u_3) &= |||u_3|||_{pl}^2 + 2\int_\Omega \frac{2E\varepsilon^3}{3(1-\nu^2)}\mathcal{L}_1(u_3)\Delta_h u_3 dv \\ &\quad + 2\int_\Omega \frac{2E\varepsilon^3}{3(1+\nu)}\mathcal{L}_2(u_3)D_h^2 u_3 dv. \end{aligned} \tag{7.65}$$

To complete the proof, it only remains to estimate the integrals appearing on the right-hand side of (7.65).

So, by employing the Cauchy-Schwarz inequality (A.12) and then the Young inequality (A.17), we can write the first integral as

$$\begin{aligned} &\int_\Omega \frac{2E\varepsilon^3}{3(1-\nu^2)}\mathcal{L}_1(u_3)\Delta_h u_3 dv \\ &\leqslant \int_\Omega \left|\frac{2E\varepsilon^3}{3(1-\nu^2)}\mathcal{L}_1(u_3)\Delta_h u_3\right| dv \\ &\leqslant \left\|\left\{\frac{2E\varepsilon^3}{3\nu(1-\nu^2)}\right\}^{1/2}\mathcal{L}_1(u_3)\right\|_\Omega \left\|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2}\Delta_h u_3\right\|_\Omega \\ &\leqslant \frac{\varepsilon_5}{2}\left\|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2}\Delta_h u_3\right\|_\Omega^2 + \frac{1}{2\varepsilon_5}\left\|\left\{\frac{2E\varepsilon^3}{3\nu(1-\nu^2)}\right\}^{1/2}\mathcal{L}_1(u_3)\right\|_\Omega^2 . \end{aligned} \tag{7.66}$$

Moreover, we shall follow the above procedure in a similar manner to estimate the second integral on the right-hand side of (7.65).

Hence, we deduce

$$\begin{aligned}\int_{\Omega} \frac{2E\varepsilon^3}{3(1+\nu)}\mathcal{L}_2(u_3)D_h^2 u_3 dv \quad \leqslant \quad & \frac{\varepsilon_6}{2}\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_{\Omega}^2 \\ & + \frac{1}{2\varepsilon_6}\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} \mathcal{L}_2(u_3)\right|\right|_{\Omega}^2 \end{aligned} \tag{7.67}$$

Thereafter, inserting the inequalities (7.66) – (7.67) on the right-hand side of (7.65), we have

$$\begin{aligned} B_{pl}(u_3, u_3) \geqslant |||u_3|||_{pl}^2 - \varepsilon_5 \left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_{\Omega}^2 \\ - \frac{1}{\varepsilon_5}\left|\left|\left\{\frac{2E\varepsilon^3}{3\nu(1-\nu^2)}\right\}^{1/2} \mathcal{L}_1(u_3)\right|\right|_{\Omega}^2 - \varepsilon_6\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_{\Omega}^2 \\ - \frac{1}{\varepsilon_6}\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} \mathcal{L}_2(u_3)\right|\right|_{\Omega}^2 . \end{aligned} \tag{7.68}$$

Next, by invoking the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$ and by applying the mathematical inequalities (7.52) together with (7.53) on the right-hand side of (7.68), we get

$$\begin{aligned} B_{pl}(u_3, u_3) \quad \geqslant \quad & |||u_3|||_{pl}^2 - \varepsilon_6 \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_{\Omega}^2 \\ & - \varepsilon_5\left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_{\Omega}^2 \\ & - \left\{\frac{2}{3\varepsilon_5(-\nu^2+\nu+1)} + \frac{2\nu(1-\nu)}{3\varepsilon_6(-\nu^2+\nu+1)}\right\} ||\gamma^{1/2}[\![u_3]\!]||_{\Gamma_0}^2 \\ & - \left\{\frac{2}{3\varepsilon_5(-\nu^2+\nu+1)} + \frac{2\nu(1-\nu)}{3\varepsilon_6(-\nu^2+\nu+1)}\right\} ||\zeta^{1/2}[\nabla u_3]||_{\Gamma_1}^2 . \end{aligned} \tag{7.69}$$

Now, by the use of energy seminorm, (7.51), and with the aid of factorization on the right-hand side of (7.69), it is clear that

$$\begin{aligned} B_{pl}(u_3, u_3) \geqslant (1-\varepsilon_6) & \left|\left| \left\{ \frac{2E\varepsilon^3}{3(1+\nu)} \right\}^{1/2} D_h^2 u_3 \right|\right|_\Omega^2 \\ & + (1-\varepsilon_5) \left|\left| \left\{ \frac{2E\varepsilon^3\nu}{3(1-\nu^2)} \right\}^{1/2} \Delta_h u_3 \right|\right|_\Omega^2 \\ & + \left\{ 1 - \frac{2}{3\varepsilon_5(-\nu^2+\nu+1)} - \frac{2\nu(1-\nu)}{3\varepsilon_6(-\nu^2+\nu+1)} \right\} ||\gamma^{1/2}[\![u_3]\!]||_{\Gamma_0}^2 \\ & + \left\{ 1 - \frac{2}{3\varepsilon_5(-\nu^2+\nu+1)} - \frac{2\nu(1-\nu)}{3\varepsilon_6(-\nu^2+\nu+1)} \right\} ||\zeta^{1/2}[\nabla u_3]||_{\Gamma_1}^2 . \end{aligned} \tag{7.70}$$

So, employing the definition of energy seminorm one more time, (7.51), on the right-hand side of (7.70) derives

$$B_{pl}(u_3, u_3) \geqslant m|||u_3|||_{pl}^2,$$

which is the desired result. We denote by the constant m the minimum of the terms enclosed into the parentheses on the right-hand side of (7.70).

In particular, we can prove (7.64) for $m = \frac{1}{9}$ if we choose

$$\varepsilon_5|_K = \varepsilon_6|_K = \frac{3}{4}.$$

□

7.6.3 Continuity of Bilinear Form

With the definition of the energy seminorm, (7.51), we have the following continuity result for the bilinear form $B_{pl}(\cdot,\cdot)$, based on the Cauchy-Schwarz inequalities (A.12) and (A.13).

Proposition 7.6.3.1. *Let $\gamma : \Gamma_0 \to \Re$ and $\zeta : \Gamma_1 \to \Re$ be piecewise constant functions. Then, the bilinear form $B_{pl}(\cdot,\cdot)$, defined in (7.48), is continuous in the sense that*

$$B_{pl}(u_3, w_3) \leqslant C|||u_3|||_{pl}|||w_3|||_{pl} \quad \forall u_3, w_3 \in \mathcal{S}, \tag{7.71}$$

where C is a positive constant depending only on the mesh parameters.

Proof. Let $u_3, w_3 \in \mathcal{S}$, we can obtain (7.71) by applying at first the triangle inequality in the bilinear form

$$\begin{aligned}B_{pl}(u_3,w_3) &\leqslant |B_{pl}(u_3,w_3)| \\ &\leqslant \int_\Omega \left|\frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 : D_h^2 w_3\right| dv + \int_\Omega \left|\frac{2E\varepsilon^3\nu}{3(1-\nu^2)} \Delta_h u_3 \Delta_h w_3\right| dv \\ &+ \int_\Omega \left|\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta_h u_3 \mathcal{L}_1(w_3)\right| dv + \int_\Omega \left|\frac{2E\varepsilon^3}{3(1-\nu^2)} \mathcal{L}_1(u_3)\Delta_h w_3\right| dv \\ &+ \int_\Omega \left|\frac{2E\varepsilon^3}{3(1+\nu)} D_h^2 u_3 \mathcal{L}_2(w_3)\right| dv + \int_\Omega \left|\frac{2E\varepsilon^3}{3(1+\nu)} \mathcal{L}_2(u_3) D_h^2 w_3\right| dv \\ &+ \int_{\Gamma_0} |\gamma [\![u_3]\!][\![w_3]\!]|\, dr + \int_{\Gamma_1} |\zeta [\![\nabla u_3]\!][\![\nabla w_3]\!]|\, dr,\end{aligned} \tag{7.72}$$

and then the Cauchy-Schwarz inequality (A.12) on the right-hand side of (7.72). For that reason, we get

$$\begin{aligned}B_{pl}(u_3,w_3) \;\leqslant\; & \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 w_3\right|\right|_\Omega \\ &+ \left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h w_3\right|\right|_\Omega \\ &+ \left|\left|\left\{\frac{2E\varepsilon^3}{3\nu(1-\nu^2)}\right\}^{1/2} \mathcal{L}_1(u_3)\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h w_3\right|\right|_\Omega \\ &+ \left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3}{3\nu(1-\nu^2)}\right\}^{1/2} \mathcal{L}_1(w_3)\right|\right|_\Omega \\ &+ \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} \mathcal{L}_2(u_3)\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 w_3\right|\right|_\Omega \\ &+ \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_\Omega \left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} \mathcal{L}_2(w_3)\right|\right|_\Omega \\ &+ ||\gamma^{1/2}[\![u_3]\!]||_{\Gamma_0} ||\gamma^{1/2}[\![w_3]\!]||_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla u_3]\!]||_{\Gamma_1} ||\zeta^{1/2}[\![\nabla w_3]\!]||_{\Gamma_1}.\end{aligned} \tag{7.73}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (7.73), we have

$$B_{pl}(u_3,w_3) \leqslant \Bigg(2\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_\Omega^2 + 2\left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_\Omega^2$$
$$+ \frac{2E\varepsilon^3}{3\nu(1-\nu^2)}||\mathcal{L}_1(u_3)||_\Omega^2 + \frac{2E\varepsilon^3}{3(1+\nu)}||\mathcal{L}_2(u_3)||_\Omega^2$$
$$+ ||\gamma^{1/2}[\![u_3]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla u_3]\!]||_{\Gamma_1}^2\Bigg)^{1/2}$$
$$\times \Bigg(2\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 w_3\right|\right|_\Omega^2 + 2\left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h w_3\right|\right|_\Omega^2$$
$$+ \frac{2E\varepsilon^3}{3\nu(1-\nu^2)}||\mathcal{L}_1(w_3)||_\Omega^2 + \frac{2E\varepsilon^3}{3(1+\nu)}||\mathcal{L}_2(w_3)||_\Omega^2$$
$$+ ||\gamma^{1/2}[\![w_3]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla w_3]\!]||_{\Gamma_1}^2\Bigg)^{1/2}. \tag{7.74}$$

Thereby, to complete the proof, it only remains to recall the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$ and therefore to employ both the mathematical expressions (7.52) together with (7.53) on the right-hand side of (7.74).

In consequence, we deduce

$$B_{pl}(u_3,w_3) \leqslant \Bigg(2\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 u_3\right|\right|_\Omega^2 + 2\left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h u_3\right|\right|_\Omega^2$$
$$+ \frac{5}{3}||\gamma^{1/2}[\![u_3]\!]||_{\Gamma_0}^2 + \frac{5}{3}||\zeta^{1/2}[\![\nabla u_3]\!]||_{\Gamma_1}^2\Bigg)^{1/2}$$
$$\times \Bigg(2\left|\left|\left\{\frac{2E\varepsilon^3}{3(1+\nu)}\right\}^{1/2} D_h^2 w_3\right|\right|_\Omega^2 + 2\left|\left|\left\{\frac{2E\varepsilon^3\nu}{3(1-\nu^2)}\right\}^{1/2} \Delta_h w_3\right|\right|_\Omega^2$$
$$+ \frac{5}{3}||\gamma^{1/2}[\![w_3]\!]||_{\Gamma_0}^2 + \frac{5}{3}||\zeta^{1/2}[\![\nabla w_3]\!]||_{\Gamma_1}^2\Bigg)^{1/2}. \tag{7.75}$$

Also, by the use of definition of energy seminorm, (7.51), on the right-hand side of (7.75), we reach to

$$B_{pl}(u_3,w_3) \leqslant C|||u_3|||_{pl}|||w_3|||_{pl},$$

which is the desired result. □

7.7 A Posteriori Error Analysis

In this section, overall our research endeavor focuses on the introduction of a suitable recovery operator and on the proof of an appropriate Lemma, for this operator, which is imperative to prove the h-version reliable a posteriori error estimate in the energy seminorm, $|||\cdot|||_{pl}$, for the interior penalty discontinuous Galerkin finite element method (7.50).

The reliability estimate is based on a suitable recovery operator, that maps discontinuous finite element spaces to V_3^0-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [84] (see also [143, 34, 132, 115] for similar constructions). Using the recovery operator, in conjuction with the incosistent formulation for the interior penalty discontinuous Galerkin method presented in the preceding section (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), reliable a posteriori error estimate of residual type can derive for the interior penalty discontinuous Galerkin method in the corresponding energy seminorm.

7.7.1 Finite Element Spaces

In this section, we will consider the finite-dimensional subspace of the broken Sobolev space $H^4(\Omega, \mathcal{T})$ which is used in the finite element approximation of the problem. Moreover, we wish to modify a little the finite element space defined in section 7.4 so as to include either triangular or quadrilateral elements.

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular or quadrilateral elements $K \in \mathcal{T}$. We assume that the elemental edges are straight line segments.

For a non-negative integer p, we denote by $\mathcal{P}_p(\hat{K})$ the set of all polynomials of total degree at most p if $\hat{K}$ is either the reference triangle or the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. For $p \geqslant 2$ we consider the finite element space

$$\mathcal{S}_h^p := \left\{ v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{P}_p(\hat{K}),\ K \in \mathcal{T} \right\}. \tag{7.76}$$

We collect the h_K into the elementwise constant function

$$\mathbf{h} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K, K \in \mathcal{T} \text{ and } \mathbf{h}|_e = \langle \mathbf{h} \rangle, \ e \subset \Gamma.$$

We shall assume throughout that the families of meshes considered are locally quasiuniform or in other words the mesh size function $\mathbf{h}$ has bounded local variation (see Remark A.3.5).

Then, the piecewise constant stabilization parameters $\gamma : \Gamma_0 \to \Re$ and $\zeta : \Gamma_1 \to \Re$ are defined by

$$\gamma = C_\gamma E \varepsilon^3 \frac{(-\nu^2 + \nu + 1)}{\nu(1 - \nu^2)} (\mathbf{h}|_e)^{-3} \tag{7.77}$$

and

$$\zeta = C_\zeta E \varepsilon^3 \frac{(-\nu^2 + \nu + 1)}{\nu(1 - \nu^2)} (\mathbf{h}|_e)^{-1}, \tag{7.78}$$

with C_γ as well as C_ζ sufficiently large positive constants.

7.7.2 Recovery Operator

The use of a recovery operator, mapping elements of S_h^p onto a C^1-conforming space consisting of macro-elements of degree $p+2$, is a significant tool helping us conduct a posteriori error analysis. The family of macro-elements considered will be higher-order versions of the classical Hsieh-Clough-Tocher macro-element, constructed in [84]. This mapping is constructed via averages of the nodal basis functions (see [143, 34, 132, 115]).

The corresponding finite element space consisting of the above macro-elements will be denoted by $\tilde{S}_h^m$.

Let us consider the standard Lagrange basis for a polynomial of degree p, where $p \geqslant 2$. A crucial observation here is that the set of the nodal points of the Lagrange basis is a subset of the set of the nodal points of the macro-elements of degree $p+2$.

Lemma 7.7.2.1. *Let us assume that the mesh $\mathcal{T}$ is constructed as in Section 7.7.1. Then, there exists an operator $E_{op} : S_h^p \to \tilde{S}_h^{p+2} \cap V_3^0$ satisfying the following bound:*

$$\begin{aligned} &\sum_{k \in \mathcal{T}} |u_{3:h} - E_{op}(u_{3:h})|_{j,K}^2 \\ &\leqslant C \left(||\mathbf{h}^{1/2-j} [\![u_{3:h}]\!]||^2_{\Gamma_{\text{int}} \cup \gamma_0 \cup \gamma_1 \cup \gamma_2} + ||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!]||^2_{\Gamma_{\text{int}} \cup \gamma_0 \cup \gamma_1 \cup \gamma_2} \right), \end{aligned} \tag{7.79}$$

with $j = 0, 1, 2$ and $C > 0$ being a constant that is independent of $\mathbf{h}$ *and* $u_{3:h}$.

Proof. For each nodal point np of the C^1-conforming finite element space $\tilde{S}_h^{p+2}$ we define ω_{np} to be the set of $K \in \mathcal{T}$ that share the nodal point np, i.e.,

$$\omega_{np} := \{K \in \mathcal{T} : np \in \mathcal{T}\}.$$

Furthermore, $|\omega_{np}|$ will denote the cardinality of ω_{np}. We note that if np located in the interior of an element, then we shall have $|\omega_{np}| = 1$.

Next, we define the operator $E_{op} : S_h^p \to \tilde{S}_h^{p+2} \cap V_3^0$ for the function, for the normal derivative and for the partial derivative respectively by

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} N_{np}(u_{3:h}|_K), & \text{if } np \notin \Gamma_c \\ 0, & \text{if } np \in \Gamma_c \end{cases} \tag{7.80}$$

or

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} u_{3:h}(np)|_K, & \text{if } np \notin \Gamma_c \\ 0, & \text{if } np \in \Gamma_c, \end{cases}$$

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} N_{np}(u_{3:h}|_K), & \text{if } np \notin \Gamma_q \\ 0, & \text{if } np \in \Gamma_q \end{cases} \tag{7.81}$$

or

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} (\nabla u_{3:h} \cdot b_K)|_K(np), & \text{if } np \notin \Gamma_q \\ 0, & \text{if } np \in \Gamma_q, \end{cases}$$

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} N_{np}(u_{3:h}|_K), & \text{if } np \notin \Gamma_q \\ 0, & \text{if } np \in \Gamma_q \end{cases} \tag{7.82}$$

or

$$N_{np}(E_{op}(u_{3:h})) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K \in \omega_{np}} \sum_{z \in \{x,y\}} (u_{3:h})_z|_K(np), & \text{if } np \notin \Gamma_q \\ 0, & \text{if } np \in \Gamma_q, \end{cases}$$

where N_{np} is any nodal variable at np and np is any nodal point of $\tilde{S}_h^{p+2}$. Note that

$$N_{np}(E_{op}(u_{3:h})) = N_{np}(u_{3:h}), \quad \text{if } np \in \text{int}K.$$

We denote by $\mathcal{N}$ the set of all nodal variables of $\tilde{S}_h^{p+2}$ defined on every element of $\mathcal{T}$, i.e., they may be discontinuous across element boundaries. Then, we can split $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1,$$

where $\mathcal{N}_0$ and $\mathcal{N}_1$ consisting of the nodal variables corresponding to the function evaluations and those involving partial and normal derivatives of the function, respectively.

The use of an inverse estimate (A.37) yields

$$\sum_{K\in\mathcal{T}} |u_{3:h} - E_{op}(u_{3:h})|_{j,K}^2 \leqslant C \left|\left|\mathbf{h}^{-j}(u_{3:h} - E_{op}(u_{3:h}))\right|\right|_{\Omega}^2, \tag{7.83}$$

with C a positive constant which is independent of $\mathbf{h}$ and $u_{3:h}$.

After that, the equivalence of norms in a finite-dimensional vector space along with a scaling argument gives

$$\begin{aligned} &\left|\left|\mathbf{h}^{-j}(u_{3:h} - E_{op}(u_{3:h}))\right|\right|_{\Omega}^2 \\ &\leqslant C \sum_{i=0}^{1} \sum_{N_{np}\in\mathcal{N}_i : np\in K} h_K^{2(i+1-j)} \left(N_{np}\left(u_{3:h} - E_{op}(u_{3:h})\right)\right)^2 . \end{aligned} \tag{7.84}$$

Now, for each nodal point np which is not on the boundary Γ_c, we consider a local numbering $K_1, \ldots, K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge or a vertex. By recalling the arithmetic-geometric mean inequality (A.15), we get

$$
\begin{aligned}
&\sum_{N_{np}\in\mathcal{N}_0:np\in K} h_K^{2(1-j)}\left(N_{np}\left(u_{3:h}-E_{op}(u_{3:h})\right)\right)^2\\
&=\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)}\left\{u_{3:h}(np)|_K-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}u_{3:h}(np)|_K\right\}^2\\
&+\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_c} h_K^{2(1-j)}\left(u_{3:h}(np)|_K\right)^2\\
&+\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_Q} h_K^{2(1-j)}\left\{u_{3:h}(np)|_K-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}u_{3:h}(np)|_K\right\}^2\\
&\leqslant C\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)}\left\{\sum_{\ell=1}^{|\omega_{np}|-1}\left(u_{3:h}|_{K_\ell}(np)-u_{3:h}|_{K_{\ell+1}}(np)\right)^2\right\}\\
&+\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_c} h_K^{2(1-j)}\left(u_{3:h}(np)|_K\right)^2\\
&+C\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_Q} h_K^{2(1-j)}\left\{\sum_{m=1}^{|\omega_{np}|-1}\left(u_{3:h}|_{K_m}(np)-u_{3:h}|_{K_{m+1}}(np)\right)^2\right\}\\
&=C\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)}\sum_{e\in\mathcal{E}_{\text{int}}}[\![u_{3:h}(np)]\!]^2\\
&+\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_c} h_K^{2(1-j)}\sum_{e\in\mathcal{E}_c}[\![u_{3:h}(np)]\!]^2\\
&+C\sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_Q} h_K^{2(1-j)}\sum_{e\in\mathcal{E}_Q}[\![u_{3:h}(np)]\!]^2.
\end{aligned}
\tag{7.85}
$$

Next in (7.85), owing to the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi-uniform, we obtain

$$\begin{aligned} &C \sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_{\text{int}}} [\![u_{3:h}(np)]\!]^2 \\ &+ \sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_c} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_c} [\![u_{3:h}(np)]\!]^2 \\ &+ C \sum_{N_{np}\in\mathcal{N}_0:np\in K\cap\Gamma_Q} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_Q} [\![u_{3:h}(np)]\!]^2 \\ &\leqslant C \sum_{e\in\mathcal{E}} ||\mathbf{h}^{1-j}[\![u_{3:h}]\!]||^2_{L^\infty(e)}. \end{aligned} \tag{7.86}$$

Then, by applying an inverse inequality in (7.86), we deduce

$$\begin{aligned} C \sum_{e\in\mathcal{E}} ||\mathbf{h}^{1-j}[\![u_{3:h}]\!]||^2_{L^\infty(e)} &\leqslant C \sum_{e\in\mathcal{E}} ||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_e \\ &= C||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_\Gamma. \end{aligned} \tag{7.87}$$

In consequence, from (7.85) – (7.87), we conclude that

$$\sum_{N_{np}\in\mathcal{N}_0:np\in K} h_K^{2(1-j)} \left(N_{np}\left(u_{3:h} - E_{op}(u_{3:h})\right)\right)^2 \leqslant C||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_\Gamma. \tag{7.88}$$

What is more, it's time for us to turn to the nodal variables in $\mathcal{N}_1$. We further split $\mathcal{N}_1$ into

$$\mathcal{N}_1 = \mathcal{N}_1^n \cup \mathcal{N}_1^p,$$

where $\mathcal{N}_1^n$ is the set of the nodal variables of normal derivatives across element edges and $\mathcal{N}_1^p$ is the set of nodal variables representing partial derivatives on elemental vertices.

Therefore, we shall follow arguments in a same way for $\mathcal{N}_1^n$ as in (7.88). For each nodal point np which is not on the boundary Γ_q, we consider a local numbering $K_1, \ldots, K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge or a vertex. Invoking the arithmetic-geometric mean inequality (A.15) derives

$$
\begin{aligned}
&\sum_{N_{np}\in\mathcal{N}_1^n:np\in K} h_K^{2(2-j)}\left(N_{np}\left(u_{3:h}-E_{op}(u_{3:h})\right)\right)^2\\
&=\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\\
&\times\left\{\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)\right\}^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_q} h_K^{2(2-j)}\left\{\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)\right\}^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_M} h_K^{2(2-j)}\\
&\times\left\{\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)\right\}^2\\
&\leqslant C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\\
&\times\left\{\sum_{\ell=1}^{|\omega_{np}|-1}\left(\left(\nabla u_{3:h}\cdot b_{K_\ell}\right)|_{K_\ell}(np)-\left(\nabla u_{3:h}\cdot b_{K_{\ell+1}}\right)|_{K_{\ell+1}}(np)\right)^2\right\}\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_q} h_K^{2(2-j)}\left\{\left(\nabla u_{3:h}\cdot b_K\right)|_K(np)\right\}^2\\
&+C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_M} h_K^{2(2-j)}\\
&\times\left\{\sum_{m=1}^{|\omega_{np}|-1}\left(\left(\nabla u_{3:h}\cdot b_{K_m}\right)|_{K_m}(np)-\left(\nabla u_{3:h}\cdot b_{K_{m+1}}\right)|_{K_{m+1}}(np)\right)^2\right\}\\
&=C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_{\rm int}}[\![\nabla u_{3:h}(np)]\!]^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_q} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_q}[\![\nabla u_{3:h}(np)]\!]^2\\
&+C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_M} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_M}[\![\nabla u_{3:h}(np)]\!]^2.
\end{aligned}
\tag{7.89}
$$

Afterwards in (7.89), in view of the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi-unirform, we get

$$\begin{aligned} &C \sum_{N_{np} \in \mathcal{N}_1^n : np \in K \cap \Gamma_{\text{int}}} h_K^{2(2-j)} \sum_{e \in \mathcal{E}_{\text{int}}} [\![\nabla u_{3:h}(np)]\!]^2 \\ &+ \sum_{N_{np} \in \mathcal{N}_1^n : np \in K \cap \Gamma_q} h_K^{2(2-j)} \sum_{e \in \mathcal{E}_q} [\![\nabla u_{3:h}(np)]\!]^2 \\ &+ C \sum_{N_{np} \in \mathcal{N}_1^n : np \in K \cap \Gamma_M} h_K^{2(2-j)} \sum_{e \in \mathcal{E}_M} [\![\nabla u_{3:h}(np)]\!]^2 \\ &\leqslant C \sum_{e \in \mathcal{E}'} ||\mathbf{h}^{2-j} [\![\nabla u_{3:h}]\!]||^2_{L^\infty(e)}. \end{aligned} \tag{7.90}$$

Also, by using an inverse inequality in (7.90), we obtain

$$\begin{aligned} C \sum_{e \in \mathcal{E}'} ||\mathbf{h}^{2-j} [\![\nabla u_{3:h}]\!]||^2_{L^\infty(e)} &\leqslant C \sum_{e \in \mathcal{E}'} ||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!]||^2_e \\ &= C ||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!]||^2_{\Gamma'}. \end{aligned} \tag{7.91}$$

Ergo, from (7.89) – (7.91), we reach the conclusion that

$$\sum_{N_{np} \in \mathcal{N}_1^n : np \in K} h_K^{2(2-j)} \left(N_{np} \left(u_{3:h} - E_{op}(u_{3:h}) \right) \right)^2 \leqslant C ||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!]||^2_{\Gamma'}. \tag{7.92}$$

Now, we shall follow the above procedure in a similar manner for $\mathcal{N}_1^p$ as both (7.88) and (7.92). For each nodal point np which is not on the boundary Γ_q, we consider a local numbering $K_1, \ldots, K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge or a vertex.

By employing the arithmetic-geometric mean inequality (A.15), we have

$$
\begin{aligned}
&\sum_{N_{np}\in\mathcal{N}_1^p:np\in K} h_K^{2(2-j)}\left(N_{np}\left(u_{3:h}-E_{op}(u_{3:h})\right)\right)^2 \\
&= \sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)} \\
&\times\left\{\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)\right\}^2 \\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_q} h_K^{2(2-j)}\left\{\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)\right\}^2 \\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_M} h_K^{2(2-j)} \\
&\times\left\{\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)\right\}^2 \\
&\leqslant C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)} \\
&\times\left\{\sum_{\ell=1}^{|\omega_{np}|-1}\left(\sum_{z\in\{x,y\}}(u_{3:h})_z|_{K_\ell}(np)-\sum_{z\in\{x,y\}}(u_{3:h})_z|_{K_{\ell+1}}(np)\right)^2\right\} \qquad (7.93)\\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_q} h_K^{2(2-j)}\left\{\sum_{z\in\{x,y\}}(u_{3:h})_z|_K(np)\right\}^2 \\
&+C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_M} h_K^{2(2-j)} \\
&\times\left\{\sum_{m=1}^{|\omega_{np}|-1}\left(\sum_{z\in\{x,y\}}(u_{3:h})_z|_{K_m}(np)-\sum_{z\in\{x,y\}}(u_{3:h})_z|_{K_{m+1}}(np)\right)^2\right\} \\
&= C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_{\text{int}}}\left[\sum_{z\in\{x,y\}}(u_{3:h})_z(np)\right]^2 \\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_q} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_q}\left[\sum_{z\in\{x,y\}}(u_{3:h})_z(np)\right]^2 \\
&+C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_M} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_M}\left[\sum_{z\in\{x,y\}}(u_{3:h})_z(np)\right]^2.
\end{aligned}
$$

Thereafter in (7.93), because of the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi-unirform, we deduce

$$\begin{aligned}
& C \sum_{N_{np}\in\mathcal{N}_1^p : np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)} \sum_{e\in\mathcal{E}_{\text{int}}} \left[\sum_{z\in\{x,y\}} (u_{3:h})_z(np) \right]^2 \\
& + \sum_{N_{np}\in\mathcal{N}_1^p : np\in K\cap\Gamma_q} h_K^{2(2-j)} \sum_{e\in\mathcal{E}_q} \left[\sum_{z\in\{x,y\}} (u_{3:h})_z(np) \right]^2 \\
& + C \sum_{N_{np}\in\mathcal{N}_1^p : np\in K\cap\Gamma_M} h_K^{2(2-j)} \sum_{e\in\mathcal{E}_M} \left[\sum_{z\in\{x,y\}} (u_{3:h})_z(np) \right]^2 \\
& \leqslant C \sum_{e\in\mathcal{E}'} \sum_{z\in\{x,y\}} ||\mathbf{h}^{2-j} [\![(u_{3:h})_z]\!] ||^2_{L^\infty(e)}.
\end{aligned} \tag{7.94}$$

Into the bargain, applying an inverse inequality in (7.94) yields

$$C \sum_{e\in\mathcal{E}'} \sum_{z\in\{x,y\}} ||\mathbf{h}^{2-j} [\![(u_{3:h})_z]\!] ||^2_{L^\infty(e)} \leqslant C \sum_{e\in\mathcal{E}'} \sum_{z\in\{x,y\}} ||\mathbf{h}^{3/2-j} [\![(u_{3:h})_z]\!] ||^2_{e}. \tag{7.95}$$

A last and necessary step is to split the partial derivatives on the right-hand side of (7.95) into normal and tangential components. The triangle inequality and subsequently the inequality (A.14) yield

$$||\mathbf{h}^{3/2-j} [\![(u_{3:h})_z]\!] ||^2_{e} \leqslant 2||\mathbf{h}^{3/2-j} [\![(u_{3:h})_t]\!] ||^2_{e} + 2||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!] ||^2_{e}. \tag{7.96}$$

Then, by using an inverse estimate (A.37) along each edge e for the tangential derivative component, together with the fact that the edges e are straight lines, we eventually conclude

$$\begin{aligned}
& 2||\mathbf{h}^{3/2-j} [\![(u_{3:h})_t]\!] ||^2_{e} + 2||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!] ||^2_{e} \\
& = 2|\mathbf{h}^{3/2-j} [\![u_{3:h}]\!] |^2_{1,e} + 2||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!] ||^2_{e} \\
& \leqslant C||\mathbf{h}^{1/2-j} [\![u_{3:h}]\!] ||^2_{e} + 2||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!] ||^2_{e}.
\end{aligned} \tag{7.97}$$

Hence, (7.96) and (7.97) entail

$$||\mathbf{h}^{3/2-j} [\![(u_{3:h})_z]\!] ||^2_{e} \leqslant C||\mathbf{h}^{1/2-j} [\![u_{3:h}]\!] ||^2_{e} + 2||\mathbf{h}^{3/2-j} [\![\nabla u_{3:h}]\!] ||^2_{e}. \tag{7.98}$$

Wherefore, from (7.93) – (7.95) and (7.98), we arrive to the conclusion

$$\sum_{N_{np}\in\mathcal{N}_1^p:np\in K} h_K^{2(2-j)}\left(N_{np}\left(u_{3:h}-E_{op}(u_{3:h})\right)\right)^2 \leqslant C\left(||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_{\Gamma'}+||\mathbf{h}^{3/2-j}[\![\nabla u_{3:h}]\!]||^2_{\Gamma'}\right). \tag{7.99}$$

After that, gathering the inequalities (7.88), (7.92) together with (7.99) and inserting them on the right-hand side of (7.84), we deduce

$$\left|\left|\mathbf{h}^{-j}(u_{3:h}-E_{op}(u_{3:h}))\right|\right|^2_{\Omega} \leqslant C\left(||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_{\Gamma}+||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_{\Gamma'}+||\mathbf{h}^{3/2-j}[\![\nabla u_{3:h}]\!]||^2_{\Gamma'}\right).$$

or in a more compressed form

$$\left|\left|\mathbf{h}^{-j}(u_{3:h}-E_{op}(u_{3:h}))\right|\right|^2_{\Omega} \leqslant C\left(||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_{\Gamma_{\text{int}}\cup\gamma_0\cup\gamma_1\cup\gamma_2}+||\mathbf{h}^{3/2-j}[\![\nabla u_{3:h}]\!]||^2_{\Gamma_{\text{int}}\cup\gamma_0\cup\gamma_1\cup\gamma_2}\right). \tag{7.100}$$

Finally, insertion of the mathematical expression (7.100) into the right-hand side of (7.83) yields

$$\sum_{K\in\mathcal{T}}|u_{3:h}-E_{op}(u_{3:h})|^2_{j,K} \leqslant C\left(||\mathbf{h}^{1/2-j}[\![u_{3:h}]\!]||^2_{\Gamma_{\text{int}}\cup\gamma_0\cup\gamma_1\cup\gamma_2}+||\mathbf{h}^{3/2-j}[\![\nabla u_{3:h}]\!]||^2_{\Gamma_{\text{int}}\cup\gamma_0\cup\gamma_1\cup\gamma_2}\right),$$

being the desired result. □

Remark 7.7.2.2. *Let $w_{3:h}\in S_h^p$, $w_3\in V_3^0$, $\eta=w_3-w_{3:h}$ and $E_{op}(u_{3:h})\in\tilde{S}_h^{p+2}\cap V_3^0$ be as in Lemma 7.7.2.1. We shall use this notation with intention to decompose the error as follows:*

$$e:=u_3-u_{3:h}=(u_3-E_{op}(u_{3:h}))+(E_{op}(u_{3:h})-u_{3:h})\equiv e^c+e^d. \tag{7.101}$$

Therefore, to establish a reliable a posteriori error estimate of residual type for the interior penalty discontinuous Galerkin method in the corresponding energy seminorm, when the analytical solution u_3 of (7.10) and (7.11) satisfies $u_3\in V_3^0$, we should estimate the terms e^c and e^d.

It is easy to estimate the term e^d by applying the Lemma 7.7.2.1. However, it is extremely difficult to prove an estimate for the term e^c, since it is imperative the inequality

$$|u|_{2,\Omega}\leqslant C||\Delta u||_{\Omega},$$

holds for the boundary value problem (7.10) – (7.11).

In case the above inequality holds for the boundary value problem (7.10) – (7.11), *it is trivial to establish a reliable a posteriori error estimate of residual type for the interior penalty discontinuous Galerkin method in the corresponding energy seminorm.*

Chapter 8

IPDGFEMs for SGE in 2-D

8.1 Preliminaries

Suppose that Ω is a bounded, open, convex domain in $\Re^2$ with boundary Γ_c. Let $\mathcal{T}$ be a subdivision of Ω into disjoint, open, convex elements domains $K = K_j$ such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K},$$

$$K_i \cap K_j = \emptyset \quad \text{for } i \neq j$$

and the intersection $\bar{K}_i \cap \bar{K}_j$ is either empty, a vertex or an edge. We define a piecewise constant mesh function $h_{\mathcal{T}}$ by

$$h_{\mathcal{T}}(x) = h_K = diam(K), \quad x \in K, \quad K \in \mathcal{T}$$

and put

$$h = \max_{K \in \mathcal{T}} h_K.$$

Let $\hat{K}$ be a fixed reference element in $\Re^2$. We shall further assume that each $K \in \mathcal{T}$ is an affine image of the reference element $\hat{K}$

$$K = F_K(\hat{K}), \quad K \in \mathcal{T}.$$

Let $\mathcal{E}$ be the set of all open one-dimensional element faces, associated with the subdivision $\mathcal{T}$. We also define a piecewise constant face-function on $\mathcal{E}$

$$h_{\mathcal{E}}(x) = h_e = diam(e), \quad x \in e, \; e \in \mathcal{E}.$$

Let us assume that the subdivision $\mathcal{T}$ is shape-regular (see either p. 124 in [55] or Remark 2.2, p. 114 in [31] or Definition A.1.7). We note that for a shape-regular family there exists a positive constant c (the shape-regularity constant), independent of h, such that

$$ch_K \leqslant h_e \leqslant h_K, \quad \forall K \in \mathcal{T},\ \forall e \in \partial K,$$

hence, for any element $K \in \mathcal{T}$, h_K and h_e are equal to within a constant.

To each $K \in \mathcal{T}$ we assign a non-negative integer p_K (the local polynomial degree) and a non-negative integer s_K (the local Sobolev space index). Then, we collect the p_K, s_K and F_K in the vectors

$$\mathbf{p} = (p_K : K \in \mathcal{T}),\ \mathbf{s} = (s_K : K \in \mathcal{T}) \text{ and } \mathbf{F} = (F_K : K \in \mathcal{T}).$$

We now return to the set $\mathcal{E}$. We also assume that $\mathcal{E}$ is decomposed into two subsets, namely $\mathcal{E}_{\text{int}}$ and $\mathcal{E}_{\partial}$, which contain the set of all elements of $\mathcal{E}$ that are not subsets of Γ_c, i.e.,

$$\mathcal{E}_{\text{int}} = \{e \in \mathcal{E} : e \subset \Omega\}$$

and the set of all elements of $\mathcal{E}$ that are subsets of Γ_c, i.e.,

$$\mathcal{E}_{\partial} = \{e \in \mathcal{E} : e \subset \Gamma_c\}.$$

For an integer m we define

$$\langle p^m \rangle_{\mathcal{E}}(x) = \langle p^m \rangle_e = \frac{p_K^m + p_{K'}^m}{2}, \quad x \in e,\ e \in \mathcal{E}_{\text{int}},$$

where the elements K and K' share the face e, as well as

$$\langle p^m \rangle_{\mathcal{E}}(x) = \langle p^m \rangle_e = p_K^m, \quad x \in e,\ e \in \mathcal{E}_{\partial},$$

where $e \subset \partial K$.

