Semigroups and stochastic evolution equations with additive noise in Hilbert spaces

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0.1 Introduction

This thesis concerns the semigroup theoretic treatment of deterministic and stochastic partial differential equations. Such equations are key tools of mathematical modelling in many different fields such as physics, chemistry, biology, population dynamics, neurophysiology, oceanography, image analysis and mathematical finance among others. Stochastic partial differential equations describe the change in time of a system in terms of the state of the system , taking additionally into account the influence of random fluctuations. There are four approaches to stochastic partial differential equations, the martingale approach [Walsh 1986], the variational approach [Pardoux 1972], [Krylov and Rozowski 1979], the wick product approach [Oksendal 1996] and the semigroup approach [Da Prato and Zabczyc 1992]. In this work we use the semigroup approach, i.e we are dealing with stochastic evolution equations in infinite dimensional Hilbert spaces.
Let X be a Banach space and U, H two separable Hilbert spaces. Firstly, we focus on the linear inhomogeneous abstract Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad t \in [0,T]; \\ u(0) &= x, \qquad\qquad\qquad . \end{aligned}$$

where A is the infinitesimal generator of strongly continuous semigroup on X and $f \in L^1([0,T], X)$.
Next we study its stochastic analogue, i.e the linear stochastic abstract Cauchy problem with additive noise of the form

$$\begin{aligned} &\mathrm{d}X(t) = (AX(t) + f(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = \xi, \end{aligned} \tag{0.1.1}$$

where $\{W(t)\}_{t\in[0,T]}$ is a U-valued Q-Wiener process and X is a random process with values in H. We will impose sufficient conditions for the existence and uniqueness of weak solutions to the above problems.
Finally this abstract theory is applied to the qualitative study of deterministic and stochastic partial differential equations. The two most important examples will be the linear heat equation with zero Dirichlet boundary conditions

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= \Delta u, && (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ u &= 0, && (t,\mathbf{x}) \in [0,\infty) \times \partial\Omega \\ u(0,\mathbf{x}) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \end{aligned}$$

and its stochastic analogue

$$\begin{aligned} d_t X(t,\xi) &= \Delta_\xi X(t,\xi)\,dt + dW(t,\xi) && t \geq 0,\ \xi \in V \\ X(t,\xi) &= 0 && t \geq 0,\ \xi \in \partial V \\ X(0,\xi) &= 0, && \xi \in V \end{aligned}$$

where Δ is the Laplacian in the space variables.

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Part I

Operator semigroup theory

Chapter 1

Semigroups of linear operators

The concept of a semigroup of linear and bounded operators is a natural extension of the exponential of a bounded linear operator to the exponential of a possible unbounded linear operator. As we will see in the first section the only continuous non trivial solutions of the Cauchy functional equation (1.1.1) are the exponential functions e^{ta} with $a \in \mathbb{R}$ and at the same time the exponential function of a $n \times n$ matrix A, $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ solves explicity the first order linear diiferential system $u' = Au$ by means of the formula $u(t) = e^{tA}u(0)$. The extension to bounded operators is not complicated (see Proposition (1.2.6)), but in order to consider the case of unbounded operators, a more elaborate theory is necessary and here is where the new notion of semigroups takes place. The theory of linear semigroups tremendously emerged berween 1930-1960 through the major contributions of Stone, Hille, Yosida, Phillips, Lumer, Miyadera. The aim of this chapter is to introduce the notion of operator semigroups and to present their most important properties. Well known books in operator semigroups are [PZ], [E-N] and [GL], while two relatively new treatises are [VR] and [BW].

1.1 Cauchy's Functional Equation

The question: Find all the continuous functions $T : \mathbb{R}^+ \to \mathbb{C}$, which satisfy the **Cauchy's functional equation**:

$$\begin{aligned} T(t+s) = T(t)T(s), \ \text{for each t,s} \in \mathbb{R}^+, \\ T(0) = 1. \end{aligned} \tag{1.1.1}$$

The answer: It is obvious that the exponential functions $T(t) = e^{at}$, for some $a \in \mathbb{C}$ satisfy the above functional equation (1.1.1). We will show that these and only these are the solutions of Cauchy's functional equation. In the sequel, we will present two propositions which form the whole answer to our initial question.

Proposition 1.1.1. *Consider the exponential function $T(t) = e^{at}$, $t \geq 0$, for some $a \in \mathbb{C}$. Then, the function T is differentiable and satisfies the following initial value problem:*

$$\begin{aligned} \frac{d}{dt}T(t) &= aT(t) \\ T(0) &= 1. \end{aligned} \tag{1.1.2}$$

Moreover, the function $T : \mathbb{R}^+ \to \mathbb{C}$, $T(t) = e^{at}$, is the only differentiable function which satisfies the initial value problem (1.1.2).

Proof: The first part of the proposition is obvious. We will now show the uniqueness. Consider another differentiable function $S : \mathbb{R}^+ \to \mathbb{C}$ that satisfies the initial value problem (1.1.2). For fixed $t \geq 0$, define the function:

$$Q : [0,t] \to \mathbb{C}, \ Q(s) = T(s)S(t-s), \ 0 \leq s \leq t.$$

The function Q is well defined and differentiable in $[0,t]$ with:

$$\frac{d}{ds}Q(s) = aT(s)S(t-s) - T(s)aS(t-s) = 0.$$

Therefore, Q is a constant function. Thus:

$$Q(0) = Q(t) \Leftrightarrow T(t) = S(t), \text{ for each t} \geq 0$$

Proposition 1.1.2. *If $T : \mathbb{R}^+ \to \mathbb{C}$ is a continuous function that satisfies the Cauchy's functional equation (1.1.1), then T is differentiable and there exists a unique $a \in \mathbb{C}$, such that: $T(t) = e^{ta}$, for each $t \geq 0$.*

Proof: Equivalently, we have to show that if a continuous function $T : \mathbb{R}^+ \to \mathbb{C}$ satisfies the Cauchy's functional equation, then it is automatically differentiable and there exists a unique $a \in \mathbb{C}$, such that the initial value problem (1.1.2) is satisfied. To this aim, define the function:

$$V : \mathbb{R}^+ \to \mathbb{C}, \quad V(t) = \int_0^t T(s)\ ds, \text{ t} \geq 0.$$

Then, V is well defined and differentiable in $\mathbb{R}^+$ with:

$$\frac{d}{dt}V(t) = T(t).$$

Moreover, we have:

$$\lim_{t\to 0}\frac{V(t)}{t} = \lim_{t\to 0}\frac{\int_0^t T(s)\ ds}{t} = \lim_{t\to 0}T(t) = T(0) = 1.$$

Thus, we can choose a suitable small $t_0 > 0$ such that: $V(t_0) \neq 0$, i.e $V(t_0)$ is an invertible real number. Therefore, by virtue of Cauchy's functional equation (1.1.1), we have:

$$\begin{aligned} T(t) &= V(t_0)^{-1}V(t_0)T(t) = V(t_0)^{-1}\int_0^{t_0} T(s)\ ds\ T(t) \\ &= V(t_0)^{-1}\int_0^{t_0} T(s+t)\ ds = V(t_0)^{-1}\int_t^{t_0+t} T(s)\ ds \\ &= V(t_0)^{-1}(V(t_0+t) - V(t)), \text{ for each t} \geq 0. \end{aligned}$$

Since V is differentiable, T is also differentiable and

$$\begin{aligned} \frac{d}{dt}T(t) &= \lim_{h\to 0}\frac{T(t+h) - T(t)T(0)}{h} \\ &= \lim_{h\to 0}T(t)\frac{T(h)-T(0)}{h} = T(t)\ \frac{d}{dt}T(t)\mid_{t=0}. \end{aligned}$$

This means that T satisfies the initial value problem (1.1.2), for $a = \frac{d}{dt}T(t)\mid_{t=0}$. Now, by virtue of proposition (1.1.1), $T(t) = e^{at}$ which completes the proof.

1.2 Semigroups Of Linear Operators

In order to understand and justify what will hapen, let us review the finite dimensional case. Consider the equation

$$\begin{cases} u'(t) = Au(t), & t \in \mathbb{R} \\ u(0) = x \in \mathbb{R}^n, & . \end{cases}$$

where $A \in B(\mathbb{R}^n)$. Observe that we can identify A with a $n \times n$ matrix $(a_{ij})_{1\leq i,j\leq n}$ where each a_{ij} is the i-th coordinate of Ae_j. Define the exponential matrix $e^{tA} = \sum_{n=0}^{\infty}\frac{t^nA^n}{n!}$. Simple computations show that $u(t) = e^{tA}x$ is the unique solution to the equation. Moreover, if we allow A to be a linear and bounded operator on a Banach space X, exactly the same calclulation (see Example (1.2.2) and Proposition (1.2.6)) shows that $e^{tA}x$ solves the equation

$$\begin{cases} u'(t) = Au(t), & t \in \mathbb{R} \\ u(0) = x \in X. & . \end{cases}$$

Then it is straghtforward to represent the solution of the equation

$$\begin{cases} u'(t) = Au(t) + f(t), \\ u(0) = x \in X, \end{cases} .$$

where $f : \mathbb{R} \to X$ is continuous, by the variation of parameter formula

$$x(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)\, ds.$$

Notice that the essential properties of the function $\mathbb{R} \ni t \to T_t \in B(X)$, given by $T_t = e^{tA}$, we used in the calculations are (compare them with the Cauchy functional equation)

$$T_0 = I$$

$$T_{t+s} = T_t T_s \quad \forall\ t, s \in \mathbb{R}$$

and the fact that the map $\mathbb{R} \ni t \to T_t x \in X$ is continuous for all $x \in X$. In order to treat more general cases, where A is not necessarily bounded, we introduce the semigroups. We begin with the following definition.

Definition 1.2.1. *Let $(X, \|\ \|_X)$ be a Banach space. A family $\{T_t : t \geq 0\} \subset B(X)$ of linear and bounded operators on X, is said to be **a semigroup of linear operators** if-f the following conditions hold:*

1. *$T_s T_t = T_{s+t}$, for each $s, t \geq 0$.*
2. *$T_0 = I_X$ (identical operator).*

Definition 1.2.2. *A semigroup of linear operators, $\{T_t : t \geq 0\} \subset B(X)$ on a Banach space X is said to be **uniformly continuous** if-f:*

$$\lim_{t \to 0^+} \| T_t - I \|_{B(X)} = 0.$$

Definition 1.2.3. *Let $(X, \|\ \|_X)$ be a Banach space. A family $\{T_t : t \in \mathbb{R}\} \subset B(X)$ of linear and bounded operators on X, is said to be **a group of linear operators** if-f the following conditions hold:*

1. *$T_s T_t = T_{s+t}$, for each $s, t \in \mathbb{R}$.*
2. *$T_0 = I_X$ (identical operator).*

*In addition, if $\lim_{t \to 0} \| T_t - I \|_{B(X)} = 0$, then we have a **uniformly continuous group**.*

Proposition 1.2.1. *If $\{T_t : t \geq 0\} \subset B(X)$ is a uniformly continuous semigroup on a Banach space X, then T_t is invertible, for each $t \geq 0$.*

Proof: By assumption, $\lim_{t \to 0^+} \| T_t - I \|_{B(X)} = 0$, thus we can choose $\delta > 0$ such that, for each $0 < t \leq \delta$: $\| T_t - I \| < 1$. By virtue of theorem (A.5.1), for each $0 < t \leq \delta$, T_t is invertible. Now, for $t > \delta$ there exists $n \in \mathbb{N}^*$ and $\eta \in [0, \delta)$ such that $t = n\delta + \eta$. Therefore, $T_t = T_\delta^n T_\eta$, so invertible.

Corollary 1.2.1. *If $\{T_t : t \geq 0\} \subset B(X)$ is a uniformly continuous semigroup of operators on a Banach space X, then it can be extended to a uniformly continuous group of operators.*

Proof: We have to show that there exists a group of operators $\{G_t : t \in \mathbb{R}\}$, such that $\lim_{t \to 0} \| G_t - I \| = 0$ and $G_t = T_t$, for each $t \geq 0$. Indeed, by virtue of proposition (1.2.1), we can define the group $(G_t)_{t \in \mathbb{R}}$ as :

$$G_t = \begin{cases} T_t & \text{when } t \geq 0, \\ (T_{-t})^{-1} & \text{when } t < 0. \end{cases}$$

Is is an easy task to verify that $\{G_t : t \in \mathbb{R}\}$ is a uniformly continuous group of operators.

Proposition 1.2.2. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a semigroup of operators on a Banach space X. The following statements are equivalent:*

1. *$\{T_t : t \geq 0\}$ is uniformly continuous*

2. *The function $[0, \infty) \ni t \to T_t \in B(X)$ is continuous in $\mathbb{R}^+$ (right continuous at 0), i.e: $\lim_{t \to s} \| T_t - T_s \|_{B(X)} = 0$, for each $s \geq 0$.*

Proof:

"$1 \Rightarrow 2$" The right continuity of $[0, \infty) \ni t \to T_t \in B(X)$ at 0 is direct from the definition. Now, by virtue of corollary (1.2.1), consider the uniformly continuous group $\{G_t : t \in \mathbb{R}\}$, which extends the semigroup $\{T_t : t \geq 0\}$. For $t > 0$ and $h \in \mathbb{R}$ such that $t + h \geq 0$, we have:

$$\begin{aligned} \| T_{t+h} - T_t \|_{B(X)} &= \| G_{t+h} - T_t \|_{B(X)} \\ &= \| T_t G_h - T_t \|_{B(X)} \leq \| T_t \|_{B(X)} \| G_h - I \|_{B(X)} . \end{aligned}$$

So, we can conclude that:

$$\lim_{h \to 0} \| T_{t+h} - T_t \|_{B(X)} = 0, \text{ for each t} > 0.$$

"$2 \Rightarrow 1$" This is direct from right continuity of $[0, \infty) \ni t \to T_t \in B(X)$ at the point 0.

Definition 1.2.4. *A semigroup of linear operators $\{T_t : t \geq 0\} \subset B(X)$ on a Banach space X is said to be **strongly continuous** or C_0**-semigroup** if-f:*

$$\lim_{t \to 0^+} T_t x = x, \text{ for each } x \in X.$$

Proposition 1.2.3. *Each uniformly continuous semigroup $\{T_t : t \geq 0\} \subset B(X)$ on a Banach space X is also strongly continuous.*

Proof: Indeed, for each $x \in X$, we have:

$$\lim_{t \to 0^+} \| T_t x - x \|_X \leq \lim_{t \to 0^+} \| T_t - I \|_{B(X)} \| x \|_X = 0.$$

Remark 1.2.1. The inverse statement is not always true as we will see in section (**??**).

Proposition 1.2.4. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a strongly continuous semigroup of operators on a Banach space X. Then, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that:*

$$\| T_t \| \leq M e^{\omega t}, \text{ for each } t \geq 0.$$

Proof: Since, $\lim_{t \to 0^+} T_t x = x$, for each x $\in X$, from Corollary (A.6.2), there exist $\delta > 0$ and $M \geq 1$ such that $\| T_t \| \leq M$, for each $0 \leq t \leq \delta$. Observe that it cannot be $M < 1$, because $\| T_0 \| = 1$. Now, for arbitrary $t \geq 0$, we can choose $n \in \mathbb{N}$ and $t' \in [0, \delta)$, such that: $t = n\delta + t'$. Therefore:

$$\begin{aligned} \| T_t \| &= \| T_{n\delta} T_{t'} \| = \| T_\delta^n T_{t'} \| \leq M^n M \\ &= M e^{n ln M} \leq M e^{(n\delta + t') \frac{ln M}{\delta}} . \end{aligned}$$

Set $\omega = \frac{ln M}{\delta}$ and the proof is complete.

Remark 1.2.2. Note that for a strongly continuous semigroup $\{T_t : t \geq 0\}$, the minimum of the set $\{\omega \in \mathbb{R} : \| T_t \| \leq M e^{\omega t}, \forall t \geq 0\}$, for some $M \geq 1$ does not always exist. Moreover, it is possible this set to be empty.

Proposition 1.2.5. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a semigroup of operators on a Banach space X. Suppose that there exist constants $0 < M < 1$ and $\omega \in \mathbb{R}$ such that:*

$$\| T_t \| \leq Me^{\omega t}, \text{ for each } t > 0.$$

Then $T_t = 0$, for each $t > 0$.

Proof: Consider the semigroup $\{S_t : t \geq 0\}$, where $S_t = e^{-\omega t}T_t$. Observe that for each $t > 0$, $\| S_t \| = e^{-\omega t} \| T_t \| \leq M$. Thus, for $t > 0$ and $n \geq 1$ we have: $\| S_t \| = \left\| S^n_{\frac{t}{n}} \right\| \leq \left\| S_{\frac{t}{n}} \right\|^n \leq M^n$. Now, taking the limits as $n \to \infty$, since $M \in [0, 1)$, it follows that $\| S_t \| = 0$, for each $t > 0$.