What is more, we define the set Γ as

$$\Gamma := \bigcup_{e \in \mathcal{E}} e$$

and the set Γ_{int} together with Γ_c as

$$\Gamma_{\text{int}} := \bigcup_{e \in \mathcal{E}_{\text{int}}} e, \quad \Gamma_c := \bigcup_{e \in \mathcal{E}_c} e,$$

all with the obvious meanings respectively.

Let $\Gamma_0 = \Gamma_{\text{int}} \cup \Gamma_c$. We define for $\mathbf{u}, \mathbf{w} \in L^2(\Gamma_0)^2$, the inner product

$$\int_{\Gamma_0} \mathbf{u}\mathbf{w} dr = \int_{\Gamma_{\text{int}}} \mathbf{u}\mathbf{w} dr + \int_{\Gamma_c} \mathbf{u}\mathbf{w} dr \tag{8.1}$$

with associated norm $||\cdot||_{\Gamma_0}$. So, it will hold as well

$$||\mathbf{u}||^2_{\Gamma_0} = ||\mathbf{u}||^2_{\Gamma_{\text{int}}} + ||\mathbf{u}||^2_{\Gamma_c} \tag{8.2}$$

or

$$\sum_{e \in \mathcal{E}_0} ||\mathbf{u}||^2_e = \sum_{e \in \mathcal{E}_{\text{int}}} ||\mathbf{u}||^2_e + \sum_{e \in \mathcal{E}_c} ||\mathbf{u}||^2_e. \tag{8.3}$$

8.2 Model Problem

Toupin and Mindlin included higher-order stresses and strains in the theory of linear elasticity, which serves today as the foundation of more advanced strain gradient elasticity and plasticity formulation [192, 157, 102], respectively. Let us introduce a two-dimensional model problem following their concepts.

Let Ω be a bounded open polygonal domain in $\Re^2$ and Γ_c its boundary. Let also Γ_∂ signify the union of one-dimensional open edges of Ω. The mechanical framework that we consider is strain gradient elasticity (or in other words dipolar gradient elasticity). The material constituting the structure is assumed to be isotropic, centrosymmetric and simplified.

We consider the equation:

$$\partial_j \left(\tau_{jk} - \partial_i \mu_{ijk}\right) + f_k - \partial_j \Phi_{jk} = 0, \tag{8.4}$$

where τ_{ij} denotes the components of the (symmetric) Cauchy stress tensor, μ_{ijk} the components of the double stress tensor, recalling the symmetry conditions

$$\mu_{ijk} = \mu_{jik} \quad \text{(Form I)},$$
$$\mu_{ijk} = \mu_{ikj} \quad \text{(Form II)},$$

f_k is a given body force per unit surface and Φ_{jk} a given body double force per unit surface, respectively.

The constitutive equations for the Cauchy stress tensor τ_{ij} and the double stress tensor μ_{ijk} can be expressed as

$$\tau_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \tag{8.5}$$

$$\mu_{ijk} = g^2 \left(2\mu\kappa_{ijk} + \lambda\kappa_{irr}\delta_{jk}\right), \tag{8.6}$$

where λ, μ are the Lamé constants, g is a length scale (which represents material length related to the volumetric elastic strain energy) and δ_{ij} is the Kronecker symbol. We also denote by

$$\varepsilon_{ij} := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) := u_{(i,j)} \tag{8.7}$$

the components of the strain tensor and by

$$\kappa_{ijk} := \partial_i \partial_j u_k \quad \text{(Form I)} \tag{8.8}$$

or by

$$\kappa_{ijk} := \partial_i \varepsilon_{jk} \quad \text{(Form II)} \tag{8.9}$$

the components of the second gradient of displacement field or of the first gradient of strain tensor, respectively.

We can rewrite (8.4) with (8.5) and (8.6) as:

$$(\lambda + \mu)\left\{\left(D^2\right)^2 \mathbf{u} - D^2\mathbf{u}\right\} + \mu\left(g^2\Delta^2\mathbf{u} - \Delta\mathbf{u}\right) = \mathbf{f} - \Phi\nabla \quad \text{in } \Omega$$

or

$$(\lambda + \mu)D^2\left(g^2 D^2\mathbf{u} - \mathbf{u}\right) + \mu\Delta\left(g^2\Delta\mathbf{u} - \mathbf{u}\right) = \mathbf{f} - \Phi\nabla \quad \text{in } \Omega, \tag{8.10}$$

where $\mathbf{f} - \Phi\nabla \in L^2(\Omega)^2$. In the above, D^2 is the symmetric Hessian matrix, Δ^2 is the biharmonic operator, Δ is the Laplace operator and $\mathbf{u}$ denotes the displacement field. In addition, we supplement the equation with the following boundary conditions

$$\begin{aligned} \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_c, \\ \nabla\mathbf{u}\cdot n &= \mathbf{0} \quad \text{on } \Gamma_c, \end{aligned} \tag{8.11}$$

where n is the unit normal to the boundary, exterior to Ω.

We mention that the boundary conditions are called homogeneous essential.

Also notice that by construction Γ_c differs from Γ_∂ on a set of one-dimensional measure zero which contains the vertices of the (polygonal) boundary of Ω.

8.3 Weak Formulation

We are ready to derive the weak formulation for the problem (8.10) – (8.11), which will lead to the discontinuous Galerkin finite element method. We shall assume for the moment that the solution $\mathbf{u}$ of the problem is a sufficiently smooth function.

For each face $e \in \mathcal{E}_{\text{int}}$, let i and j be such indices that $i > j$ and the elements $K := K_i$ and $K' := K_j$ share the face e. Let us define the jump across e and the mean value on e of $\mathbf{u} \in H^1(\Omega, \mathcal{T})^2$ by

$$[\![\mathbf{u}]\!]_e := \mathbf{u}|_{\partial K \cap e} - \mathbf{u}|_{\partial K' \cap e} \quad \text{and} \quad \langle \mathbf{u} \rangle_e := \frac{1}{2}\left(\mathbf{u}|_{\partial K \cap e} + \mathbf{u}|_{\partial K' \cap e}\right),$$

respectively.

For the sake of convenience, we extend the definitions of the jump and of the mean value to faces $e \in \mathcal{E}_\partial$ by letting:

$$[\![\mathbf{u}]\!]_e = \mathbf{u}|_e \quad \text{and} \quad \langle \mathbf{u} \rangle_e = \mathbf{u}|_e.$$

In these definitions, the subscript e will be supressed when no confusion is likely to occur. With each face $e \in \mathcal{E}_{\text{int}}$ we associate the unit normal vector $n = n_{K_i}$ to e, pointing from element K_i to K_j when $i > j$, and with each $e \in \mathcal{E}_\partial$ we associate the external unit normal vector $n = n_K$, where $e \subset \partial K$.

Since the method will be non-conforming, we shall use the broken Sobolev space $H^4(\Omega, \mathcal{T})^2$ as trial space. We multiply the equation, (8.10), by a test function $\mathbf{w} \in H^4(\Omega, \mathcal{T})^2$ and integrate over Ω

$$\int_\Omega (\lambda + \mu) D^2 \left(g^2 D^2 \mathbf{u} - \mathbf{u}\right) \mathbf{w} dv + \int_\Omega \mu \Delta \left(g^2 \Delta \mathbf{u} - \mathbf{u}\right) \mathbf{w} dv$$
$$= \int_\Omega (\mathbf{f} - \Phi \nabla) \mathbf{w} dv.$$

Afterwards, we split the integrals

$$\sum_{K\in\mathcal{T}}\int_K (\lambda+\mu)D^2\left(g^2D^2\mathbf{u}-\mathbf{u}\right)\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K \mu\Delta\left(g^2\Delta\mathbf{u}-\mathbf{u}\right)\mathbf{w}dv \\ =\sum_{K\in\mathcal{T}}\int_K\left(\mathbf{f}-\Phi\nabla\right)\mathbf{w}dv \tag{8.12}$$

and applying the Green theorem on every elemental integral, using the anti-clockwise orientation, so we get

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K (\lambda+\mu)g^2D^2\mathbf{u}D^2\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)\nabla\mathbf{u}\cdot\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_K \mu g^2\Delta\mathbf{u}\Delta\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K\mu\nabla\mathbf{u}:\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr-\sum_{K\in\mathcal{T}}\int_{\partial K}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr+\sum_{K\in\mathcal{T}}\int_{\partial K}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr-\sum_{K\in\mathcal{T}}\int_{\partial K}(\mu\nabla\mathbf{u}\cdot n)\mathbf{w}dr\\
&=\sum_{K\in\mathcal{T}}\int_K\left(\mathbf{f}-\Phi\nabla\right)\mathbf{w}dv,
\end{aligned}$$

where n denotes the outward normal to each element edge.

Now, we split the boundary terms as follows

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)g^2D^2\mathbf{u}D^2\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)\nabla\mathbf{u}\cdot\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_K\mu g^2\Delta\mathbf{u}\Delta\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K\mu\nabla\mathbf{u}:\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr+\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr-\sum_{K\in\mathcal{T}}\int_{\partial K\cap\Gamma_c}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr\\
&=\sum_{K\in\mathcal{T}}\int_K\left(\mathbf{f}-\Phi\nabla\right)\mathbf{w}dv,
\end{aligned}
$$

and hence we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)g^2D^2\mathbf{u}D^2\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)\nabla\mathbf{u}\cdot\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_K\mu g^2\Delta\mathbf{u}\Delta\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K\mu\nabla\mathbf{u}:\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr+\int_{\Gamma_c}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr-\int_{\Gamma_c}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr-\int_{\Gamma_c}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr\\
&+\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr+\int_{\Gamma_c}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr-\int_{\Gamma_c}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\sum_{K\in\mathcal{T}}\int_{\partial K\setminus\Gamma_\partial}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr-\int_{\Gamma_c}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr=\sum_{K\in\mathcal{T}}\int_K\left(\mathbf{f}-\Phi\nabla\right)\mathbf{w}dv.
\end{aligned}\tag{8.13}$$

The fifth, the seventh, the ninth, the eleventh, the thirteenth and the fifteenth term respectively on the left-hand side of (8.13) contain the boundary integrals over the interior element edges, i.e. the edges $e\in\Gamma_{\text{int}}$. Consequently, in this sum of boundary integrals, we have two integrals over every interior edge.

Remark 8.3.0.1. *Let us note that, for a given face $e\in\mathcal{E}_{int}$ shared by two adjacent elements K_i and K_j $(i>j)$, we can write*

$$(\nabla\mathbf{u}_{K_i}\cdot n_{K_i})\mathbf{w}_{K_i}+(\nabla\mathbf{u}_{K_j}\cdot n_{K_j})\mathbf{w}_{K_j}=(\nabla\mathbf{u}_{K_i}\cdot n)\mathbf{w}_{K_i}-(\nabla\mathbf{u}_{K_j}\cdot n)\mathbf{w}_{K_j}$$

Hence, by analogy with the formula

$$ac-bd=\frac{1}{2}(a+b)(c-d)+\frac{1}{2}(a-b)(c+d)\quad\forall a,b,c,d\in\Re,$$

we get

$$\begin{aligned}(\nabla\mathbf{u}_{K_i}\cdot n_{K_i})\mathbf{w}_{K_i}+(\nabla\mathbf{u}_{K_j}\cdot n_{K_j})\mathbf{w}_{K_j}\;&=\;\langle\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]+[\![\nabla\mathbf{u}\cdot n]\!]\langle\mathbf{w}\rangle\\
&\qquad\forall\mathbf{u},\mathbf{w}\in H^1(\Omega,\mathcal{T})^2.\end{aligned}\tag{8.14}$$

In order to evaluate these integrals, we always use the interior trace of the test function $\mathbf{w}$. Taking into account the Remark 8.3.0.1 (together with the orientation convention that we have adopted) and applying (8.14), we can see that the fifth, the seventh, the ninth, the eleventh, the thirteenth and the fifteenth term respectively, on the left-hand side of (8.13), can be rewritten as follows

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)g^2D^2\mathbf{u}D^2\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)\nabla\mathbf{u}\cdot\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_K\mu g^2\Delta\mathbf{u}\Delta\mathbf{w}dv+\sum_{K\in\mathcal{T}}\int_K\mu\nabla\mathbf{u}:\nabla\mathbf{w}dv\\
&+\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr+\int_{\Gamma_{\text{int}}}[\![(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n]\!]\langle\mathbf{w}\rangle dr\\
&+\int_{\Gamma_c}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr-\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)g^2D^2\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr\\
&-\int_{\Gamma_{\text{int}}}[\![(\lambda+\mu)g^2D^2\mathbf{u}]\!]\langle\nabla\mathbf{w}\cdot n\rangle dr-\int_{\Gamma_c}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr-\int_{\Gamma_{\text{int}}}[\![(\lambda+\mu)\nabla\mathbf{u}\cdot n]\!]\langle\mathbf{w}\rangle dr \qquad (8.15)\\
&-\int_{\Gamma_c}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr+\int_{\Gamma_{\text{int}}}\langle\mu g^2\nabla\Delta\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr\\
&+\int_{\Gamma_{\text{int}}}[\![\mu g^2\nabla\Delta\mathbf{u}\cdot n]\!]\langle\mathbf{w}\rangle dr+\int_{\Gamma_c}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr\\
&-\int_{\Gamma_{\text{int}}}\langle\mu g^2\Delta\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr-\int_{\Gamma_{\text{int}}}[\![\mu g^2\Delta\mathbf{u}]\!]\langle\nabla\mathbf{w}\cdot n\rangle dr\\
&-\int_{\Gamma_c}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr-\int_{\Gamma_{\text{int}}}\langle\mu\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr\\
&-\int_{\Gamma_{\text{int}}}[\![\mu\nabla\mathbf{u}\cdot n]\!]\langle\mathbf{w}\rangle dr-\int_{\Gamma_c}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr=\sum_{K\in\mathcal{T}}\int_K(\mathbf{f}-\Phi\nabla)\,\mathbf{w}dv.
\end{aligned}$$

By noting that the fluxes

$$(\lambda+\mu)\left(g^2\nabla D^2\mathbf{u}\cdot n-\nabla\mathbf{u}\cdot n\right)+\mu\left(g^2\nabla\Delta\mathbf{u}\cdot n-\nabla\mathbf{u}\cdot n\right)$$

and

$$(\lambda+\mu)g^2D^2\mathbf{u}+\mu g^2\Delta\mathbf{u}$$

are continuous across the element faces $e \in \mathcal{E}_{\text{int}}$ (e.g., when the exact solution $\mathbf{u} \in H^4(\Omega)^2$), we have

$$\int_{\Gamma_{\text{int}}} [\![(\lambda+\mu)\left(g^2\nabla D^2\mathbf{u}\cdot n - \nabla\mathbf{u}\cdot n\right) + \mu\left(g^2\nabla\Delta\mathbf{u}\cdot n - \nabla\mathbf{u}\cdot n\right)]\!]\langle\mathbf{w}\rangle dr = 0,$$

$$\int_{\Gamma_{\text{int}}} [\![(\lambda+\mu)g^2D^2\mathbf{u} + \mu g^2\Delta\mathbf{u}]\!]\langle\nabla\mathbf{w}\cdot n\rangle dr = 0 \quad \forall\mathbf{w}\in H^4(\Omega,\mathcal{T})^2.$$

Then, (8.15) reduces to

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)g^2D^2\mathbf{u}D^2\mathbf{w}dv + \sum_{K\in\mathcal{T}}\int_K(\lambda+\mu)\nabla\mathbf{u}\cdot\nabla\mathbf{w}dv\\
&+\sum_{K\in\mathcal{T}}\int_K\mu g^2\Delta\mathbf{u}\Delta\mathbf{w}dv + \sum_{K\in\mathcal{T}}\int_K\mu\nabla\mathbf{u}:\nabla\mathbf{w}dv\\
&+\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr + \int_{\Gamma_c}(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\mathbf{w}dr\\
&-\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)g^2D^2\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr - \int_{\Gamma_c}(\lambda+\mu)g^2D^2\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\int_{\Gamma_{\text{int}}}\langle(\lambda+\mu)\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr - \int_{\Gamma_c}(\lambda+\mu)\nabla\mathbf{u}\cdot n\mathbf{w}dr\\
&+\int_{\Gamma_{\text{int}}}\langle\mu g^2\nabla\Delta\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr + \int_{\Gamma_c}\mu g^2\nabla\Delta\mathbf{u}\cdot n\mathbf{w}dr\\
&-\int_{\Gamma_{\text{int}}}\langle\mu g^2\Delta\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr - \int_{\Gamma_c}\mu g^2\Delta\mathbf{u}(\nabla\mathbf{w}\cdot n)dr\\
&-\int_{\Gamma_{\text{int}}}\langle\mu\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr - \int_{\Gamma_c}\mu\nabla\mathbf{u}\cdot n\mathbf{w}dr = \sum_{K\in\mathcal{T}}\int_K\left(\mathbf{f}-\Phi\nabla\right)\mathbf{w}dv.
\end{aligned} \tag{8.16}$$

Next, we multiply the boundary condition $\mathbf{u}=\mathbf{0}$, on Γ_c, by

$$\begin{aligned}
&-\theta(\lambda+\mu)g^2\nabla D^2\mathbf{w}\cdot n + \gamma_c\mathbf{w}, \quad \theta(\lambda+\mu)\nabla\mathbf{w}\cdot n + \xi_c\mathbf{w},\\
&-\theta\mu g^2\nabla\Delta\mathbf{w}\cdot n + \alpha_c\mathbf{w}, \quad \theta\mu\nabla\mathbf{w}\cdot n + \delta_c\mathbf{w}.
\end{aligned}$$

Then, integrating over Γ_c, we get

$$\begin{aligned} -\int_{\Gamma_c} \theta(\lambda+\mu) g^2 \nabla D^2 \mathbf{w} \cdot n \mathbf{u} dr + \int_{\Gamma_c} \gamma_c \mathbf{u}\mathbf{w} dr = 0, \\ \int_{\Gamma_c} \theta(\lambda+\mu) \nabla \mathbf{w} \cdot n \mathbf{u} dr + \int_{\Gamma_c} \xi_c \mathbf{u}\mathbf{w} = 0, \\ -\int_{\Gamma_c} \theta\mu g^2 \nabla \Delta \mathbf{w} \cdot n \mathbf{u} dr + \int_{\Gamma_c} \alpha_c \mathbf{u}\mathbf{w} dr = 0, \\ \int_{\Gamma_c} \theta\mu \nabla \mathbf{w} \cdot n \mathbf{u} dr + \int_{\Gamma_c} \delta_c \mathbf{u}\mathbf{w} = 0, \end{aligned} \tag{8.17}$$

where θ is the symmetrization parameter. We restrict ourselves to the case $\theta \in \{-1, 1\}$. The non-negative piecewise continuous functions γ_c, ξ_c, α_c and δ_c, defined on Γ_c, are referred to as the stabilization parameters.

In addition, $\mathbf{u}$ is continuous on Ω, in that case the jump $[\![\mathbf{u}]\!]$ vanishes, i.e., $[\![\mathbf{u}]\!] = 0$. If we choose

$$\begin{aligned} &-\theta\langle(\lambda+\mu) g^2 \nabla D^2 \mathbf{w} \cdot n\rangle + \gamma[\![\mathbf{w}]\!], \quad \theta\langle(\lambda+\mu)\nabla \mathbf{w} \cdot n\rangle + \xi[\![\mathbf{w}]\!], \\ &-\theta\langle\mu g^2 \nabla \Delta \mathbf{w} \cdot n\rangle + \alpha[\![\mathbf{w}]\!], \quad \theta\langle\mu \nabla \mathbf{w} \cdot n\rangle + \delta[\![\mathbf{w}]\!] \end{aligned}$$

as test functions and integrate over Γ_{int}, we shall deduce

$$\begin{aligned} -\int_{\Gamma_{\text{int}}} \theta\langle(\lambda+\mu) g^2 \nabla D^2 \mathbf{w} \cdot n\rangle [\![\mathbf{u}]\!] dr + \int_{\Gamma_{\text{int}}} \gamma [\![\mathbf{u}]\!][\![\mathbf{w}]\!] dr = 0, \\ \int_{\Gamma_{\text{int}}} \theta\langle(\lambda+\mu) \nabla \mathbf{w} \cdot n\rangle [\![\mathbf{u}]\!] dr + \int_{\Gamma_{\text{int}}} \xi [\![\mathbf{u}]\!][\![\mathbf{w}]\!] = 0, \\ -\int_{\Gamma_{\text{int}}} \theta\langle\mu g^2 \nabla \Delta \mathbf{w} \cdot n\rangle [\![\mathbf{u}]\!] dr + \int_{\Gamma_{\text{int}}} \alpha [\![\mathbf{u}]\!][\![\mathbf{w}]\!] dr = 0, \\ \int_{\Gamma_{\text{int}}} \theta\langle\mu \nabla \mathbf{w} \cdot n\rangle [\![\mathbf{u}]\!] dr + \int_{\Gamma_{\text{int}}} \delta [\![\mathbf{u}]\!][\![\mathbf{w}]\!] = 0, \end{aligned} \tag{8.18}$$

where γ, ξ, α and δ are non-negative piecewise continuous functions, defined on Γ_{int}, which are referred to as stabilization parameters.

Moreover, from the boundary condition $\nabla \mathbf{u} \cdot n = \mathbf{0}$, on Γ_c, upon multiplying by

$$\theta(\lambda+\mu) g^2 D^2 \mathbf{w} + \zeta_c \nabla \mathbf{w} \cdot n, \quad \theta\mu g^2 \Delta \mathbf{w} + \beta_q \nabla \mathbf{w} \cdot n$$

and integrating over Γ_c, we have

$$\begin{aligned}\int_{\Gamma_c} \theta(\lambda+\mu)g^2 D^2\mathbf{w}(\nabla\mathbf{u}\cdot n)dr + \int_{\Gamma_c} \zeta_c \nabla\mathbf{u}\cdot n\nabla\mathbf{w}\cdot n dr = 0,\\ \int_{\Gamma_c} \theta\mu g^2 \Delta\mathbf{w}(\nabla\mathbf{u}\cdot n)dr + \int_{\Gamma_c} \beta_c \nabla\mathbf{u}\cdot n\nabla\mathbf{w}\cdot n dr = 0.\end{aligned} \tag{8.19}$$

The non-negative piecewise continuous functions ζ_c and β_c, defined on Γ_c, are referred to as the stabilization parameters.

To boot, $\nabla\mathbf{u}\cdot n$ is continuous on Ω, then it follows that the jump $[\![\nabla\mathbf{u}\cdot n]\!]$ vanishes, i.e. $[\![\nabla\mathbf{u}\cdot n]\!] = 0$. If we choose

$$\theta\langle(\lambda+\mu)g^2 D^2\mathbf{w}\rangle + \zeta[\![\nabla\mathbf{w}\cdot n]\!], \quad \theta\langle\mu g^2\Delta\mathbf{w}\rangle + \beta[\![\nabla\mathbf{w}\cdot n]\!]$$

as test functions and integrate over Γ_{int}, it will give

$$\begin{aligned}\int_{\Gamma_{\text{int}}} \theta\langle(\lambda+\mu)g^2 D^2\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr + \int_{\Gamma_{\text{int}}} \zeta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr = 0,\\ \int_{\Gamma_{\text{int}}} \theta\langle\mu g^2\Delta\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr + \int_{\Gamma_{\text{int}}} \beta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr = 0,\end{aligned} \tag{8.20}$$

where ζ and β are non-negative continuous functions, defined on Γ_{int}, which are referred to as the stabilization parameters.

Now adding (8.16) – (8.20) and using $\Gamma_0 = \Gamma_{\text{int}} \cup \Gamma_c$, we get the discontinuous Galerkin weak formulation of the problem in a more compressed form

$$
\begin{aligned}
&\int_{\Omega}(\lambda+\mu)g^2 D_h^2\mathbf{u}D_h^2\mathbf{w}dv+\int_{\Omega}(\lambda+\mu)\nabla_h\mathbf{u}\cdot\nabla_h\mathbf{w}dv\\
&+\int_{\Omega}\mu g^2\Delta_h\mathbf{u}\Delta_h\mathbf{w}dv+\int_{\Omega}\mu\nabla_h\mathbf{u}:\nabla_h\mathbf{w}dv\\
&+\int_{\Gamma_0}\langle(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr-\int_{\Gamma_0}\theta\langle(\lambda+\mu)g^2\nabla D^2\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr\\
&-\int_{\Gamma_0}\langle(\lambda+\mu)g^2 D^2\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr+\int_{\Gamma_0}\theta\langle(\lambda+\mu)g^2 D^2\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr\\
&-\int_{\Gamma_0}\langle(\lambda+\mu)\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr+\int_{\Gamma_0}\theta\langle(\lambda+\mu)\nabla\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr\\
&+\int_{\Gamma_0}\langle\mu g^2\nabla\Delta\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr-\int_{\Gamma_0}\theta\langle\mu g^2\nabla\Delta\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr\\
&-\int_{\Gamma_0}\langle\mu g^2\Delta\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr+\int_{\Gamma_0}\theta\langle\mu g^2\Delta\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr\\
&-\int_{\Gamma_0}\langle\mu\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr+\int_{\Gamma_0}\theta\langle\mu\nabla\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr\\
&+\int_{\Gamma_0}\gamma[\![\mathbf{u}]\!][\![\mathbf{w}]\!]+\int_{\Gamma_0}\zeta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr+\int_{\Gamma_0}\xi[\![\mathbf{u}]\!][\![\mathbf{w}]\!]\\
&+\int_{\Gamma_0}\alpha[\![\mathbf{u}]\!][\![\mathbf{w}]\!]dr+\int_{\Gamma_0}\beta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr+\int_{\Gamma_0}\delta[\![\mathbf{u}]\!][\![\mathbf{w}]\!]\\
&=\int_{\Omega}(\mathbf{f}-\Phi\nabla)\,\mathbf{w}dv,
\end{aligned}
\tag{8.21}
$$

where D_h^2 defines the broken Hessian matrix, ∇_h defines the broken divergence (second integral) as well as the broken gradient (fourth integral) and Δ_h defines the broken Laplacian with respect to the subdivision $\mathcal{T}$, respectively.

The bilinear form $B_{sg}(\cdot,\cdot)$ is defined as

$$\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{w}) &:= \int_\Omega (\lambda+\mu)g^2 D_h^2\mathbf{u} D_h^2\mathbf{w}dv + \int_\Omega (\lambda+\mu)\nabla_h\mathbf{u}\cdot\nabla_h\mathbf{w}dv \\
&+ \int_\Omega \mu g^2\Delta_h\mathbf{u}\Delta_h\mathbf{w}dv + \int_\Omega \mu\nabla_h\mathbf{u} : \nabla_h\mathbf{w}dv \\
&+ \int_{\Gamma_0} \langle(\lambda+\mu)g^2\nabla D^2\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr - \int_{\Gamma_0} \theta\langle(\lambda+\mu)g^2\nabla D^2\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr \\
&- \int_{\Gamma_0} \langle(\lambda+\mu)g^2 D^2\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr + \int_{\Gamma_0} \theta\langle(\lambda+\mu)g^2 D^2\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr \\
&- \int_{\Gamma_0} \langle(\lambda+\mu)\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr + \int_{\Gamma_0} \theta\langle(\lambda+\mu)\nabla\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr \\
&+ \int_{\Gamma_0} \langle\mu g^2\nabla\Delta\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr - \int_{\Gamma_0} \theta\langle\mu g^2\nabla\Delta\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr \\
&- \int_{\Gamma_0} \langle\mu g^2\Delta\mathbf{u}\rangle[\![\nabla\mathbf{w}\cdot n]\!]dr + \int_{\Gamma_0} \theta\langle\mu g^2\Delta\mathbf{w}\rangle[\![\nabla\mathbf{u}\cdot n]\!]dr \\
&- \int_{\Gamma_0} \langle\mu\nabla\mathbf{u}\cdot n\rangle[\![\mathbf{w}]\!]dr + \int_{\Gamma_0} \theta\langle\mu\nabla\mathbf{w}\cdot n\rangle[\![\mathbf{u}]\!]dr \\
&+ \int_{\Gamma_0} \gamma[\![\mathbf{u}]\!][\![\mathbf{w}]\!] + \int_{\Gamma_0} \zeta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr + \int_{\Gamma_0} \xi[\![\mathbf{u}]\!][\![\mathbf{w}]\!] \\
&+ \int_{\Gamma_0} \alpha[\![\mathbf{u}]\!][\![\mathbf{w}]\!]dr + \int_{\Gamma_0} \beta[\![\nabla\mathbf{u}\cdot n]\!][\![\nabla\mathbf{w}\cdot n]\!]dr + \int_{\Gamma_0} \delta[\![\mathbf{u}]\!][\![\mathbf{w}]\!].
\end{aligned} \tag{8.22}$$

We introduce the linear functional $L_{sg}(\cdot)$ on $H^4(\Omega,\mathcal{T})^2$

$$L_{sg}(\mathbf{w}) := \int_\Omega (\mathbf{f} - \Phi\nabla)\,\mathbf{w}dv. \tag{8.23}$$

The stabilization parameters γ, ζ, ξ, α, β and δ depend on the discretization parameters h and p for the hp-method, in a manner that will be specified later in the text.

Then the broken weak formulation of the problem (8.10) – (8.11) reads as follows:

$$\text{Find } \mathbf{u} \in bSs \text{ such that } B_{sg}(\mathbf{u},\mathbf{w}) = L_{sg}(\mathbf{w}) \quad \forall\mathbf{w}\in H^4(\Omega,\mathcal{T})^2, \tag{8.24}$$

where by bSs we denote the following function space

$$\begin{aligned} bSs &= \Big\{ \mathbf{u} \in H^4(\Omega, \mathcal{T})^2 : \mathbf{u}, \nabla \mathbf{u} \cdot n, \\ &\quad (\lambda + \mu) \left(g^2 \nabla D^2 \mathbf{u} \cdot n - \nabla \mathbf{u} \cdot n \right) + \mu \left(g^2 \nabla \Delta \mathbf{u} \cdot n - \nabla \mathbf{u} \cdot n \right), \\ &\quad (\lambda + \mu) g^2 D^2 \mathbf{u} + \mu g^2 \Delta \mathbf{u} \text{ are continuous across } e \in \mathcal{E}_{\text{int}} \Big\}. \end{aligned}$$

Note that for $\theta = -1$ the bilinear form $B_{sg}(\cdot, \cdot)$ is symmetric, whereas for $\theta = 1$ it is not symmetric.

We shall associate with the bilinear form $B_{sg}(\cdot, \cdot)$ the energy seminorm, $|||\cdot|||_{sg}$, defined by

$$\begin{aligned} |||\mathbf{u}|||_{sg} &= \Big(||\{(\lambda + \mu) g^2\}^{1/2} D_h^2 \mathbf{u}||_\Omega^2 + ||(\lambda + \mu)^{1/2} \nabla_h \mathbf{u}||_\Omega^2 \\ &\quad + ||(\mu g^2)^{1/2} \Delta_h \mathbf{u}||_\Omega^2 + ||\mu^{1/2} \nabla_h \mathbf{u}||_\Omega^2 \\ &\quad + ||\gamma^{1/2} [\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2} [\![\nabla \mathbf{u}]\!]||_{\Gamma_0}^2 + ||\xi^{1/2} [\![\mathbf{u}]\!]||_{\Gamma_0}^2 \\ &\quad + ||\alpha^{1/2} [\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta^{1/2} [\![\nabla \mathbf{u}]\!]||_{\Gamma_0}^2 + ||\delta^{1/2} [\![\mathbf{u}]\!]||_{\Gamma_0}^2 \Big)^{1/2}, \\ &\quad \mathbf{u} \in H^2(\Omega, \mathcal{T})^2. \end{aligned} \tag{8.25}$$

Notice that $[\![\nabla \mathbf{u}]\!] \equiv [\![\nabla \mathbf{u} \cdot n]\!]$ and $\langle \nabla \mathbf{u} \rangle \equiv \langle \nabla \mathbf{u} \cdot n \rangle$.

Proposition 8.3.0.2. *If $\gamma, \zeta, \xi, \alpha, \beta, \delta > 0$, then $|||\cdot|||_{sg}$ is a seminorm on $H^2(\Omega, \mathcal{T})^2$.*

We note in passing that since $H^4(\Omega, \mathcal{T})^2 \subset H^2(\Omega, \mathcal{T})^2$, then $|||\cdot|||_{sg}$ is also a seminorm on $H^4(\Omega, \mathcal{T})^2$.

8.4 Finite Element Spaces

In this section, we will consider the finite-dimensional subspace of the broken Sobolev space $H^4(\Omega, \mathcal{T})^2$ which is used in the finite element approximation of the problem.

For a non-negative integer p, we denote by $\mathcal{Q}_p(\hat{K})$ the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. We collect the h_K and p_K into the element-wise constant functions

$$\mathbf{h}, \mathbf{p} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K \text{ and } \mathbf{p}|_K = p_K, \; K \in \mathcal{T},$$

respectively. We consider the finite element space

$$\mathcal{S}_1 \equiv \mathcal{S}^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})^2 := \left\{ \mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}|_K \circ F_K \in \mathcal{Q}_{p_K}(\hat{K})^2,\ K \in \mathcal{T} \right\}. \tag{8.26}$$

We shall assume throughout that the mesh size function $\mathbf{h}$ and polynomial degree function $\mathbf{p}$, with $p_K \geqslant 2$ for each $K \in \mathcal{T}$, have bounded local variation (see Remark A.3.5). What's more, we will refer to the functions in $\mathcal{S}_1$ as test functions. We note that the test functions are discontinuous along the edges of the mesh.

8.5 DGFEM with Lifting Operators

We would like to present the interior penalty discontinuous Galerkin method by using appropriate lifting operators for the problem (8.10) – (8.11). We shall employ the weak formulation which derives in Section 8.3 and the finite element space $\mathcal{S}_1$ constructed in the above section.

Let us first introduce the following functional space

$$H_0^2(\Omega)^2 = \left\{ \mathbf{u} |\ \mathbf{u} \in H^2(\Omega)^2 : \mathbf{u} = \mathbf{0},\ \nabla \mathbf{u} \cdot n = \mathbf{0} \text{ on } \Gamma_c \right\}, \tag{8.27}$$

which is equipped with the norm induced by the Sobolev space $H^2(\Omega)^2$.

Next, we introduce the lifting operators $\mathcal{L}_i : \mathcal{S}^2 := \mathcal{S}_1 + H_0^2(\Omega)^2 \to \mathcal{S}_1$, $i = 1, 2, 3, 4$ by

$$\int_\Omega \mathcal{L}_1(\mathbf{u}) \mathbf{w} dv = \int_{\Gamma_0} \left([\![\mathbf{u}]\!] \langle \nabla \mathbf{w} \rangle - \langle \mathbf{w} \rangle [\![\nabla \mathbf{u}]\!] \right) dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{8.28}$$

$$\int_\Omega \mathcal{L}_2(\mathbf{u}) \mathbf{w} dv = \int_{\Gamma_0} [\![\mathbf{u}]\!] \langle \mathbf{w} \rangle dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{8.29}$$

$$\int_\Omega \mathcal{L}_3(\mathbf{u}) \mathbf{w} dv = \int_{\Gamma_0} \left([\![\mathbf{u}]\!] \langle \nabla \mathbf{w} \rangle - \langle \mathbf{w} \rangle [\![\nabla \mathbf{u}]\!] \right) dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{8.30}$$

and

$$\int_\Omega \mathcal{L}_4(\mathbf{u}) \mathbf{w} dv = \int_{\Gamma_0} [\![\mathbf{u}]\!] \langle \mathbf{w} \rangle dr \quad \forall \mathbf{w} \in \mathcal{S}_1. \tag{8.31}$$

Now, we can rewrite the discontinuous Galerkin weak formulation, (8.21), of the problem (8.10) – (8.11), by employing the lifting operators $\mathcal{L}_i$, as

$$\begin{aligned}
&\int_\Omega (\lambda+\mu) g^2 D_h^2 \mathbf{u} D_h^2 \mathbf{w} dv + \int_\Omega (\lambda+\mu)\nabla_h \mathbf{u}\cdot\nabla_h \mathbf{w} dv \\
&+ \int_\Omega \mu g^2 \Delta_h \mathbf{u}\Delta_h \mathbf{w} dv + \int_\Omega \mu \nabla_h \mathbf{u} : \nabla_h \mathbf{w} dv \\
&+ \int_\Omega \left\{(\lambda+\mu) g^2 D_h^2 \mathbf{u}\mathcal{L}_1(\mathbf{w}) - \theta\mathcal{L}_1(\mathbf{u})(\lambda+\mu) g^2 D_h^2 \mathbf{w}\right\} dv \\
&- \int_\Omega \left\{(\lambda+\mu)\nabla_h \mathbf{u}\mathcal{L}_2(\mathbf{w}) - \theta\mathcal{L}_2(\mathbf{u})(\lambda+\mu)\nabla_h \mathbf{w}\right\} dv \\
&+ \int_\Omega \left\{\mu g^2 \Delta_h \mathbf{u}\mathcal{L}_3(\mathbf{w}) - \theta\mathcal{L}_3(\mathbf{u})\mu g^2 \Delta_h \mathbf{w}\right\} dv \\
&- \int_\Omega \left\{\mu \nabla_h \mathbf{u}\mathcal{L}_4(\mathbf{w}) - \theta\mathcal{L}_4(\mathbf{u})\mu\nabla_h \mathbf{w}\right\} dv + \int_{\Gamma_0} \gamma [\![\mathbf{u}]\!][\![\mathbf{w}]\!] \\
&+ \int_{\Gamma_0} \zeta [\![\nabla \mathbf{u}\cdot n]\!][\![\nabla \mathbf{w}\cdot n]\!] dr + \int_{\Gamma_0} \xi [\![\mathbf{u}]\!][\![\mathbf{w}]\!] + \int_{\Gamma_0} \alpha [\![\mathbf{u}]\!][\![\mathbf{w}]\!] dr \\
&+ \int_{\Gamma_0} \beta [\![\nabla \mathbf{u}\cdot n]\!][\![\nabla \mathbf{w}\cdot n]\!] dr + \int_{\Gamma_0} \delta [\![\mathbf{u}]\!][\![\mathbf{w}]\!] = \int_\Omega \left(\mathbf{f} - \Phi\nabla\right) \mathbf{w} dv.
\end{aligned} \tag{8.32}$$

The bilinear form $B_{sg} : \mathcal{S}^2 \times \mathcal{S}^2 \to \Re$ is defined as

$$
\begin{aligned}
B_{sg}(\mathbf{u}, \mathbf{w}) \; := \; & \int_\Omega (\lambda+\mu) g^2 D_h^2 \mathbf{u} D_h^2 \mathbf{w} dv + \int_\Omega (\lambda+\mu) \nabla_h \mathbf{u} \cdot \nabla_h \mathbf{w} dv \\
& + \int_\Omega \mu g^2 \Delta_h \mathbf{u} \Delta_h \mathbf{w} dv + \int_\Omega \mu \nabla_h \mathbf{u} : \nabla_h \mathbf{w} dv \\
& + \int_\Omega \left\{ (\lambda+\mu) g^2 D_h^2 \mathbf{u} \mathcal{L}_1(\mathbf{w}) - \theta \mathcal{L}_1(\mathbf{u})(\lambda+\mu) g^2 D_h^2 \mathbf{w} \right\} dv \\
& - \int_\Omega \left\{ (\lambda+\mu) \nabla_h \mathbf{u} \mathcal{L}_2(\mathbf{w}) - \theta \mathcal{L}_2(\mathbf{u})(\lambda+\mu) \nabla_h \mathbf{w} \right\} dv \\
& + \int_\Omega \left\{ \mu g^2 \Delta_h \mathbf{u} \mathcal{L}_3(\mathbf{w}) - \theta \mathcal{L}_3(\mathbf{u}) \mu g^2 \Delta_h \mathbf{w} \right\} dv \\
& - \int_\Omega \left\{ \mu \nabla_h \mathbf{u} \mathcal{L}_4(\mathbf{w}) - \theta \mathcal{L}_4(\mathbf{u}) \mu \nabla_h \mathbf{w} \right\} dv \\
& + \int_{\Gamma_0} \gamma [\![\mathbf{u}]\!][\![\mathbf{w}]\!] + \int_{\Gamma_0} \zeta [\![\nabla \mathbf{u} \cdot n]\!][\![\nabla \mathbf{w} \cdot n]\!] dr + \int_{\Gamma_0} \xi [\![\mathbf{u}]\!][\![\mathbf{w}]\!] \\
& + \int_{\Gamma_0} \alpha [\![\mathbf{u}]\!][\![\mathbf{w}]\!] dr + \int_{\Gamma_0} \beta [\![\nabla \mathbf{u} \cdot n]\!][\![\nabla \mathbf{w} \cdot n]\!] dr + \int_{\Gamma_0} \delta [\![\mathbf{u}]\!][\![\mathbf{w}]\!],
\end{aligned}
\tag{8.33}
$$

for any $\mathbf{u}, \mathbf{w} \in \mathcal{S}^2$.

The linear form $L_{sg} : \mathcal{S}^2 \to \Re$ is given by

$$L_{sg}(\mathbf{w}) := \int_\Omega (\mathbf{f} - \Phi \nabla) \mathbf{w} dv, \tag{8.34}$$

for any $\mathbf{w} \in \mathcal{S}^2$.

Then, the interior penalty discontinuous Galerkin method of the problem (8.10) – (8.11), reads as folllows:

$$\text{Find } \mathbf{u}_{DG} \in \mathcal{S}_1 \text{ such that } \; B_{sg}(\mathbf{u}_{DG}, \mathbf{w}) = L_{sg}(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{S}_1. \tag{8.35}$$

We shall allude to the discontinuous Galerkin finite element method with $\theta = -1$ as the symmetric interior penalty Galerkin (SIPG), whereas for $\theta = 1$ the discontinuous Galerkin finite element method will be referred to as the non-symmetric interior penalty Galerkin (NIPG).

We notice that this formulation is incosistent for trial and test functions belonging either to the solution space $\mathcal{S}^2$ or to the solution space $H_0^2(\Omega)^2$.

In practice, the right-hand side is approximated by the L^2-projection of the source of the function $\mathbf{f}$ onto the finite element space $\mathcal{S}_1$. We denote the L^2-projection of $\mathbf{f}$ onto $\mathcal{S}_1$ by $\Pi\mathbf{f}$.

We shall associate with the bilinear form $B_{sg}(\cdot,\cdot)$, (8.33), the energy seminorm be denoted in (8.25).

8.5.1 Stability Bounds of Lifting Operators

In this section, our main concern is to derive the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$.

Lemma 8.5.1.1. *Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ be the trace liftings defined in* (8.28), *in* (8.29), *in* (8.30) *as well as in* (8.31), *respectively. Then, for $\mathbf{u} \in \mathcal{S}^2$, the following bounds hold:*

$$||\mathcal{L}_1(\mathbf{u})||^2_\Omega \leqslant C_1(\lambda,\mu,g^2)\left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||^2_{\Gamma_0}\right), \tag{8.36}$$

$$||\mathcal{L}_2(\mathbf{u})||^2_\Omega \leqslant C_2(\lambda,\mu)||\xi_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0}, \tag{8.37}$$

$$||\mathcal{L}_3(\mathbf{u})||^2_\Omega \leqslant C_3(\mu,g^2)\left(||\alpha_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} + ||\beta_1^{1/2}[\![\nabla\mathbf{u}]\!]||^2_{\Gamma_0}\right), \tag{8.38}$$

$$||\mathcal{L}_4(\mathbf{u})||^2_\Omega \leqslant C_4(\mu)||\delta_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0}, \tag{8.39}$$

where

$$C_1(\lambda,\mu,g^2) = \frac{1}{(\lambda+\mu)g^2},\ C_2(\lambda,\mu) = \frac{1}{\lambda+\mu},\ C_3(\mu,g^2) = \frac{1}{\mu g^2},\ C_4(\mu) = \frac{1}{\mu} \tag{8.40}$$

are positive constants, that are independent of $\mathbf{u}$ and of discretization parameters. We denote by $\gamma_1 : \Gamma_0 \to \Re$, $\zeta_1 : \Gamma_0 \to \Re$, $\xi_1 : \Gamma_0 \to \Re$, $\alpha_1 : \Gamma_0 \to \Re$, $\beta_1 : \Gamma_0 \to \Re$ and $\delta_1 : \Gamma_0 \to \Re$ piecewise constant functions, defined by

$$\gamma_1 = C_{\gamma_1}(\lambda+\mu)g^2\left\langle\frac{\mathbf{p^6}}{\mathbf{h^3}}\right\rangle, \quad \zeta_1 = C_{\zeta_1}(\lambda+\mu)g^2\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle, \quad \xi_1 = C_{\xi_1}(\lambda+\mu)\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle,$$

$$\alpha_1 = C_{\alpha_1}\mu g^2\left\langle\frac{\mathbf{p^6}}{\mathbf{h^3}}\right\rangle, \quad \beta_1 = C_{\beta_1}\mu g^2\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle, \quad \delta_1 = C_{\delta_1}\mu\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle,$$

with C_{γ_1}, C_{ζ_1}, C_{ξ_1}, C_{α_1}, C_{β_1} as well as C_{δ_1} sufficiently large positive constants depending only on the mesh parameters.

Proof. We denote by $\Pi : L^2(\Omega)^2 \to \mathcal{S}_1$ the (orthogonal) L^2-projection operator onto the finite element $\mathcal{S}_1$. By invoking the definition of the L^2-norm, the orthogonality of the L^2-projection operator and the definition of the trace lifting $\mathcal{L}_1$, we get

$$\begin{aligned} ||\mathcal{L}_1(\mathbf{u})||_\Omega &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_\Omega \mathcal{L}_1(\mathbf{u})\mathbf{z}dv}{||\mathbf{z}||_\Omega} \\ &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_\Omega \mathcal{L}_1(\mathbf{u})\Pi\mathbf{z}dv}{||\mathbf{z}||_\Omega} \\ &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_{\Gamma_0} \left([\![\mathbf{u}]\!]\langle\nabla(\Pi\mathbf{z})\rangle - \langle\Pi\mathbf{z}\rangle[\![\nabla\mathbf{u}]\!]\right) dr}{||\mathbf{z}||_\Omega}. \end{aligned} \tag{8.41}$$

By recalling the Cauchy-Schwarz inequality (A.12) and then the Cauchy-Schwarz discrete inequality (A.13) in (8.41), we obtain

$$\begin{aligned} &\sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_{\Gamma_0} \left([\![\mathbf{u}]\!]\langle\nabla(\Pi\mathbf{z})\rangle - \langle\Pi\mathbf{z}\rangle[\![\nabla\mathbf{u}]\!]\right) dr}{||\mathbf{z}||_\Omega} \\ &\leqslant \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0} + ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}}{||\mathbf{z}||_\Omega} \\ &\leqslant \sup_{\mathbf{z}\in L^2(\Omega)} \frac{\left(||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0}^2 + ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}^2\right)^{\frac{1}{2}}}{||\mathbf{z}||_\Omega} \\ &\times \left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{8.42}$$

As a consequence, from (8.41) – (8.42), we deduce

$$\begin{aligned} ||\mathcal{L}_1(\mathbf{u})||_\Omega \leqslant &\sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\left(||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0}^2 + ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}^2\right)^{\frac{1}{2}}}{||\mathbf{z}||_\Omega} \\ &\times \left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{8.43}$$

Thereby, to complete the proof, it only remains to estimate each of the mean value terms appearing on the right-hand side of (8.43).

Hence, by applying the mean value inequality (A.19), we can write the first mean value term as

$$\begin{aligned}
&||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0}^2 \\
&= \sum_{e\in\mathcal{E}_0} ||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_e^2 \\
&\leqslant \sum_{e\in\mathcal{E}_{\text{int}}} \left(||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})^+||_e^2 + ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})^-||_e^2\right) + \sum_{e\in\mathcal{E}_c} ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_e^2 \\
&\leqslant \sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial} \left(||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_{\partial K'}^2 + ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_{\partial K}^2\right) \\
&+ \sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c} ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_{\partial K}^2 \\
&\leqslant \sum_{K\in\mathcal{T}} ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_{\partial K}^2.
\end{aligned} \tag{8.44}$$

Afterwards, by using the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions on the finite element space $\mathcal{S}_1$, as well as the inverse inequality (A.21) in (8.44), we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} ||\gamma_1^{-1/2}\nabla(\Pi\mathbf{z})||_{\partial K}^2 \\
&\leqslant \frac{1}{\tilde{C}C_{\gamma_1}(\lambda+\mu)g^2} \sum_{K\in\mathcal{T}} \frac{h_K^3}{p_K^6} ||\nabla(\Pi\mathbf{z})||_{\partial K}^2 \\
&\leqslant \frac{c_1}{\tilde{C}C_{\gamma_1}(\lambda+\mu)g^2} \sum_{K\in\mathcal{T}} ||\mathbf{z}||_K^2 \\
&\leqslant \frac{1}{2(\lambda+\mu)g^2} ||\mathbf{z}||_\Omega^2,
\end{aligned} \tag{8.45}$$

where $\tilde{C} = \tilde{C}(\eta, \rho)$ is a positive constant and $C_{\gamma_1} \geqslant \frac{2c_1}{\tilde{C}}$.

Therefore, from (8.44) – (8.45), we reach the conclusion that the first mean value term, on the right-hand side of (8.43), can be bounded as

$$||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0}^2 \leqslant \frac{1}{2(\lambda+\mu)g^2} ||\mathbf{z}||_\Omega^2. \tag{8.46}$$

In addition, we shall follow the above series of steps in the same way to estimate the remaining mean value term, on the right-hand side of (8.43).

By employing the mean value inequality (A.19), we conclude

$$\begin{aligned}
&||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}^2 \\
&= \sum_{e\in\mathcal{E}_0} ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_e^2 \\
&\leqslant \sum_{e\in\mathcal{E}_{\text{int}}} \left(||\zeta_1^{-1/2}(\Pi\mathbf{z})^+||_e^2 + ||\zeta_1^{-1/2}(\Pi\mathbf{z})^-||_e^2\right) + \sum_{e\in\mathcal{E}_c} ||\zeta_1^{-1/2}\Pi\mathbf{z}||_e^2 \\
&\leqslant \sum_{K',K\in\mathcal{T}:\partial K',\partial K\backslash\Gamma_\partial} \left(||\zeta_1^{-1/2}\Pi\mathbf{z}||_{\partial K'}^2 + ||\zeta_1^{-1/2}\Pi\mathbf{z}||_{\partial K}^2\right) \\
&+ \sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c} ||\zeta_1^{-1/2}\Pi\mathbf{z}||_{\partial K}^2 \\
&\leqslant \sum_{K\in\mathcal{T}} ||\zeta_1^{-1/2}\Pi\mathbf{z}||_{\partial K}^2.
\end{aligned} \tag{8.47}$$

Next, by invoking the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions on the finite element space $\mathcal{S}_1$, as well as the inverse inequality (A.20) in (8.47), we get

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} ||\zeta_1^{-1/2}\Pi\mathbf{z}||_{\partial K}^2 \\
&\leqslant \frac{1}{CC_{\zeta_1}(\lambda+\mu)g^2} \sum_{K\in\mathcal{T}} \frac{h_K}{p_K^2} ||\Pi\mathbf{z}||_{\partial K}^2 \\
&\leqslant \frac{c_0}{CC_{\zeta_1}(\lambda+\mu)g^2} \sum_{K\in\mathcal{T}} ||\mathbf{z}||_K^2 \\
&\leqslant \frac{1}{2(\lambda+\mu)g^2} ||\mathbf{z}||_\Omega^2,
\end{aligned} \tag{8.48}$$

where $C = C(\eta,\rho)$ is a positive constant and $C_{\zeta_1} \geqslant \frac{2c_0}{C}$.

Ergo, from (8.47) – (8.48), we arrive to the conclusion that the second mean value term, on the right-hand side of (8.43), can subsequently be estimated as

$$||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}^2 \leqslant \frac{1}{2(\lambda+\mu)g^2} ||\mathbf{z}||_\Omega^2. \tag{8.49}$$

To boot, inserting the inequalities (8.46) – (8.49) on the right-hand side of (8.43) yields

$$||\mathcal{L}_1(\mathbf{u})||_\Omega^2 \leqslant \frac{1}{(\lambda+\mu)g^2} \left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right),$$

which is one of the desired results.

What is more, by following the above procedure step by step, we shall bound the trace lifitng $\mathcal{L}_2$ as

$$||\mathcal{L}_3(\mathbf{u})||_\Omega^2 \leqslant \frac{1}{\mu g^2}\left(||\alpha_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right),$$

being one of the desired result, too.

It is time for us to bound the trace lifting $\mathcal{L}_2$. By recalling the definition of the L^2-norm, the orthogonality of the L^2-projection operator and the definition of the trace lifting $\mathcal{L}_2$, we have

$$\begin{aligned} ||\mathcal{L}_2(\mathbf{u})||_\Omega &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_\Omega \mathcal{L}_2(\mathbf{u})\mathbf{z}dv}{||\mathbf{z}||_\Omega} \\ &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_\Omega \mathcal{L}_2(\mathbf{u})\Pi\mathbf{z}dv}{||\mathbf{z}||_\Omega} \\ &= \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_{\Gamma_0} [\![\mathbf{u}]\!]\langle\Pi\mathbf{z}\rangle dr}{||\mathbf{z}||_\Omega}. \end{aligned} \tag{8.50}$$

Application of the Cauchy-Schwarz inequality (A.12) in (8.50) gives

$$\sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\int_{\Gamma_0} [\![\mathbf{u}]\!]\langle\Pi\mathbf{z}\rangle dr}{||\mathbf{z}||_\Omega} \leqslant \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{||\xi_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}||\xi_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}}{||\mathbf{z}||_\Omega}. \tag{8.51}$$

In consequence, from (8.50) – (8.51), we arrive at

$$||\mathcal{L}_2(\mathbf{u})||_\Omega \leqslant \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{||\xi_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}||\xi_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}}{||\mathbf{z}||_\Omega}. \tag{8.52}$$

Thus, to complete the proof, a last step is to estimate the mean value term appearing on the right-hand side of (8.52).

At this point, by using the mean value inequality (A.19), we obtain

$$\begin{aligned}
&||\xi_1^{-1/2}\langle \Pi\mathbf{z}\rangle||^2_{\Gamma_0} \\
&= \sum_{e\in\mathcal{E}_0} ||\xi_1^{-1/2}\langle \Pi\mathbf{z}\rangle||^2_e \\
&\leqslant \sum_{e\in\mathcal{E}_{\text{int}}} \left(||\xi_1^{-1/2}(\Pi\mathbf{z})^+||^2_e + ||\xi_1^{-1/2}(\Pi\mathbf{z})^-||^2_e\right) + \sum_{e\in\mathcal{E}_c} ||\xi_1^{-1/2}\Pi\mathbf{z}||^2_e \\
&\leqslant \sum_{K',K\in\mathcal{T}:\partial K',\partial K\setminus\Gamma_\partial} \left(||\xi_1^{-1/2}\Pi\mathbf{z}||^2_{\partial K'} + ||\xi_1^{-1/2}\Pi\mathbf{z}||^2_{\partial K}\right) \\
&+ \sum_{K\in\mathcal{T}:\partial K\cap\Gamma_c} ||\xi_1^{-1/2}\Pi\mathbf{z}||^2_{\partial K} \\
&\leqslant \sum_{K\in\mathcal{T}} ||\xi_1^{-1/2}\Pi\mathbf{z}||^2_{\partial K}.
\end{aligned} \tag{8.53}$$

After that, by employing the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions on the finite element space $\mathcal{S}_1$, together with the inverse inequality (A.20) in (8.53), we conclude

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} ||\xi_1^{-1/2}\Pi\mathbf{z}||^2_{\partial K} \\
&\leqslant \frac{1}{\hat{C}C_{\xi_1}(\lambda+\mu)} \sum_{K\in\mathcal{T}} \frac{h_K}{p_K^2}||\Pi\mathbf{z}||^2_{\partial K} \\
&\leqslant \frac{c_0}{\hat{C}C_{\xi_1}(\lambda+\mu)} \sum_{K\in\mathcal{T}} ||\mathbf{z}||^2_K \\
&\leqslant \frac{1}{(\lambda+\mu)}||\mathbf{z}||^2_\Omega,
\end{aligned} \tag{8.54}$$

where $\hat{C} = \hat{C}(\eta,\rho)$ is a positive constant and $C_{\xi_1} \geqslant \frac{c_0}{\hat{C}}$.

Wherefore, from (8.53) – (8.54), we reach the conclusion that the mean value term, on the right-hand side of (8.52), can be bounded as follows

$$||\xi_1^{-1/2}\langle \Pi\mathbf{z}\rangle||^2_{\Gamma_0} \leqslant \frac{1}{(\lambda+\mu)}||\mathbf{z}||^2_\Omega. \tag{8.55}$$

Also, insertion of the inequality (8.55) on the right-hand side of (8.52) entails

$$||\mathcal{L}_2(\mathbf{u})||^2_\Omega \leqslant \frac{1}{(\lambda+\mu)}||\xi_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0},$$

which is one of the desired results.

Furthemore, as above, we shall use arguments in a similar manner to bound the trace lifitng $\mathcal{L}_4$ as

$$||\mathcal{L}_4(\mathbf{u})||_{\Omega}^2 \leqslant \frac{1}{\mu}||\delta_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2,$$

being one of the desired result, too. □

In the following sections, we are going to prove the coercivity and the continuity property of the bilinear form for the symmetric interior penalty discontinuous Galerkin method.

8.5.2 Coercivity of Bilinear Form

In this section, our goal is to examine the coercivity of the bilinear form $B_{sg}(\cdot,\cdot)$ for the symmetric interior penalty discontinuous Galerkin finite element method.

We showed earlier that $|||\cdot|||_{sg}$, (8.25), is a seminorm on the space $H^4(\Omega,\mathcal{T})^2$, thereby, since $\mathcal{S}_1 \subset H^4(\Omega,\mathcal{T})^2$, we get that $|||\cdot|||_{sg}$ is also a seminorm on $\mathcal{S}_1$.

Proposition 8.5.2.1. *Let $\gamma:\Gamma_0\to\Re$, $\zeta:\Gamma_0\to\Re$, $\xi:\Gamma_0\to\Re$, $\alpha:\Gamma_0\to\Re$, $\beta:\Gamma_0\to\Re$ and $\delta:\Gamma_0\to\Re$ be piecewise constant functions, such that $\gamma>2\gamma_1$, $\zeta>2\zeta_1$, $\xi>2\xi_1$, $\alpha>2\alpha_1$, $\beta>2\beta_1$ as well as $\delta>2\delta_1$. Then, the bilinear form $B_{sg}(\cdot,\cdot)$, defined in (8.33), is coercive in the sense that*

$$B_{sg}(\mathbf{u},\mathbf{u}) \geqslant m|||\mathbf{u}|||_{sg}^2 \quad \forall \mathbf{u}\in\mathcal{S}_1, \tag{8.56}$$

where m is a positive constant depending only on the mesh parameters.

Proof. Substituting $\mathbf{u}$ for $\mathbf{w}$ in the bilinear form, (8.33), and for $\theta=-1$, we obtain

$$\begin{aligned} B_{sg}(\mathbf{u},\mathbf{u}) \;=\;& |||\mathbf{u}|||_{sg}^2 + 2\int_\Omega \mathcal{L}_1(\mathbf{u})(\lambda+\mu)g^2 D_h^2\mathbf{u}dv \\ & -2\int_\Omega \mathcal{L}_2(\mathbf{u})(\lambda+\mu)\nabla_h\mathbf{u}dv + 2\int_\Omega \mathcal{L}_3(\mathbf{u})\mu g^2\Delta_h\mathbf{u}dv \\ & -2\int_\Omega \mathcal{L}_4(\mathbf{u})\mu\nabla_h\mathbf{u}dv \end{aligned} \tag{8.57}$$

To complete the proof, it only remains to estimate the integrals appearing on the right-hand side of (8.57).

So, by applying the Cauchy-Schwarz inequality (A.12) and then the Young inequality (A.17) for $\varepsilon = 1$, we can write the first integral as

$$\begin{aligned}
&\int_\Omega \mathcal{L}_1(\mathbf{u})(\lambda+\mu)g^2 D_h^2\mathbf{u}dv \\
&\leqslant \int_\Omega \left|\mathcal{L}_1(\mathbf{u})(\lambda+\mu)g^2 D_h^2\mathbf{u}\right| dv \\
&\leqslant ||\{2(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u})||_\Omega||\{\frac{1}{2}(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega \\
&\leqslant ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u})||_\Omega^2 + \frac{1}{4}||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2.
\end{aligned} \tag{8.58}$$

Moreover, we shall follow the above procedure in a similar manner to estimate the second, the third and the fourth integral on the right-hand side of (8.57). Hence, we deduce

$$\begin{aligned}
\int_\Omega \mathcal{L}_2(\mathbf{u})(\lambda+\mu)\nabla_h\mathbf{u}dv &\leqslant ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u})||_\Omega^2 + \frac{1}{4}||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2, \\
\int_\Omega \mathcal{L}_3(\mathbf{u})\mu g^2\Delta_h\mathbf{u}dv &\leqslant ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u})||_\Omega^2 + \frac{1}{4}||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2, \\
\int_\Omega \mathcal{L}_4(\mathbf{u})\mu\nabla_h\mathbf{u}dv &\leqslant ||\mu^{1/2}\mathcal{L}_4(\mathbf{u})||_\Omega^2 + \frac{1}{4}||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2.
\end{aligned} \tag{8.59}$$

Thereafter, inserting the inequalities (8.58) – (8.59) on the right-hand side of (8.57), we have

$$\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{u}) \;\geqslant\; & |||\mathbf{u}|||_{sg}^2 - 2||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u})||_\Omega^2 \\
& - \frac{1}{2}||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2 - 2||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u})||_\Omega^2 \\
& - \frac{1}{2}||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2 - 2||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u})||_\Omega^2 \\
& - \frac{1}{2}||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 - 2||\mu^{1/2}\mathcal{L}_4(\mathbf{u})||_\Omega^2 - \frac{1}{2}||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2.
\end{aligned} \tag{8.60}$$

Next, by invoking the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and by using the mathematical inequalities (8.36) – (8.39) on the right-hand side of

(8.60), we get

$$\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{u}) \;\geqslant\;& |||\mathbf{u}|||_{sg}^2 - 2\left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right)\\
&- \frac{1}{2}||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2 - 2||\xi_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2\\
&- \frac{1}{2}||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2 - 2\left(||\alpha_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right)\\
&- \frac{1}{2}||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 - 2||\delta_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 - \frac{1}{2}||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2. \quad (8.61)
\end{aligned}$$

Now, by the use of energy seminorm, (8.25), and with the aid of factorization on the right-hand side of (8.61), it is clear that

$$\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{u}) \;\geqslant\;& \frac{1}{2}||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2 + \frac{1}{2}||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2\\
&+ \frac{1}{2}||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 + \frac{1}{2}||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2\\
&+ ||(\gamma-2\gamma_1)^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||(\zeta-2\zeta_1)^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\\
&+ ||(\xi-2\xi_1)^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||(\alpha-2\alpha_1)^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2\\
&+ ||(\beta-2\beta_1)^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + ||(\delta-2\delta_1)^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2. \quad (8.62)
\end{aligned}$$

Since we assumed $\gamma > 2\gamma_1$, $\zeta > 2\zeta_1$, $\xi > 2\xi_1$, $\alpha > 2\alpha_1$, $\beta > 2\beta_1$ and $\delta > 2\delta_1$, coercivity follows, i.e.

$$B_{sg}(\mathbf{u},\mathbf{u}) \geqslant m|||\mathbf{u}|||_{sg}^2,$$

which is the desired result. We denote by the constant m the minimum of the coefficients on the right-hand side of (8.62). □

8.5.3 Continuity of Bilinear Form

With the definition of the energy seminorm, (8.25), we have the following continuity result for the bilinear form $B_{sg}(\cdot,\cdot)$, based on the Cauchy-Schwarz inequalities (A.12) and (A.13).

Proposition 8.5.3.1. *Let $\gamma : \Gamma_0 \to \Re$, $\zeta : \Gamma_0 \to \Re$, $\xi : \Gamma_0 \to \Re$, $\alpha : \Gamma_0 \to \Re$, $\beta : \Gamma_0 \to \Re$ and $\delta : \Gamma_0 \to \Re$ be piecewise constant functions, such that $\gamma > 2\gamma_1$, $\zeta > 2\zeta_1$, $\xi > 2\xi_1$, $\alpha > 2\alpha_1$, $\beta > 2\beta_1$ as well as $\delta > 2\delta_1$. Then, the bilinear form $B_{sg}(\cdot,\cdot)$, defined in* (8.33)*, is continuous in the sense that*

$$B_{sg}(\mathbf{u},\mathbf{w}) \leqslant C|||\mathbf{u}|||_{sg}|||\mathbf{w}|||_{sg} \quad \forall \mathbf{u},\mathbf{w} \in \mathcal{S}^2, \quad (8.63)$$

where C is a positive constant depending only on the mesh parameters.

Proof. Let $\mathbf{u}, \mathbf{w} \in \mathcal{S}^2$, we can obtain (8.63) by applying at first the triangle inequality in the bilinear form and then the Cauchy-Schwarz inequality (A.12). For that reason, we get

$$\begin{aligned}
B_{sg}(\mathbf{u}, \mathbf{w}) \leqslant\ & |B_{sg}(\mathbf{u}, \mathbf{w})| \\
\leqslant\ & ||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{u}||_\Omega ||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{w}||_\Omega \\
& + ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{w}||_\Omega \\
& + ||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega ||(\mu g^2)^{1/2}\Delta_h\mathbf{w}||_\Omega \\
& + ||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega ||\mu^{1/2}\nabla_h\mathbf{w}||_\Omega \\
& + ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u})||_\Omega ||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{w}||_\Omega \\
& + ||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{u}||_\Omega ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{w})||_\Omega \\
& + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u})||_\Omega ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{w}||_\Omega \\
& + ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{w})||_\Omega \\
& + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u})||_\Omega ||(\mu g^2)^{1/2}\Delta_h\mathbf{w}||_\Omega \\
& + ||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{w})||_\Omega \\
& + ||\mu^{1/2}\mathcal{L}_4(\mathbf{u})||_\Omega ||\mu^{1/2}\nabla_h\mathbf{w}||_\Omega + ||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega ||\mu^{1/2}\mathcal{L}_4(\mathbf{w})||_\Omega \\
& + ||\gamma^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0} ||\gamma^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0} + ||\zeta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0} ||\zeta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0} \\
& + ||\xi^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0} ||\xi^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0} + ||\alpha^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0} ||\alpha^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0} \\
& + ||\beta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0} ||\beta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0} + ||\delta^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0} ||\delta^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}.
\end{aligned} \tag{8.64}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (8.64), we have

$$
\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{w}) \leqslant\ & \big(2||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\
& + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\
& + ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u})||_\Omega^2 + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u})||_\Omega^2 \\
& + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u})||_\Omega^2 + ||\mu^{1/2}\mathcal{L}_4(\mathbf{u})||_\Omega^2 \\
& + ||\gamma^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\xi^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 \\
& + ||\alpha^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\delta^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2\big)^{1/2} \\
& \times \big(2||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{w}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\
& + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{w}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\
& + ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{w})||_\Omega^2 + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{w})||_\Omega^2|| \\
& + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{w})||_\Omega^2 + ||\mu^{1/2}\mathcal{L}_4(\mathbf{w})||_\Omega^2 \\
& + ||\gamma^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + ||\xi^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 \\
& + ||\alpha^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + ||\beta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + ||\delta^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2\big)^{1/2}.
\end{aligned}
\tag{8.65}
$$

Thereby, to complete the proof, a last step remaining is to recall the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and therefore to employ the mathematical expressions (8.36) – (8.39) on the right-hand side of (8.65).

In consequence, we deduce

$$
\begin{aligned}
B_{sg}(\mathbf{u},\mathbf{w}) \leqslant\ & \big(2||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\
& + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\
& + \frac{1}{2}||\gamma^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\zeta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\xi^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 \\
& + \frac{1}{2}||\alpha^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\beta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\delta^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2\Big)^{1/2} \\
& \times \big(2||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{w}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\
& + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{w}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\
& + \frac{1}{2}||\gamma^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\zeta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\xi^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 \\
& + \frac{1}{2}||\alpha^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\beta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\delta^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2\Big)^{1/2}.
\end{aligned}
\tag{8.66}
$$

Also, by the use of definition of energy seminorm, (8.25), on the right-hand side of (8.66), we reach to

$$B_{sg}(\mathbf{u}, \mathbf{w}) \leqslant C|||\mathbf{u}|||_{pl}|||\mathbf{w}|||_{sg},$$

being the desired result. □

8.6 A Posteriori Error Analysis

In this section, we want to conduct an error analysis for interior penalty discontinuous Galerkin finite element method (8.35). Specifically, overall our research endeavor focuses on the introduction of a suitable recovery operator, on the proof of an appropriate Lemma for this operator and on the proof of h-version reliable a posteriori error estimate in the energy seminorm, $|||\cdot|||_{sg}$, for the interior penalty discontinuous Galerkin method.

The reliability estimate is based on a suitable recovery operator, that maps discontinuous finite element spaces to H_0^2-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [84] (see also [143, 34, 132, 115] for similar constructions). Using the recovery operator, in conjuction with the incosistent formulation for the interior penalty discontinuous Galerkin method presented in the preceding section (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), reliable a posteriori error estimate of residual type can derive for the interior penalty discontinuous Galerkin method in the corresponding energy seminorm.

8.6.1 Finite Element Spaces

In this section, we will consider the finite-dimensional subspace of the broken Sobolev space $H^4(\Omega, \mathcal{T})^2$ being used in the finite element approximation of the problem. Moreover, we wish to modify a little the finite element space, defined in section 8.4, so that it can include either triangular or quadrilateral elements.

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular or quadrilateral elements $K \in \mathcal{T}$. We assume that the elemental edges are straight line segments.

For a non-negative integer p, we denote by $\mathcal{P}_p(\hat{K})$ the set of all polynomials of total degree at most p if $\hat{K}$ is either the reference triangle or the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. For $p \geqslant 2$ we consider the finite element space

$$\mathcal{S}_1 \equiv [\mathcal{S}_h^p]^2 := \left\{ \mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}|_K \circ F_K \in \mathcal{P}_p(\hat{K})^2, \; K \in \mathcal{T} \right\}. \tag{8.67}$$

We collect the h_K into the elementwise constant function

$$\mathbf{h} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K, K \in \mathcal{T} \text{ and } \mathbf{h}|_e = \langle \mathbf{h} \rangle, e \subset \Gamma_0.$$

We shall assume throughout that the families of meshes considered are locally quasiuniform or in other words the mesh size function $\mathbf{h}$ has bounded local variation (see Remark A.3.5).

Then, the piecewise constant stabilization parameters $\gamma : \Gamma_0 \to \Re$, $\zeta : \Gamma_0 \to \Re$, $\xi : \Gamma_0 \to \Re$, $\alpha : \Gamma_0 \to \Re$, $\beta : \Gamma_0 \to \Re$ and $\delta : \Gamma_0 \to \Re$ are defined by

$$\gamma = C_\gamma(\lambda + \mu) g^2 (\mathbf{h}|_e)^{-3}, \quad \zeta = C_\zeta(\lambda + \mu) g^2 (\mathbf{h}|_e)^{-1}, \quad \xi = C_\xi(\lambda + \mu)(\mathbf{h}|_e)^{-1}, \tag{8.68}$$

$$\alpha = C_\alpha \mu g^2 (\mathbf{h}|_3)^{-3}, \quad \beta = C_\beta \mu g^2 (\mathbf{h}|_e)^{-1}, \quad \delta = C_\delta \mu (\mathbf{h}|_e)^{-1}, \tag{8.69}$$

with C_γ, C_ζ, C_ξ, C_α, C_β as well as C_δ sufficiently large positive constants.

8.6.2 Recovery Operator

The use of a recovery operator, mapping elements of $\mathcal{S}_1$ onto a C^1-conforming space consisting of macro-elements of degree $p+2$, is a significant tool helping us conduct a posteriori error analysis. The family of macro-elements considered will be higher-order versions of the classical Hsieh-Clough-Tocher macro-element, constructed in [84] (see A.1.8). This mapping is constructed via averages of the nodal basis functions (see [143, 34, 132, 115]).

The corresponding finite element space consisting of the above macro-elements will be denoted by $\mathcal{S}_2 \equiv [\tilde{\mathcal{S}}_h^m]^2$.

Let us consider the standard Lagrange basis for a polynomial of degree p, where $p \geqslant 2$. A crucial observation here is that the set of the nodal points of the Lagrange basis is a subset of the set of the nodal points of the macro-elements of degree $p+2$. In that case, the corresponding finite element space $\mathcal{S}_2 \equiv \left[\tilde{\mathcal{S}}_h^{p+2}\right]^2$ and it will be used in the following Lemma.

Lemma 8.6.2.1. *Let us assume that the mesh $\mathcal{T}$ is constructed as in Section 8.6.1. Then, there exists an operator $\mathbf{E}_{op} : \mathcal{S}_1 \to \mathcal{S}_2 \cap H_0^2(\Omega)^2$ satisfying the following error bounds:*

$$\sum_{k\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|^2_{j,K} \leqslant C_1 \left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + ||\mathbf{h}^{3/2-j}[\![\nabla \mathbf{u}_h]\!]||^2_{\Gamma_0} \right), \quad (8.70)$$

with $j = 2$ and

$$\sum_{k\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|^2_{j,K} \leqslant C_2 ||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0}, \quad (8.71)$$

with $j = 1$. We denote by $C_1, C_2 > 0$ some constants that are independent of $\mathbf{h}$ and $\mathbf{u}_h$.

Proof. For each nodal point np of the C^1-conforming finite element space $\mathcal{S}_2$, we define ω_{np} to be the set of $K \in \mathcal{T}$ that share the nodal point np, i.e.,

$$\omega_{np} := \{K \in \mathcal{T} : np \in \mathcal{T}\}.$$

Furthermore, $|\omega_{np}|$ will denote the cardinality of ω_{np}. We note that if np located in the interior of an element, then we shall have $|\omega_{np}| = 1$.

Next, we define the operator $\mathbf{E}_{op} : \mathcal{S}_1 \to \mathcal{S}_2 \cap H_0^2(\Omega)^2$ by

$$N_{np}(\mathbf{E}_{op}(\mathbf{u}_h)) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K\in\omega_{np}} N_{np}(\mathbf{u}_h|_K), & \text{if } np \notin \Gamma_c \\ 0, & \text{if } np \in \Gamma_c, \end{cases} \quad (8.72)$$

where N_{np} is any nodal variable at np and np is any nodal point of $\mathcal{S}_2$. Note that

$$N_{np}(\mathbf{E}_{op}(\mathbf{u}_h)) = N_{np}(\mathbf{u}_h), \quad \text{if } np \in \text{int}K.$$

We denote by $\mathcal{N}$ the set of all nodal variables of $\mathcal{S}_2$ defined on every element of $\mathcal{T}$, i.e., they may be discontinuous across element boundaries. Then, we can split $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1,$$

where $\mathcal{N}_0$ and $\mathcal{N}_1$ consisting of the nodal variables corresponding to the function evaluations and those involving partial and normal derivatives of the function, respectively.

The use of an inverse estimate (A.37) yields

$$\sum_{K\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(u_h)|^2_{j,K} \leqslant C \left|\left|\mathbf{h}^{-j}(\mathbf{u}_h - \mathbf{E}_{op}(u_h))\right|\right|^2_{\Omega}, \quad (8.73)$$

with C a positive constant which is independent of $\mathbf{h}$ and $\mathbf{u}_h$.