Theorem 1.2.1. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a semigroup of operators on a Banach space X. The following statements are equivalent:*

(i) *The semigroup $\{T_t : t \geq 0\}$ is strongly continuous.*

(ii) *For each $x \in X$ the trajectory, $\mathbb{R}^+ \ni t \to T_t x \in X$ is continuous in $\mathbb{R}^+$ (right continuous at 0).*

Proof: Suppose that $\{T_t : t \geq 0\}$ is strongly continuous and $x \in X$. The right continuity of $\mathbb{R}^+ \ni t \to T_t x \in X$ at 0 is direct from the definition. Furthermore, for $t > 0$ we have:

$$\lim_{h\to 0^+} T_{t+h}x = \lim_{h\to 0^+} T_h T_t x = T_t x, \text{ since } T_t x \in X.$$

This shows the right continuity of the trajectory. Now, let $\omega \in \mathbb{R}$ and $M \geq 1$ such that: $\| T_t \| \leq Me^{\omega t}$, for each $t \geq 0$. Then for $0 < h < t$ we have:

$$\| T_{t-h}x - T_t x \|_X = \| T_{t-h}(Ix - T_h x) \|_X \leq Me^{\omega(t-h)} \| T_h x - x \|_X .$$

Thus $\lim_{h\to 0^+} \| T_{t-h}x - T_t x \|_X = 0$, which proves the left continuity of the trajectory.

Remark 1.2.3. By virtue of Proposition (1.2.2) and Theorem (1.2.1), a semigroup $\{T_t : t \geq 0\}$ is uniformly continuous if and only if the function $[0, \infty) \ni t \to T_t \in B(X)$ is continuous in the uniform topology on $B(X)$, while is strongly continuous if and only if the function $[0, \infty) \ni t \to T_t \in B(X)$ is continuous in the strong operator topology on $B(X)$.

Example 1.2.1 (Exponential function on $\mathbb{R}$)**.** Consider the space $X = \mathbb{R}$ and $a \in \mathbb{R}$. For each $t \in \mathbb{R}$, define the function $T_t : \mathbb{R} \to \mathbb{R}$, where $T_t(x) = e^{ta}x$. It is easy to verify that $(T_t)_{t\in\mathbb{R}^+}$ and $(T_t)_{t\in\mathbb{R}}$ are a semigroup and a group of operators respectively. Moreover, $\| T_t - I \| = |e^{at} - 1|$, thus $\lim_{t\to 0} \| T_t - I \| = 0$. Therefore, $\{T_t : t \geq 0\}$ is a uniformly continuous semigroup.

Example 1.2.2 (Generalization of previous Example)**.** Let $(X, \| \, \|_X)$ be a Banach space and $A \in B(X)$. Define, $A^n = A \circ A^{n-1}$, $n \geq 2$. By induction, it is easy to verify that $\| A^n \|_{B(X)} \leq \| A \|^n_{B(X)}$, for each $n \geq 2$. The series $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ is convergent in $B(X)$, for each $t \in \mathbb{R}$, since the sequence of partial sums $(S_n)_n$, $S_n = \sum_{i=0}^{n} \frac{t^i A^i}{i!}$, is Cauchy in $(B(X), \| \, \|_{B(X)})$. Indeed, for positive integers $m > n$ we have:

$$\| S_n - S_m \| = \left\| \sum_{i=n+1}^{m} \frac{t^i A^i}{i!} \right\| \leq \sum_{i=n+1}^{m} \frac{|t|^i \| A \|^i}{i!} \to 0, \text{ as n,m} \to \infty,$$

because $\sum_{n=0}^{\infty} \frac{|t|^n \| A \|^n}{n!} = e^{|t| \| A \|}$. Thus, for each $t \in \mathbb{R}$ we can define the function $T_t : X \to X$, where $T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. It is an easy task to verify that $T_t \in B(X)$ and $\| T_t \|_{B(X)} \leq e^{|t| \| A \|}$, for each $t \in \mathbb{R}$. Indeed,

$$\begin{aligned} \| T_t x \|_X &= \left\| \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k A^k(x)}{k!} \right\|_X \leq \lim_{n\to\infty} \sum_{k=0}^{n} \frac{|t|^k \| A \|^k_{B(X)} \| x \|_X}{k!} \\ &= \| x \|_X \sum_{n=0}^{\infty} \frac{|t|^n \| A \|^n_{B(X)}}{n!} = \| x \|_X \, e^{|t| \| A \|_{B(X)}}. \end{aligned}$$

Moreover, for $s, t \in \mathbb{R}$ we have:

$$\begin{aligned} e^{tA}e^{sA} &= \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \sum_{n=0}^{\infty} \frac{s^n A^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^{n-k} A^{n-k}}{(n-k)!} \frac{s^k A^k}{k!} \\ &= \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!} = e^{(t+s)A}. \end{aligned}$$

Therefore, $T_t T_s = T_{t+s}$, for each $t, s \in \mathbb{R}$. So, the family $\{e^{tA} : t \in \mathbb{R}\}$ consists a group of operators on X. In addition, for $t \in \mathbb{R}$ we have:

$$\begin{aligned} \| e^{tA} - I \| &= \left\| \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} - I \right\| = \left\| \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} \right\| \\ &\leq \sum_{n=1}^{\infty} \frac{|t|^n \| A \|^n}{n!} = e^{|t| \| A \|} - 1. \end{aligned}$$

So, $\lim_{t \to 0} \| e^{tA} - I \| = 0$. Thus, $(T_t)_{t \in \mathbb{R}}$ and $(T_t)_{t \in \mathbb{R}^+}$ are uniformly continuous group and semigroup respectively.

Remark 1.2.4. Example (1.2.2) is a generalization of example (1.2.1). Indeed, the semigroup $\{T_t : t \geq 0\}$ on $\mathbb{R}$, where $T_t x = e^{at}x$, $x \in \mathbb{R}$ can be written in the form $T_t = e^{tA}$, for $A \in B(\mathbb{R})$, where $Ax = ax$, $x \in \mathbb{R}$. As we have already discuss in section (1.1), the semigroup of example (1.2.1) is related to the Cauchy's functional equation. In the sequel, we will prove the relevant propositions for the semigroup $\{e^{tA} : t \geq 0\}$ and we will conclude that every uniformly continuous semigroup on a Banach space X is of the form $\{e^{tA} : t \geq 0\}$, for some $A \in B(X)$.

Proposition 1.2.6. *Let $(X, \| \|_X)$ be a Banach space. Consider the semigroup $\{T_t : t \geq 0\}$, where $T_t = e^{tA}$, $t \geq 0$ for some $A \in B(X)$. Then the function*

$$[0, \infty) \ni t \to T_t \in B(X)$$

is differentiable and satisfies the following initial value problem:

$$\begin{aligned} \frac{d}{dt} T_t = A T_t &= T_t A \\ T_0 &= I \end{aligned} \tag{1.2.1}$$

Conversely, every differentiable function $T : [0, \infty) \to B(X)$ that satisfies the initial value problem (1.2.1), has the form $T_t = e^{tA}$, $t \geq 0$ and $A = \frac{d}{dt} T_t \mid_{t=0}$.

Proof: From example (1.2.2), we know that $\{T_t\}_{t \in \mathbb{R}}$ is a group of operators. Thus, for each $t, h \in \mathbb{R}$ we have:

$$\frac{T_{t+h} - T_t}{h} = T_t \frac{T_h - I}{h} = \frac{T_h - I}{h} T_t.$$

This means that in order to prove that (1.2.1) is satisfied (see also Remark (A.3.5)), it suffices to show that:

$$\lim_{h \to 0} \frac{T_h - I}{h} = A.$$

To this end, observe that,

$$\begin{aligned} \left\| \frac{T_h - I}{h} - A \right\| &= \frac{1}{|h|} \left\| \sum_{k=2}^{\infty} \frac{h^k A^k}{k!} \right\| \leq \frac{1}{|h|} \sum_{k=2}^{\infty} \frac{|h|^k \| A \|^k}{k!} \\ &= \frac{1}{|h|} (e^{\| A \| |h|} - \| A \| \, |h| - 1) \to 0, \text{ as h} \to 0 \end{aligned}$$

Uniqueness can be proved with the same way as in Proposition (1.1.1). Finally, observe that $A = \frac{d}{dt} T_t \mid_{t=0}$.