After that, the equivalence of norms in a finite-dimensional vector space along with a scaling argument gives

$$\begin{aligned} & \left|\left|\mathbf{h}^{-j}(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h))\right|\right|_{\Omega}^2 \\ & \leqslant C \sum_{i=0}^{1} \sum_{N_{np}\in\mathcal{N}_i : np\in K} h_K^{2(i+1-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 . \end{aligned} \tag{8.74}$$

Now, for each nodal point np which is not on the boundary Γ_c, we consider a local numbering $K_1, \ldots, K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge. By recalling the arithmetic-geometric mean inequality (A.15), we get

$$\begin{aligned} & \sum_{N_{np}\in\mathcal{N}_0 : np\in K} h_K^{2(1-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 \\ & = \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)} \left\{ \mathbf{u}_h(np)|_K - \frac{1}{|\omega_{np}|} \sum_{K\in\omega_{np}} \mathbf{u}_h(np)|_K \right\}^2 \\ & + \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_c} h_K^{2(1-j)} \left(\mathbf{u}_h(np)|_K\right)^2 \\ & \leqslant C \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)} \left\{ \sum_{\ell=1}^{|\omega_{np}|-1} \left(\mathbf{u}_h|_{K_\ell}(np) - \mathbf{u}_h|_{K_{\ell+1}}(np)\right)^2 \right\} \\ & + \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_c} h_K^{2(1-j)} \left\{\mathbf{u}_h(np)|_K\right\}^2 \\ & = C \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_{\text{int}}} [\![\mathbf{u}_h(np)]\!]^2 \\ & + \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_c} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_c} [\![\mathbf{u}_h(np)]\!]^2 . \end{aligned} \tag{8.75}$$

Next in (8.75), owing to the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi unirform, we obtain

$$\begin{aligned} & C \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_{\text{int}}} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_{\text{int}}} [\![\mathbf{u}_h(np)]\!]^2 \\ & + \sum_{N_{np}\in\mathcal{N}_0 : np\in K\cap\Gamma_c} h_K^{2(1-j)} \sum_{e\in\mathcal{E}_c} [\![\mathbf{u}_h(np)]\!]^2 \leqslant C \sum_{e\in\mathcal{E}_0} ||\mathbf{h}^{1-j}[\![\mathbf{u}_h]\!]||_{L^\infty(e)}^2 . \end{aligned} \tag{8.76}$$

Then, by applying an inverse inequality in (8.76), as a result we deduce

$$C \sum_{e \in \mathcal{E}_0} ||\mathbf{h}^{1-j} [\![\mathbf{u}_h]\!]||^2_{L^\infty(e)} \leqslant C \sum_{e \in \mathcal{E}_0} ||\mathbf{h}^{1/2-j} [\![\mathbf{u}_h]\!]||^2_e \\ = C||\mathbf{h}^{1/2-j} [\![\mathbf{u}_h]\!]||^2_{\Gamma_0}. \tag{8.77}$$

In consequence, from (8.75) – (8.77), we conclude that

$$\sum_{N_{np} \in \mathcal{N}_0 : np \in K} h_K^{2(1-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 \leqslant C||\mathbf{h}^{1/2-j} [\![\mathbf{u}_h]\!]||^2_{\Gamma_0}. \tag{8.78}$$

What is more, it's time for us to turn to the nodal variables in $\mathcal{N}_1$. We further split $\mathcal{N}_1$ into

$$\mathcal{N}_1 = \mathcal{N}_1^n \cup \mathcal{N}_1^p,$$

where $\mathcal{N}_1^n$ is the set of the nodal variables of normal derivatives across element edges and $\mathcal{N}_1^p$ is the set of nodal variables representing partial derivatives on elemental vertices.

Hence, we shall follow arguments in a same way for $\mathcal{N}_1^n$ as in (8.78). For each nodal point np which is not on the boundary Γ_c, we consider a local numbering $K_1, \ldots, K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge. By invoking the arithmetic-geometric mean inequality (A.15) derives

$$
\begin{aligned}
&\sum_{N_{np}\in\mathcal{N}_1^n:np\in K} h_K^{2(2-j)}\left(N_{np}\left(\mathbf{u}_h-\mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2\\
&=\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\\
&\times\left\{\left(\nabla\mathbf{u}_h\cdot n_K\right)|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\left(\nabla\mathbf{u}_h\cdot n_K\right)|_K(np)\right\}^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_c} h_K^{2(2-j)}\left\{\left(\nabla\mathbf{u}_h\cdot n_K\right)|_K(np)\right\}^2\\
&\leqslant C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\\
&\times\left\{\sum_{\ell=1}^{|\omega_{np}|-1}\left(\left(\nabla\mathbf{u}_h\cdot n_{K_\ell}\right)|_{K_\ell}(np)-\left(\nabla\mathbf{u}_h\cdot n_{K_{\ell+1}}\right)|_{K_{\ell+1}}(np)\right)^2\right\}\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_c} h_K^{2(2-j)}\left\{\left(\nabla\mathbf{u}_h\cdot n_K\right)|_K(np)\right\}^2\\
&=C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_{\rm int}}[\![\nabla\mathbf{u}_h(np)]\!]^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_c} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_c}[\![\nabla\mathbf{u}_h(np)]\!]^2.
\end{aligned}
\tag{8.79}
$$

Afterwards in (8.79), in view of the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi unirform, we get

$$
\begin{aligned}
&C\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_{\rm int}} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_{\rm int}}[\![\nabla\mathbf{u}_h(np)]\!]^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^n:np\in K\cap\Gamma_c} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_c}[\![\nabla\mathbf{u}_h(np)]\!]^2\leqslant C\sum_{e\in\mathcal{E}_0}||\mathbf{h}^{2-j}[\![\nabla\mathbf{u}_h]\!]||^2_{L^\infty(e)}.
\end{aligned}
\tag{8.80}
$$

Also, by using an inverse inequality in (8.80), we obtain

$$
\begin{aligned}
C\sum_{e\in\mathcal{E}_0}||\mathbf{h}^{2-j}[\![\nabla\mathbf{u}_h]\!]||^2_{L^\infty(e)}&\leqslant C\sum_{e\in\mathcal{E}_0}||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e\\
&=C||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}.
\end{aligned}
\tag{8.81}
$$

Ergo, from (8.79) – (8.81), we reach the conclusion that

$$\sum_{N_{np}\in\mathcal{N}_1^n:np\in K} h_K^{2(2-j)}\left(N_{np}\left(\mathbf{u}_h-\mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 \leqslant C||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}. \tag{8.82}$$

Now, we shall follow the above procedure in a similar manner for $\mathcal{N}_1^p$ as in both (8.78) and (8.82). For each nodal point np which is not on the boundary Γ_c, we consider a local numbering $K_1,\ldots,K_{|\omega_{np}|-1}$ of the elements in ω_{np}, so that each consecutive pair K_ℓ and $K_{\ell+1}$ shares an edge. By employing the arithmetic-geometric mean inequality (A.15), we have

$$\begin{aligned}
&\sum_{N_{np}\in\mathcal{N}_1^p:np\in K} h_K^{2(2-j)}\left(N_{np}\left(\mathbf{u}_h-\mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2\\
&=\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)}\\
&\times\left\{\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_K(np)-\frac{1}{|\omega_{np}|}\sum_{K\in\omega_{np}}\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_K(np)\right\}^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_c} h_K^{2(2-j)}\left\{\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_K(np)\right\}^2\\
&\leqslant C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)}\\
&\times\left\{\sum_{\ell=1}^{|\omega_{np}|-1}\left(\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_{K_\ell}(np)-\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_{K_{\ell+1}}(np)\right)^2\right\}\\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_c} h_K^{2(2-j)}\left\{\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z|_K(np)\right\}^2\\
&=C\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_{\text{int}}}\left[\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z(np)\right]^2\\
&+\sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_c} h_K^{2(2-j)}\sum_{e\in\mathcal{E}_c}\left[\sum_{z\in\{x,y\}}(\mathbf{u}_h)_z(np)\right]^2.
\end{aligned}\tag{8.83}$$

Thereafter in (8.83), because of the fact that the subdivision $\mathcal{T}$ of Ω is locally quasi unirform, we deduce

$$\begin{aligned} & C \sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_{\text{int}}} h_K^{2(2-j)} \sum_{e\in\mathcal{E}_{\text{int}}} \left[\sum_{z\in\{x,y\}} (\mathbf{u}_h)_z(np)\right]^2 \\ & + \sum_{N_{np}\in\mathcal{N}_1^p:np\in K\cap\Gamma_c} h_K^{2(2-j)} \sum_{e\in\mathcal{E}_c} \left[\sum_{z\in\{x,y\}} (\mathbf{u}_h)_z(np)\right]^2 \\ & \leqslant C \sum_{e\in\mathcal{E}_0} \sum_{z\in\{x,y\}} ||\mathbf{h}^{2-j}[\![(\mathbf{u}_h)_z]\!]||^2_{L^\infty(e)}. \end{aligned} \tag{8.84}$$

Into the bargain, applying an inverse inequality in (8.84) yields

$$C \sum_{e\in\mathcal{E}_0} \sum_{z\in\{x,y\}} ||\mathbf{h}^{2-j}[\![(\mathbf{u}_h)_z]\!]||^2_{L^\infty(e)} \leqslant C \sum_{e\in\mathcal{E}_0} \sum_{z\in\{x,y\}} ||\mathbf{h}^{3/2-j}[\![(\mathbf{u}_h)_z]\!]||^2_e. \tag{8.85}$$

A last and imperative step remaining is to split the partial derivatives on the right-hand side of (8.85) into normal and tangential components. Employing at the same time the triangle inequality and subsequently (A.14) entails

$$||\mathbf{h}^{3/2-j}[\![(\mathbf{u}_h)_z]\!]||^2_e \leqslant 2||\mathbf{h}^{3/2-j}[\![(\mathbf{u}_h)_t]\!]||^2_e + 2||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e. \tag{8.86}$$

Then, by using an inverse estimate (A.37) along each edge e for the tangential derivative component, together with the fact that the edges e are straight lines, we eventually conclude

$$\begin{aligned} & 2||\mathbf{h}^{3/2-j}[\![(\mathbf{u}_h)_t]\!]||^2_e + 2||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e \\ & = 2|\mathbf{h}^{3/2-j}[\![\mathbf{u}_h]\!]|^2_{1,e} + 2||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e \\ & \leqslant C||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||^2_e + 2||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e. \end{aligned} \tag{8.87}$$

Hence, (8.86) and (8.87) imply that

$$||\mathbf{h}^{3/2-j}[\![(\mathbf{u}_h)_z]\!]||^2_e \leqslant C||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||^2_e + 2||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_e. \tag{8.88}$$

Wherefore, from (8.83) – (8.85) and (8.88), we arrive to the conclusion

$$\begin{aligned} & \sum_{N_{np}\in\mathcal{N}_1^p:np\in K} h_K^{2(2-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 \\ & \leqslant C\left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + ||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}\right). \end{aligned} \tag{8.89}$$

After that, gathering the inequalities (8.78), (8.82) together with (8.89) and inserting them on the right-hand side of (8.74), we deduce

$$\left|\left|\mathbf{h}^{-j}(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h))\right|\right|_{\Omega}^2 \leqslant C\left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + ||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right). \tag{8.90}$$

Finally, insertion of the mathematical expression (8.90) into the right-hand side of (8.73) yields

$$\sum_{K\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|_{j,K}^2 \leqslant C\left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + ||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right),$$

being the desired result for $j = 2$.

Since the set of nodal points of the Lagrange basis is a subset of the set of nodal points of macro-elements and since $H_0^2(\Omega)^2 \subseteq H_0^1(\Omega)^2$, we analogously prove the inequality (8.71). In this case, the set, $\mathcal{N}$, of all nodal variables of $\mathcal{S}_2$ defined on every element of $\mathcal{T}$ is equivalent with $\mathcal{N}_0$, i.e., $\mathcal{N} \equiv \mathcal{N}_0$. □

8.6.3 A Posteriori Error Estimates

In this section, overall our research endeavor focuses mainly on establishing a reliable a posteriori error estimate of residual type for the (symmetric) interior penalty discontinuous Galerkin method in the corresponding energy seminorm, when the analytical solution $\mathbf{u}$ of (8.10) – (8.11) satisfies $\mathbf{u} \in H_0^2(\Omega)^2$.

Theorem 8.6.3.1. *Let* $\mathbf{u} \in H_0^2(\Omega)^2$ *be the solution to* (8.10) – (8.11), $\mathbf{u}_h \in \mathcal{S}_1$ *be the approximate solution obtained by the interior penalty discontinuous Galerkin method and* γ, ζ, ξ, α, β *together with* δ *as in* (8.68) – (8.69). *Then, there exists a positive constant* C, *independent of* $\mathbf{h}$, $\mathbf{u}$ *and*

$\mathbf{u}_h$, *so that*

$$
\begin{aligned}
&|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \\
&\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}}-(\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h+(\lambda+\mu)D_h^2\mathbf{u}_h-\mu g^2\Delta_h^2\mathbf{u}_h+\mu\Delta_h\mathbf{u}_h\}||_\Omega^2 \\
&+C_p^2\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+(\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right. \\
&+(\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 \\
&\left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\} \\
&+(\lambda+\mu)g^2||\mathbf{h}^{3/2}[\![\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2+(\lambda+\mu)g^2||\mathbf{h}^{1/2}[\![D^2\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2 \\
&+(\lambda+\mu)||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2+\mu g^2||\mathbf{h}^{3/2}[\![\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2 \\
&+\mu g^2||\mathbf{h}^{1/2}[\![\Delta\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2+\mu||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\mathrm{int}}}^2\Big),
\end{aligned}
\tag{8.91}
$$

where $C_p := \max\{C_\gamma, C_\zeta, C_\xi, C_\alpha, C_\beta, C_\delta\}$ *and* $\tilde{\mathbf{f}} = \mathbf{f} - \Phi\nabla$.

Proof. Let $\mathbf{w}_h \in \mathcal{S}_1$, $\mathbf{w} \in H_0^2(\Omega)^2$, $\eta = \mathbf{w} - \mathbf{w}_h$ and $\mathbf{E}_{op}(\mathbf{u}_h) \in \mathcal{S}_2 \cap H_0^2(\Omega)^2$ be as in Lemma 8.6.2.1. We shall use this notation with intention to decompose the error as follows:

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{E}_{op}(\mathbf{u}_h)) + (\mathbf{E}_{op}(\mathbf{u}_h) - \mathbf{u}_h) \equiv \mathbf{e}^c + \mathbf{e}^d. \tag{8.92}$$

Since $\mathbf{u}$ is the solution to the weak problem, we get

$$B_{sg}(\mathbf{u},\mathbf{w}) = L_{sg}(\mathbf{w}) \quad \text{as} \quad \mathcal{L}_i(\mathbf{u}) = \mathcal{L}_i(\mathbf{w}) = 0 \quad \forall i = 1,2,3,4.$$

As a consequence,

$$
\begin{aligned}
B_{sg}(\mathbf{e},\mathbf{w}) &= B_{sg}(\mathbf{u},\mathbf{w}) - B_{sg}(\mathbf{u}_h,\mathbf{w}) \\
&= L_{sg}(\mathbf{w}) - B_{sg}(\mathbf{u}_h,\mathbf{w}) \\
&= L_{sg}(\mathbf{w}) - B_{sg}(\mathbf{u}_h,\mathbf{w}-\mathbf{w}_h) - B_{sg}(\mathbf{u}_h,\mathbf{w}_h) \\
&= L_{sg}(\mathbf{w}) - B_{sg}(\mathbf{u}_h,\eta) - L_{sg}(\mathbf{w}_h) \\
&= L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta)
\end{aligned}
\tag{8.93}
$$

and it also holds that

$$B_{sg}(\mathbf{e},\mathbf{w}) = B_{sg}(\mathbf{e}^c,\mathbf{w}) + B_{sg}(\mathbf{e}^d,\mathbf{w}). \tag{8.94}$$

Thereby, (8.93) and (8.94) entail that

$$B_{sg}(\mathbf{e}^c,\mathbf{w}) = L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta) - B_{sg}(\mathbf{e}^d,\mathbf{w}). \tag{8.95}$$

After that, by employing the definition of energy seminorm, (8.25), the decomposition of the error, (8.92), and then (A.14), we obtain for the energy seminorm of the error $|||\mathbf{u}-\mathbf{u}_h|||_{sg}$

$$\begin{aligned}
|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \leqslant\ & 2\left(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + ||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{e}^d||_\Omega^2\right.\\
& + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
& + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2 + ||(\mu g^2)^{1/2}\Delta_h\mathbf{e}^d||_\Omega^2\\
& \left.+||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2 + ||\mu^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\right)\\
& + C_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\
& + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
& + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
& \left.+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}.
\end{aligned} \tag{8.96}$$

Thus, to complete the proof, it only remains to estimate the terms of $\mathbf{e}^c$ and $\mathbf{e}^d$, enclosed into the parenthesis on the right-hand side of (8.96).

For the terms of $\mathbf{e}^d$, we have

$$\begin{aligned}
& ||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{e}^d||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
& + ||(\mu g^2)^{1/2}\Delta_h\mathbf{e}^d||_\Omega^2 + ||\mu^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
& \leqslant 2(\lambda+\mu)g^2\sum_{K\in\mathcal{T}}|\mathbf{E}_{op}(\mathbf{u}_h)-\mathbf{u}_h|_{2,K}^2 + 2(\lambda+\mu)\sum_{K\in\mathcal{T}}|\mathbf{E}_{op}(\mathbf{u}_h)-\mathbf{u}_h|_{1,K}^2\\
& + 2\mu g^2\sum_{K\in\mathcal{T}}|\mathbf{E}_{op}(\mathbf{u}_h)-\mathbf{u}_h|_{2,K}^2 + \mu\sum_{K\in\mathcal{T}}|\mathbf{E}_{op}(\mathbf{u}_h)-\mathbf{u}_h|_{1,K}^2.
\end{aligned} \tag{8.97}$$

By recalling the Lemma 8.6.2.1 on the right-hand side of (8.97), we deduce

$$\begin{aligned}
& ||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{e}^d||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
& + ||(\mu g^2)^{1/2}\Delta_h\mathbf{e}^d||_\Omega^2 + ||\mu^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
& \leqslant C\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\
& + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
& \left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}.
\end{aligned} \tag{8.98}$$

Therefore, by inserting the inequality (8.98) on the right-hand side of

(8.96), we reach the conclusion for the energy seminorm of the error

$$
\begin{aligned}
|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \leqslant\ & 2\left(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2+||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2\right.\\
&\left.+||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2+||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2\right)\\
&+C_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\
&+(\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2+(\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+\mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&\left.+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}. \qquad (8.99)
\end{aligned}
$$

Now, it only remains to estimate the terms of $\mathbf{e}^c$.

Next, we notice that $\mathcal{L}_i(\mathbf{e}^c)=0$, $i=1,2,3,4$, since $\mathbf{e}^c\in H_0^2(\Omega)^2$. Ergo, upon setting $\mathbf{w}=\mathbf{e}^c$ in (8.95), we deduce

$$
B_{sg}(\mathbf{e}^c,\mathbf{e}^c)=L_{sg}(\eta)-B_{sg}(\mathbf{u}_h,\eta)-B_{sg}(\mathbf{e}^d,\mathbf{e}^c). \qquad (8.100)
$$

Wherefore, recalling the triangle inequality on the right-hand side of (8.100) derives

$$
B_{sg}(\mathbf{e}^c,\mathbf{e}^c)\leqslant|L_{sg}(\eta)-B_{sg}(\mathbf{u}_h,\eta)|+|B_{sg}(\mathbf{e}^d,\mathbf{e}^c)|, \qquad (8.101)
$$

where

$$
\begin{aligned}
B_{sg}(\mathbf{e}^c,\mathbf{e}^c)\ =\ & ||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2+||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2\\
&+||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2+||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2. \qquad (8.102)
\end{aligned}
$$

So, the following step will be to estimate the terms on the right-hand side of (8.101). We shall initially bound the third factor. Since $\mathbf{e}^c\in H_0^2(\Omega)$,

$$
\mathcal{L}_i(\mathbf{e}^c)=[\![\mathbf{e}^c]\!]=[\![\nabla\mathbf{e}^c]\!]=0\quad\forall i=1,2,3,4,
$$

by applying at first the triangle inequality in the bilinear form (8.33), we can obtain

$$
\begin{aligned}
|B_{sg}(\mathbf{e}^d,\mathbf{e}^c)|\ \leqslant\ & \int_\Omega\left|(\lambda+\mu)g^2D_h^2\mathbf{e}^dD^2\mathbf{e}^c\right|dv+\int_\Omega\left|(\lambda+\mu)\nabla_h\mathbf{e}^d\cdot\nabla\mathbf{e}^c\right|dv\\
&+\int_\Omega\left|\mu g^2\Delta_h\mathbf{e}^d\Delta\mathbf{e}^c\right|dv+\int_\Omega\left|\mu\nabla_h\mathbf{e}^d:\nabla\mathbf{e}^c\right|dv\\
&+\int_\Omega\left|\mathcal{L}_1(\mathbf{e}^d)(\lambda+\mu)g^2D^2\mathbf{e}^c\right|dv+\int_\Omega\left|\mathcal{L}_2(\mathbf{e}^d)(\lambda+\mu)\nabla\mathbf{e}^c\right|dv\\
&+\int_\Omega\left|\mathcal{L}_3(\mathbf{e}^d)\mu g^2\Delta\mathbf{e}^c\right|dv+\int_\Omega\left|\mathcal{L}_4(\mathbf{e}^d)\mu\nabla\mathbf{e}^c\right|dv.
\end{aligned}
\qquad (8.103)
$$

Then, by recalling the Cauchy-Schwarz inequality (A.12) on the right-hand side of (8.103), we consequently get

$$\begin{aligned} |B_{sg}(\mathbf{e}^d, \mathbf{e}^c| \ \leqslant \ & ||\{(\lambda+\mu)g^2\}^{1/2} D_h^2 \mathbf{e}^d||_\Omega ||\{(\lambda+\mu)g^2\}^{1/2} D^2 \mathbf{e}^c||_\Omega \\ & + ||(\lambda+\mu)^{1/2}\nabla_h \mathbf{e}^d||_\Omega ||(\lambda+\mu)^{1/2}\nabla \mathbf{e}^c||_\Omega \\ & + ||(\mu g^2)^{1/2}\Delta_h \mathbf{e}^d||_\Omega ||(\mu g^2)^{1/2}\Delta \mathbf{e}^c||_\Omega \\ & + ||\mu^{1/2}\nabla_h \mathbf{e}^d||_\Omega ||\mu^{1/2}\nabla \mathbf{e}^c||_\Omega \\ & + ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{e}^d)||_\Omega ||\{(\lambda+\mu)g^2\}^{1/2} D^2 \mathbf{e}^c||_\Omega \\ & + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{e}^d)||_\Omega ||(\lambda+\mu)^{1/2}\nabla \mathbf{e}^c||_\Omega \\ & + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{e}^d)||_\Omega ||(\mu g^2)^{1/2}\Delta \mathbf{e}^c||_\Omega \\ & + ||\mu^{1/2}\mathcal{L}_4(\mathbf{e}^d)||_\Omega ||\mu^{1/2}\Delta \mathbf{e}^c||_\Omega. \end{aligned} \tag{8.104}$$

Using the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side of (8.104), we have

$$\begin{aligned} |B_{sg}(\mathbf{e}^d, \mathbf{e}^c)| \ \leqslant \ & \big(||\{(\lambda+\mu)g^2\}^{1/2} D_h^2 \mathbf{e}^d||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla_h \mathbf{e}^d||_\Omega^2 \\ & + ||(\mu g^2)^{1/2}\Delta_h \mathbf{e}^d||_\Omega^2 + ||\mu^{1/2}\nabla_h \mathbf{e}^d||_\Omega^2 \\ & + ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{e}^d)||_\Omega^2 + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{e}^d)||_\Omega^2 \\ & + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{e}^d)||_\Omega^2 + ||\mu^{1/2}\mathcal{L}_4(\mathbf{e}^d)||_\Omega^2\big)^{1/2} \\ & \times \big(2||\{(\lambda+\mu)g^2\}^{1/2} D^2 \mathbf{e}^c||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla \mathbf{e}^c||_\Omega \\ & + 2||(\mu g^2)^{1/2}\Delta \mathbf{e}^c||_\Omega^2 + 2||\mu^{1/2}\Delta \mathbf{e}^c||_\Omega^2\big)^{1/2}. \end{aligned} \tag{8.105}$$

Afterwards, by invoking the stability of lifting operators, (8.36) – (8.39), and by inserting the inequality (8.98) on the right-hand side of (8.105), we reach to

$$\begin{aligned} |B_{sg}(\mathbf{e}^d, \mathbf{e}^c)| \ \leqslant \ & C_p^{1/2}\big\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla \mathbf{u}_h]\!]||_{\Gamma_0}^2 \\ & + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 \\ & + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla \mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\big\}^{1/2} \\ & \times \big(2||\{(\lambda+\mu)g^2\}^{1/2} D^2 \mathbf{e}^c||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla \mathbf{e}^c||_\Omega \\ & + 2||(\mu g^2)^{1/2}\Delta \mathbf{e}^c||_\Omega^2 + 2||\mu^{1/2}\Delta \mathbf{e}^c||_\Omega^2\big)^{1/2}. \end{aligned} \tag{8.106}$$

To proceed, we shall estimate the first two terms on the right-hand side of (8.101), hence we obtain

$$
\begin{aligned}
& L_{sg}(\eta) - B_{sg}(\mathbf{u}_h, \eta) \\
& = \int_\Omega \tilde{\mathbf{f}} \eta dv - \int_\Omega \left\{ (\lambda+\mu) g^2 D_h^2 \mathbf{u}_h D_h^2 \eta + (\lambda+\mu) \nabla_h \mathbf{u}_h \cdot \nabla_h \eta \right\} dv \\
& - \int_\Omega \left(\mu g^2 \Delta_h \mathbf{u}_h \Delta_h \eta + \mu \nabla_h \mathbf{u}_h : \nabla_h \eta \right) dv \\
& - \int_\Omega \left\{ \mathcal{L}_1(\eta)(\lambda+\mu) g^2 D_h^2 \mathbf{u}_h + \mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu) g^2 D_h^2 \eta \right\} dv \\
& + \int_\Omega \{ \mathcal{L}_2(\eta)(\lambda+\mu) \nabla_h \mathbf{u}_h + \mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu) \nabla_h \eta \} dv \\
& - \int_\Omega \left\{ \mathcal{L}_3(\eta) \mu g^2 \Delta_h \mathbf{u}_h + \mathcal{L}_3(\mathbf{u}_h) \mu g^2 \Delta_h \eta \right\} dv \\
& + \int_\Omega \{ \mathcal{L}_4(\eta) \mu \nabla_h \mathbf{u}_h + \mathcal{L}_4(\mathbf{u}_h) \mu \nabla_h \eta \} dv \\
& - \int_{\Gamma_0} \gamma [\![\mathbf{u}_h]\!] [\![\eta]\!] dr - \int_{\Gamma_0} \zeta [\![\nabla \mathbf{u}_h]\!] [\![\nabla \eta]\!] dr - \int_{\Gamma_0} \xi [\![\mathbf{u}_h]\!] [\![\eta]\!] dr \\
& - \int_{\Gamma_0} \alpha [\![\mathbf{u}_h]\!] [\![\eta]\!] dr - \int_{\Gamma_0} \beta [\![\nabla \mathbf{u}_h]\!] [\![\nabla \eta]\!] dr - \int_{\Gamma_0} \delta [\![\mathbf{u}_h]\!] [\![\eta]\!] dr.
\end{aligned}
\tag{8.107}
$$

A next step is to perform integration by parts on the right-hand side of

(8.107), thereby we arrive at

$$
\begin{aligned}
& L_{sg}(\eta) - B_{sg}(\mathbf{u}_h, \eta) \\
& = \int_\Omega \left\{ \tilde{\mathbf{f}} - (\lambda+\mu) g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu) D_h^2 \mathbf{u}_h - \mu g^2 \Delta_h^2 \mathbf{u}_h + \mu \Delta_h \mathbf{u}_h \right\} \eta dv \\
& - \sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu) g^2 D^2 \mathbf{u}_h (\nabla\eta\cdot n) dr + \sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h \cdot n\eta dr \\
& - \sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu) \nabla \mathbf{u}_h \cdot n\eta dr - \sum_{K\in\mathcal{T}} \int_{\partial K} \mu g^2 \Delta \mathbf{u}_h (\nabla\eta\cdot n) dr \\
& + \sum_{K\in\mathcal{T}} \int_{\partial K} \mu g^2 \nabla\Delta \mathbf{u}_h \cdot n\eta dr - \sum_{K\in\mathcal{T}} \int_{\partial K} \mu \nabla \mathbf{u}_h \cdot n\eta dr \\
& - \int_\Omega \left\{ \mathcal{L}_1(\eta)(\lambda+\mu) g^2 D_h^2 \mathbf{u}_h + \mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu) g^2 D_h^2 \eta \right\} dv \\
& + \int_\Omega \left\{ \mathcal{L}_2(\eta)(\lambda+\mu) \nabla_h \mathbf{u}_h + \mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu) \nabla_h \eta \right\} dv \\
& - \int_\Omega \left\{ \mathcal{L}_3(\eta)\mu g^2 \Delta_h \mathbf{u}_h + \mathcal{L}_3(\mathbf{u}_h)\mu g^2 \Delta_h \eta \right\} dv \\
& + \int_\Omega \left\{ \mathcal{L}_4(\eta)\mu \nabla_h \mathbf{u}_h + \mathcal{L}_4(\mathbf{u}_h)\mu \nabla_h \eta \right\} dv \\
& - \int_{\Gamma_0} \gamma [\![\mathbf{u}_h]\!][\![\eta]\!] dr - \int_{\Gamma_0} \zeta [\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!] dr - \int_{\Gamma_0} \xi [\![\mathbf{u}_h]\!][\![\eta]\!] dr \\
& - \int_{\Gamma_0} \alpha [\![\mathbf{u}_h]\!][\![\eta]\!] dr - \int_{\Gamma_0} \beta [\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!] dr - \int_{\Gamma_0} \delta [\![\mathbf{u}_h]\!][\![\eta]\!] dr.
\end{aligned} \tag{8.108}
$$

Thanks to the fact that $\mathbf{u}_h, \mathbf{w}_h \in \mathcal{S}_1$ and $\mathbf{w} \in H_0^2(\Omega)^2$, we can use the definitions of the lifting operators, (8.28) – (8.31), to deduce

$$
\begin{aligned}
\int_\Omega \mathcal{L}_1(\eta)(\lambda+\mu) g^2 D_h^2 \mathbf{u}_h dv &= \int_{\Gamma_0} [\![\eta]\!] \langle (\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h \rangle dr \\
&\quad - \int_{\Gamma_0} \langle (\lambda+\mu) g^2 D^2 \mathbf{u}_h \rangle [\![\nabla\eta]\!] dr,
\end{aligned} \tag{8.109}
$$

$$
\int_\Omega \mathcal{L}_2(\eta)(\lambda+\mu) \nabla_h \mathbf{u}_h dv = \int_{\Gamma_0} [\![\eta]\!] \langle (\lambda+\mu) \nabla \mathbf{u}_h \rangle dr, \tag{8.110}
$$

$$
\int_\Omega \mathcal{L}_3(\eta)\mu g^2 \Delta_h \mathbf{u}_h dv = \int_{\Gamma_0} [\![\eta]\!] \langle \mu g^2 \nabla\Delta \mathbf{u}_h \rangle dr - \int_{\Gamma_0} \langle \mu g^2 \Delta \mathbf{u}_h \rangle [\![\nabla\eta]\!] dr \tag{8.111}
$$

and

$$\int_{\Omega} \mathcal{L}_4(\eta)\mu\nabla_h \mathbf{u}_h dv = \int_{\Gamma_0} [\![\eta]\!]\langle \mu\nabla \mathbf{u}_h\rangle dr. \tag{8.112}$$

As showed at the beginning of this chapter, the integrals on ∂K on the right-hand side of (8.108) can be written as

$$\begin{aligned}\sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu)g^2 D^2\mathbf{u}_h(\nabla\eta\cdot n)dr &= \int_{\Gamma_0} \langle(\lambda+\mu)g^2 D^2\mathbf{u}_h\rangle[\![\nabla\eta]\!]dr \\ &\quad + \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu)g^2 D^2\mathbf{u}_h]\!]\langle\nabla\eta\rangle dr,\end{aligned} \tag{8.113}$$

$$\begin{aligned}\sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu)g^2 \nabla D^2\mathbf{u}_h\cdot n\eta dr &= \int_{\Gamma_0} \langle(\lambda+\mu)g^2 \nabla D^2\mathbf{u}_h\rangle[\![\eta]\!]dr \\ &\quad + \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu)g^2 \nabla D^2\mathbf{u}_h]\!]\langle\eta\rangle dr,\end{aligned} \tag{8.114}$$

$$\begin{aligned}\sum_{K\in\mathcal{T}} \int_{\partial K} (\lambda+\mu)\nabla\mathbf{u}_h\cdot n\eta dr &= \int_{\Gamma_0} \langle(\lambda+\mu)\nabla\mathbf{u}_h\rangle[\![\eta]\!]dr \\ &\quad + \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu)\nabla\mathbf{u}_h]\!]\langle\eta\rangle dr,\end{aligned} \tag{8.115}$$

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \mu g^2\Delta\mathbf{u}_h(\nabla\eta\cdot n)dr = \int_{\Gamma_0} \langle\mu g^2\Delta\mathbf{u}_h\rangle[\![\nabla\eta]\!]dr + \int_{\Gamma_{\text{int}}} [\![\mu g^2\Delta\mathbf{u}_h]\!]\langle\nabla\eta\rangle dr, \tag{8.116}$$

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \mu g^2\nabla\Delta\mathbf{u}_h\cdot n\eta dr = \int_{\Gamma_0} \langle\mu g^2\nabla\Delta\mathbf{u}_h\rangle[\![\eta]\!]dr + \int_{\Gamma_{\text{int}}} [\![\mu g^2\nabla\Delta\mathbf{u}_h]\!]\langle\eta\rangle dr \tag{8.117}$$

and

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \mu\nabla\mathbf{u}_h\cdot n\eta dr = \int_{\Gamma_0} \langle\mu\nabla\mathbf{u}_h\rangle[\![\eta]\!]dr + \int_{\Gamma_{\text{int}}} [\![\mu\nabla\mathbf{u}_h]\!]\langle\eta\rangle dr. \tag{8.118}$$

The substitution of the mathematical expressions (8.109) – (8.118) on the right-hand side of (8.108) yields

$$
\begin{aligned}
&L_{sg}(\eta) - B_{sg}(\mathbf{u}_h, \eta) \\
&= \int_\Omega \left\{ \tilde{\mathbf{f}} - (\lambda+\mu) g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu) D_h^2 \mathbf{u}_h - \mu g^2 \Delta_h^2 \mathbf{u}_h + \mu \Delta_h \mathbf{u}_h \right\} \eta dv \\
&- \int_\Omega \mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu) g^2 D_h^2 \eta dv + \int_\Omega \mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu) \nabla_h \eta dv \\
&- \int_\Omega \mathcal{L}_3(\mathbf{u}_h) \mu g^2 \Delta_h \eta dv + \int_\Omega \mathcal{L}_4(\mathbf{u}_h) \mu \nabla_h \eta dv + \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h]\!] \langle \eta \rangle dr \\
&- \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu) g^2 D^2 \mathbf{u}_h]\!] \langle \nabla \eta \rangle dr - \int_{\Gamma_{\text{int}}} [\![(\lambda+\mu) \nabla \mathbf{u}_h]\!] \langle \eta \rangle dr \\
&+ \int_{\Gamma_{\text{int}}} [\![\mu g^2 \nabla \Delta \mathbf{u}_h]\!] \langle \eta \rangle dr - \int_{\Gamma_{\text{int}}} [\![\mu g^2 \Delta \mathbf{u}_h]\!] \langle \nabla \eta \rangle dr - \int_{\Gamma_{\text{int}}} [\![\mu \nabla \mathbf{u}_h]\!] \langle \eta \rangle dr \\
&- \int_{\Gamma_0} \gamma [\![\mathbf{u}_h]\!] [\![\eta]\!] dr - \int_{\Gamma_0} \zeta [\![\nabla \mathbf{u}_h]\!] [\![\nabla \eta]\!] dr - \int_{\Gamma_0} \xi [\![\mathbf{u}_h]\!] [\![\eta]\!] dr \\
&- \int_{\Gamma_0} \alpha [\![\mathbf{u}_h]\!] [\![\eta]\!] dr - \int_{\Gamma_0} \beta [\![\nabla \mathbf{u}_h]\!] [\![\nabla \eta]\!] dr - \int_{\Gamma_0} \delta [\![\mathbf{u}_h]\!] [\![\eta]\!] dr
\end{aligned}
\tag{8.119}
$$

and by using the triangle inequality on the right-hand side of (8.119), we have

$$
\begin{aligned}
&|L_{sg}(\eta) - B_{sg}(\mathbf{u}_h, \eta)| \\
&\leqslant \int_\Omega \left| \left\{ \tilde{\mathbf{f}} - (\lambda+\mu) g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu) D_h^2 \mathbf{u}_h - \mu g^2 \Delta_h^2 \mathbf{u}_h + \mu \Delta_h \mathbf{u}_h \right\} \eta \right| dv \\
&+ \int_\Omega \left| \mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu) g^2 D_h^2 \eta \right| dv + \int_\Omega |\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h \eta|\, dv \\
&+ \int_\Omega \left| \mathcal{L}_3(\mathbf{u}_h) \mu g^2 \Delta_h \eta \right| dv + \int_\Omega |\mathcal{L}_4(\mathbf{u}_h)\mu \nabla_h \eta|\, dv \\
&+ \int_{\Gamma_{\text{int}}} \left| [\![(\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h]\!] \langle \eta \rangle \right| dr + \int_{\Gamma_{\text{int}}} \left| [\![(\lambda+\mu) g^2 D^2 \mathbf{u}_h]\!] \langle \nabla \eta \rangle \right| dr \\
&+ \int_{\Gamma_{\text{int}}} |[\![(\lambda+\mu)\nabla \mathbf{u}_h]\!]\langle \eta\rangle|\, dr + \int_{\Gamma_{\text{int}}} \left| [\![\mu g^2 \nabla \Delta \mathbf{u}_h]\!] \langle \eta \rangle \right| dr \\
&+ \int_{\Gamma_{\text{int}}} \left| [\![\mu g^2 \Delta \mathbf{u}_h]\!] \langle \nabla \eta \rangle \right| dr + \int_{\Gamma_{\text{int}}} |[\![\mu \nabla \mathbf{u}_h]\!]\langle \eta \rangle|\, dr \\
&+ \int_{\Gamma_0} |\gamma [\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\zeta [\![\nabla \mathbf{u}_h]\!][\![\nabla \eta]\!]|\, dr + \int_{\Gamma_0} |\xi [\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \\
&+ \int_{\Gamma_0} |\alpha [\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\beta [\![\nabla \mathbf{u}_h]\!][\![\nabla \eta]\!]|\, dr + \int_{\Gamma_0} |\delta [\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr.
\end{aligned}
\tag{8.120}
$$

Fix $\mathbf{w}_h$ to be the elementwise linear approximation to $\mathbf{e}^c$ such that

$$|\mathbf{e}^c - \mathbf{w}_h|_{j,K} \leqslant C h_K^{m-j} |\mathbf{e}^c|_{m,K} \tag{8.121}$$

for $C > 0$, independent of $\mathcal{T}$, for $0 \leqslant j \leqslant m \leqslant 2$ and $K \in \mathcal{T}$ (see [55]). We shall employ this to bound the terms on the right-hand side of (8.120).