Proposition 1.2.7. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a uniformly continuous semigroup of operators on a Banach space X. Then, there exists $A \in B(X)$ such that $T_t = e^{tA}$, for each $t \geq 0$. Moreover, the bounded linear operator A is given by the formula:*

$$A = \lim_{h \to 0^+} \frac{T_h - I}{h}.$$

Proof: Equivalently, we have to show that if a function $[0, \infty) \ni t \to T_t \in B(X)$ is continuous and satisfies:

$$\begin{aligned} T_{t+s} &= T_t T_s \\ T_0 &= I \end{aligned}$$

then it is automatically differentiable and there exists a unique operator $A \in B(X)$ such that the initial value problem (1.2.1) is satisfied. To this aim, define the function,

$$V : \mathbb{R}^+ \to B(X), \quad V_t = \int_0^t T_s \; ds, \; \text{t} \geq 0.$$

By virtue of Theorem (A.3.1), V is well defined, since the function $t \to T_t$ is continuous. Moreover, by virtue of Theorem (A.3.2) V is differentiable in $\mathbb{R}^+$ with

$$\frac{d}{dt} V_t = T_t.$$

It is an easy task to verify that,

$$\lim_{t \to 0^+} \frac{V_t}{t} = \lim_{t \to 0^+} \frac{\int_0^t T_s \; ds}{t} = \lim_{t \to 0^+} T_t = T_0 = I_X.$$

Indeed,

$$\begin{aligned} \lim_{t \to 0^+} \left\| \frac{\int_0^t T_s \; ds}{t} - I \right\| &\leq \lim_{t \to 0^+} \frac{1}{|t|} \int_0^t \| T_s - I \| \; ds \\ &= \lim_{t \to 0^+} \| T_t - I \| = 0, \end{aligned}$$

where in the last identity we used the uniform continuity of the semigroup. This means. that we can choose a small enough $t_0 > 0$, such that $\left\| \frac{V_{t_0}}{t_0} - I \right\| < 1$. By virtue of Theorem (A.5.1), $\frac{V_{t_0}}{t_0}$ is invertible, thus V_{t_0} is also invertible. So, we have:

$$\begin{aligned} T_t &= V_{t_0}^{-1} V_{t_0} T_t = V_{t_0}^{-1} \int_0^{t_0} T_s \; ds \;\; T_t \\ &= V_{t_0}^{-1} \int_0^{t_0} T_{s+t} \; ds = V_{t_0}^{-1} \int_t^{t_0+t} T_s \; ds \\ &= V_{t_0}^{-1} (V_{t_0+t} - V_t), \;\; \text{for each t} \geq 0, \end{aligned}$$

where in the third identity we used Proposition (A.3.5) and in the last identity we used Corollary (A.3.2). Since V is differentiable, T is also differentiable and

$$\begin{aligned} \frac{d}{dt} T_t &= \lim_{h \to 0^+} \frac{T_{t+h} - T_t}{h} \\ &= \lim_{h \to 0^+} \frac{T_h - I}{h} T_t = (\lim_{h \to 0^+} \frac{T_h - I}{h}) T_t = \frac{d}{dt} T(t) \mid_{t=0} \; T_t, \end{aligned}$$

where in the third identity we used Remark (A.3.5). Now, by Proposition (1.2.6), we derive that $T_t = e^{tA}$, for each $t \geq 0$, where $A = \frac{d}{dt} T(t) \mid_{t=0}$.

1.3 More Examples

Consider the following Banach spaces:

(a) The space $(BC(\mathbb{R}^+), \| \|_\infty)$ of continuous and bounded functions $x : \mathbb{R}^+ \to \mathbb{C}$, with $\| x \|_\infty = \sup\{|x(\tau)| : \tau \in \mathbb{R}^+\}$.

(b) The space $(C_0(\mathbb{R}^+), \| \|_\infty)$ of continuous functions $x : \mathbb{R}^+ \to \mathbb{C}$ that vanish to infinity, i.e $\lim_{\tau\to\infty} x(\tau) = 0$, with $\| x \|_\infty = \sup\{|x(\tau)| : \tau \in \mathbb{R}^+\}$.

(c) The space $(BC_u(\mathbb{R}^+), \| \|_\infty)$ of uniformly continuous and bounded functions $x : \mathbb{R}^+ \to \mathbb{C}$, with $\| x \|_\infty = \sup\{|x(\tau)| : \tau \in \mathbb{R}^+\}$.

(d) The space $(C_l(\mathbb{R}^+), \| \|_\infty)$ of continuous functions $x : \mathbb{R}^+ \to \mathbb{C}$, such that there exists $a \in \mathbb{C}$: $\lim_{\tau\to\infty} x(\tau) = a$, with $\| x \|_\infty = \sup\{|x(\tau)| : \tau \in \mathbb{R}^+\}$.

(e) The space $(L^p(\mathbb{R}^+), \| \|_p)$, $1 \le p < \infty$ of (equivalent classes of) Lebesgue measurable functions $x : \mathbb{R}^+ \to \mathbb{C}$, such that $\| x \|_p^p = \int_{\mathbb{R}^+} |x(\tau)|^p \, dm(\tau) < \infty$, where m is the Lebesgue measure.

Example 1.3.1 (Left translation semigroup)**.** *Let $(X, \| \|_X)$ be any of $BC(\mathbb{R}^+)$, $C_0(\mathbb{R}^+)$, $BC_u(\mathbb{R}^+)$, $C_l(\mathbb{R}^+)$. For each $t \ge 0$, define the function $T_t : X \to X$, such that for each $x \in X$, $T_t x(\tau) = x(\tau + t)$, $\tau \in \mathbb{R}^+$. Then, $\{T_t : t \ge 0\}$ is a semigroup of operators on X.*

Proof: First of all, for any one of the above cases, the operator T_t is well defined, $T_t \in B(X)$ and $\| T_t \| = 1$, for each $t \ge 0$. For example, if $X = BC_u(\mathbb{R}^+)$, $x \in BC_u(\mathbb{R}^+)$ and $\varepsilon > 0$, then there exists $\delta > 0$ such that, for each $\tau, \sigma \in \mathbb{R}^+$ with $|\tau - \sigma| < \delta$, it holds that $|x(\tau) - x(\sigma)| < \varepsilon$. Therefore, for each $\tau, \sigma \in \mathbb{R}^+$ with $|\tau - \sigma| < \delta$, it also holds that $|T_t x(\tau) - T_t x(\sigma)| < \varepsilon$. This means, that $T_t x \in BC_u(\mathbb{R}^+)$. In addition, $\| T_t x \|_\infty = \sup\{|x(\tau + t)| : \tau \in \mathbb{R}^+\} = \sup\{|x(\tau)| : \tau \in \mathbb{R}^+\} = \| x \|_\infty$. Secondly, observe that for $s, t \ge 0$ and $x \in X$, we have: $T_s T_t x(\tau) = T_t x(\tau + s) = x(\tau + s + t) = T_{s+t} x(\tau)$, for each $\tau \in \mathbb{R}^+$. Thus, the semigroup property of the definition is satisfied. Finally, $T_0 x(\tau) = x(\tau)$, for each $x \in X$ and $\tau \in \mathbb{R}^+$.

Proposition 1.3.1. *The left translation semigroup is not uniformly continuous on each of the Banach spaces $BC(\mathbb{R}^+)$, $C_0(\mathbb{R}^+)$, $BC_u(\mathbb{R}^+)$, $C_l(\mathbb{R}^+)$.*

Proof: Indeed, we will show that in all cases, $\| T_t - I \|_{B(X)} \ge 2$, for each $t \ge 0$. We begin with the case $X = BC(\mathbb{R}^+)$ or $C_0(\mathbb{R}^+)$ or $C_l(\mathbb{R}^+)$. For $t \ge 0$, consider the function:

$$x(\tau) = \begin{cases} 1 - \frac{2\tau}{t} & \text{when } 0 \le \tau \le \frac{t}{2}, \\ 0 & \text{when } \tau > \frac{t}{2}. \end{cases}$$

Observe, that x is an element of $BC(\mathbb{R}^+)$, $C_0(\mathbb{R}^+)$ and $C_l(\mathbb{R}^+)$ and $\| x \|_\infty = 1$. Moreover, we have $\| T_t x - x \|_\infty = 2$, which means that $\| T_t - I \|_{B(X)} \ge 2$. For X=$BC_u(\mathbb{R}^+)$, take the function:

$$x(\tau) = 1 - \frac{2\tau}{t}, \tau \in \mathbb{R}^+,$$

which is bounded and uniformly continuous, as Lipshitz. Continue with the same arguments as before.

Proposition 1.3.2. *The left translation semigroup is strongly continuous on each of the Banach spaces $C_0(\mathbb{R}^+)$, $BC_u(\mathbb{R}^+)$, $C_l(\mathbb{R}^+)$.*

Proof: First Case: X=$BC_u(\mathbb{R}^+)$. Consider $x \in BC_u(\mathbb{R}^+)$ and $\varepsilon > 0$. So, we can choose $\delta > 0$ such that, for each $\tau, \sigma \ge 0$ with $|\tau - \sigma| < \delta$, it holds that $|x(\tau) - x(\sigma)| < \varepsilon$. Thus, for $0 < t < \delta$ we have:

$$\| T_t x - x \|_\infty = \sup_{\tau \in \mathbb{R}^+} |x(\tau + t) - x(\tau)| \le \varepsilon.$$

This means, that $\lim_{t\to 0^+} \| T_t x - x \| = 0$, as desired.

Second case:$X = C_l(\mathbb{R}^+)$. Let $x \in C_l(\mathbb{R}^+)$. Consider $a \in \mathbb{C}$, such that $\lim_{\tau\to\infty} x(\tau) = a$. This means, that for $\varepsilon > 0$, there exists $M > 0$, such that, for each $\tau \in \mathbb{R}^+$ with $\tau \geq M$, it holds that $|x(\tau) - a| < \frac{\varepsilon}{2}$. Therefore, for each $\tau \geq M$ and $t \geq 0$ we have:

$$|x(\tau + t) - x(\tau)| \leq |x(\tau + t) - a| + |x(\tau) - a)| < \varepsilon.$$

Thus, for $t \geq 0$ it follows:

$$\sup_{\tau\in[M,\infty)} |x(\tau + t) - x(\tau)| \leq \varepsilon. \tag{1.3.1}$$

On the other hand, x is uniformly continuous in the compact set $[0, M]$. So, with the same arguments as in the first case, we can choose $\delta > 0$ such that, for each $0 < t < \delta$:

$$\sup_{\tau\in[0,M]} |x(\tau + t) - x(\tau)| \leq \varepsilon. \tag{1.3.2}$$

As a result of (1.3.1) and (1.3.2), for each $0 < t < \delta$ we have:

$$\| T_t x - x \|_\infty \leq \varepsilon,$$

which means that $\lim_{t\to 0^+} \| T_t x - x \| = 0$, as desired.

Third case: X=$C_0(\mathbb{R}^+)$. Use the same arguments as in the second case.

Example 1.3.2 (Right translation semigroup)**.** *Consider the space* $X = (L^p(\mathbb{R}^+), \| \|_p)$*,* $1 \leq p < \infty$*. For each* $t \geq 0$*, define the function* $T_t : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$*, such that for each* $x \in L^p(\mathbb{R}^+)$*:*

$$T_t x(\tau) = \begin{cases} 0 & \textit{when } \tau < t, \\ x(\tau - t) & \textit{when } \tau \geq t. \end{cases}$$

Then, $\{T_t : t \geq 0\}$ *is a semigroup of operators on* $L^p(\mathbb{R}^+)$*.*

Proof: First of all, observe that if $x, y \in \mathcal{L}^p(\mathbb{R}^+)$, then:

$$\int_0^\infty |T_t x(\tau) - T_t y(\tau)|^p \, dm(\tau) = \int_t^\infty |x(\tau - t) - y(\tau - t)|^p \, dm(\tau) = \int_0^\infty |x(\tau) - y(\tau)|^p \, dm(\tau) \tag{1.3.3}$$

Thus, if $x = y$ almost everywhere in $\mathbb{R}^+$, then $T_t x = T_t y$ a.e and T_t is well defined. Moreover, from (1.3.3) we deduce that $T_t \in B(L^p(\mathbb{R}^+))$ and $\| T_t \| = 1$, for each $t \geq 0$. In order to prove the semigroup property, we will use the shorthand: $T_t x(\tau) = x(\tau - t)\mathcal{X}_{[t,\infty)}(\tau)$, although it is not totally correct, since $x(\tau - t)$ is not defined for $\tau < t$. So, for $x \in L^p(\mathbb{R}^+)$ and $s, t \geq 0$ we have:

$$\begin{aligned} T_s T_t x(\tau) &= \mathcal{X}_{[s,\infty)}(\tau) T_t x(\tau - s) = \mathcal{X}_{[s,\infty)}(\tau)\mathcal{X}_{[t,\infty)}(\tau - s)x(\tau - s - t) \\ &= \mathcal{X}_{[s+t,\infty)}(\tau)x(\tau - s - t) = T_{s+t}x(\tau), \text{ for each } \tau \geq 0. \end{aligned}$$

Proposition 1.3.3. *The right translation semigroup on* $L^p(\mathbb{R}^+)$ *is strongly continuous.*

Proof: Since $\| T_t \| = 1$ for each $t \geq 0$, by virtue of Proposition (A.7.1), it is enough to show that $\lim_{t\to 0^+} T_t x = x$, for each x in a dense subset of $L^p(\mathbb{R}^+)$. To this aim, consider the space $C_c(0, \infty)$ of continuous functions $x : (0, \infty) \to \mathbb{C}$ with compact support (see [BR] chapter IV). Now, let $x \in C_c(0, \infty)$, then x is also uniformly continuous and with a similar argument as in Proposition (1.3.2), we can show that $\lim_{t\to 0^+} \| T_t x - x \|_\infty = 0$. Now, if the support of x is contained in the interval $(0, k)$ for some $k > 0$, then:

$$\| T_t x - x \|_p \leq k \| T_t x - x \|_\infty ,$$

from which we deduce that $\lim_{t\to 0^+} \| T_t x - x \|_p = 0$ and the proof is complete.

Example 1.3.3 (Product Semigroups)**.** *Let* $(X, \| \|_X)$ *be a Banach space and* $\{T_t : t \geq 0\}$*,* $\{S_t : t \geq 0\}$ *two strongly continuous semigroups on* X *such that:*

$$S_t T_t = T_t S_t, \ \textit{for each } t \geq 0.$$

Then $\{U_t : t \geq 0\}$*, where* $U_t = S_t T_t$*,* $t \geq 0$ *is a strongly continuous semigroup on* X*.*

Proof: We will show the strong continuity. Since $\lim_{t\to 0^+} S_t x = x$, by Corollary (A.6.2), there exists $\delta > 0$ such that: $\sup_{0\le t\le\delta} \| S_t \| < \infty$. Therefore, for each $0 \le t \le \delta$ and $x \in X$ we have:

$$\begin{aligned} \| U_t x - x \| &\le \| S_t T_t x - S_t x \| + \| S_t x - x \| \\ &\le \| S_t \| \, \| T_t x - x \| + \| S_t x - x \| \\ &\le (\sup_{0\le t\le\delta} \| S_t \|) \, \| T_t x - x \| + \| S_t x - x \| \end{aligned}$$

Example 1.3.4 (Rescaled semigroup). *Let $\{T_t : t \ge 0\} \subset B(X)$ be a strongly continuous semigroup on a Banach space $(X, \| \, \|_X)$ and $\omega \in \mathbb{R}$. Then $\{U_t : t \ge 0\}$, where $U_t = e^{\omega t} T_t$, $t \ge 0$ is a strongly continuous semigroup on X.*

Proof: It is an easy task to show the semigroup properties. For the strong continuity, observe that:

$$\begin{aligned} \| U_t x - x \| &\le \left\| e^{\omega t} T_t x - e^{\omega t} x \right\| + \left\| e^{\omega t} x - x \right\| \\ &\le e^{\omega t} \| T_t x - x \| + |e^{\omega t} - 1| \, \| x \| \end{aligned}$$

1.4 The Infinitesimal Generator of a C_0-Semigroup

Until now, we have seen that every uniformly continuous semigroup $\{T_t : t \ge 0\}$ on a Banach space X is characterised by a linear and bounded operator $A \in B(X)$, such that $T_t = e^{tA}$, for all $t \ge 0$. We can say that A generates in some way the semigroup. Moreover, Proposition (1.2.7) tells us that this A is given by the formula

$$A = \lim_{h\to 0^+} \frac{T_h - I}{h}.$$

The question now is if there is a similar situation for strongly continuous semigroups. In this section we wiil show that there exists a densely defined linear operator $A : X \supset D(A) \to X$, generally not bounded but always closed, which characterises uniquely the semigroup.