First, we shall estimate the sixth integral on the right-hand side of (8.120). By applying the Cauchy-Schwarz inequality (A.12), we conclude

$$
\begin{aligned}
&\int_{\Gamma_{\text{int}}} \left| [\![(\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h]\!] \langle \eta \rangle \right| dr \\
&\leqslant ||\gamma^{-1/2} [\![(\lambda+\mu) g^2 \nabla D^2 \mathbf{u}_h]\!]||_{\Gamma_{\text{int}}} ||\gamma^{1/2} \langle \eta \rangle||_{\Gamma_{\text{int}}}.
\end{aligned}
\tag{8.122}
$$

Now, we shall bound the second factor on the right-hand side of (8.122). By recalling the mean value inequality (A.19), we get

$$\begin{aligned}||\gamma^{1/2}\langle\eta\rangle||^2_{\Gamma_{\rm int}} &\leqslant \sum_{e\in\mathcal{E}_{\rm int}}\gamma\left(||\eta^+||^2_e+||\eta^-||^2_e\right)\\ &\leqslant C_\gamma(\lambda+\mu)g^2\sum_{K\in\mathcal{T}}h_K^{-3}||\eta||^2_{\partial K}.\end{aligned} \tag{8.123}$$

After that, employing the trace inequality (A.38) on the right-hand side of (8.123) implies

$$||\gamma^{1/2}\langle\eta\rangle||^2_{\Gamma_{\rm int}} \leqslant CC_\gamma(\lambda+\mu)g^2\sum_{K\in\mathcal{T}}h_K^{-3}\left(h_K^{-1}||\eta||^2_K+h_K|\eta|^2_{1,K}\right). \tag{8.124}$$

Then, by invoking the mathematical inequality (8.121) on the right-hand side of (8.124), we have

$$||\gamma^{1/2}\langle\eta\rangle||^2_{\Gamma_{\rm int}} \leqslant CC_\gamma(\lambda+\mu)g^2|\mathbf{e}^c|^2_{2,\Omega} \tag{8.125}$$

Inserting (8.125) on the right-hand side of (8.122), we arrive to the conclusion that the sixth integral can be bounded as follows

$$\int_{\Gamma_{\rm int}}\left|[\![(\lambda+\mu)g^2\nabla D^2\mathbf{u}_h]\!]\langle\eta\rangle\right|dr \leqslant C||\mathbf{h}^{3/2}[\![(\lambda+\mu)g^2\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{2,\Omega}. \tag{8.126}$$

What is more, we shall analogously estimate the seventh up to eleventh integral on the right-hand side of (8.120). In consequence, we reach to

$$\int_{\Gamma_{\rm int}}\left|[\![(\lambda+\mu)g^2D^2\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right|dr \leqslant C||\mathbf{h}^{1/2}[\![(\lambda+\mu)g^2D^2\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{2,\Omega}, \tag{8.127}$$

$$\int_{\Gamma_{\rm int}}|[\![(\lambda+\mu)\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\,dr \leqslant C||\mathbf{h}^{1/2}[\![(\lambda+\mu)\nabla\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{1,\Omega}, \tag{8.128}$$

$$\int_{\Gamma_{\rm int}}\left|[\![\mu g^2\nabla\Delta\mathbf{u}_h]\!]\langle\eta\rangle\right|dr \leqslant C||\mathbf{h}^{3/2}[\![\mu g^2\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{2,\Omega}, \tag{8.129}$$

$$\int_{\Gamma_{\rm int}}\left|[\![\mu g^2\Delta\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right|dr \leqslant C||\mathbf{h}^{1/2}[\![\mu g^2\Delta\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{2,\Omega}, \tag{8.130}$$

and

$$\int_{\Gamma_{\rm int}}|[\![\mu\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\,dr \leqslant C||\mathbf{h}^{1/2}[\![\mu\nabla\mathbf{u}_h]\!]||_{\Gamma_{\rm int}}|\mathbf{e}^c|_{1,\Omega}. \tag{8.131}$$

In the meanwhile, adding the mathematical expressions (8.126) – (8.131) and then applying the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side derives

$$\begin{aligned}
&\int_{\Gamma_{\text{int}}} \left|[\![(\lambda+\mu)g^2\nabla D^2\mathbf{u}_h]\!]\langle\eta\rangle\right| dr + \int_{\Gamma_{\text{int}}} \left|[\![(\lambda+\mu)g^2 D^2\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right| dr\\
&\int_{\Gamma_{\text{int}}} |[\![(\lambda+\mu)\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\, dr + \int_{\Gamma_{\text{int}}} \left|[\![\mu g^2\nabla\Delta\mathbf{u}_h]\!]\langle\eta\rangle\right| dr\\
&+ \int_{\Gamma_{\text{int}}} \left|[\![\mu g^2\Delta\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right| dr + \int_{\Gamma_{\text{int}}} |[\![\mu\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\, dr\\
&\leqslant C\left(||\mathbf{h}^{3/2}[\![\{(\lambda+\mu)g^2\}^{1/2}\nabla D^2\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{1/2}[\![\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}}\right.\\
&+ ||\mathbf{h}^{1/2}[\![(\lambda+\mu)^{1/2}\nabla\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{3/2}[\![(\mu g^2)^{1/2}\nabla\Delta\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}}\\
&\left.+||\mathbf{h}^{1/2}[\![(\mu g^2)^{1/2}\Delta\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{1/2}[\![\mu^{1/2}\nabla\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}}\right)^{1/2}\\
&\times\left(|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|^2_{2,\Omega} + |(\lambda+\mu)^{1/2}\mathbf{e}^c|^2_{1,\Omega} + |(\mu g^2)^{1/2}\mathbf{e}^c|^2_{2,\Omega} + |\mu^{1/2}\mathbf{e}^c|^2_{1,\Omega}\right)^{1/2}.
\end{aligned} \tag{8.132}$$

Thereafter, we shall follow quite similar series of steps to estimate the twelfth integral, contained the stabilization parameter, on the right-hand side of (8.120). Employing the Cauchy-Schwarz inequality (A.12) gives

$$\int_{\Gamma_0} |\gamma[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \leqslant ||\gamma^{1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}||\gamma^{1/2}[\![\eta]\!]||_{\Gamma_0}. \tag{8.133}$$

Wherefore, it is important to bound the second factor on the right-hand side of (8.133). By applying the jump inequality (A.18), we obtain

$$\begin{aligned}
||\gamma^{1/2}[\![\eta]\!]||^2_{\Gamma_0} &\leqslant \sum_{e\in\mathcal{E}_0} 2\gamma\left(||\eta^+||^2_e + ||\eta^-||^2_e\right)\\
&\leqslant 2C_\gamma(\lambda+\mu)g^2\sum_{K\in\mathcal{T}} h_K^{-3}||\eta||^2_{\partial K}.
\end{aligned} \tag{8.134}$$

Recalling subsequently the trace inequality (A.38) on the right-hand side of (8.134) entails

$$||\gamma^{1/2}[\![\eta]\!]||^2_{\Gamma_0} \leqslant CC_\gamma(\lambda+\mu)g^2\sum_{K\in\mathcal{T}} h_K^{-3}\left(h_K^{-1}||\eta||^2_K + h_K|\eta|^2_{1,K}\right). \tag{8.135}$$

Now, by using the mathematical inequality (8.121) on the right-hand side of (8.135), we conclude

$$||\gamma^{1/2}[\![\eta]\!]||^2_{\Gamma_0} \leqslant CC_\gamma(\lambda+\mu)g^2|\mathbf{e}^c|^2_{2,\Omega}. \tag{8.136}$$

At this point, insertion of (8.136) on the right-hand side of (8.133) yields

$$\int_{\Gamma_0} |\gamma[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \leqslant C||\gamma^{1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}\{C_\gamma(\lambda+\mu)g^2\}^{1/2}|\mathbf{e}^c|_{2,\Omega}. \tag{8.137}$$

In addition, by following the above procedure step by step, we shall achieve to estimate the integrals, which contain the remaining stabilization parameters, on the right-hand side of (8.120). As a consequence, we deduce

$$\int_{\Gamma_0} |\zeta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr \leqslant C||\zeta^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}\{C_\zeta(\lambda+\mu)g^2\}^{1/2}|\mathbf{e}^c|_{2,\Omega}, \tag{8.138}$$

$$\int_{\Gamma_0} |\xi[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \leqslant C||\xi^{1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}\{C_\xi(\lambda+\mu)\}^{1/2}|\mathbf{e}^c|_{1,\Omega}, \tag{8.139}$$

$$\int_{\Gamma_0} |\alpha[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \leqslant C||\alpha^{1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}(C_\alpha\mu g^2)^{1/2}|\mathbf{e}^c|_{2,\Omega}, \tag{8.140}$$

$$\int_{\Gamma_0} |\beta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr \leqslant C||\beta^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}(C_\beta\mu g^2)^{1/2}|\mathbf{e}^c|_{2,\Omega}, \tag{8.141}$$

and

$$\int_{\Gamma_0} |\delta[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \leqslant C||\delta^{1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}(C_\delta\mu)^{1/2}|\mathbf{e}^c|_{1,\Omega}. \tag{8.142}$$

Furthermore, by adding the mathematical expressions (8.137) – (8.142), containing the stabilization parameters, and then by invoking the Cauchy-Schwarz discrete inequality (A.13) on the right-hand side, we conclude

$$\begin{aligned}
&\int_{\Gamma_0} |\gamma[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\zeta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr + \int_{\Gamma_0} |\xi[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \\
&+ \int_{\Gamma_0} |\alpha[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\beta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr + \int_{\Gamma_0} |\delta[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \\
&\leqslant CC_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}\right. \\
&+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0} \\
&\left.+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0}\right\}^{1/2} \\
&\times\left\{|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|^2_{2,\Omega} + |(\lambda+\mu)^{1/2}\mathbf{e}^c|^2_{1,\Omega} + |(\mu g^2)^{1/2}\mathbf{e}^c|^2_{2,\Omega} + |\mu^{1/2}\mathbf{e}^c|^2_{1,\Omega}\right\}^{1/2}.
\end{aligned} \tag{8.143}$$

Moreover, we shall estimate the rest of the terms on the right-hand side of (8.120). It is obvious that by using the Cauchy-Schwarz inequality (A.12), we arrive at

$$
\begin{aligned}
&\int_\Omega \left|\left\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\right\}\eta\right| dv \\
&+ \int_\Omega \left|\mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu)g^2 D_h^2\eta\right| dv + \int_\Omega \left|\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h\eta\right| dv \\
&+ \int_\Omega \left|\mathcal{L}_3(\mathbf{u}_h)\mu g^2\Delta_h\eta\right| dv + \int_\Omega \left|\mathcal{L}_4(\mathbf{u}_h)\mu\nabla_h\eta\right| dv \\
&\leqslant ||\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h||_\Omega||\eta||_\Omega \\
&+ ||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u}_h)||_\Omega||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\eta||_\Omega \\
&+ ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u}_h)||_\Omega||(\lambda+\mu)^{1/2}\nabla_h\eta||_\Omega \\
&+ ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u}_h)||_\Omega||(\mu g^2)^{1/2}\Delta_h\eta||_\Omega + ||\mu^{1/2}\mathcal{L}_4(\mathbf{u}_h)||_\Omega||\mu^{1/2}\nabla_h\eta||_\Omega.
\end{aligned}
\tag{8.144}
$$

Next, employing the inequality (8.121) on the right hand of (8.144) implies

$$
\begin{aligned}
&\int_\Omega \left|\left\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\right\}\eta\right| dv \\
&+ \int_\Omega \left|\mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu)g^2 D_h^2\eta\right| dv + \int_\Omega \left|\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h\eta\right| dv \\
&+ \int_\Omega \left|\mathcal{L}_3(\mathbf{u}_h)\mu g^2\Delta_h\eta\right| dv + \int_\Omega \left|\mathcal{L}_4(\mathbf{u}_h)\mu\nabla_h\eta\right| dv \\
&\leqslant C||\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h||_\Omega \\
&\times \mathbf{h}^2|\mathbf{e}^c|_{2,\Omega} \\
&+ C||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u}_h)||_\Omega|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|_{2,\Omega} \\
&+ C||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u}_h)||_\Omega|(\lambda+\mu)^{1/2}\mathbf{e}^c|_{1,\Omega} \\
&+ C||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u}_h)||_\Omega|(\mu g^2)^{1/2}\mathbf{e}^c|_{2,\Omega} + C||\mu^{1/2}\mathcal{L}_4(\mathbf{u}_h)||_\Omega|\mu^{1/2}\mathbf{e}^c|_{1,\Omega}.
\end{aligned}
\tag{8.145}
$$

Thereafter, by recalling the Cauchy-Schwarz discrete inequality (A.13) on the right hand of (8.145), we reach to

$$\int_\Omega \left|\left\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\right\}\eta\right| dv$$
$$+ \int_\Omega \left|\mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu)g^2 D_h^2\eta\right| dv + \int_\Omega \left|\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h\eta\right| dv$$
$$+ \int_\Omega \left|\mathcal{L}_3(\mathbf{u}_h)\mu g^2\Delta_h\eta\right| dv + \int_\Omega \left|\mathcal{L}_4(\mathbf{u}_h)\mu\nabla_h\eta\right| dv$$
$$\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||_\Omega^2$$
$$||\{(\lambda+\mu)g^2\}^{1/2}\mathcal{L}_1(\mathbf{u}_h)||_\Omega^2 + ||(\lambda+\mu)^{1/2}\mathcal{L}_2(\mathbf{u}_h)||_\Omega^2 + ||(\mu g^2)^{1/2}\mathcal{L}_3(\mathbf{u}_h)||_\Omega^2$$
$$+||\mu^{1/2}\mathcal{L}_4(\mathbf{u}_h)||_\Omega^2\Big)^{1/2}$$
$$\times\left(|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|_{2,\Omega}^2 + |(\lambda+\mu)^{1/2}\mathbf{e}^c|_{1,\Omega}^2 + |(\mu g^2)^{1/2}\mathbf{e}^c|_{2,\Omega}^2 + |\mu^{1/2}\mathbf{e}^c|_{1,\Omega}^2\right)^{1/2}. \tag{8.146}$$

Now, we can invoke the stability of the lifting operators (8.36) – (8.39), on the right-hand side of (8.146), so we arrive to the conclusion

$$\int_\Omega \left|\left\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\right\}\eta\right| dv$$
$$+ \int_\Omega \left|\mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu)g^2 D_h^2\eta\right| dv + \int_\Omega \left|\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h\eta\right| dv$$
$$+ \int_\Omega \left|\mathcal{L}_3(\mathbf{u}_h)\mu g^2\Delta_h\eta\right| dv + \int_\Omega \left|\mathcal{L}_4(\mathbf{u}_h)\mu\nabla_h\eta\right| dv$$
$$\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2 \left(D_h^2\right)^2 \mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||_\Omega^2$$
$$+ C_p\big\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2$$
$$+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2$$
$$+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\big\}\Big)^{1/2}$$
$$\times\left(|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|_{2,\Omega}^2 + |(\lambda+\mu)^{1/2}\mathbf{e}^c|_{1,\Omega}^2 + |(\mu g^2)^{1/2}\mathbf{e}^c|_{2,\Omega}^2 + |\mu^{1/2}\mathbf{e}^c|_{1,\Omega}^2\right)^{1/2}. \tag{8.147}$$

We note that it holds

$$|\{(\lambda+\mu)g^2\}^{1/2}\mathbf{e}^c|_{2,\Omega}^2 = ||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2, \tag{8.148}$$

$$|\mu^{1/2}\mathbf{e}^c|_{1,\Omega}^2 = ||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2, \tag{8.149}$$

for the gradient ∇ of vector $\mathbf{e}^c$ and

$$|(\lambda+\mu)^{1/2}\mathbf{e}^c|^2_{1,\Omega} \leqslant C||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||^2_{\Omega}, \tag{8.150}$$

for the divergence ∇ of vector $\mathbf{e}^c$ from Poincaré's inequality (A.22).

In that point, we shall assume that it also holds

$$|(\mu g^2)^{1/2}\mathbf{e}^c|^2_{2,\Omega} \leqslant C||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||^2_{\Omega}. \tag{8.151}$$

Now, by inserting the mathematical expressions (8.148) – (8.151) on the right-hand side of (8.147), of (8.132) as well as of (8.143) respectively, we deduce

$$\begin{aligned}
&\int_\Omega \left|\left\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\right\}\eta\right| dv\\
&+ \int_\Omega \left|\mathcal{L}_1(\mathbf{u}_h)(\lambda+\mu)g^2D_h^2\eta\right| dv + \int_\Omega |\mathcal{L}_2(\mathbf{u}_h)(\lambda+\mu)\nabla_h\eta|\, dv\\
&+ \int_\Omega \left|\mathcal{L}_3(\mathbf{u}_h)\mu g^2\Delta_h\eta\right| dv + \int_\Omega |\mathcal{L}_4(\mathbf{u}_h)\mu\nabla_h\eta|\, dv\\
&\leqslant C\left(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||^2_\Omega\right.\\
&+ C_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}\right.\\
&+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}\\
&\left.\left. +\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0}\right\}\right)^{1/2}\\
&\times\left(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||^2_\Omega + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c|^2_\Omega + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c|^2_\Omega\right.\\
&\left. +||\mu^{1/2}\nabla\mathbf{e}^c||^2_\Omega\right)^{1/2},
\end{aligned} \tag{8.152}$$

$$
\begin{aligned}
&\int_{\Gamma_{\text{int}}} \left|[\![(\lambda+\mu)g^2\nabla D^2\mathbf{u}_h]\!]\langle\eta\rangle\right| dr + \int_{\Gamma_{\text{int}}} \left|[\![(\lambda+\mu)g^2 D^2\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right| dr \\
&\int_{\Gamma_{\text{int}}} |[\![(\lambda+\mu)\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\, dr + \int_{\Gamma_{\text{int}}} \left|[\![\mu g^2\nabla\Delta\mathbf{u}_h]\!]\langle\eta\rangle\right| dr \\
&+ \int_{\Gamma_{\text{int}}} \left|[\![\mu g^2\Delta\mathbf{u}_h]\!]\langle\nabla\eta\rangle\right| dr + \int_{\Gamma_{\text{int}}} |[\![\mu\nabla\mathbf{u}_h]\!]\langle\eta\rangle|\, dr \\
&\leqslant C\left(||\mathbf{h}^{3/2}[\![\{(\lambda+\mu)g^2\}^{1/2}\nabla D^2\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{1/2}[\![\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}}\right. \\
&+ ||\mathbf{h}^{1/2}[\![(\lambda+\mu)^{1/2}\nabla\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{3/2}[\![(\mu g^2)^{1/2}\nabla\Delta\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} \\
&\left.+||\mathbf{h}^{1/2}[\![(\mu g^2)^{1/2}\Delta\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}} + ||\mathbf{h}^{1/2}[\![\mu^{1/2}\nabla\mathbf{u}_h]\!]||^2_{\Gamma_{\text{int}}}\right)^{1/2} \\
&\times\left(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||^2_\Omega + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||^2_\Omega + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c|^2_\Omega\right. \\
&\left.+||\mu^{1/2}\nabla\mathbf{e}^c||^2_\Omega\right)^{1/2},
\end{aligned}
\tag{8.153}
$$

and

$$
\begin{aligned}
&\int_{\Gamma_0} |\gamma[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\zeta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr + \int_{\Gamma_0} |\xi[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \\
&+ \int_{\Gamma_0} |\alpha[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr + \int_{\Gamma_0} |\beta[\![\nabla\mathbf{u}_h]\!][\![\nabla\eta]\!]|\, dr + \int_{\Gamma_0} |\delta[\![\mathbf{u}_h]\!][\![\eta]\!]|\, dr \\
&\leqslant CC_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0}\right. \\
&+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0} \\
&\left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||^2_{\Gamma_0} + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||^2_{\Gamma_0}\right\}^{1/2} \\
&\times\left\{||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||^2_\Omega + |(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||^2_\Omega + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c|^2_\Omega\right. \\
&\left.+||\mu^{1/2}\nabla\mathbf{e}^c||^2_\Omega\right\}^{1/2}.
\end{aligned}
\tag{8.154}
$$

Now, we shall insert (8.152) – (8.154) on the right-hand side of (8.120). Afterwards, by combining the deriving inequality with the mathematical expressions (8.101), (8.102), (8.106) and the Cauchy-Schwarz discrete inequality (A.13), we obtain

$$
\begin{aligned}
&||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + |(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2 + |(\mu g^2)^{1/2}\Delta\mathbf{e}^c|_\Omega^2 + ||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2 \\
&\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||_\Omega^2 \\
&+ C_p^2\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right. \\
&+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 \\
&\left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\} \\
&+ (\lambda+\mu)g^2||\mathbf{h}^{3/2}[\![\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + (\lambda+\mu)g^2||\mathbf{h}^{1/2}[\![D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 \\
&+ (\lambda+\mu)||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + \mu g^2||\mathbf{h}^{3/2}[\![\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 \\
&+\mu g^2||\mathbf{h}^{1/2}[\![\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + \mu||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\Big)\,.
\end{aligned}
\tag{8.155}
$$

Finally, insertion of inequality (8.155) on the right-hand side of (8.99) gives

$$
\begin{aligned}
&|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \\
&\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||_\Omega^2 \\
&+ C_p^2\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right. \\
&+ (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 \\
&\left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\} \\
&+ (\lambda+\mu)g^2||\mathbf{h}^{3/2}[\![\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + (\lambda+\mu)g^2||\mathbf{h}^{1/2}[\![D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 \\
&+ (\lambda+\mu)||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma\ \text{int}}^2 + \mu g^2||\mathbf{h}^{3/2}[\![\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma\ \text{int}}^2 \\
&+\mu g^2||\mathbf{h}^{1/2}[\![\Delta\mathbf{u}_h]\!]||_{\Gamma\ \text{int}}^2 + \mu||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\Big)\,,
\end{aligned}
$$

which is the desired result. □

Chapter 9

Numerical Validation

In this section, we numerically test the interior penalty discontinuous Galerkin finite element methods for the one-dimensional Toupin-Mindlin strain gradient theory of the previous section. Thus, we go on introducing a specific boundary value problem and performing a convergence study for different orders of interpolation and discretization. We end this section with a comparison of the numerical findings with our analytical results.

9.1 Boundary Layer Problem with the SGE

We wish to consider a Toupin-Mindlin boundary layer as model problem to validate our interior penalty discontinuous Galerkin methods for the strain gradient elasticity. We proceed with the introduction of the model problem, its exact solution (which will be used for assessing the accuracy of the numerical method), the presentation of a convergence study and the comparison of the numerical findings with our results of the error analysis.

9.1.1 Model Problem

We want to simulate a prismatic bar, with a cross-section A, which is fixed on its left and upon which an axial tensile load $\bar{P}$ acts on its right. In particular, it is assumed that the length from the attachment to where the axial tensile load acts is L ($L = 1$) in this problem. Furthermore, an axially distributed load $\bar{f}$ is applied along the bar. Therefore, the problem can be formulated as

$$g^2 u^{IV} - u'' = \frac{\bar{f}}{AE} = f \quad \text{in } \Omega = (0,1), \tag{9.1}$$

$$\begin{aligned} u(0) &= 0, \\ u'(1) &= \varepsilon_1, \\ R(0) &= \bar{R}, \\ P(1) &= \bar{P}, \end{aligned} \tag{9.2}$$

and the exact solution to this problem can be expressed as

$$\begin{aligned} u(x) \quad = \quad & \frac{AEc - R + Px + R\cosh(x/g)}{AE} \\ & - \frac{sech(1/g)(g(P - AE\varepsilon_1) + R\sinh(1/g))\sinh(x/g)}{AE}, \end{aligned} \tag{9.3}$$

if $\bar{f} = 0$.
Then, we select the values of the constants: $g = 0.1$, $\varepsilon_1 = 0.6$, $AE = 1$, $\bar{R} = 0$ and $\bar{P} = 1$.

9.1.2 Convergence Study

We present a series of numerical experiments to confirm the a priori error estimates stated in the Theorems 4.5.1.2, 4.5.1.3, 4.5.1.5 and 4.5.1.6.

We emphasize that in order to ensure that the a priori error estimates for the symmetric method (SIPG) are valid, from the Chapter 4, the selected values of the stabilization constants must be large enough. These constants depend on the constants in the inverse inequalities, the shape regularity of the mesh and the degree of the approximation polynomial, and are difficult to determine in practice. For the present model problem, for reasons of consistency for each of the methods considered, we used the values $C_\alpha = C_\beta = 12$ and $C_\delta = 6$.

We have employed a uniform mesh, which has been successively refined. We also note that $N_{el} = \frac{1}{h}$ serves as a parameter to indicate the degree of mesh refinement.

One can show that

$$|||u - u^h|||_{sb} = Ch^{k_1} \quad \text{and} \quad ||u - u^h||_\Omega = Ch^{k_2},$$

where C is a constant independent of h. The convergence rate of the method in the energy seminorm, respectively in the L^2-norm, is then defined to be the power k_1, respectively k_2. Assuming that the solution is smooth, the mesh is uniform, and discontinuous piecewise polynomials of degree k are employed, the convergence rates are summarized in Tables 9.1 – 9.4. These rates can be proven theoretically, and they are obtained numerically for h sufficiently small by applying the formulas

$$k_1 = \frac{1}{\ln 2} \ln \left(\frac{|||u - u^h|||_{sb}}{|||u - u^{h/2}|||_{sb}} \right), \quad k_2 = \frac{1}{\ln 2} \ln \left(\frac{||u - u^h||_\Omega}{||u - u^{h/2}||_\Omega} \right). \tag{9.4}$$

One can also show that

$$|||u - u^p|||_{sb} = Cp^{-k_3} \quad \text{and} \quad ||u - u^p||_\Omega = Cp^{-k_4},$$

where C is a constant independent of p. The convergence rate of the method in the energy seminorm, respectively in the L^2-norm, is then defined to be the power k_3, respectively k_4. Assuming that the solution is smooth, the p-refinement is uniform and discontinuous piecewise polynomials of degree k are employed, the convergence rates are summarized in Tables 9.5 – 9.6. These rates can be proven theoretically, and they are obtained numerically for p by applying the formulas

$$\begin{aligned} k_3 &= \frac{1}{\ln(p+1) - \ln p} \ln \left(\frac{|||u - u^p|||_{sb}}{|||u - u^{p+1}|||_{sb}} \right), \\ k_4 &= \frac{1}{\ln(p+1) - \ln p} \ln \left(\frac{||u - u^p||_\Omega}{||u - u^{p+1}||_\Omega} \right). \end{aligned} \tag{9.5}$$

In Figures 9.1 – 9.2, we first display the convergence of the h-version SIPG as well as the h-version NIPG in the energy seminorm under h-refinement, respectively. With exact words, we present a comparison of the energy seminorm, $|||\cdot|||_{sb}$, of the error in the approximation to u with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 6$. Moreover, we observe that $|||u - u^h|||_{sb}$ converges to zero, for each fixed p, at the optimal rate $O(h^{p-1})$ as the mesh is refined. Thus, Theorems 4.5.1.2 together with 4.5.1.3 are confirmed.

What is more, in Figures 9.3 – 9.4, we exhibit the convergence of the h-version SIPG as well as the h-version NIPG in the L^2-norm under h-refinement, respectively. To make simple, we plot the L^2-norm of the error in the approximation to u with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 6$. For

$p \geqslant 2$, we notice optimal rates of convergence as the mesh parameter, N_{el}, increases. Especially, in case of SIPG method, $||u - u^h||_\Omega$ converges to zero at the rate $O(h^{p+1})$, as N_{el} tends to infinity, for each fixed p. Nevertheless, in case of NIPG method, $||u - u^h||_\Omega$ converges to zero with a suboptimal rate, as N_{el} tends to infinity, for each fixed p.

Moreover, Figures 9.5 – 9.8 display the convergence of the *hp*-version SIPG and the *hp*-version NIPG in the energy seminorm under h-enrichment, respectively. To clarify, in Figures 9.5 – 9.6, we present a comparison of the energy seminorm, $||| \cdot |||_{sb}$, of the error in the approximation to u with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 4$. On the contrary, in Figures 9.7 – 9.8, we exhibit a comparison of the energy seminorm of the error with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 5$. We conclude that $|||u - u_{DG}|||_{sb}$ converges to zero, for each fixed p, at the optimal rate $O(h^{p-1})$ as the mesh is refined. Therefore, Theorems 4.5.1.5 and 4.5.1.6 are confirmed.

Furthermore, in Figures 9.9 – 9.12, we display the convergence of the *hp*-version SIPG and the *hp*-version NIPG in the L^2-norm under h-enrichment, respectively. With exact words, in Figures 9.9 – 9.10, we plot the L^2-norm of the error in the approximation to u with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 4$. On the other hand, in Figures 9.11 – 9.12, we plot the L^2-norm of the error with the mesh parameter, N_{el}, for $2 \leqslant p \leqslant 5$. For $p \geqslant 2$, in case of SIPG method, we notice optimal rates of convergence as the mesh parameter, N_{el}, increases. In particular, $||u - u_{DG}||_\Omega$ converges to zero at the rate $O(h^{p+1})$, as N_{el} tends to infinity, for each fixed p. However, in case of NIPG method, $||u - u_{DG}||_\Omega$ converges to zero with a suboptimal rate, as N_{el} tends to infinity, for each fixed p.

Figures 9.13 – 9.20 display the convergence with p-refinement for fixed N_{el} of the energy seminorm and the L^2-norm of the error for the *hp*-version SIPG as well as *hp*-version NIPG, respectively. Since the solution u of the test problem is a (real) analytic function, an exponential rate of the convergence under p-enrichment is expected. Indeed, we observe that on a linear-log scale, the convergence plots become straight lines as the degree of the approximating polynomial increases, hence indicating exponential convergence in p. Wherefore, Theorems 4.5.1.5 and 4.5.1.6 are confirmed.

Figures 9.21 – 9.22 display the analytical displacement of the boundary value problem (9.1) – (9.2) and the IPDG approximate displacement, either deriving from *hp*-version SIPG or *hp*-version NIPG method, respectively. Specifically, we show four-element uniform meshes generated using discontinuous piecewise cubic polynomials. Into the bargain, Figures 9.23 – 9.24

display a comparison of the analytical Cauchy stress with the IPDG approximate Cauchy stress, either obtaining from *hp*-version SIPG or *hp*-version NIPG method, respectively. In particular, we exhibit four-element uniform meshes generated using discontinuous piecewise polynomials of fourth-degree. Figures 9.25 – 9.26 display a comparison of the analytical double stress with the IPDG approximate double stress, either deriving from *hp*-version SIPG or *hp*-version NIPG method, respectively. Especially, we present eight-element uniform meshes generated using discontinuous piecewise polynomials of fifth-degree. We notice that all the IPDG numerical solutions, either obtaining from *hp*-version SIPG or *hp*-version NIPG method, converge to the exact solutions.

In Tables 9.1 – 9.4, we present a comparison of the energy seminorm and L^2-norm of the error in the approximation to u, with the mesh function, N_{el}, on a sequence of uniform subdivisions for $2 \leqslant p \leqslant 6$, respectively. In each case, we exhibit the number of elements in the computational mesh, the corresponding energy seminorm and L^2-norm of the error as well as their respective computed rates of convergence k_1 and k_2. Here, we observe that energy seminorms, $|||u - u^h|||_{sb}$ and $|||u - u_{DG}|||_{sb}$, tend to zero, as N_{el} increases (or h tends to zero). On the other hand, the L^2-norms, $||u - u^h||_\Omega$ and $||u - u_{DG}||_\Omega$, of the error are also observed to tend to zero, as the mesh is enriched.

In Table 9.1, we notice that the values of convergence rate, k_1, present something strange for both fixed $p = 3$ and $p = 4$. By watching the values of Table 9.1, someone can think that the numerical solution has not converged yet, so increase of the mesh function, N_{el}, is required. After refining the mesh, the deriving values of convergence rate, k_1, present the same behavior for the above fixed p. We observed through numerical experiments performed that the h-version SIPG would present unusual values for the convergence rate, k_1, for fixed $p = 3$ and $p = 4$, if the selected values of the stabilization constants C_α, C_β belonged to the interval $[10, 20)$. However, if the selected values of the stablization constants $C_\alpha, C_\beta \in [20, 99)$, the convergence rate, k_1, would diverge for fixed $p = 4$ and so on.

In Tables 9.5 – 9.6, we show a comparison of the energy seminorm and L^2-norm of the error in the approximation to u, with the polynomial degree p on a sequence of uniform enrichments for fixed $N_{el} = 2, 4, 8, 16$, respectively. In each case, we present the polynomial degree in the computational mesh, the corresponding energy seminorm and L^2-norm of the error as well as their respective computed rates of convergence k_3 and k_4. We observe that energy seminorm, $|||u - u_{DG}|||_{sb}$, tends to zero, as p increases. On the other hand, the L^2-norm, $||u - u_{DG}||_{\Omega}$, of the error is observed to tend to zero under p-refinement.

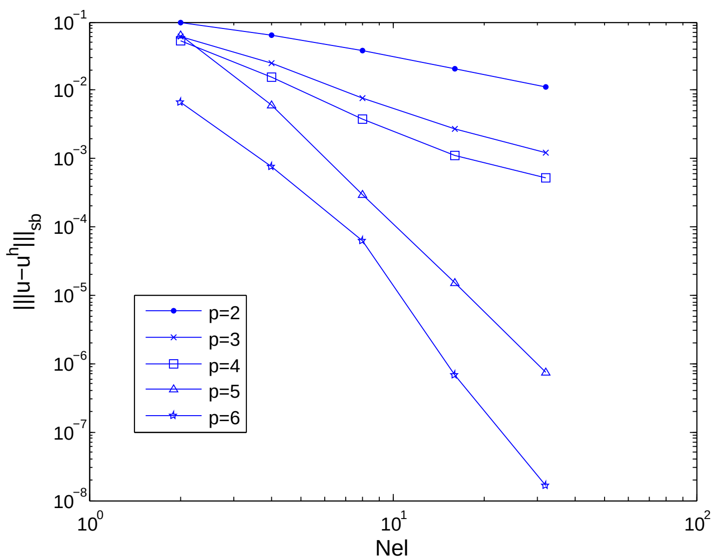

Figure 9.1: Convergence of the h-version SIPG in the energy seminorm under h-refinement.

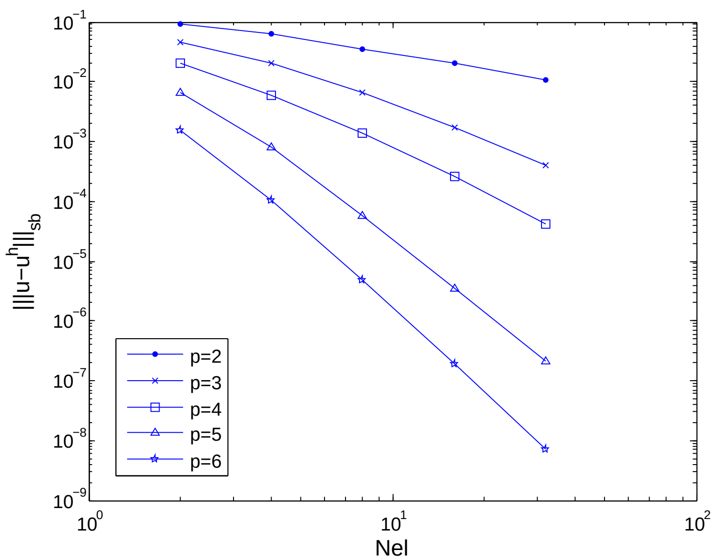

Figure 9.2: Convergence of the h-version NIPG in the energy seminorm under h-refinement.

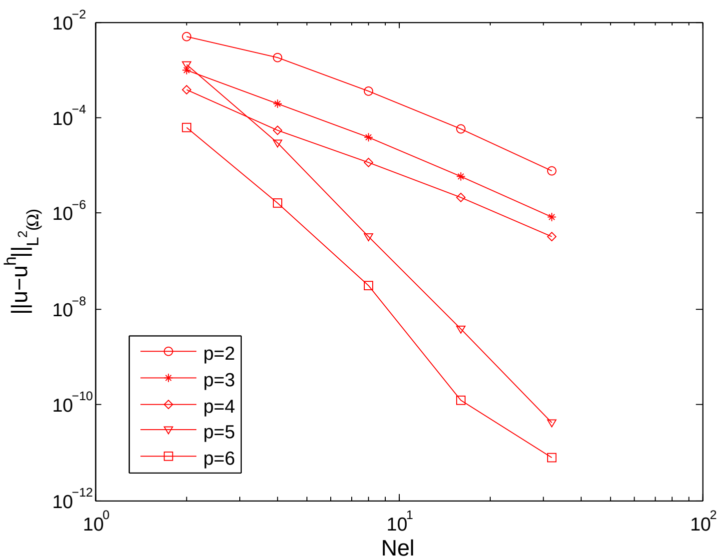

Figure 9.3: Convergence of the h-version SIPG in the L^2-norm under h-refinement.

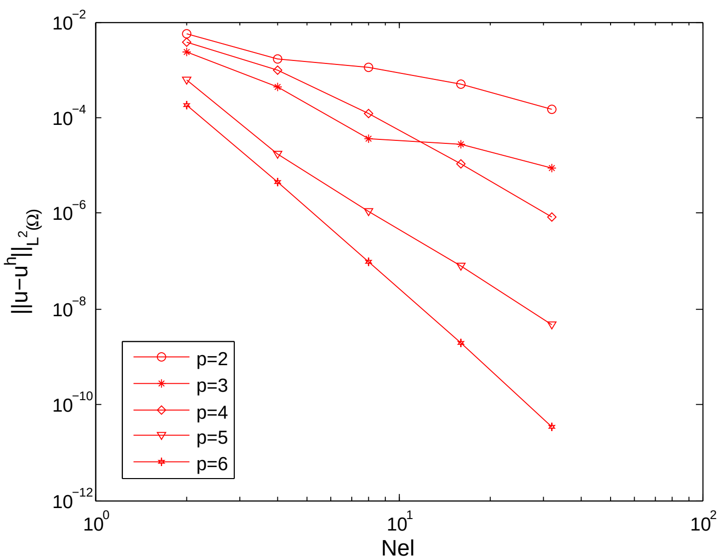

Figure 9.4: Convergence of the h-version NIPG in the L^2-norm under h-refinement.

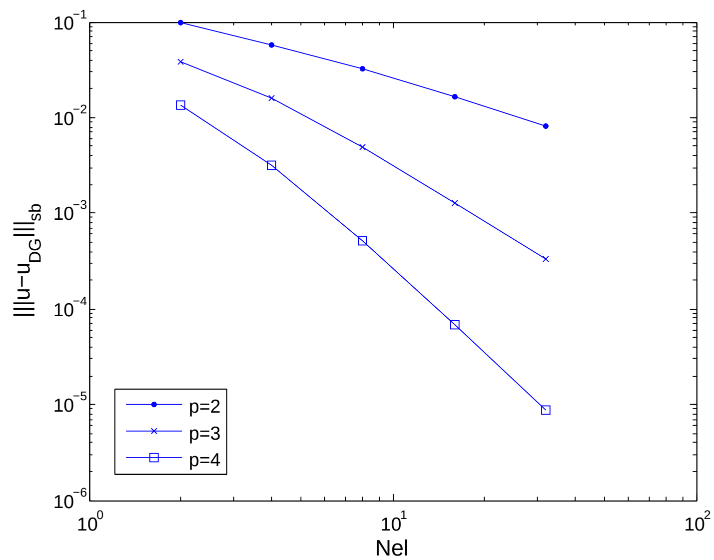

Figure 9.5: Convergence of the hp-version SIPG in the energy seminorm under h-refinement.

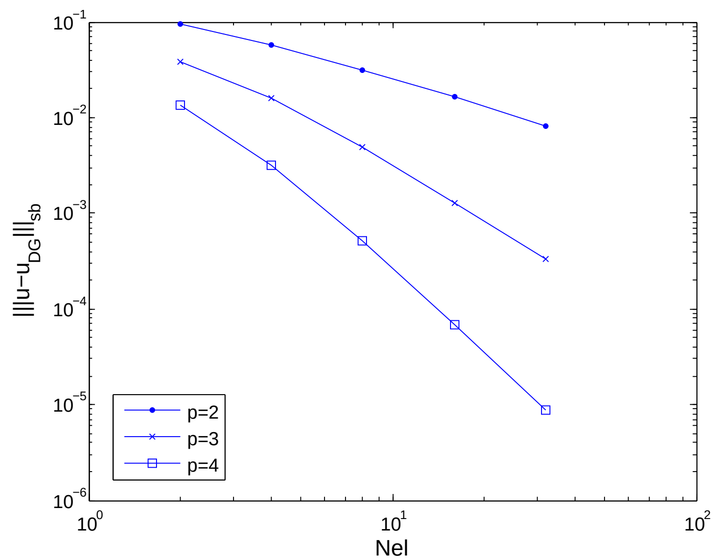

Figure 9.6: Convergence of the *hp*-version NIPG in the energy seminorm under *h*-refinement.

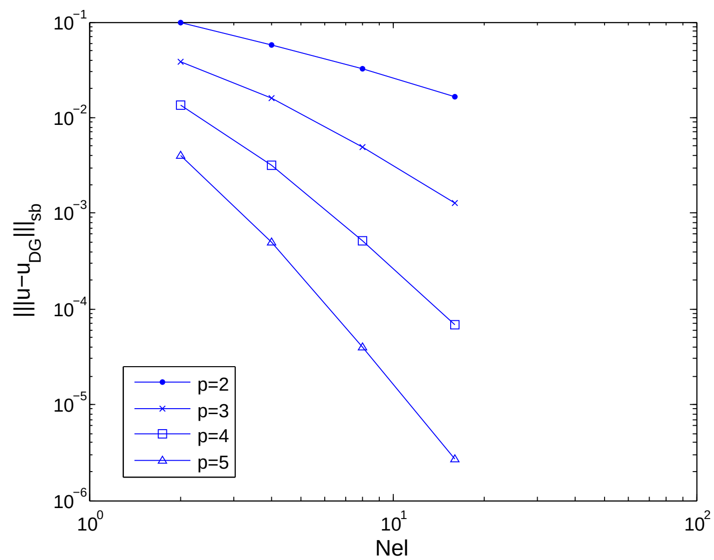

Figure 9.7: Convergence of the *hp*-version SIPG in the energy seminorm under *h*-refinement.

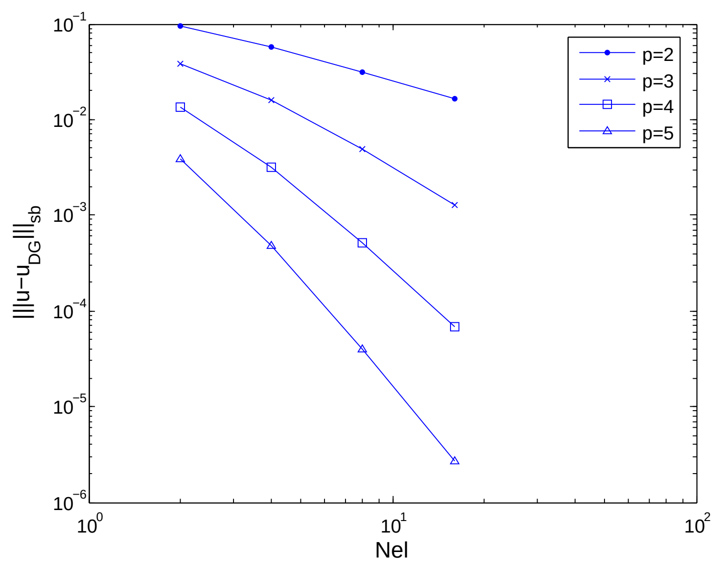

Figure 9.8: Convergence of the *hp*-version NIPG in the energy seminorm under *h*-refinement.

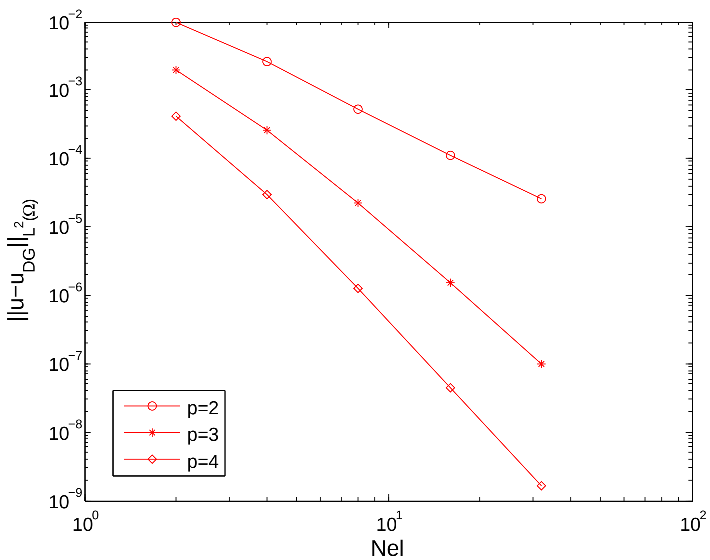

Figure 9.9: Convergence of the hp-version SIPG in the L^2-norm under h-refinement.

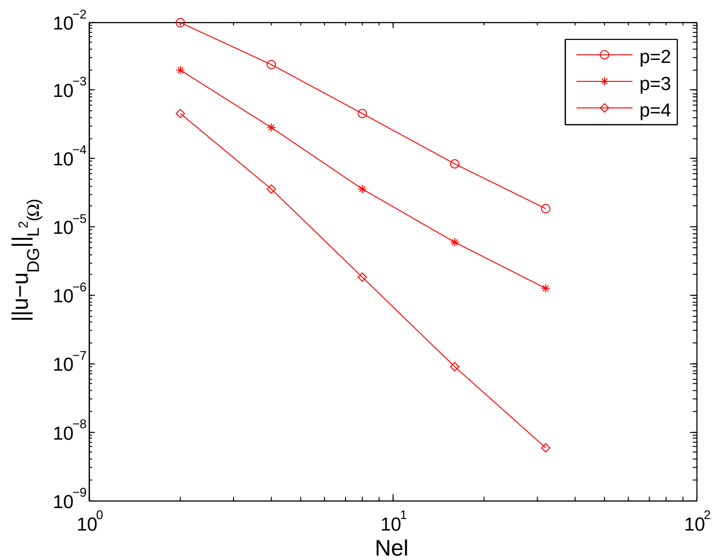

Figure 9.10: Convergence of the hp-version NIPG in the L^2-norm under h-refinement.

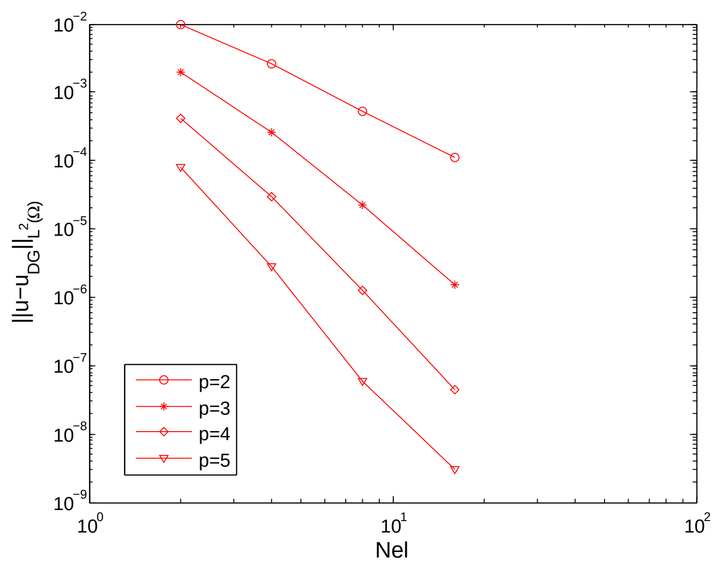

Figure 9.11: Convergence of the hp-version SIPG in the L^2-norm under h-refinement.

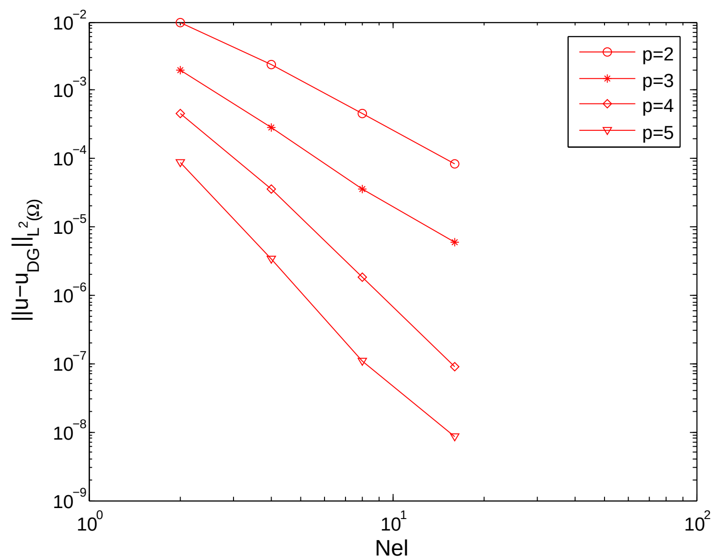

Figure 9.12: Convergence of the hp-version NIPG in the L^2-norm under h-refinement.

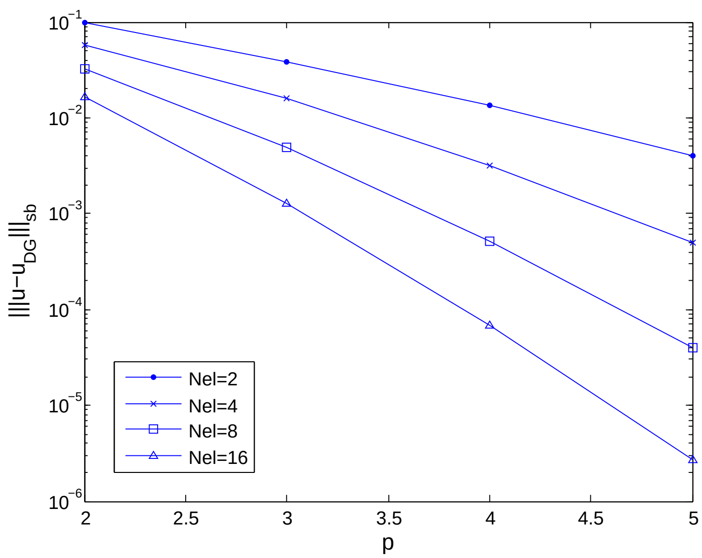

Figure 9.13: Convergence of the *hp*-version SIPG in the energy seminorm under *p*-refinement.

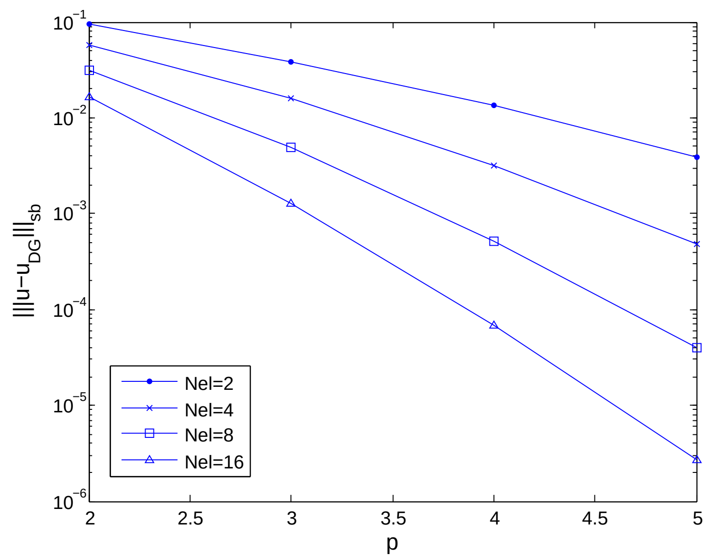

Figure 9.14: Convergence of the *hp*-version NIPG in the energy seminorm under *p*-refinement.

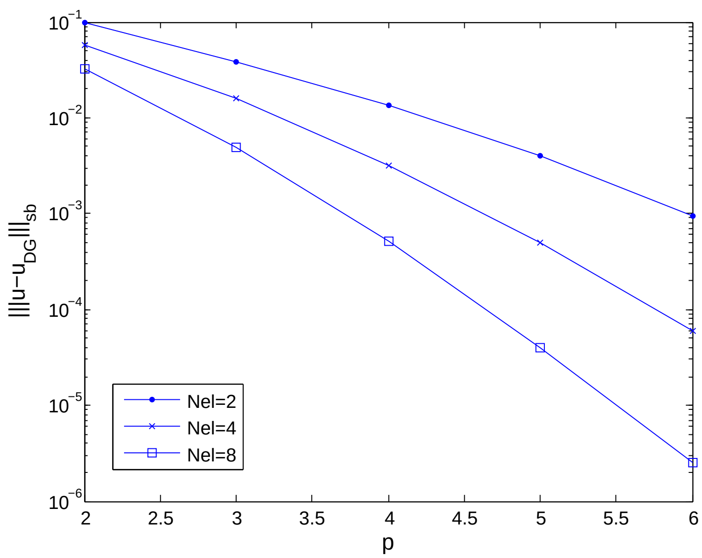

Figure 9.15: Convergence of the hp-version SIPG in the energy seminorm under p-refinement.

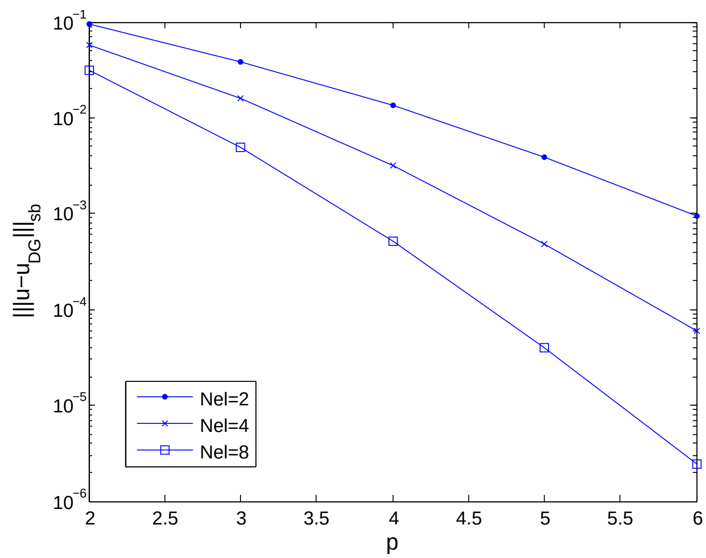

Figure 9.16: Convergence of the *hp*-version NIPG in the energy seminorm under *p*-refinement.

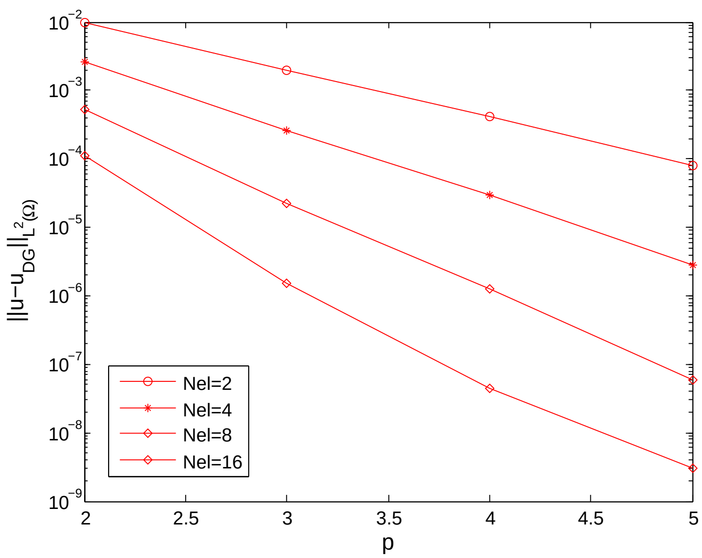

Figure 9.17: Convergence of the hp-version SIPG in the L^2-norm under p-refinement.

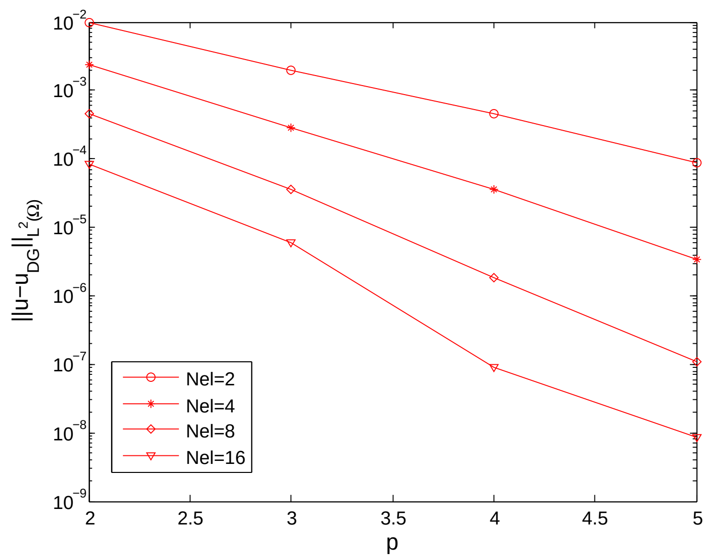

Figure 9.18: Convergence of the *hp*-version NIPG in the L^2-norm under p-refinement.

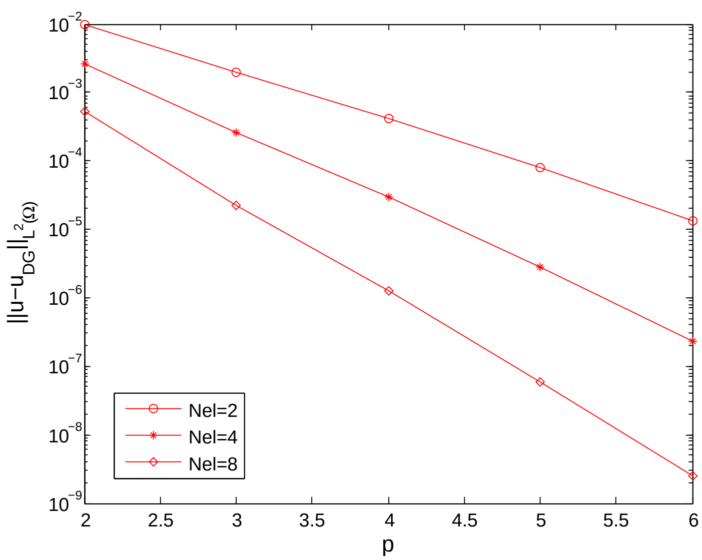

Figure 9.19: Convergence of the hp-version SIPG in the L^2-norm under p-refinement.

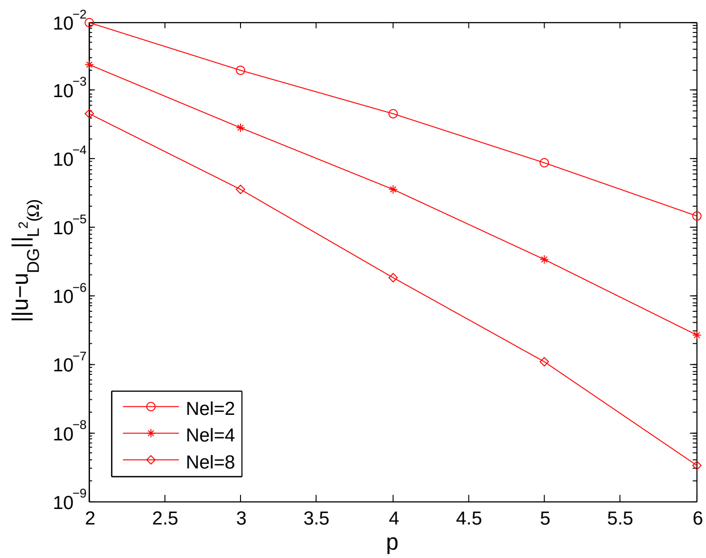

Figure 9.20: Convergence of the *hp*-version NIPG in the L^2-norm under p-refinement.

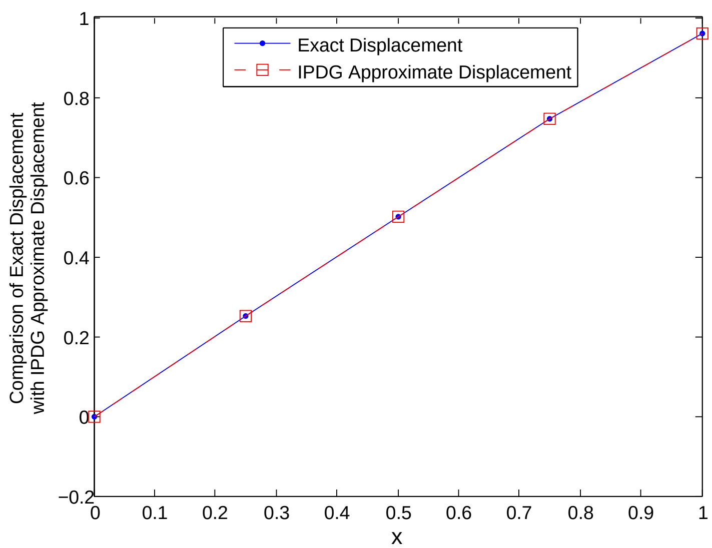

Figure 9.21: Comparison of Exact Displacement with hp-version SIPG Approximate Displacement ($p = 3$, $N_{el} = 4$)

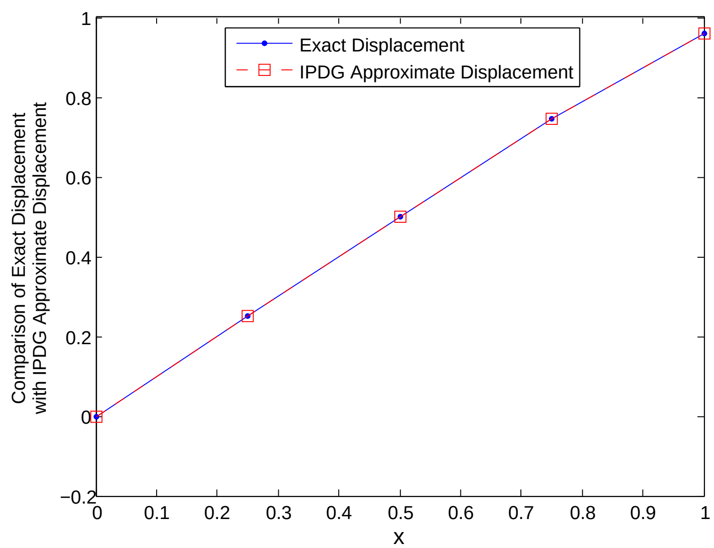

Figure 9.22: Comparison of Exact Displacement with *hp*-version NIPG Approximate Displacement ($p = 3$, $N_{el} = 4$)

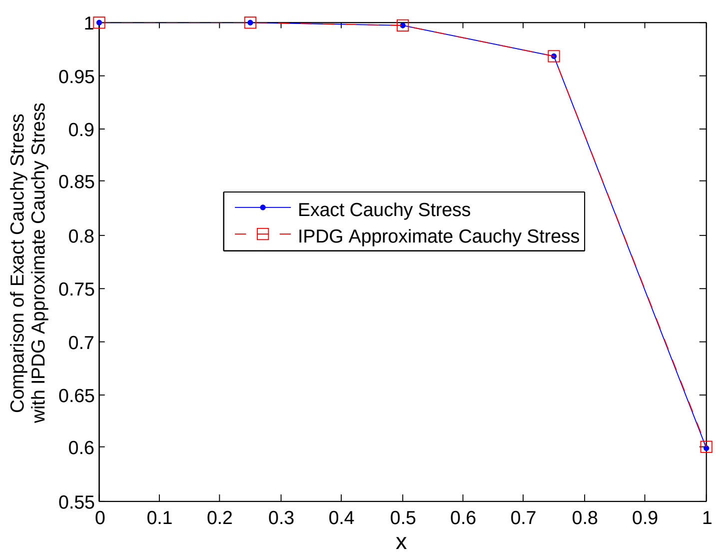

Figure 9.23: Comparison of Exact Cauchy Stress with *hp*-version SIPG Approximate Cauchy Stress ($p = 4$, $N_{el} = 4$)

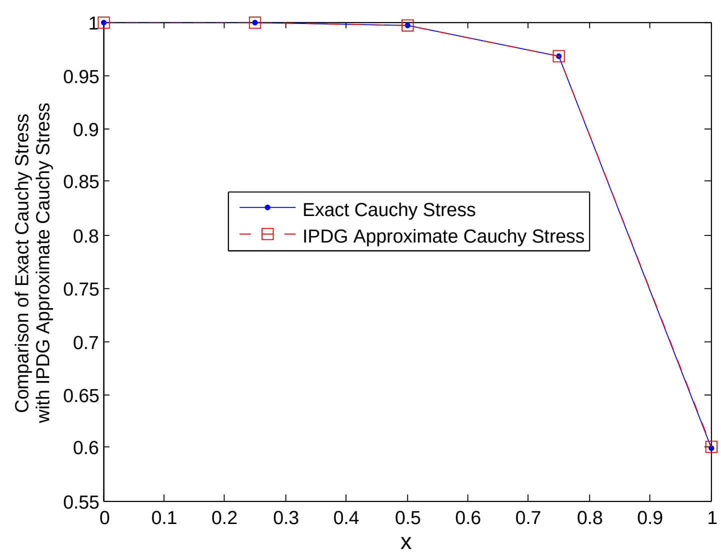

Figure 9.24: Comparison of Exact Cauchy Stress with *hp*-version NIPG Approximate Cauchy Stress ($p = 4,\ N_{el} = 4$)

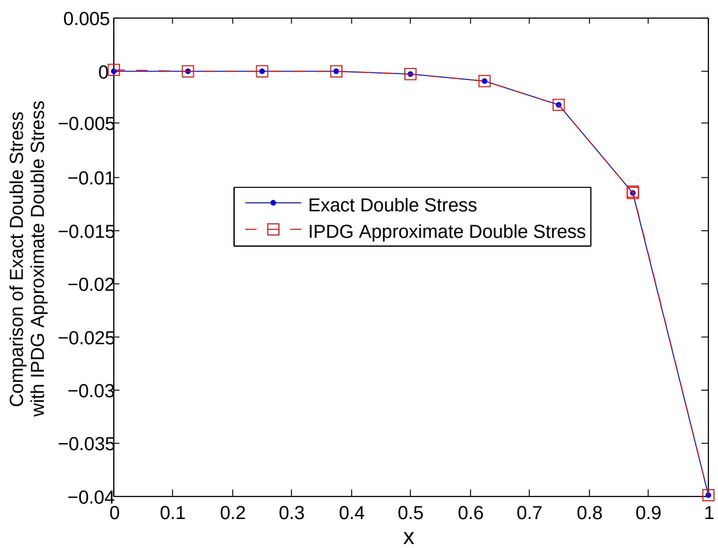

Figure 9.25: Comparison of Exact Double Stress with *hp*-version SIPG Approximate Double stress ($p = 5$, $N_{el} = 8$)

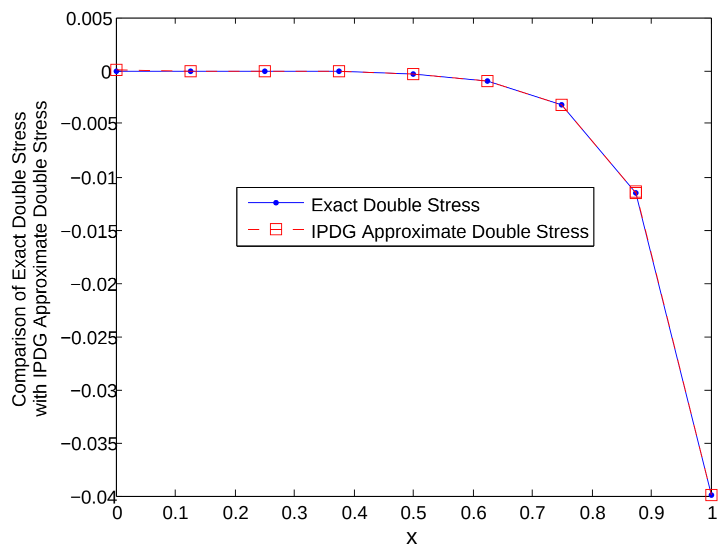

Figure 9.26: Comparison of Exact Double Stress with hp-version NIPG Approximate Double stress ($p = 5$, $N_{el} = 8$)

Method	N_{el}	$\|\|\|u-u^h\|\|\|_{sb}$	k_1	$\|\|u-u^h\|\|_\Omega$	k_2
SIPG, $p=2$	2	9.6145×10^{-2}		5.0177×10^{-3}	
	4	6.5260×10^{-2}	0.5590	1.7634×10^{-3}	1.5087
	8	3.7850×10^{-2}	0.7859	3.6687×10^{-4}	2.2650
	16	2.0812×10^{-2}	0.8629	5.6871×10^{-5}	2.6895
	32	1.0961×10^{-2}	0.9250	7.8874×10^{-6}	2.8501
	64	5.6071×10^{-3}	0.9671	1.0398×10^{-6}	2.9232
SIPG, $p=3$	2	6.2303×10^{-2}		9.5772×10^{-4}	
	4	2.5035×10^{-2}	1.3154	1.8906×10^{-4}	2.3408
	8	7.5467×10^{-3}	1.7300	3.9271×10^{-5}	2.2673
	16	2.6736×10^{-3}	1.4971	6.0611×10^{-6}	2.6958
	32	1.2376×10^{-3}	1.1112	8.1828×10^{-7}	2.8889
	64	6.1789×10^{-4}	1.0021	1.0561×10^{-7}	2.9600
SIPG, $p=4$	2	5.3852×10^{-2}		3.7989×10^{-4}	
	4	1.5715×10^{-2}	1.7769	5.3915×10^{-5}	2.8168
	8	3.7964×10^{-3}	2.0494	1.1289×10^{-5}	2.2558
	16	1.1318×10^{-3}	1.7460	2.0740×10^{-6}	2.4444
	32	5.1385×10^{-4}	1.1391	3.2940×10^{-7}	2.6545
	64	3.1011×10^{-4}	0.7286	5.3199×10^{-8}	2.6304
SIPG, $p=5$	2	6.4190×10^{-2}		1.3353×10^{-3}	
	4	5.9576×10^{-3}	3.4295	3.0175×10^{-5}	5.4677
	8	2.9909×10^{-4}	4.3161	3.1670×10^{-7}	6.5741
	16	1.5203×10^{-5}	4.2982	3.7174×10^{-9}	6.4127
	32	7.4803×10^{-7}	4.3412	4.1854×10^{-11}	6.4728
	64	3.7696×10^{-8}	4.3106	4.3479×10^{-12}	3.2670
SIPG, $p=6$	2	6.6645×10^{-3}		6.1414×10^{-5}	
	4	7.4765×10^{-4}	3.1561	1.6663×10^{-6}	5.2038
	8	6.1368×10^{-5}	3.6068	3.1114×10^{-8}	5.7429
	16	6.8504×10^{-7}	6.4852	1.2239×10^{-10}	7.9899
	32	1.6169×10^{-8}	5.4049	7.8475×10^{-12}	3.9631

Table 9.1: Numerical errors and convergence rates for the h-version SIPG method under uniform mesh refinement.

Method	N_{el}	$\|\|\|u - u^h\|\|\|_{sb}$	k_1	$\|\|u - u^h\|\|_\Omega$	k_2
NIPG, $p = 2$	2	9.1853×10^{-2}		5.6175×10^{-3}	
	4	6.1835×10^{-2}	0.5709	1.7116×10^{-3}	1.7146
	8	3.5760×10^{-2}	0.7901	1.0959×10^{-3}	0.6432
	16	2.0093×10^{-2}	0.8317	4.8805×10^{-4}	1.1670
	32	1.0779×10^{-2}	0.8985	1.4763×10^{-4}	1.7250
	64	5.5630×10^{-3}	0.9543	3.9015×10^{-5}	1.9199
NIPG, $p = 3$	2	4.5954×10^{-2}		2.2629×10^{-3}	
	4	2.0325×10^{-2}	1.1469	4.4506×10^{-4}	2.3461
	8	6.5473×10^{-3}	1.6343	3.6518×10^{-5}	3.6073
	16	1.6614×10^{-3}	1.9785	2.8344×10^{-5}	0.3656
	32	4.0048×10^{-4}	2.0526	9.0382×10^{-6}	1.6489
	64	9.7793×10^{-5}	2.0339	2.3100×10^{-6}	1.9681
NIPG, $p = 4$	2	1.9864×10^{-2}		3.7970×10^{-3}	
	4	6.0348×10^{-3}	1.7188	9.5894×10^{-4}	1.9853
	8	1.3626×10^{-3}	2.1469	1.2265×10^{-4}	2.9669
	16	2.5352×10^{-4}	2.4262	1.1051×10^{-5}	3.4723
	32	4.0869×10^{-5}	2.6330	8.2821×10^{-7}	3.7380
	64	5.9354×10^{-6}	2.7836	5.6132×10^{-8}	3.8810
NIPG, $p = 5$	2	6.5896×10^{-3}		6.0626×10^{-4}	
	4	8.0393×10^{-4}	3.0350	1.6818×10^{-5}	5.1719
	8	5.8821×10^{-5}	3.7727	1.0806×10^{-6}	3.9601
	16	3.5699×10^{-6}	4.0424	7.6842×10^{-8}	3.8138
	32	2.1473×10^{-7}	4.0553	4.7567×10^{-9}	4.0139
	64	1.3168×10^{-8}	4.0274	4.1207×10^{-10}	3.5290
NIPG, $p = 6$	2	1.5338×10^{-3}		1.8109×10^{-4}	
	4	1.0264×10^{-4}	3.9014	4.5774×10^{-6}	5.3060
	8	4.8420×10^{-6}	4.4058	9.7539×10^{-8}	5.5524
	16	1.9454×10^{-7}	4.6375	1.8959×10^{-9}	5.6850
	32	7.1150×10^{-9}	4.7731	3.3952×10^{-11}	5.8032

Table 9.2: Numerical errors and convergence rates for the h-version NIPG method under uniform mesh refinement.

Method	N_{el}	$\|\|\|u - u_{DG}\|\|\|_{sb}$	k_1	$\|\|u - u_{DG}\|\|_{\Omega}$	k_2
SIPG, $p = 2$	2	9.6847×10^{-2}		9.8950×10^{-3}	
	4	5.7654×10^{-2}	0.7749	2.5710×10^{-3}	1.9443
	8	3.1853×10^{-2}	0.8560	5.1782×10^{-4}	2.3118
	16	1.6397×10^{-2}	0.9580	1.0991×10^{-4}	2.2361
	32	8.2159×10^{-3}	0.9970	2.5880×10^{-5}	2.0864
SIPG, $p = 3$	2	3.8846×10^{-2}		2.0084×10^{-3}	
	4	1.5930×10^{-2}	1.2860	2.5977×10^{-4}	2.95073
	8	4.9341×10^{-3}	1.6909	2.1949×10^{-5}	3.5650
	16	1.3138×10^{-3}	1.9090	1.5074×10^{-6}	3.8640
	32	3.3315×10^{-4}	1.9795	9.6920×10^{-8}	4.0186
SIPG, $p = 4$	2	1.3701×10^{-2}		4.1314×10^{-4}	
	4	3.2005×10^{-3}	2.0980	2.9156×10^{-5}	3.8248
	8	5.1096×10^{-4}	2.6470	1.2909×10^{-6}	4.4973
	16	6.8483×10^{-5}	2.8994	4.5264×10^{-8}	4.8339
	32	8.6915×10^{-6}	2.9781	1.6057×10^{-9}	4.8171
SIPG, $p = 5$	2	3.9569×10^{-3}		7.9679×10^{-5}	
	4	4.9169×10^{-4}	3.0085	2.7877×10^{-6}	4.8371
	8	3.9931×10^{-5}	3.6222	5.9016×10^{-8}	5.5618
	16	2.6932×10^{-6}	3.8901	2.9983×10^{-9}	4.2989

Table 9.3: Numerical errors and convergence rates for the *hp*-version SIPG method under uniform mesh refinement.