Definition 1.4.1. *Let $\{T_t : t \ge 0\} \subset B(X)$ be a strongly continuous semigroup on a Banach space $(X, \| \, \|_X)$. We call **infinitesimal generator** of the semigroup, the operator $A : X \supset D(A) \to X$, where:*

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists}\}$$

and

$$A(x) = \lim_{t\to 0^+} \frac{T_t x - x}{t}, \quad x \in D(A).$$

Remark 1.4.1. The set $D(A)$ is a linear subspace of X and A is a linear operator. Indeed, for $x, y \in D(A)$ and $\lambda, \mu \in \mathbb{C}$ we have:

$$\lim_{h\to 0^+} \frac{T_h(\lambda x + \mu y) - (\lambda x + \mu y)}{h} = \lambda \lim_{h\to 0^+} \frac{T_h x - x}{h} + \mu \lim_{h\to 0^+} \frac{T_h y - y}{h}.$$

Thus, $\lambda x + \mu y \in D(A)$ and $A(\lambda x + \mu y) = \lambda A x + \mu A y$.

Remark 1.4.2. The generator of a uniformly continuous semigroup $\{T_t : t \ge 0\}$ on a Banach space X is the (unique) $A \in B(X)$ such that: $T_t = e^{tA}, t \ge 0$ (see Prop. (1.2.7)). Indeed, as we have already shown in Proposition (1.2.6)

$$\lim_{h\to 0^+} \frac{T_h - I}{h} = A.$$

Theorem 1.4.1. *Let $\{T_t : t \ge 0\} \subset B(X)$ be a strongly continuous semigroup on a Banach space $(X, \| \, \|_X)$ and $A : X \supset D(A) \to X$ its infinitesimal generator. Then the following are valid:*

(i) *If $x \in D(A)$ and $t \ge 0$ then $T_t x \in D(A)$ and $\frac{d}{dt} T_t x = T_t A x = A T_t x$.*

(ii) *For each $t \geq 0$ and $x \in X$, we have that $\int_0^t T_s x\, ds \in D(A)$ and $A(\int_0^t T_s x\, ds) = T_t x - x$.*

Proof:

(i) Let $x \in D(A)$ and $t \geq 0$. Then,

$$\lim_{h\to 0^+} \frac{T_h T_t x - T_t x}{h} = T_t \lim_{h\to 0^+} \frac{T_h x - x}{h} = T_t A x,$$

which means that:

$$T_t x \in D(A) \text{ and } AT_t x = T_t A x = (\frac{d}{dt})^+ T_t x.$$

It remains to show that:

$$AT_t x = T_t A x = (\frac{d}{dt})^- T_t x.$$

To this end, for $0 < h < t$ we have :

$$\begin{aligned}
\|\frac{T_{t-h}x - T_t x}{-h} - T_t A x\|_X &= \left\| T_{t-h}(\frac{x - T_h x}{-h} - Ax) + (T_{t-h}Ax - T_t A x) \right\|_X \\
&\leq \| T_{t-h} \|_{B(X)} \left\| \frac{x - T_h x}{-h} - Ax \right\|_X + \| T_{t-h}Ax - T_t A x \|_X \\
&\leq M e^{\omega(t-h)} \left\| \frac{x - T_h x}{-h} - Ax \right\|_X + \| T_{t-h}Ax - T_t A x \|_X ,
\end{aligned}$$

for some $\omega \in \mathbb{R}$ and $M \geq 1$. Since $x \in D(A)$ and $\{T_t : t \geq 0\}$ is strongly continuous, it follows that the last expression tends to zero as $h \to 0^+$.

(ii) For each $x \in X$ and $t \geq 0$ by virtue of Theorem (1.2.1) the integral $\int_0^t T_s x\, ds$ is well defined and

$$\begin{aligned}
\frac{T_h \int_0^t T_s x\, ds - \int_0^t T_s x\, ds}{h} &= \frac{\int_0^t T_{h+s} x\, ds - \int_0^t T_s x\, ds}{h} \\
&= \frac{\int_h^{t+h} T_s x\, ds - \int_0^t T_s x\, ds}{h} = \frac{\int_t^{t+h} T_s x\, ds - \int_0^h T_s x\, ds}{h},
\end{aligned}$$

where in the first identity we used Proposition (A.4.2). But from the strong continuity of the semigroup, we deduce that

$$\lim_{h\to 0^+} \frac{\int_t^{t+h} T_s x\, ds - \int_0^h T_s x\, ds}{h} = \lim_{h\to 0^+} (T_{t+h}x - T_h x) = T_t x - x.$$

As a result, $\int_0^t T_s x\, ds \in D(A)$ and $A(\int_0^t T_s x\, ds) = T_t x - x$.

Corollary 1.4.1. *Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space $(X, \| \|_X)$ and $A : X \supset D(A) \to X$ its infinitesimal generator. Then,*

(i) *For each $x \in D(A)$ the trajectory $[0, \infty) \ni t \to T_t x \in X$ is of class $C^1[0, \infty)$.*

(ii) *$D(A) = \{x \in X : [0, \infty) \ni t \to T_t x \in X$ is differentiable in $[0, \infty)\}$*

(iii) *For each $x \in D(A)$ and $t \geq 0$ it holds that $T_t x - x = \int_0^t T_s A x\, ds = \int_0^t A T_s x\, ds$.*

Proof:

(i) It is a direct consequence of (i) of Theorem(1.4.1) and of the fact that the function $[0, \infty) \ni t \to T_t A x \in X$ is continuous in its domain, since $(T_t)_{t\geq 0}$ is a strongly continuous semigroup.

(ii) Again from (i) of Theorem(1.4.1) we have:

$$x \in D(A) \Leftrightarrow t \to T_t x \text{ right differentiable at } 0 \Leftrightarrow t \to T_t x \text{ differentiable in } [0, \infty)$$

(iii) Integrate both sides of the relationship of (i) of Theorem(1.4.1) and apply Corollary (A.3.2).

Proposition 1.4.1. *Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space $(X, \| \|_X)$ and $A : X \supset D(A) \to X$ its infinitesimal generator. Then, $D(A)$ is dense in X and A is closed.*

Let $x \in X$ and $t > 0$. From (ii) of Theorem(1.4.1) $\int_0^t T_s x \, ds \in D(A)$ and $D(A)$ is a linear space, so $\frac{\int_0^t T_s x \, ds}{t} \in D(A)$. Observe now that,

$$\lim_{t \to 0^+} \frac{\int_0^t T_s x \, ds}{t} = T_0 x = x,$$

from where we deduce that $D(A)$ is dense in X. Now consider a sequence $\{x_n\}_{n=1}^{\infty} \subset D(A)$ such that: $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Ax_n = y \in X$. From (iii) of Corollary (1.4.1) we have that:

$$T_h x_n - x_n = \int_0^h T_s A x_n \, ds, \text{ for each } n \in \mathbb{N} \text{ and } h > 0.$$

Observe now that $\lim_{n\to\infty} T_s A x_n = T_s y$ and the convergence is uniform in $[0, h]$. For the last one, fix an $\varepsilon > 0$. For each $x \in X$, the map $[0, h] \ni t \to T_t x \in X$ is continuous and since [0,h] is compact we deduce that $\sup_{t\in[0,h]} \| T_t x \| < \infty$. Therefore, by the Banach Steinhauss Theorem we get $M = \sup_{t\in[0,h]} \| T_t \| < \infty$. Now choose $n_0 \in \mathbb{N}$ such that $\| Ax_n - y \| < \frac{\varepsilon}{M}$, for all $n \geq n_0$. Then, $\| T_s A x_n - T_s y \| \leq M \| Ax_n - y \| < \varepsilon$, for all $n \geq n_0$ and $s \in [0, h]$. Thus, from the Uniform Convergence Theorem (A.3.3), taking the limits as $n \to \infty$ we deduce:

$$T_h x - x = \int_0^h T_s y \, ds, \text{ for each } h > 0.$$

Therefore

$$\lim_{h\to 0^+} \frac{T_h x - x}{h} = y.$$

This means that $x \in D(A)$ and $Ax = y$.

Proposition 1.4.2. *Let $A : X \supset D(A) \to X$ be the infinitesimal generator of two strongly continuous semigroups $\{T_t : t \geq 0\}$ and $\{S_t : t \geq 0\}$ on a Banach space $(X, \| \|_X)$. Then $S_t = T_t$, for each $t \geq 0$.*

Proof: Let $x \in D(A)$ and $t > 0$. Define the function $f : [0, t] \to X$, where

$$f_s = S_{t-s} T_s x, \ s \in [0, t].$$

By Theorem (1.4.1), f is differentiable in $[0, t]$ and

$$f'_s = -AS_{t-s}T_s x + S_{t-s}AT_s x = -AS_{t-s}T_s x + AS_{t-s}T_s x = 0, \text{ for each } s \in [0, t].$$

Therefore, from Proposition (A.3.1) it follows that f is constant. Thus,

$$f_0 = f_t \Leftrightarrow S_t x = T_t x$$

Until now we have shown that $S_t x = T_t x$, for each $x \in D(A)$ and $t \geq 0$. Because $D(A)$ is dense in X and S_t, T_t are linear bounded operators we get easily that $S_t x = T_t x$, for each $x \in X$ and $t \geq 0$.