Method	N_{el}	$\|\|\|u-u_{DG}\|\|\|_{sb}$	k_1	$\|\|u-u_{DG}\|\|_{\Omega}$	k_2
NIPG, $p=2$	2	9.6414×10^{-2}		9.7798×10^{-3}	
	4	5.7421×10^{-2}	0.7477	2.4246×10^{-3}	2.0121
	8	3.1758×10^{-2}	0.8545	4.4646×10^{-4}	2.4411
	16	1.6364×10^{-2}	0.9566	8.4208×10^{-5}	2.4065
	32	8.2058×10^{-3}	0.9958	1.8400×10^{-5}	2.1943
NIPG, $p=3$	2	3.8677×10^{-2}		2.0331×10^{-3}	
	4	1.5865×10^{-2}	1.2856	2.8997×10^{-4}	2.8097
	8	4.9183×10^{-3}	1.6896	3.5703×10^{-5}	3.0218
	16	1.3107×10^{-3}	1.9078	5.9009×10^{-6}	2.5970
	32	3.3258×10^{-4}	1.9786	1.2478×10^{-6}	2.2415
NIPG, $p=4$	2	1.3646×10^{-2}		4.5116×10^{-4}	
	4	3.1875×10^{-3}	2.0980	3.5130×10^{-5}	3.6829
	8	5.0927×10^{-4}	2.6459	1.8090×10^{-6}	4.2794
	16	6.8321×10^{-5}	2.8980	8.8179×10^{-8}	4.3586
	32	8.6785×10^{-6}	2.9768	5.9169×10^{-9}	3.8975
NIPG, $p=5$,	2	3.9404×10^{-3}		8.7614×10^{-5}	
	4	4.8976×10^{-4}	3.0082	3.3424×10^{-6}	4.7122
	8	3.9796×10^{-5}	3.6214	1.1042×10^{-7}	4.9198
	16	2.6858×10^{-6}	3.8892	8.3901×10^{-9}	3.7182

Table 9.4: Numerical errors and convergence rates for the *hp*-version NIPG method under uniform mesh refinement.

Method	p	$\|\|\|u - u_{DG}\|\|\|_{sb}$	k_3	$\|\|u - u_{DG}\|\|_{\Omega}$	k_4
SIPG, $N_{el} = 2$	2	9.6847×10^{-2}		9.8950×10^{-3}	
	3	3.8846×10^{-2}	2.2530	2.0084×10^{-3}	3.9330
	4	1.3701×10^{-2}	3.6225	4.1314×10^{-4}	5.4967
	5	3.9569×10^{-3}	5.5660	7.9679×10^{-5}	7.3754
	6	9.4191×10^{-4}	7.8724	1.3425×10^{-5}	9.7678
SIPG, $N_{el} = 4$	2	5.7654×10^{-2}		2.5710×10^{-3}	
	3	1.5930×10^{-2}	3.1723	2.5977×10^{-4}	5.6534
	4	3.2005×10^{-3}	5.5787	2.9156×10^{-5}	7.6026
	5	4.9169×10^{-4}	8.3947	2.7877×10^{-6}	10.5199
	6	6.0749×10^{-5}	11.4693	2.3293×10^{-7}	13.6146
SIPG, $N_{el} = 8$	2	3.1853×10^{-2}		5.1782×10^{-4}	
	3	4.9341×10^{-3}	4.5996	2.1949×10^{-5}	7.7958
	4	5.1096×10^{-4}	7.8824	1.2909×10^{-6}	9.8490
	5	3.9931×10^{-5}	11.4238	5.9016×10^{-8}	13.8265
	6	2.4886×10^{-6}	15.2227	2.4782×10^{-9}	17.3884
SIPG, $N_{el} = 16$	2	1.6397×10^{-2}		1.0991×10^{-4}	
	3	1.3138×10^{-3}	6.2254	1.5074×10^{-6}	10.5787
	4	6.8483×10^{-5}	10.2686	4.5264×10^{-8}	12.1858
	5	2.6932×10^{-6}	14.5012	2.9983×10^{-9}	12.1647

Table 9.5: Numerical errors and convergence rates for the hp-version SIPG method under p-refinement.

Method	p	$\|\|\|u - u_{DG}\|\|\|_{sb}$	k_3	$\|\|u - u_{DG}\|\|_{\Omega}$	k_4
NIPG, $N_{el} = 2$	2	9.6414×10^{-2}		9.7798×10^{-3}	
	3	3.8677×10^{-2}	2.2527	2.0331×10^{-3}	3.8740
	4	1.3646×10^{-2}	3.6214	4.5116×10^{-4}	5.2332
	5	3.9404×10^{-3}	5.5667	8.7614×10^{-5}	7.3445
	6	9.3807×10^{-4}	7.8719	1.4592×10^{-5}	9.8314
NIPG, $N_{el} = 4$	2	5.7421×10^{-2}		2.4246×10^{-3}	
	3	1.5865×10^{-2}	3.1724	2.8997×10^{-4}	5.2376
	4	3.1875×10^{-3}	5.5787	3.5130×10^{-5}	7.3370
	5	4.8976×10^{-4}	8.3940	3.3424×10^{-6}	10.5419
	6	6.0532×10^{-5}	11.4673	2.6809×10^{-7}	13.8389
NIPG, $N_{el} = 8$	2	3.1758×10^{-2}		4.4646×10^{-4}	
	3	4.9183×10^{-3}	4.6001	3.5703×10^{-5}	6.2302
	4	5.0927×10^{-4}	7.8828	1.8090×10^{-6}	10.3672
	5	3.9796×10^{-5}	11.4241	1.1042×10^{-7}	12.5311
	6	2.4814×10^{-6}	15.2201	3.3666×10^{-9}	19.1441
NIPG, $N_{el} = 16$	2	1.6364×10^{-2}		8.4208×10^{-5}	
	3	1.3107×10^{-3}	6.2262	5.9009×10^{-6}	6.5559
	4	6.8321×10^{-5}	10.2686	8.8179×10^{-8}	14.6116
	5	2.6858×10^{-6}	14.5029	8.3901×10^{-9}	10.5417

Table 9.6: Numerical errors and convergence rates for the *hp*-version NIPG method under *p*-refinement.

Chapter 10

Concluding Remarks

10.1 Conclusions

This dissertation primarily engaged with the development of both h- and hp-version interior penalty discontinuous Galerkin finite element methods for boundary value problems of strain gradient elasticity and of plate theory. However, its scope also extended to the development of both h- and hp-version continuous interior penalty finite element method for one-dimensional boundary value problems of strain gradient elasticity. Overall, our research endeavor focused on conducting either a priori error analysis for one-dimensional problems or a posteriori error analysis for higher dimensional problems.

For this purpose, we presented a functional, analytic framework using broken Sobolev spaces as well as corresponding finite element spaces for the above methods and establishing a priori error estimates for one-dimensional problems of SGE on regular families of subdivisions. A priori error estimates of the h-version were optimal in h, irrespective of the applied method. In addition, a priori error estimates of the hp-version were optimal in h, but were p-suboptimal. To the best of our knowledge, it was the first time that the h- and hp-version interior penalty discontinuous Galerkin finite element methods were applied for a fourth-order elliptic problem of strain gradient elasticity in 1-D. The hp-version continuous interior penalty finite element method had never been applied for a fourth-order elliptic problem of strain gradient elasticity in 1-D. It was also the first time that the h- and hp-version continuous interior penalty finite element method was applied in a sixth-order elliptic problem.

By using lifting operators, we developed the interior penalty discontinuous Galerkin methods for the Kirchhoff-Love plate model problem of linear elasticity, which was equipped with essential and complicated natural boundary conditions. Then, we introduced a recovery operator that mapped discontinuous finite element spaces to C^1-conforming finite element spaces. Next, we presented a technical lemma about this recovery operator. The use of lifting operators, application of IPDG methods to the above boundary value problem and the development of a technical lemma of a recovery operator for that kind of fourth-order elliptic problem is also a original contribution to the field.

Furthermore, by employing lifting operators, we developed interior penalty discontinuous Galerkin methods for a system of partial differential equations of strain gradient elasticity with respect to the displacement. The problem was equipped with essential boundary conditions (clamped type) to which we introduced a recovery operator that mapped discontinuous finite element spaces to C^1-conforming finite element spaces. After that, presentation of a technical lemma about the recovery operator followed. Next, using this technical lemma, we established a reliable a posteriori error estimate of residual type for the symmetric interior penalty discontinuous Galerkin method in the corresponding energy seminorm. To the best of our knowledge, it was also the first time that a posteriori error analysis conducted for a system of partial differential equations of strain gradient elasticity in 2-D, by applying the interior penalty discontinuous Galerkin methods, which led to a reliable a posteriori error estimate.

The theoretical findings of the interior penalty discontinuous Galerkin finite element methods of strain gradient elasticity in one-dimension were tested through numerical experiments; we employed a one-dimensional boundary value problem of strain gradient elasticity whose analytical solution presented a boundary layer. We mention that the numerical solutions (i.e., approximate solutions, convergence rates etc.) assured the analytical findings of the IPDG methods developed.

In a nutshell, there is a thriving interest in interior penalty discontinuous Galerkin and continuous interior penalty methods since they have been proven to provide flexible and accurate discretisations to many problems of practical interest.

10.2 Directions For Future Work

We briefly reflect on some interesting open problems concerning these methods.

- **Extensions of the theory.** In this work, only elliptic boundary value problems were considered.

  1. It would be fairly challenging to extend the design of continuous interior penalty method to non-linear, hyperbolic and parabolic problems.
  2. It would be worthy of further research to study the interior penalty discontinuous Galerkin methods, for the problems presented in this work, by employing shape irregular (anisotropic) elements in the subdivision of Ω.
  3. What is more, the development of interior penalty discontinuous Galerkin and continuous interior penalty methods for other higher dimensional problems of strain gradient elasticity and plasticity would be of great importance.
  4. The development of dynamic analysis of interior penalty discontinuous Galerkin and continuous interior penalty finite element methods for evolutionary differential equations would be both significant and challenging.

- **Numerical validation.** The numerical experiments of the higher dimensional problems, for finite element methods developed in this dissertation, would be crucial so that the theoretical findings could be confirmed.

In any case, there exist quite a few unanswered questions regarding the interior penalty discontinuous Galerkin and continuous interior penalty methods as well as their possible applications to various problems.

Appendix A

Inequalities

Appendix A initially states the basic definitions of a finite element, of degrees of freedom, of basis functions, of an interpolant as well as of affine-equivalent finite elements, needed for the definition of regularity, of high-order C^1-conforming macro-element, of the continuity of a functional and finally the Riesz representation theorem. Then, interpolation estimates are given. Appendix A is concluded with important inverse estimates, basic trace inequalities and useful integration formulas. The interpolation estimates, the inverse estimates and the trace inequalities are extensively used in the proofs of the coercivity, of the continuity and of the error analyses throughout this dissertation. The references for the following definitions and theorems can be found in the book of Brenner and Scott [36] and of Ciarlet [56], respectively.

A.1 Basic Definitions

To set the stage for the development in this appendix, we first wish to define Lipschitz boundary, a finite element, degrees of freedom, basis functions, an interpolant, affine-equivalent elements as well as regularity.

Definition A.1.1. *We say Ω has a Lipschitz boundary $\partial\Omega$ provided there exists a collection of open sets O_i, a positive parameter ϵ, an integer N and a finite number M, such that for all $x \in \partial\Omega$ the ball of radius ϵ centered at x is contained in some O_i, no more than N of the sets O_i intersect nontrivially, and each domain $O_i \cap \Omega = O_i \cap \Omega_i$ where Ω_i is a domain whose boundary is a graph of a Lipschitz function ϕ_i (i.e., $\Omega_i = \{(x, y) \in \Re^n : x \in \Re^{n-1}, y < \phi_i(x)\}$) satisfying $||\phi_i||_{Lip(\Re^{n-1})} \leqslant M$.*

Defintion A.1.2. *A finite element in $\Re^d$ is a triple $(K, \mathcal{P}, \mathcal{N})$ where*

- *$\Omega \subset \Re^d$ is closed and has a non-empty interior and a Lipschitz-continuous boundary,*
- *$\mathcal{P}$ is a finite-dimensional space of real-valued functions on K with $dim\mathcal{P} = k$,*
- *$\mathcal{N}$ is a set of k linear forms ϕ_i, $1 \leqslant i \leqslant k$, defined over the space $\mathcal{P}$, and it is assumed that the set $\mathcal{N}$ is $\mathcal{P}$-unisolvent, i.e., for given any real scalars c_i, $1 \leqslant i \leqslant k$, there exists a unique function $\upsilon \in \mathcal{P}$ that satisfies $\phi(\upsilon) = c_i$, $1 \leqslant i \leqslant k$.*

Definition A.1.3. *The linear forms ϕ_i, $1 \leqslant i \leqslant k$, are called degrees of freedom of the finite element and, due to the $\mathcal{P}$-unisolvency of $\mathcal{N}$, it is easy to see that they are linearly independent. Therefore, $\mathcal{N}$ is a basis for the dual of $\mathcal{P}$, which we will denote by $\mathcal{P}'$.*

Definition A.1.4. *Always because of the $\mathcal{P}$-unisolvency of $\mathcal{N}$, it is clear that there exist k functions $w_i \in \mathcal{P}$, $1 \leqslant i \leqslant k$, that satisfy $\phi_j(\upsilon_i) = \delta_{ij}$, $1 \leqslant j \leqslant k$. Ergo, the identity*

$$\upsilon = \sum_{i=1}^{k} \phi_i(\upsilon)\upsilon_i \quad \forall \upsilon \in \mathcal{P} \tag{A.1}$$

holds. The functions υ_i, $1 \leqslant i \leqslant k$, are called basis functions of the finite element.

We can now state the definition of an interpolant, which has great importance in both interpolation and approximation theory.

Definition A.1.5. *Given $(K, \mathcal{P}, \mathcal{N})$ and a function $w : K \to \Re$ smooth enough to have the degrees of freedom $\phi_i(\upsilon)$, $1 \leqslant i \leqslant k$, well defined, we define the $\mathcal{P}$-interpolant of the function w as*

$$\mathcal{I}w = \sum_{i=1}^{k} \phi_i(w)\upsilon_i. \tag{A.2}$$

Due to the $\mathcal{P}$-unisolvency of $\mathcal{N}$, the $\mathcal{P}$-interpolant is the unique function satisfying

$$\mathcal{I}w \in \mathcal{P} \quad \text{and} \quad \phi_i(\mathcal{I}w) = \phi_i(w), \quad 1 \leqslant i \leqslant k. \tag{A.3}$$

Definition A.1.6. *For* $\mathbf{x} \in \Re^d$, *let* $F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ *be an affine map (A non-singular). We say that the finite elements* $(K, \mathcal{P}, \mathcal{N})$ *and* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *are affine-equivalent if*

$$\hat{K} = F(K),$$
$$\hat{\mathcal{P}} = \left\{ \hat{\upsilon} : \hat{K} \to \Re | \hat{\upsilon} = \upsilon \circ F, \upsilon \in \mathcal{P} \right\},$$
$$\hat{\mathcal{N}} = \left\{ \hat{\phi}_i \in \hat{\mathcal{P}}' | \hat{\phi}_i(\upsilon \circ F) = \phi_i(\upsilon), \upsilon \in \mathcal{P} \right\}.$$

Definition A.1.7. *Given a family of finite elements* $(\mathcal{T}, \mathcal{P}_{\mathcal{T}}, \mathcal{N}_{\mathcal{T}})$ *where* $K \in \mathcal{T}$, *let*

$$h_K = diam(K), \tag{A.4}$$

and

$$\rho_K = \sup \{ diam(S), S \text{ is a ball contained in } K \}. \tag{A.5}$$

We say that the family of finite elements is regular if

- *there exists a constant* c *such that*

$$\frac{h_K}{\rho_K} \leqslant c, \tag{A.6}$$

- *the family* (h_K) *is bounded and* 0 *is its only accumulation point, which we indicate, with an abuse of notation, as*

$$h_K \to 0. \tag{A.7}$$

Definition A.1.8. *Let element* $K \in \mathcal{T}$. *For* $m \geqslant 4$ *a macro-element of degree* m *is a nodal finite element* $(K, \tilde{\mathcal{P}}_m, \tilde{\mathcal{N}}_m)$ *consisting of subtriangles* K_i, *where* $i = 1, 2, \ldots, s$, *with* $s = 3$ *if* K *is a triangle or* $s = 4$ *if* K *is a quadrilateral. The local element space* $\tilde{\mathcal{P}}_m$ *is defined by*

$$\tilde{\mathcal{P}}_m := \left\{ \upsilon \in C^1(K) : \upsilon|_{K_i} \in \mathcal{P}_m(K_i), i = 1, \ldots, s \right\}.$$

The degrees of freedom $\tilde{\mathcal{N}}_m$ *are defined as follows:*

- *the value and the first (partial) derivatives at the vertices of* K*;*
- *the value at* $m - 3$ *distinct points in the interior of each exterior edge of* K*;*

- *the normal derivative at $m-2$ distinct points in the interior of each exterior edge of K;*
- *the value and the first (partial) derivatives at the common vertex of all K_i, where $i = 1, \ldots, s$;*
- *the value at $m-4$ distinct points in the interior of each edge of the K_i, where $i = 1, \ldots, s$, that is not an edge of K;*
- *if K is a triangle then the normal derivative at $m-4$ distinct points in the interior of each edge of the K_i, where $i = 1, \ldots, 3$, that is not an edge of K, and if K is a quadrilateral then the normal derivative at $m-4$ distinct points in the interior of each edge of the K_i, where $i = 1, \ldots, 4$, that is not an edge of K and an extra normal derivative at a point in the interior of just one of the edges of the K_i that is not an edge of K;*
- *the value at $(m-4)(m-5)/2$ distinct points in the interior of each K_i chosen so that, if a polynomial of degree $m-6$ vanishes at those points, then it vanishes identically.*

See [84].

Proposition A.1.9. *A linear functional, L, on a Banach space, B, is continuous if and only if it is bounded, i.e., if there is a finite constant C such that*

$$|L(v)| \leqslant C||v||_B \quad \forall v \in B. \tag{A.8}$$

For a continuous linear functional, L, on a Banach space, B, the proposition states that the following quantity is always finite:

$$||L||_{B'} := \sup_{0 \neq v \in B} \frac{L(v)}{||v||_B}, \tag{A.9}$$

where B' is the dual space.

Riesz Representation Theorem A.1.10. *Any continuous linear functional, L, on a Hilbert space, H, can be represented uniquely as*

$$L(v) = (u, v) \tag{A.10}$$

for some $u \in H$. Furthermore, we have

$$||L||_{H'} = ||u||_H. \tag{A.11}$$

A.2 Basic Inequalities

Cauchy-Schwarz's inequality A.2.1. *If $u, \upsilon \in L^2(\Omega)$ then $u\upsilon \in L^1(\Omega)$ and*

$$|(u, \upsilon)_\Omega| \leqslant ||u||_{L^2(\Omega)} ||\upsilon||_{L^2(\Omega)}. \tag{A.12}$$

Cauchy-Schwarz's discrete inequality A.2.2. *If $a_1, \ldots, a_k, b_1, \ldots, b_k$ are $2k$ real numbers, then*

$$\sum_{i=1}^{k} |a_i b_i| \leqslant \left(\sum_{i=1}^{k} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{k} |b_i|^2 \right)^{1/2} \tag{A.13}$$

Algebra's inequality A.2.3. *For any non-negative real numbers a, b and p, the following inequality holds*

$$(a+b)^p \leqslant 2^{p-1} \left(a^p + b^p \right). \tag{A.14}$$

Arithmetic-geometric mean inequality A.2.4. *Given N real numbers $\{\alpha_1, \ldots, \alpha_N\}$ let $\beta = \frac{1}{N} \sum_{j=1}^{N} \alpha_j$. Then,*

$$\sum_{j=1}^{N} |\alpha_j - \beta|^2 \leqslant C \sum_{j=1}^{N-1} |\alpha_{j+1} - \alpha_j|^2, \tag{A.15}$$

where C depends only on N.

See [143].

Minkowski's inequality A.2.5. *For $1 \leqslant p \leqslant \infty$ and $u, \upsilon \in L^p(\Omega)$, we have*

$$||u + \upsilon||_{L^p(\Omega)} \leqslant ||u||_{L^p(\Omega)} + ||\upsilon||_{L^p(\Omega)}. \tag{A.16}$$

Young's inequality A.2.6.

$$\forall \varepsilon > 0, \quad \forall a, b \in \Re, \quad ab \leqslant \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2. \tag{A.17}$$

Jump inequality A.2.7.

$$\forall \upsilon \in L^2(\Psi), \quad ||[\![\upsilon]\!]||^2_{L^2(\Psi)} \leqslant 2 \left(||\upsilon^+||^2_{L^2(\Psi)} + ||\upsilon^-||^2_{L^2(\Psi)} \right). \tag{A.18}$$

See [86].

Mean value inequality A.2.8.

$$\forall \upsilon \in L^2(\Psi), \quad ||\langle \upsilon \rangle||^2_{L^2(\Psi)} \leqslant ||\upsilon^+||^2_{L^2(\Psi)} + ||\upsilon^-||^2_{L^2(\Psi)}. \tag{A.19}$$

See [86].

Inverse inequalities A.2.9. *Let υ be a polynomial of degree p in the finite element K and let e either a vertex or an edge or a surface of K with $h_e = diam(e)$. Then, there exist constants $c_0 < \infty$ and $c_1 < \infty$ such that*

$$||\upsilon||^2_{L^2(\partial K)} \leqslant c_0 \frac{p_K^2}{h_K} ||\upsilon||^2_{L^2(K)} \quad \forall \upsilon \in \mathcal{Q}_{p_K}(K) \tag{A.20}$$

and

$$||\nabla \upsilon||^2_{L^2(\partial K)} \leqslant c_1 \frac{p_K^6}{h_K^3} ||\upsilon||^2_{L^2(K)} \quad \forall \upsilon \in \mathcal{Q}_{p_K}(K), \tag{A.21}$$

with the constants c_0, c_1 depending only on the shape-regularity constant (see Theorem 4.76 in [185]).

Poincaré's inequality A.2.10. *Suppose that the open domain Ω is bounded. Then, there exists a constant, $C < \infty$ (which is dependent on Ω and p), such that*

$$||\upsilon||_{W^1_p(\Omega)} \leqslant C|\upsilon|_{W^1_p(\Omega)} \quad \upsilon \in \dot{W}^1_p(\Omega). \tag{A.22}$$

Sobolev inequality A.2.11. *Let Ω be an n-dimensional domain with Lipschitz boundary, let k be a positive integer and let p be a real number in the range $1 \leqslant p < \infty$ such that*

$$k \geqslant n \quad when \quad p = 1,$$
$$k > \frac{n}{p} \quad when \quad p > 1.$$

Then, there is a constant C such that for all $\upsilon \in W^k_p(\Omega)$

$$||\upsilon||_{L^\infty(\Omega)} \leqslant C||\upsilon||_{W^k_p(\Omega)}. \tag{A.23}$$

Moreover there is a continuous function in the $L^\infty(\Omega)$ equivalence class of υ.

h-optimal A.2.12. *The a priori error estimate is optimal in h when the subsequent inequality holds*

$$||\upsilon - \upsilon^{ap}||_{H^s(\Omega)} \leqslant Ch^{t-s}||\upsilon||_{H^t(\Omega)}, \tag{A.24}$$

where υ is the analytical solution of the problem and υ^{ap} is the approximate solution to υ.

p-optimal A.2.13. *The a priori error estimate is optimal in p when the following inequality holds*

$$||\upsilon - \upsilon^{ap}||_{H^s(\Omega)} \leqslant Cp^{-(t-s)}||\upsilon||_{H^t(\Omega)}, \tag{A.25}$$

where υ is the analytical solution of the problem and υ^{ap} is the approximate solution to υ.

A.3 Interpolation Estimates

Theorem A.3.1. *Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ be a finite element and s the greatest order of the partial derivatives occuring in the definition of $\hat{\mathcal{N}}$. Furthermore, assume that for some integers $m \geqslant 0$ and $k \geqslant 0$ it is also*

$$H^{k+1}(\hat{K}) \hookrightarrow C^s(\hat{K}), \tag{A.26}$$

$$H^{k+1}(\hat{K}) \hookrightarrow H^m(\hat{K}), \tag{A.27}$$

$$P_k(\hat{K}) \subset \hat{\mathcal{P}} \subset H^m(\hat{K}), \tag{A.28}$$

where $C^s(K)$ denotes the space of all real-valued s-times continuously differentiable functions on $K \subset \Re^d$, $\hookrightarrow$ is the symbol indicating inclusion with continuous injection, and $P_k(\hat{K})$ is the space of all polynomials of degree $\leqslant k$ defined over $\hat{K}$. Then, there exists a constant C, depending only on $\hat{K}$, $\hat{\mathcal{P}}$ and $\hat{\mathcal{N}}$, such that for all affine-equivalent finite elements in the family $(\mathcal{T}, \mathcal{P}_{\mathcal{T}}, \mathcal{N}_{\mathcal{T}})$, i.e. $\forall K \in \mathcal{T}$, we have

$$|\upsilon - \mathcal{I}_K \upsilon|_{H^m(K)} \leqslant C \frac{h_K^{k+1}}{\rho_K^m} |\upsilon|_{H^{k+1}(K)} \quad \upsilon \in H^{k+1}(K), \tag{A.29}$$

where $\mathcal{I}_K$ indicates the $\mathcal{P}$-interpolant of υ.

A simpler version of this theorem can also be proven by considering the case of a regular affine family of elements as in Definition A.1.7. We have in this case

Theorem A.3.2. *Let $(\mathcal{T}, \mathcal{P}_{\mathcal{T}}, \mathcal{N}_{\mathcal{T}})$ be an affine family of finite elements whose reference finite elements $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ satisfies conditions* (A.26) – (A.28), *and assume in addition that the family is regular. Then, there exists a constant C, depending only on $\hat{K}$, $\hat{\mathcal{P}}$ and $\hat{\mathcal{N}}$, such that for all finite elements in the family we have*

$$|\upsilon - \mathcal{I}_K \upsilon|_{H^m(K)} \leqslant C h_K^{k+1-m} |\upsilon|_{H^{k+1}(K)} \quad \upsilon \in H^{k+1}(K). \tag{A.30}$$

Remarks A.3.3. *1. Under the given assumptions, it is possible to prove a stronger version of A.3.2, for which* (A.30) *is replaced by*

$$||\upsilon - \mathcal{I}_K \upsilon||_{H^m(K)} \leqslant C h_K^{k+1-m} |\upsilon|_{H^{k+1}(K)} \quad \upsilon \in H^{k+1}(K). \tag{A.31}$$

2. The above theorems are the particularization of the analogous results hoding in Sobolev spaces, i.e., for $\upsilon \in W_p^{k+1}(K), p \in [1, \infty]$, the proof of which can be found in Ciarlet [56].

Lemma A.3.4. *Suppose that a triangulation $\mathcal{K}_h$ of Ω consists of d-dimensional simplices or parallelepipeds. Then, for every $u \in H^{\mathbf{t}}(\Omega, \mathcal{K}_h)$, $\mathbf{t} = (t_K : K \in \mathcal{K}_h)$, and for each $\mathbf{p} = (p_K : K \in \mathcal{K}_h, p_K \in \aleph)$, there exists a projector*

$$\Pi_{\mathbf{p}}^{\mathbf{h}} : H^{\mathbf{t}}(\Omega, \mathcal{K}_h) \to S^{\mathbf{p}}(\Omega, \mathcal{K}_h, \mathbf{F}), \quad \left(\Pi_{\mathbf{p}}^{\mathbf{h}} u\right)|_K = \Pi_{p_K}^{h_K}(u|_K),$$

such that, for $0 \leqslant q \leqslant t_K$,

$$||u - \Pi_{p_K}^{h_K} u||_{H^q(K)} \leqslant C \frac{h_K^{s_K - q}}{p_K^{t_K - q}} ||u||_{H^{t_K}(K)} \quad \forall K \in \mathcal{K}_h \tag{A.32}$$

and, for $0 \leqslant q \leqslant t_K - 1$,

$$||D^a(u - \Pi_{p_K}^{h_K} u)||_{L^2(\partial K)} \leqslant C \frac{h_K^{s_K - q - \frac{1}{2}}}{p_K^{t_K - q - \frac{1}{2}}} ||u||_{H^{t_K}(K)}, \quad |a| = q \quad \forall K \in \mathcal{K}_h, \tag{A.33}$$

where $s_K = \min(p_K + 1, t_K)$ and C is a constant independent of u, h_K and p_K, but dependent on $t = \max_{K \in \mathcal{K}_h} t_K$.

See Lemma 8 of Suli and Mozolevki [190].

Remark A.3.5. *We shall assume that the mesh size vector* $\mathbf{h}$ *and the polynomial degree vector* $\mathbf{p}$*, with $p_K \in \aleph$, have bounded local variation, that is there exist constants $\eta, \rho \geqslant 1$ independent of* $\mathbf{h}$ *and* $\mathbf{p}$ *such that, for any pair of elements K and K' which share some face $e \in \mathcal{E}$, one has*

$$\eta^{-1} h_{K'} \leqslant h_K \leqslant \eta h_{K'}.$$

$$\rho^{-1} p_{K'} \leqslant p_K \leqslant \rho p_{K'}.$$

We shall collectively to η and ρ as mesh parameters.

A.4 Inverse Estimates

We now examine the relationship between different seminorms on a finite element space. For some local results, see Brenner and Scott [36], as Ciarlet [56] reports only global results, from which these here need to be extracted.

Theorem A.4.1. *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a finite element,* $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ *its reference element and* l*,* m *two positive integers such that*

$$l \leqslant m, \tag{A.34}$$

$$\hat{\mathcal{P}} \subset H^l(\hat{K}) \cap H^m(\hat{K}). \tag{A.35}$$

Then there exists a constant C_I*, depending only on* $\hat{K}$*,* $\hat{\mathcal{P}}$*,* l *and* m *such that*

$$|v|_{H^m(K)} \leqslant C_I h_K^{l-m} |v|_{H^l(K)} \quad v \in \mathcal{P}. \tag{A.36}$$

Remarks A.4.2. 1. *As in the section on interpolation estimates, under the same assumptions, a stronger theorem, holding for norms rather than seminorms, can be proven, viz.*

$$||v||_{H^m(K)} \leqslant C_I h_K^{l-m} ||v||_{H^l(K)} \quad v \in \mathcal{P}. \tag{A.37}$$

2. *Also, again in the previous section, the above theorem is the particularization of the analogous result holding in Sobolen rather than in Hilbert spaces.*

A.5 Trace Inequalities and Trace Theorem

To conclude, we report two important lemmas [16, 4] giving relevant and very useful bounds for the L^2-norm of a function, and its normal derivative respectively, on an element boundary in terms of norms of the same function in the interior of the element.

Lemma A.5.1. *Let* $\Omega \subset \Re^d$ *have a Lipschitz boundary. Then there is a constant* $0 < C < \infty$ *such that*

$$||v||^2_{L^2(\partial K)} \leqslant C \left(h_K^{-1} ||v||^2_{L^2(K)} + h_K |v|^2_{H^1(K)} \right) \quad \forall v \in H^1(K). \tag{A.38}$$

$$\left|\left| \frac{\partial v}{\partial n} \right|\right|^2_{L^2(\partial K)} \leqslant C \left(h_K^{-1} |v|^2_{H^1(K)} + h_K |v|^2_{H^2(K)} \right) \quad \forall v \in H^2(K). \tag{A.39}$$

See [16, 4] for more details.

Theorem A.5.2. *Suppose that* Ω *has a Lipschitz boundary, and that* p *is a real number in the range* $1 \leqslant p \leqslant \infty$*. Then, there is a constant,* C*, such that*

$$||v||_{L^p(\partial\Omega)} \leqslant C||v||^{1-1/p}_{L^p(\partial\Omega)}||v||^{1/p}_{W^1_p(\Omega)} \quad v \in W^1_p(\Omega). \tag{A.40}$$

We will use the notation $\dot{W}$ to denote the subset of $W^1_p(\Omega)$, consisting of functions whose trace on $\partial\Omega$ is zero, that is

$$\dot{W}^1_p(\Omega) = \left\{ v \in W^1_p(\Omega) : v|_{\partial\Omega} = 0 \in L^2(\partial\Omega) \right\}. \tag{A.41}$$

Simirarly, we let $\dot{W}^k_p(\Omega)$ denote the subset of $W^k_p(\Omega)$ consisting of functions whose derivatives of order $k-1$ are in $\dot{W}^1_p(\Omega)$, i.e.

$$\dot{W}^k_p(\Omega) = \left\{ v \in W^k_p(\Omega) : v^{(a)}|_{\partial\Omega} = 0 \in L^2(\partial\Omega)\ \forall |a| < k \right\}. \tag{A.42}$$

See [36] for more information.

Appendix B

Integration Formulas

Appendix B presents some useful integration formulas, which are used extensively in the variational form of the finite element methods. The references for the following definitions and theorems can be found in the book of Destuynder and Salaun [80] and of Rivière [180], respectively.

B.1 Green's Theorem

Green's theorem B.1.1. *Given K a bounded domain and n_K the outward normal vector to ∂K, we have for all $v \in H^2(K)$ and $w \in H^1(K)$*

$$-\int_K w\Delta v = \int_K \nabla v \cdot \nabla w - \int_{\partial K} \nabla v \cdot n_K w, \tag{B.1}$$

where $\Delta w = \nabla \cdot \nabla w = \sum_{i=1}^{d} \frac{\partial^2 w}{\partial x_i^2}$. A more generalized Green's theorem is

$$-\int_K w\nabla \cdot \mathbf{F}\nabla v = \int_K \mathbf{F}\nabla v \cdot \nabla w - \int_{\partial K} \mathbf{F}\nabla v \cdot n_K w, \tag{B.2}$$

where $\mathbf{F}$ is a matrix-value function.

See [180].

B.2 Double Stokes Formula for Plates

Double Stokes formula for plates B.2.1. *Let u_3 be a smooth function defined over the open set ω. The boundary of ω, say γ, is supposed to be C^1*

(i.e. there exists a mapping which describes γ and is C^1). Then one has for any smooth function v_3 defined over ω, by applying Stokes formula:

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}v_3 dv = -\int_{\partial\omega} m_{\alpha\beta}b_\alpha\partial_\beta v_3 dr + \int_\omega \partial_\alpha m_{\alpha\beta}\partial_\beta v_3 dv \tag{B.3}$$

or else applying one more time Stokes formula:

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}v_3 dv = -\int_{\partial\omega} m_{\alpha\beta}b_\alpha\partial_\beta v_3 dr + \int_{\partial\omega} \partial_\alpha m_{\alpha\beta}b_\beta v_3 dr - \int_\omega \partial_{\alpha\beta}m_{\alpha\beta}v_3 dv. \tag{B.4}$$

This is the so called "double" Stokes formula for plates. There is another way to write relation (B.4)*. Let us notice that on the boundary $\partial\omega$ of ω one has:*

$$\partial_\beta v_3 = \frac{\partial v_3}{\partial s}a_\beta + \frac{\partial v_3}{\partial b}b_\beta$$

which is pricesely the definition of the tangential (respectively normal) derivative $\frac{\partial v_3}{\partial s}$ (respectively $\frac{\partial v_3}{\partial b}$) with respect to the curvilinear abscissa s (respactively the normal coordinate along b). Hence one has from (B.4)*:*

$$\begin{aligned}-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}v_3 dv \quad &= \quad -\int_{\partial\omega} m_{\alpha\beta}b_\alpha b_\beta\frac{\partial v_3}{\partial b}dr - \int_{\partial\omega} m_{\alpha\beta}b_\alpha a_\beta\frac{\partial v_3}{\partial s}dr \\ &\quad + \int_{\partial\omega}\partial_\alpha m_{\alpha\beta}b_\beta v_3 dr - \int_\omega \partial_{\alpha\beta}m_{\alpha\beta}v_3 dv.\end{aligned}$$

Let us notice that:

$$-\int_{\partial\omega} m_{\alpha\beta}b_\alpha a_\beta\frac{\partial v_3}{\partial s}dr = \int_{\partial\omega}\frac{\partial}{\partial s}\left(m_{\alpha\beta}b_\alpha a_\beta\right)v_3 dr.$$

Therefore:

$$\begin{aligned}-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}v_3 dv \quad &= \quad -\int_{\partial\omega} m_{\alpha\beta}b_\alpha b_\beta\frac{\partial v_3}{\partial b}dr \\ &\quad + \int_{\partial\omega}\left[\frac{\partial}{\partial s}(m_{\alpha\beta}b_\alpha a_\beta) + \partial_\alpha m_{\alpha\beta}b_\beta\right]v_3 dr \\ &\quad - \int_\omega \partial_{\alpha\beta}m_{\alpha\beta}v_3 dv,\end{aligned} \tag{B.5}$$

which is the most useful expression of the "double" Stokes formula for plates.

See [80] for more details.

Bibliography

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] E. Amanatidou and N. Aravas, *Mixed finite element formulations of strain-gradient elasticity problems*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 1723-1751.

[3] D. N. Arnold, *An interior Penalty Finite Element Method with Discontinuous Elements. Ph.D. thesis*, (1979), The University of Chicago, Chicago, IL.

[4] D. N. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742 –760.

[5] D. Arnold, F. Brezzi, B. Cockburn, and D. Marini, *Discontinuous Galerkin methods for elliptic problems, in Discontinuous Galerkin Methods. Theory, computation and Applications. B. Cockburn, G. E. Karniadakis, and C. W. Shu, eds*, Lecture Notes in Comput. Sci. Engrg. 11 (2000), Springer - Verlag, New York, pp.89–101.

[6] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., Vol. 39, No. 5 (2002), pp. 1749–1779.

[7] H. Askes, J. Pamin, and R. de Borst, *Dispersion analysis and element-free Galerkin solutions of second- and fourth-order gradient-enhanced damage models*, Int. J. Numer. Methods Engrg., 49 (2000), pp. 811–832.

[8] I. Babuška, *The finite element method with penalty*, Math. Comp., 27 (1973), pp. 221– 228.