Corollary 1.4.2. *Let $(X, \| \|_X)$ be a Banach space.An operator $A : X \supset D(A) \to X$ is the infinitesimal operator of a uniformly continuous semigroup on X if and only if $D(A) = X$ and $A \in B(X)$.*

Proof: Assume that $\{T_t : t \geq 0\}$ is a uniformly continuous semigroup on X. As we have already mention in Remark (1.4.2), its generator is the bounded operator $A \in B(X)$ such that $T_t = e^{At}$ and $A = \lim_{h\to 0^+} \frac{T_h - I}{h}$. Conversely, suppose that the linear and bounded operator $A : X \to X$ is the generator of a semigroup $\{T_t : t \geq 0\}$. Then, A is also the generator of the semigroup $\{e^{At} : t \geq 0\}$. By virtue of Proposition (1.4.2), $T_t = e^{At}$, $t \geq 0$.

Remark 1.4.3. *Theorem (1.4.1)(i) and Proposition (1.4.2) assert that for each $x \in D(A)$, the function $u : [0, \infty) \to X$, defined by $u(t) = T_t x$, for each $t \geq 0$ is the unique classical solution (see Definition (5.3.1)) of the Cauchy problem*

$$\begin{cases} u'(t) \;=\; Au(t), & t \in [0, \infty) \\ u(0) \;=\; x, & . \end{cases}$$

Example 1.4.1 (Left translation semigroup revisited)**.** *Let $(X, \| \|_X) = (BC_u(\mathbb{R}^+), \| \|_\infty)$, $\{T_t : t \geq 0\} \subset B(X)$ the left translation semigroup on $BC_u(\mathbb{R}^+)$ (see Example (1.3.1)) and $A : X \supset D(A) \to X$ its infinitesimal generator.Then,*

$$D(A) = \{x : \mathbb{R}^+ \to \mathbb{C} : \textit{x differentiable and } x' \in BC_u(\mathbb{R}^+)\}$$

and $Ax = x'$, for each $x \in D(A)$.

Proof: Assume that $x \in D(A)$.Then, the limit $\lim_{t \to 0^+} \frac{T_t x - x}{t}$ exists in $BC_u(\mathbb{R}^+)$.Now, since the convergence on $BC_u(\mathbb{R}^+)$ with respect to the norm $\| \|_\infty$ is equivalent to the uniform convergence we get that:

$$\lim_{t \to 0^+} \frac{T_t x(\tau) - x(\tau)}{t} = \lim_{t \to 0^+} \frac{x(\tau + t) - x(\tau)}{t} = (\frac{d}{dt})^+ x(\tau) \in BC_u(\mathbb{R}^+).$$

Therefore, $D(A) \subset \{x : \mathbb{R}^+ \to \mathbb{C} : x \, \text{differentiable and } x' \in BC_u(\mathbb{R}^+)\}$, since x is uniformly continuous.For the inverse inclusion, let $x : \mathbb{R}^+ \to \mathbb{C}$ be a differentiable function with $x' \in BC_u(\mathbb{R}^+)$.From the mean value theorem, we deduce that for each $\tau \geq 0$, $h > 0$, there exists $\theta \in [0, 1]$ such that $x(\tau + h) - x(\tau) = hx'(\tau + \theta h)$.Moreover, since x' is uniformly continuous, for each $\varepsilon > 0$, there exists $\delta > 0$ such that: for each $\tau, \sigma \geq 0$ with $|\tau - \sigma| < \delta$, it holds that $|x'(\tau) - x'(\sigma)| < \varepsilon$.Therefore for $0 < h < \delta$ we have:

$$\left\| \frac{T_h x - x}{h} - x' \right\|_\infty = \sup_{\tau \in \mathbb{R}^+} \left| \frac{T_h x(\tau) - x(\tau)}{h} - x'(\tau) \right| = \sup_{\tau \in \mathbb{R}^+} |x'(\tau + \theta h - x'(\tau)| \leq \varepsilon.$$

This means that $x \in D(A)$ and $Ax = x'$.

Example 1.4.2 (Rescaled semigroups revisited)**.** *Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space X and $A : X \supset D(A) \to X$ its infinitesimal generator.Define the rescaled semigroup $\{S_t : t \geq 0\}$, where $S_t = e^{-\lambda t} T_t, t \geq 0$, for some $\lambda \in \mathbb{R}$.Then, $B : D(A) \to X$, $B = A - \lambda I$ is its infinitesimal generator.*

Proof: Observe that for each $x \in X$:

$$\begin{aligned} \lim_{t \to 0^+} \left\{ \frac{S_t x - x}{t} - \frac{T_t x - x}{t} \right\} &= \lim_{t \to 0^+} \frac{S_t x - T_t x}{t} \\ &= \lim_{t \to 0^+} \frac{(e^{-\lambda t} - 1) T_t x}{t} = -\lambda x \end{aligned}$$

Thus, the limit $\lim_{t \to 0^+} \frac{S_t x - x}{t}$ exists if and only if the limit $\lim_{t \to 0^+} \frac{T_t x - x}{t}$ exists and in this case, $Bx = Ax - \lambda x$.

Example 1.4.3 (Isomorphic semigroups)**.** *Let X , Y be two Banach spaces and $J : X \to Y$ an isometric isomorphism.If $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on X, then $\{S_t : t \geq 0\}$, where*

$$S_t = J T_t J^{-1}, \; t \geq 0,$$

is a strongly continuous semigroup on Y.Moreover, if $A : X \supset D(A) \to X$ is the infinitesimal generator of $\{T_t : t \geq 0\}$ and $B : X \supset D(B) \to Y$ is the generator of $\{S_t : t \geq 0\}$, then

$$y \in D(B) \Leftrightarrow J^{-1} \in D(A) \;\; \textit{and} \;\; B = JAJ^{-1}.$$

Proof: It is an easy to task to verify that $\{S_t : t \geq 0\}$ is a semigroup on Y.Furthermore, if $y \in Y$, then $J^{-1}y \in X$.So, from strong continuity of $\{T_t : t \geq 0\}$ we get that

$$\lim_{t \to 0^+} T_t J^{-1} y = J^{-1} y.$$

Therefore,

$$\lim_{t \to 0^+} J T_t J^{-1} y = J J^{-1} y = y,$$

which proves that $\{S_t : t \geq 0\}$ is strongly continuous.In addition, $\lim_{h \to 0^+} \frac{J T_t J^{-1} y - y}{h}$ exists if and only if $\lim_{h \to 0^+} \frac{T_t J^{-1} y - J^{-1} y}{h}$ exists.Thus, $y \in D(B) \Leftrightarrow J^{-1} \in D(A)$ and $B = JAJ^{-1}$.

Example 1.4.4 (Restriction of a semigroup)**.** *Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space X.If X_1 is a linear Banach subspace of X, such that $T_t X_1 \subset X_1$, then $\{S_t : t \geq 0\}$, where $S_t = T_{t|X_1}, t \geq 0$ is a strongly continuous semigroup on X_1.Moreover, if $A : X \supset D(A) \to X$ is the infinitesimal generator of $\{T_t : t \geq 0\}$ and $B : X \supset D(B) \to Y$ is the generator of $\{S_t : t \geq 0\}$, then*

$$D(B) = D(A) \cap X_1 \text{ and } Bx = Ax, \text{,for each } x \in D(B).$$

Proof: It is enough to observe that X_1 is closed.So, if $x \in D(A) \cap X_1$, then the limit $\lim_{h \to 0^+} \frac{S_h x - x}{h} = Ax \in X_1$.This means that $D(A) \cap X_1 \subset D(B)$.The inverse inclusion is direct.