[9] BABUŠKA AND M. ZLÁMAL, *Nonconforming elements in the finite element method with penalty*, SIAM J. Numer. Anal., 10 (1973), pp. 863–875.

[10] I. BABUŠKA, J. OSBORN, AND J. PITKÄRANTA *Analysis of mixed methods using mesh dependent norms*, Math. Comp., 35 (1980), pp. 1039–1062.

[11] I. BABUŠKA, B. A. SZABO, AND I. N. KATZ, *The p-vesrion of the finite element method*, SIAM J. Numer. Anal. 18, 3 (1981), pp. 515-545.

[12] I. BABUŠKA AND M. SURI *The h-p version of the finite element method with quasiuniform meshes*, Math. Model. Numer. Anal., 21 (1987), pp. 199–238.

[13] I. BABUŠKA AND M. SURI, *The optimal convergence rate of the p-version of the finite element method*, SIAM J. Numer. Anal. 24, 4 (1987), pp. 750-776.

[14] I. BABUŠKA, A. GRAIG, J. MANDEL, AND J. PITKÄRANTA, *Efficient preconditionning for the p-version finite element method in two dimensions*, SIAM J. Mumer. Anal. 28, 3 (1991), pp. 624–661.

[15] I. BABUŠKA, C. BAUMANN, AND J. T. ODEN, *A discontinuous hp-finite element method for diffusion problems: 1-D analysis*, Comput. Math. Appl., 37 (1999), pp. 103–122.

[16] G. A. BAKER, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp., 31 (1977), pp. 45–59.

[17] G. A. BAKER, W. N. JUREIDINI, AND O. A. KARAKASHIAN, *Piecewise solenoidal vector fields and the Stokes problem*, SIAM J. Numer. Anal., 27 (1990), pp. 1466–1485.

[18] F. BASSI AND S. REBAY, *A high-order accurate discontinuous finite element solution of the 2D Euler equations*, J. Comput. Phys., 138 (1997), pp. 251–285.

[19] F. BASSI AND S. REBAY, *A highorder accurate discontinuous finite element method for the numerical solution of the compressible Navier Stokes equations*, J. Comput. Phys., 131 (1997), pp. 267–279.

[20] F. Bassi and S. Rebay, *A high order discontinuous Galerkin method for compressible turbulent flows*, in: Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin, 2000, pp. 77–88.

[21] C. Baumann, *An hp-Adaptive Discontinuous Finite Element Method for Computational Fluid Dynamics*, Ph.D. thesis, The University of Texas at Austin, 1997.

[22] C. E. Baumann and J. T. Oden, *A discontinuous hp finite element method for the Euler and Navier Stokes equations*, Internat. J. Numer. Methods Fluids, 31 (1999), pp. 79–95.

[23] C. E. Baumann and J. T. Oden, *A discontinuous hp finite element method for convection diffusion problems*, Comput. Methods Appl. Mech. Engrg., 175 (1999), pp. 311–341.

[24] R. Becker and P. Hansbo, *A Finite Element Method for Domain Decomposition with Non Matching Grids*, Tech. Report 3613, (1999), INRIA, LE Chesnay, France.

[25] R. Becker and P. Hansbo, *Discontinuous Galerkin methods for convection-diffusion problems with arbitrary Péclet number, in Numerical Mathematics and Advanced Applications, Proceedings of ENUMATH 99, P. Neittaanmaki, T. Tiihonen and P. Tarvainen, eds.*, World Scientific, River Edge, NJ, (2000), pp. 100 –109.

[26] C. Bernardi, N. Debit, and Y. Maday, *Coupling finite element and spectral methods: First results*, Math. Comp., 54 (1990), pp. 21–39.

[27] C. Bernardi, Y. Maday, and A. T. Patera, *Domain decomposition by the mortar element method in Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters. H. G. Kaper and M. Garbey, eds.*, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1993), pp. 269–286.

[28] C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: The mortar element method in Nonlinear Partial Differential Equations and Their Applications. College de France Seminar, vol. XI., H. Brezis and J.L.*

Lions, eds., Pitman Res. Notes Math. Ser., 299 (1994), Longman Sci. Tech., Harlow, pp. 13–51.

[29] K. S. Bey and J. T. Oden, *hp-version discontinuous Galerkin methods for hyperbolic conservation laws*, Comput. Methods Appl. Mech. Engrg., 133 (1996), pp. 259–286.

[30] J. L. Bleustein, *A note on the boundary conditions of Toupin's strain-gradient theory*, Int. J. Solids Struct., vol. 3 (1967), pp. 1053–1057.

[31] D. Braess, *Finite elements. Theory, fast solvers and application in solid mechanics*, Cambridge University Press, United Kingdom, 1997.

[32] J. H. Bramble and R. Falk, *Two mixed finite element methods for the simply supported plate problem*, R.A.I.R.O. Anal. Numer., 17 (1983), pp. 337–384.

[33] S. C. Brenner, K. Wang, and J. Zhao, *Poincaré-Friedrichs Inequalities for Piecewise H^2 Functions*, Numerical Functional Analysis and Optimization, 25 (5–6) (2004), pp. 463–478.

[34] S. C. Brenner and L. Y. Sung, *C^0-interior penalty methods for fourth order elliptic boundary value problems on polygonal domains*, J. Sci. Comput., 22/23 (2005), pp. 83–118.

[35] S. C. Brenner and L. Y. Sung, *Multigrid algorithms for C^0-interior penalty methods*, SIAM J. Numer. Anal., 44 (2006), pp. 199–223.

[36] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Springer, New York, 2008.

[37] S. C. Brenner, T. Gudi, and L. Y. Sung, *An a posteriori error estimator for a quadratic C^0-interior penalty method for the biharmonic problem*, IMA J. Numer. Anal. 30 (2010), pp. 777–798.

[38] S. C. Brenner and M. Neilan, *A C^0-interior penalty method for a fourth order elliptic singular perturbation problem*, SIAM J. Numer. Anal., vol. 49, No. 2 (2011), pp. 869–892.

[39] S. C. Brenner, S. Gu, T. Gudi, and L. Y. Sung, *A quadratic C^0-interior penalty method for linear fourth order boundary value problems with boundary conditions of the Chan-Hilliard type*, SIAM J. Numer. Anal., vol. 50, No.4 (2012), pp. 2088–2110.

[40] S. C. Brenner, M. Neilan, and L. Y. Sung, *Isoparametric C^0-interior penalty methods for plate bending problems on smooth domains*, Calcolo, 49 (2012), pp. 1–33.

[41] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Masson, éditeur, Paris, 1983.

[42] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Verlag, New York, 1991.

[43] F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, Math. Models Methods Appl. Sci., Vol. 4, No. 4 (1994), pp. 581–587.

[44] F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, *Stabilization techniques and subgrid scales capturing, in: The state of the art in numerical analysis*, Oxford University Press, New York, 1997, pp. 391–406.

[45] F. Brezzi, L . P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., 145 (1997), pp. 329–339.

[46] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo, *Discontinuous finite elements for diffusion problems*, in Atti Convegno in onore di F. Brioschi (Milan, 1997), Istituto Lombardo, Academia di Scienze e Lettere, Milan, Italy, 1999, pp.197–217.

[47] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo, *Discontinuous Galerkin approximations for elliptic problems, Numer. Methods PDE*, 16 (2000), pp. 365–378.

[48] A. N. Brooks and T. J. R. Hughes, *Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the imcompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199–259.

[49] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, *An a priori error analysis of the local discontinuous Galerkin Method for elliptic problems*, SIAM J. Numer. Anal., 38 (2001), pp. 1676–1706.

[50] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab, *Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems*, Math. Comp., to appear.

[51] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, *Local discontinuous Galerkin methods for elliptic problems*, Commun. Numer.Meth. Engrg., 18 (2002), pp. 69–75.

[52] Z. Chen, B. Cockburn, C. Gardner, and J. Jerome, *Quantum hydrodynamics simulation of hysteresis in the resonant tunnelling diode*, J. Comput., Phys., 117 (1995), pp. 274–280.

[53] Z. Chen, B. Cockburn, J. Jerome, and C. W. Shu, *Mixed RKDG finite element methods for the 2-D hydrodynamic model for semiconductor device simulation*, VLSI Design, 3 (1995), pp. 145–158.

[54] J. S. Chen, C. T. Wu, and T. Belytschko, *Regularization of material instabilities by mesh free approximations with intrinsic length scales*, Int. J. Numer. Methods Engrg., 47 (2000), pp. 1303–1322.

[55] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Company, Amsterdam, 1978.

[56] P. G. Ciarlet, *Basic error estimates for elliptic problems*, in: P. G. Ciarlet, J. L. Lions (Eds.), Finite Element Methods (Part 1), Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, 1991.

[57] B. Cockburn and C. -W. Shu, *Runge Kutta discontinuous Galerkin methods for convection dominated problems*, J. Sci. Comput., to appear.

[58] B. Cockburn, S. Y. Lin, and C. W. Shu, *TVB Runge Kutta local projection discontinuous Galerkin finite element method for conservation laws III: One dimensional systems*, J. Comput. Phys., 84 (1989), pp. 90–113.

[59] B. Cockburn and C. -W. Shu, *TVB Runge Kutta local projection discontinuous Galerkin finite element method for scalar conservation laws II: General framework*, Math. Comp., 52 (1989), pp. 411–435.

[60] B. Cockburn, S. Hou, and C. -W. Shu, *TVB Runge Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case*, Math. Comp., 54 (1990), pp. 545–581.

[61] B. Cockburn and C. -W. Shu, *The Runge Kutta local projection P^1-discontinuous Galerkin method for scalar conservation laws*, RAIRO Model Math. Anal. Numer., 25 (1991), pp. 337–361.

[62] B. Cockburn and C. W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463.

[63] B. Cockburn and C. -W. Shu, *The Runge Kutta discontinuous Galerkin finite element method for conservation laws V: Multidimensional systems*, J. Comput. Phys., 141 (1998), pp. 199–224.

[64] B. Cockburn, *Discontinuous Galerkin methods for convection dominated problems, in High-Order Methods for computational Physics, T. Barth and H. Deconink, eds.*, Lecture Notes in Comput. Sci. Engrg. 9 (1999), Springer - Verlag, New York, pp. 69–224.

[65] B. Cockburn and C. Dawson, *Some extensions of the local discontinuous Galerkin method for convection-diffusion equations in multidimensions, in Proceedings of the Conference on the Mathematics of Finite Elements and Applications: MAFELAP X, J. R. Whitteman, ed.*, Elsevier, New York (2000), pp. 225–238.

[66] B. Cockburn, J. Jerome, and C. -W. Shu, *The utility of modelling and simulation in determining performance and symmetry properties of semiconductors in Discontinuous Galerkin Methods.*

Theory, Computation and Applications, B. Cockburn, G. E. Karniadakis, and C.-W. Shu, eds., Lecture Notes in Comput. Sci. Engrg. 11 (2000), Springer-Verlag, New York, pp. 147–156.

[67] B. Cockburn, G. E. Karniadakis, and C. -W. Shu, *The development of discontinuous Galerkin methods, in Discontinuous Galerkin Methods. Theory, Computation and Applications, B. Cockburn, G. E. Karniadakis, and C. W. Shu, eds.*, Lecture Notes in Comput. Sci. Engrg. 11 (2000), Springer Verlag, New York, pp. 3–50.

[68] B. Cockburn, *Devising discontinuous Galerkin methods for non-linear hyperbolic conservation laws*, J. Comput. Appl. Math., 128 (2001), pp. 187–204.

[69] B. Cockburn, G. Kanschat, I. Perugia, and D. Schötzau *Superconvergence of the local discontinuous Galerkin method for elliptic problems on cartesian grids*, SIAM J. Numer. Anal., Vol. 39, No. 1 (2002), pp. 264–285.

[70] E. Cosserat and F. Cosserat, *Sur la mécanique génerale*, Comptes Rendus de l 'Academie des Sciences, Serie I, Mathématique 145 (1907), pp. 1139–1142.

[71] E. Cosserat and F. Cosserat, *Theorie des corps déformables*, Herman, Paris, 1909.

[72] F. Costanzo and H. Huang, *Proof of unconditional stability for a single-field discontinuous Galerkin finite element formulation for linear elasto-dynamics*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 2059–2076.

[73] C. N. Dawson and Godunov, *Mixed methods for advection flow problems equations in one space dimension*, SIAM J. Numer. Anal., 28 (1991), pp. 1282–1309.

[74] C. Dawson and Godunov, *Mixed methods for advection-diffusion equations in multidimensions*, SIAM J. Numer. Anal., 30 (1993), pp. 1315–1332.

[75] C. Dawson, *Analysis of an upwind mixed finite element method for linear contaminant transport equations*, SIAM J. Numer. Anal., 35 (1998), pp.1709–1724.

[76] C. N. Dawson and V. Aizinger, *Upwind-mixed methods for transport equations*, Comp. Geosci., 3 (1999), pp. 93–110.

[77] L. M. Delves and C. A. Hall, *An implicit matching principle for global element calculations*, J. Inst. Math. Appl. 23 (2) (1979), pp. 223–234.

[78] B. Despres, *Entropy inequality for high-order discontinuous Galerkin approximation of Euler equations*, in: Hyperbolic problems: theory, numerics, applications, Vol. I (Zürich, 1998). Birkhäuser, Basel (1999), pp. 225–231.

[79] B. Despres, *Discontinuous Galerkin method in the numerical solution of Euler equations in axisymmetric geometry*, in: Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin (2000), pp. 315-320.

[80] P. Destuynder and M. Salaun, *Mathematical Analysis of Thin Plate Models, mathématiques and applications 24*, Springer-Verlag, New York, 1996.

[81] V. Dolejǰsí, M. Feistauer, and J. Hozman, *Analysis of semi-implicit DGFEM for non-linear convection-diffusion problems on non-conforming meshes*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 2813–2827.

[82] J. Douglas, Jr. and T. Dupont, *Interior Penalty Procedures for Eliptic and Parabolic Galerkin Methods*, in: Comput. Methods Appl. Sci., Lecture Notes in Phys.,Vol. 58 (1976), Springer - Verlag, Berlin, pp. 207–216.

[83] J. Douglas, Jr., B. L. Darlow, R. P. Kendall, and M. F. Wheeler, *Self-adaptive Galerkin methods for one-dimensional, two-phase immiscible flow*, in Proceedings AIME Fifth Symposium on Reservoir Simulation (Denver, Colorado), Society of Petroleum Engineers, Dallas, TX 1979, pp. 65–72.

[84] J. Douglas, Jr., T. Dupont, P. Percell, and R. Scott, *A family of C^1-finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems*, RAIRO Analy. Numér., 13 (1979), pp. 227–255.

[85] E. G. Dutra Do Carmo and A. V. C. Duarte, *A discontinuous finite element-based domain decomposition method*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 825–843.

[86] G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Methods Appl. Mech. Eng., 191 (2002), pp. 3669–3750.

[87] K. G. Eptaimeros, *Continuous Interior Penalty Finite Element Methods for the biharmonic problem*, Graduate Thesis, National Technical University of Athens, Greece, 2006.

[88] K. G. Eptaimeros, *Discontinuous Galerkin Finite Element Methods for Strain Gradient Elasticity*, MSc Thesis, National Technical University of Athens, Greece, 2008.

[89] K. G. Eptaimeros and G. J. Tsamasphyros, *Interior penalty discontinuous Galerkin finite element methods for strain gradient elasticity*, Proccedings of the 4th International conference from Scientific Computing to Computational Engineering (4th IC- SCCE), Athens, Greece, 7–10 July 2010, Vol. 1, pp. 343–351.

[90] K. G. Eptaimeros and G. J. Tsamasphyros, *Continuous interior penalty finite element method for strain gradient elasticity*, Proccedings of the 7th GRACM International Congress on Computational Mechanics, Athens, Greece, 30 June–2 July 2011.

[91] A. C. Eringen, *Continuum physics: Polar and non-local theories*, Vol. 4, New York, Academic Press, 1976.

[92] D. A. Fafalis, S. P. Filopoulos, and G. J. Tsamasphyros, *On the capability of generalized continuum theories to capture disper-*

sion characteristics at the atomic scale, European J. Mech. A/Solids., 36 (2012), pp. 25–37.

[93] R. S. Falk and G. R. Richter, *Local error estimates for a finite element method for hyperbolic and convection-diffusion equations*, SIAM J. Numer. Anal. 29, 3 (1992), pp. 730–754.

[94] R. S. Falk and G. R. Richter, *Explicit finite element methods for symmetric hyperbolic equations*, SIAM J. Numer. Anal. 36, 3 (1999), pp. 935–952 (electronic).

[95] R. S. Falk, *Analysis of finite element methods for linear hyperbolic problems*, In Discontinuous Galerkin methods (Newport, RI, !999), Springer, Berlin (2000), pp. 103–112.

[96] X. Feng and O. A. Karakashian, *Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems*, SIAM J. Numer. Anal.,Vol. 39, No. 4 (2001), pp. 1343–1365.

[97] X. Feng and O. A. Karakashian, *Two-level non-overlapping additive Schwarz methods for a discontinuous Galerkin approximation of the biharmonic problem*, J. Scient. Comput., 22 (2005), pp. 299–324.

[98] X. Feng and O. A. Karakashian, *Fully discrete dynamic mesh discontinuous Galerkin methods for the Cahn-Hilliard equation of phase transition*, Math. of computation, Vol. 76, No. 259 (2007), pp. 1093–1117.

[99] S. P. Filopoulos, Th. K. Papathanasiou, and G. J. Tsamasphyros, *A finite element model for calculating the stresses in bars with microstructure loaded by ultra-short laser pulses*, J. Therm. Stresses, 32 (2009), pp. 905–922.

[100] S. P. Filopoulos, Th. K. Papathanasiou, S. I. Markolefas, and G. J. Tsamasphyros, *Dynamic finite element analysis of a gradient elastic bar with micro-inertia*, Comput. Mech., 45 (2010), pp. 311-319.

[101] N. A. Fleck, G. M. Muller, M. F. Ashby, and J. W. Hutchinson, *Strain gradient plasticity: Theory and experiment*, Acta Metallurgica et Materialia, 42 (1994), pp. 475–487.

[102] N. A. Fleck and J. W. Hutchinson, *Strain gradient plasticity*, Advances Appl. Mech., vol. 33 (1997), pp. 295–361.

[103] L. P. Franca, S. L. Frey, and T. J. R. Hughes, *Stabilized finite element methods: I. Application to the advective-diffusive model*, Comput. Methods Appl. Mech. Engrg., 95 (1992), pp. 253–276.

[104] H. Gao, Y. Huang, and W. D. Nix, *Mechanism-based strain gradient plasticity. I. Theory*, J. Mech. Phys. Solids, 47 (1999), pp. 1239–1263.

[105] H. G. Georgiadis and I. Vardoulakis, *Anti-plane shear Lamb's problem treated by gradient elasticity with surface energy*, Wave Motion, 28 (1998), pp. 353–366.

[106] H. G. Georgiadis, I. Vardoulakis, and G. Lykotrafitis, *Torsional surface waves in a gradient-elastic half-space*, Wave Motion, 31 (2000), pp. 333–348.

[107] H. G. Georgiadis, *The mode III crack problem in microstructured solids governed by dipolar gradient elasticity: Static and dynamic analysis*, J. Appl. Mech., 70 (2003), pp. 517–530.

[108] H. G. Georgiadis and E. G. Velgaki, *High-frequency Rayleigh waves in materials with microstructure and couple-stress effects*, Int. J. Solids Struct., 40 (2003), pp. 2501–2520.

[109] H. G. Georgiadis, I. Vardoulakis, and E. G. Velgaki, *Dispersive Rayleigh-wave propagation in microstructured solids characterized by dipolar gradient elasticity*, J. Elast., 74 (2004), pp. 17–45.

[110] H. G. Georgiadis and C. G. Grentzelou, *Energy theorems and the J-integral in dipolar gradient elasticity*, Int. J. Solids Struct., 43 (2006), pp. 5690–5712.

[111] E. H. Georgoulis, *Discontinuous Galerkin Methods on Shape-Regular and Anisotropic Meshes*, Ph.D. thesis, University of Oxford, 2003.

[112] E. H. Georgoulis and E. Süli, *Optimal error estimates for the hp-version interior penalty discontinuous Galerkin finite element method*, IMA J. Numer. Anal., 25 (2005), pp. 205–220.

[113] E. H. Georgoulis, *hp-version interior penalty discontinuous Galerkin finite element methods on anisotropic meshes*, Int. J. Numer. Anal. Model., Vol. 3, No. 1 (2006), pp. 52–79.

[114] E. H. Georgoulis and P. Houston, *Discontinuous Galerkin methods for the biharmonic problem*, IMA J. Numer. Anal., 29 (2009), pp. 573–594.

[115] E. H. Georgoulis, P. Houston, and J. Virtanen, *An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems*, IMA J. Numer. Anal., (2009), pp. 1–18.

[116] A. E. Giannakopoulos and K. Stamoulis, *Structural analysis of gradient elastic components*, Int. J. Solids Struct., vol. 44 (2007), pp. 3440–3451.

[117] J. Gopalakrishnan and G. Kanschat, *Mutli-level preconditioners for the interior penalty method*, Proceedings of ENUMATH 01, 2002.

[118] J. Gopalakrishnan and G. Kanschat, *A multilevel discontinuous galerkin method*, Numer. Math. to appear.

[119] J. Greenstadt, *An abridged history of cell discretization, in Discontinuous Galerkin Methods. Theory, Computation and Appllications, B. Cockburn, G. E. Karniadakis, and C. W. Shu, eds.*, Lecture Notes in Comput. Sci. Engrg. 11 (2000), Springer Verlag, New York, pp. 337–342.

[120] P. Grisvard, *Elliptic Problems in Non Smooth Domains*, Pitman, Boston, 1985.

[121] T. Gudi, *A new error analysis for discontinuous finite element methods for linear elliptic problems*, Math. Comp., vol. 79, No. 272 (2010), pp. 2169–2189.

[122] P. Hansbo and M. G. Larson, *A posteriori error estimates for continuous/discontinuous Galerkin approximations of the Kirchhoff-Love plate*, Preprint2008:10, Cöteborg (2008), Sweden, Chalmers University of Technology.

[123] I. Harari and T. J. R. Hughes, *What are C and h?: Inequalities for the analysis and design of finite element methods*, Comput. Methods Appl. Mech. Engrg., 97 (1992), pp. 157–192.

[124] P. W. Hemker, W. Hoffmann, and M. H. van Raalte, *Fourier two-level analysis for higher dimensional discontinuous Galerkin*, CWI report MAS-R0227, Amsterdam, 2002 .

[125] P. W. Hemker and M. H. Raalte, *Fourier two-level analysis for discontinuous Galerkin discretization with linear elements*, CWI report MAS-R0217, Amsterdam, 2002 .

[126] J. A. Hendry and L. M. Delves, *The global element method applied to a harmonic mixed boundary value problem*, J Comp. Phys., Vol. 33 (1979), Issue 1, pp. 33–44.

[127] D. J. Higham and N. J. Higham, *Matlab Guide*, 2nd ed., SIAM, 2005.

[128] P. Houston, C. Schwab, and E. Süli, *Stabilized hp-finite element methods for first-order hyperbolic problems*, Technical Report 98/14, Oxford University Comput. Lab., Oxford, England, November 1998.

[129] P. Houston, C. Schwab, and E. Süli, *Discontinuous hp-finite element methods for advection-diffusion problems*, Tech. Report NA 00–15 (2000), Oxford University Computing Laboratory, Oxford, UK.

[130] P. Houston, C. Schwab, and E. Süli, *Discontinuous hp-finite element methods for advection-diffusion problems*, SIAM J. Numer. Anal., 39 (6) (2002), pp. 2133–2163.

[131] P. Houston, M. Jensen, E. Süli, *hp-discontinuous Galerkin finite element methods with least-squares stabilization*, In Proceedings

of the Fifth International Conference on Spectral and High-Order Methods (ICOSAHOM-01), (Uppsala), Vol. 17 (2002), pp. 3–25.

[132] P. Houston, D. Schötzau, T. P. Wihler, *Energy norm a posteriori error estimation of hp-adaptive discontinuous Galerkin methods for elliptic problems*, Math. Models Methods Appl. Sci., 17 (2007), pp. 33–62.

[133] Y. Huang, H. Gao, W. D. Nix, and J. W. Hutchinson, Mechanism-based strain gradient plasticity. II. Analysis, J. Mech. Phys. Solids, 48 (2000), pp. 99–128.

[134] T. J. R. Hughes and A. Brooks, *A theoretical framework for Petrov-Galerkin methods with discontinuous weighting functions: application to the streamline-upwind procedure*, in: Finite elements in fluids., vol. 4 of A Wiley-Interscience Publicatio, John Wiley and Sons Inc., New York (1982), pp. 47-65.

[135] T. J. R. Hughes, *The finite element method: Linear, static and Dynamic F. E. analysis*, Prentice-Hall, 1987.

[136] T. J. R. Hughes, L. P. Franca, and G. M. Hulbert, *A new finite element formulation for computational fluid dynamics. VIII. The Galerkin/least-squares method for the advective-dffusive equations*, Comput. Methods Appl. Mech. Engrg., Vol. 73, No. 2 (1989), pp. 173–189.

[137] T. J. R. Hughes, *Mutliscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods*, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.

[138] C. Johnson, U. Nävert, and J. Pitkäranta, *Finite element methods for linear hyperbolic problems*, Comput. Methods Appl. Mech. Engrg. 45, 1–3 (1984), pp. 285–312.

[139] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp. 46, 173 (1986), pp. 1–26.

[140] C. Johnson and J. Saranen, *Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations*, Math. Comp. 47, 175 (1986), pp. 1-18.

[141] G. Kanschat, *Preconditioning methods for local discontinuous Galerkin discretizations*, Preprint 2002-18, SFB 359, Heidelberg, 2002.

[142] O. A. Karakashian and T. Katsaounis, *A discontinuous Galerkin method for the incompressible Navier-Stokes equations*, in: Discontinuous Galerkin methods (Newport, RI, 1999). Springer, Berlin, 2000, pp. 157–166.

[143] O. A. Karakashian and F. Pascal *A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems*, SIAM J. Numer. Anal., vol. 41 (6) (2003), pp. 2374–2399.

[144] R. B. Kellogg, *Higher-order singularities for interface problems*, In Proceedings of a Symposium Held at the University of Maryland Baltimore, Maryland, June 26-30, 1972., Vol. 15, Academic Press, New York and London (1972), pp. 589–602.

[145] F. Kikuchi and Y. Ando, *A new variational functional for the finite element method*, Nucl. Engrg Design, 21 (1972), pp. 95–113.

[146] W. T. Koiter, *Couple stresses in the theory of elasticity, I and II*, Proceedings of the Koninkliske Nederlandse Akademie van Wetenschappen (B), vol. 67 (1964), pp. 17–44.

[147] M. G. Larson and T. J. Barth, *A posteriori error estimation for adaptive discontinuous Galerkin approximations of hyperbolic systems*, In Discontinuous Galerkin methods (Newport, RI, 1999), Berlin (2000), pp. 363–368.

[148] P. Lesaint and P. -A. Raviart, *On a finite element method for solving the neutron transport equation*, in: C. de Boor (Ed.), Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York (1974), pp. 89–123.

[149] P. Lesaint, *On the convergence of Wilson's non-conforming element for solving the elastic problem*, Comput. Methods Appl. Mech. Engrg., 7 (1976), pp. 1–16.

[150] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, vol. III, Springer, New York, 1973.

[151] I. Lomttev, C. W. Quillen, and G. E. Karniadakis, *Spectral hp-methods for viscous compressible flows on unstructured 2-D meshes*, J. Comput. Phys., 144 (1998), pp. 325–357.

[152] M. Lyly, *Stabilized finite element methods for plate bending problems*, Ph.D. Thesis, Helsiski University of Technology, Espoo, Finland, 1999.

[153] S.I. Markolefas, D.A. Tsouvalas, and G.I. Tsamasphyros, *Theoretical analysis of a class of mixed, C^0-continuity formulations for general dipolar Gradient Elasticity boundary value problems*, Int. J. Solids Struct., 44 (2007), pp. 546–572.

[154] S.I. Markolefas, D.A. Tsouvalas, and G.J. Tsamasphyros, *Some C^0-continuity mixed formulations for general dipolar linear Gradient Elasticity boundary value problems and the associated energy theorems*, Int. J. Solids Struct., vol. 45 (2008), pp. 3255–3281.

[155] S.I. Markolefas, D.A. Tsouvalas, and G.J. Tsamasphyros, *Mixed finite element formulation for the general anti-plane shear problem, including mode III crack computations, in the framework of dipolar linear gradient elasticity*, Comput. Mech., 43 (2009), pp. 715-730.

[156] A.T. McBride and B. D. Reddy *A discontinuous Galerkin formulation of a model of gradient plasticity at finite strains*, Comput. Methods Appl. Mech. Engrg., vol. 198 (2009), pp. 1805–1820.

[157] R. D. Mindlin, *Micro-structure in Linear Elasticity*, Archive for Rational Mech. Anal., vol. 16 (1964), pp. 51–78.

[158] R. D. Mindlin and N. N. Eshel, *On First Strain-Gradient Theories In Linear Elasticity*, Int. J. Solids Struct., vol. 4 (1968), pp. 109–124.

[159] P. Monk, *A mixed finite element method for the biharmonic equation*, SIAM J. Numer. Anal., 24 (1987), pp. 737–749.

[160] I. Mozolevski and P. R. Bösing, *Sharp expressions for the stabilization parameters in symmetric interior penalty discontinuous Galerkin finite element approximations of fourth-order elliptic problems*, Comput. Methods Appl. Math., vol. 7 (4) (2007), pp. 365–375.

[161] S. A. Nazarov and B. A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, de Gruyter, Berlin, New York, 1994.

[162] N. C. Nguyen, J. Peraire, and B. Cockburn, *An implicit high-order hybridizable discontinuous Galerkin method for linear convection-diffusion equations*, J. Comp. Phys., vol. 228 (2009), pp. 3232–3254.

[163] J. A. Nitsche, *Ube rein Variationsprizip zur Losung Dirichlet Problemen bei Verwendurg von Teilraumen, die keinen Randbedingungen unteworfen sind*, Abh. Math. Sem. Univ. Hamburg, 36 (1971), pp. 9–15.

[164] J. T. oden, I. Babuska, and C. E. Baumann, *A discontinuous hp-finite element method for diffusion problems*, J. Comput. Phys., 146 (1998), pp. 491–519.

[165] J. T. Oden, I. Babuska, and C. Baumann, *The local discontinuous Galerkin finite element method for diffusion problems*, J. Comput. Phys., vol. 146 (1998), pp. 491–519.

[166] J. T. Oden and C. E. Baumann, *A conservative discontinuous Galerkin method for convection-diffusion and Navier-Stokes problems*, in: Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin, 2000, pp. 179–196.

[167] S. Papargyri-Beskou, K. G. Tsepoura, D. Polyzos, and D. E. Beskos, *Bending and stability analysis of gradient elastic beams*, Int. J. Solids Struct., vol. 40 (2003), pp. 385–400.

[168] P. Percell and M. F. Wheeler, *A local residual finite element procedure for elliptic equations*, SIAM J. Numer. Anal., 15 (4) (1978), pp. 705–714.

[169] I. Perugia and D. Schötzau, *An hp-analysis of the local discontinuous Galerkin method for diffusion problems*, J. Sci. Comput., 17 (2002), pp. 561–571.

[170] S. Prudhomme, F. Pascal, J. T. Oden, and A. Romkes, *A priori error estimate for the Baumann-Oden version of the discontinuous Galerkin method*, Comptes Rendus de l' Academie des Sciences, Serie I, Mathématique 332 (2001), pp. 851–856.

[171] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Clarendon Press, Oxford, 1999.

[172] J. N. Reddy, *An introduction to the finite element method*, 2nd ed., 1993.

[173] W. H. Reed and T. R. Hill *Triangular mesh methods for the neutron transport equation*, Technical Report LA-UR-73-479 (1973), Los Alamos Scientific Laboratory, Los Alamos.

[174] G. R. Richter, *An optimal-order error estimate for the discontinuous Galerkin method*, Math. Comp. 50, 181 (1988), pp. 75–88.

[175] G. R. Richter, *The discontinuous Galerkin method with diffusion*, Math. Comp., 58 (1992), pp. 631–643.

[176] B. Rivière and M. F. Wheeler, *A discontinuous Galerkin method applied to non-linear parabolic equations*, Lecture Notes Comput. Sci. Engrg., 11 (1999), pp. 231–244.

[177] B. Rivière, M. F. Wheeler, and V. Girault, *Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems I*, Comput. Geosci., 3 (1999), pp. 337–360.

[178] B. Rivière, M. F. Wheeler, *Optimal error estimates for discontinuous Galerkin methods applied to linear elasticity problems*, TICAM, Technical Report, no. 00-30 (2000).

[179] B. Rivière, M. F. Wheeler, and V. Girault, *A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems*, SIAM J. Numer. Anal., vol. 39, No. 3 (2001), pp. 902–931.

[180] B. Rivière, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations, Theory and Implementation*, S.I.A.M., Society for Industrial and Applied Mathematics Philadelphia, 2008.

[181] A. Romkes, S. Prudhomme, and J. T. Oden, *A priori error analyses of stabilized discontinuous Galerkin method*, Comput. Math. Appl., 46 (2003), pp. 1289–1311.

[182] T. Rustten, P. S. Vassilevski, and R. Winther, *Interior penalty preconditioners for mixed finite element approximations of elliptic problems*, Math. Comp., 65 (1996), pp. 447–466.

[183] M. Scheghter, *A generalization of the problem of transmission*, Ann. Scuola Norm. Sup. Pisa, vol. 14 (1960), pp. 207–236.

[184] D. Schötzau, C. Schwab, and A. Toselli, *Mixed hp-DGFEM for imcompressible flows II: Geometric edges meshes*, IMA J. Numer. Anal., 24 (2004), pp. 273–308.

[185] C. Schwab, *p- and hp-finite element methods. Theory and applications to solid and fluid mechanics*, Oxford University Press, 1998.

[186] C. - W. Shu, *Different formulations of the discontinuous Galerkin method for the viscous terms, in Proccedings of Conference in Honor of Professor H. C. Huang on the Occasion of His Retirement, W. M. Xue, Z.C. Shi, M. Mu, and J. Zou, eds.*, Science press, New York, Beijing (2000), pp. 14–45.

[187] J. S. Stolken and A. G. Evans, *Microbend test method for measuring the plasticity length scale*, Acta Materialia, 46 (1998), pp. 5109–5115.

[188] E. Süli, C. Schwab, and P. Houston, *hp-DGFEM for partial differential equations with nonnegative characteristic form*, Lecture Notes in Comput. Sci. Engrg., 11 (1999), Springer, Berlin, pp. 221–230.

[189] E. Süli, P. Houston, and C. Schwab, *hp-finite element methods for hyperbolic problems, in Proceedings of the Conference on the Mathematics of Finite Elements and Applications: MAFELAP X, j. R. Whiteman, ed.*, Elsevier, New York (2000), pp. 143–162.

[190] E. Süli and I. Mozolevski, *hp-version interior penalty finite element DGFEMs for the biharmonic equation*, Comput. Methods Appl. Mech. Eng., 196 (2007), pp. 1851–1863.

[191] H. Swann, *The cell discretization algorithm: An overview, in Discontinuous Galerkin Methods. Theory, Computation nad Applications, B. Cockburn, G. E. Karniadakis, and C. W. Shu, eds.*, Lecture Notes in Comput. Sci. Engrg. 11 (2000), SpringerVerlag, New York, pp. 433–438.

[192] R. A. Toupin, *Elastic materials with couple stress*, Archives for Rational Mechanics and Analysis, vol. 11 (1962), pp. 385–414.

[193] C. Truesdell and R. A. Toupin, *Principles of classical mechanics and field theory*, in Handbuch der Physik, Vol. III/1, Springer, Berlin, 1960.

[194] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, In Handbuch der Physik, Vol. III/3 (1965), Springer, Berlin.

[195] G.I. Tsamasphyros, S.I. Markolefas, and D.A. Tsouvalas, *Convergence and performance of the h- and p-extensions with mixed finite element C^0-continuity formulations, for tension and buckling of a gradient elastic beam*, Int. J. Solids Struct., 44 (2007), pp. 5056–5074.

[196] G. J. Tsamasphyros and C. D. Vrettos, *A mixed finite volume formulation for the solution of gradient elasticity problems*, Arch. Appl. Mech., 80 (2010), pp. 609–627.

[197] K. G. Tsepoura, S. Papargyri Beskou, D. Polyzos, and D. E. Beskos, *Static and Dynamic Analysis of gradient Elastic Bar in Tension.*, Archive of Applied Mechanics, 72 (2002), pp. 483–497.

[198] I. Vardoulakis and J. Sulem, *Bifurcation analysis in geomechanics*, Blackie/Chapman and Hall, London, 1995.

[199] I. VARDOULAKIS AND H. G. GEORGIADIS, *sh surface waves in a homogeneous gradient-elastic half-space with surface energy*, J. Elast., 47 (1997), pp. 147–165.

[200] T. WARBURTON, *Application of the discontinuous Galerkin method to Maxwells equations using unstructured polymorphic hp-finite elements*, in: Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin (2000), pp. 451-458.

[201] G. N. WELLS, K. GARIKIPATI, AND L. MOLARI, *A discontinuous Galerkin formulation for a strain gradient-dependent damage problem*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 3633–3645.

[202] M. F. WHEELER, *An elliptic collocation finite element method with interior penalties*, SIAM J. Numer. Anal., 15 (1978), pp. 152–161.

[203] T. P. WIHLER, *Locking-free DGFEM for elasticity problems in polygons*, IMA J. Mumer. Anal., 24 (2004), pp. 45–75.

[204] E. L. WILSON, R. L. TAYLOR, W. P. DOHERTY, AND J. GHABOUSSI, *Incompatible displacement problems*, in: S. J. Ferves, N. Perrone, A. R. Robbinson, W. C. Schnobrick (Eds.), Numer. Comput. Models Struct. Mech., Academic Press, New York (1973), pp. 43–57.

[205] Z. C. XIA AND J. W. HUTCHINSON, *Crack tip fields in strain gradient plasticity*, J. Mech. Phys. Solids, 44 (1996), pp. 1621–1648.

[206] O. C. ZIENKIEWICZ AND R. L. TAYLOR, *The finite element method*, McGraw-Hill Book Company, London, 1991.