Example 1.4.5 (Semigroup restricted to the domain of its generator)**.** Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space X and $A : D(A) \to X$ its infinitesimal generator.From Proposition (1.4.1), A is closed.So from Proposition (A.4.3) the space $(A, \| \, \|_A)$ is Banach.Thus, we can define the restriction semigroup $\{S_t : t \geq 0\}$, where $S_t = T_{t|D(A)}$, on $(D(A), \| \, \|_A)$.Observe that:

$$\begin{aligned} \| S_t x \|_A &= \| S_t x \|_X + A S_t x X = \| T_t x \|_X + \| T_t A x \|_X \\ &\leq \| T_t \|_{B(X)} \| x \|_A , \text{ for each } x \in D(A). \end{aligned}$$

Thus, $\| S_t \| \leq \| T_t \|$.In addition, if $B : X \supset D(B) \to Y$ is the infinitesimal generator of $\{S_t : t \geq 0\}$ then $D(B) = \{x \in D(A) : Ax \in D(A)\}$ and $Bx = Ax$, for each $x \in D(B)$.

Chapter 2

Generation Theorems

In this chapter we will present the most powerful Theorem in the theory of semigroups, the so-called Hille-Yosida Theorem and its variants. All these theorems give necessary and sufficient conditions that a linear operator $A : X \supset D(A) \to X$ on a Banach space X must satisfy in order to be the infinitesimal generator of a strongly continuous semigroup. In the first section we state and prove the Hille-Yosida Theorem which refers to C_0-semigroups of contractions. In section (2.3.1) we study the Feller-Miyadera-Phillips Theorem which is the generalisation of the Hille-Yosida Theorem for arbitrary C_0-semigroups. Finally in section (2.5) we prove the two versions of the Lumer-Phillips Theorem which is a very useful reformulation of the Hille-Yosida Theorem. In most cases it is easier to verify that an operator satisfies the conditions of the Lumer-Phillips Theorem. This is why in many classic books of applied functional analysis for partial differential equations,(e.g [BR] th.VII.4) , the Lumer-Phillips theorem plays the central role. As we will see in the next chapters (see Chapter 7) generation theorems presented here combined with existence and uniqueness results of solutions of determinitstic (stochastic) abstract Cauchy problems (e.g see Remark (1.4.3), Chapter 5 and Chapter 6) are a powerful tool for the qualitative study of deterministic (stochastic)partial differential equations.

2.1 Elements of Spectral Analysis

Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup on a Banach space X. Then there exists its (unique) infinitesimal generator $A : X \supset D(A) \to X$, with the characteristics that we have already discussed in the previous chapter. In order to retrieve the semigroup $\{T_t : t \geq 0\}$ from its infinitesimal generator, we need a third object which is called the resolvent. In this section we will examine some definitions and results from Spectral analysis, which will be very useful in the sequel.

Definition 2.1.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X. Then we define:*

(i) ***the resolvent set of A****:*

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective and } (\lambda I - A)^{-1} \in B(X)\}.$$

(ii) ***the spectrum of A****:*

$$\sigma(A) := \mathbb{C} \backslash \rho(A)$$

(iii) ***the resolvent operator of A at a point $\boldsymbol{\lambda} \in \boldsymbol{\rho(A)}$****:*

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

Remark 2.1.1. Observe that for $\rho(A) \neq \emptyset$, it is necessary A to be closed. Indeed, if $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1}$ is a closed operator, since it is bounded. By virtue of Proposition (A.4.4), $(\lambda I - A)$ is also closed. Finally, by Corollary (A.4.2), A is closed.

Remark 2.1.2. From Proposition (A.4.4) and the Closed Graph Theorem (see also Corollary (A.4.1)) we deduce that if A is closed, then

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective }\}$$

Remark 2.1.3. It is an easy task to verify that for $\lambda, \mu \in \rho(A)$, the operators $R(\lambda, A), R(\mu, A)$ commute, i.e $R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A)$.

Lemma 2.1.1 (Resolvent Equation)**.** *Let $A : X \supset D(A) \to X$ be a closed linear operator on a Banach space X. Then, for each $\lambda, \mu \in \rho(A)$, it holds that:*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

Proof: From definition (2.1.1) we have that:

$$\begin{aligned} [\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) &= R(\mu, A) \\ \mu R(\mu, A) - AR(\mu, A)]R(\lambda, A) &= R(\lambda, A). \end{aligned}$$

By subtraction and Remark (2.1.3), we get the desired result.

Theorem 2.1.1. *Let $A : X \supset D(A) \to X$ be a closed linear operator on a Banach space X. Then, $\rho(A)$ is an open subset in $\mathbb{C}$.*

Proof: Let $\mu \in \rho(A)$. We want to find an $\varepsilon > 0$ such that the open ball $B(\mu, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \mu| < \varepsilon\} \subset \rho(A)$. To this end, consider

$$S_\lambda = R(\mu, A) \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^n. \tag{2.1.1}$$

Clearly as we have shown in Theorem (A.5.1), for each $\lambda \in \mathbb{C}$, with $|\lambda - \mu| \parallel R(\mu, A) \parallel_{B(X)} < 1$, the series in (2.1.1) is $\parallel \parallel_{B(X)}$- convergent. Moreover, again from Theorem (A.5.1), we deduce that for $\lambda \in B(\mu, \frac{1}{\parallel R(\mu,A) \parallel})$ it holds that

$$\begin{aligned} S_\lambda &= (\mu I - A)^{-1}[I - (\mu - \lambda)R(\mu, A)]^{-1} \\ &= [(I - (\mu - \lambda)R(\mu, A))(\mu I - A)]^{-1} \\ &= (\lambda I - A)^{-1} = R(\lambda, A). \end{aligned}$$

Thus, $\lambda \in \rho(A)$ and the proof is complete.

Remark 2.1.4. By virtue of Theorem (2.1.1), we deduce that the resolvent map $\lambda \to R(\lambda, A)$ is locally analytic and

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1},$$

for all $\lambda \in \rho(A)$ and $n \in \mathbb{N}$.

Lemma 2.1.2. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a C_0-semigroup on a Banach space X and $A : X \supset D(A) \to X$ its generator. Then, for each $t \geq 0$ and $\lambda \in \rho(A)$ it holds that*

$$R(\lambda, A)T_t = T_t R(\lambda, A).$$

Proof: Let $\lambda \in \rho(A)$, $t \geq 0$, $x \in X$. Set $y = R(\lambda, A)x \in D(A)$. By virtue of Theorem (1.4.1) we obtain

$$T_t x = T_t(\lambda I - A)y = \lambda T_t y - AT_t y = (\lambda I - A)T_t y = (\lambda I - A)T_t R(\lambda, A)x.$$

Therefore,

$$R(\lambda, A)T_t x = T_t R(\lambda, A)x.$$

2.2 The Hille-Yosida Theorem

Theorem 2.2.1 (Hille-Yosida Theorem). *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is the infinitesimal generator of a C_0-semigroup of contractions if and only if :*

(1) *$D(A)$ is dense and A is closed.*

(2) *$(0, \infty) \subset \rho(A)$ and for each $\lambda > 0$ it holds that:*

$$\| R(\lambda, A) \|_{B(X)} \leq \frac{1}{\lambda}.$$

Remark 2.2.1. Observe that $(\lambda I - A)^{-1} = \frac{1}{\lambda}(I - \lambda^{-1}A)^{-1}$, whenever at least one of the two inverses is well defined. This means that (2) of Theorem (2.2.1) can be replaced by:

(2') For each $\lambda > 0$, $I - \lambda^{-1}A$ is bijective and $\left\| (I - \lambda^{-1}A)^{-1} \right\|_{B(X)} \leq 1$

Remark 2.2.2. If A is densely defined and

(2") For each $\lambda > 0$, $\lambda I - A$ is bijective, $(\lambda I - A)^{-1} \in B(X)$ and $\left\| (\lambda I - A)^{-1} \right\|_{B(X)} \leq \frac{1}{\lambda}$

then conditions (1), (2) of Theorem (2.2.1) are satisfied. See also Remark (2.1.1).

Proof (Necessity): Let $A : X \supset D(A) \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $\{S_t : t \geq 0\} \subset B(X)$. By virtue of Proposition (1.4.1), A is a densely defined closed operator. So it remains to show condition (2). To this end, for each $\lambda > 0$, define:

$$R_\lambda x = \int_0^\infty e^{-\lambda t} S_t x \; dt, \;\; x \in X.$$

We claim that, $R_\lambda : X \to X$, is well defined. Indeed, from the strong continuity of the semigroup it follows that $t \to S_t x$ is continuous , so Riemann integrable, for each $x \in X$. Moreover, for each $a, b \geq 0$, $a \leq b$, we have:

$$\begin{aligned} \left\| \int_a^b e^{-\lambda t} S_t x \; dt \right\|_X &\leq \int_a^b e^{-\lambda t} \| S_t \|_{B(X)} \| x \|_X \; dt \\ &\leq \| x \|_X \int_a^b e^{-\lambda t} \; dt = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda} \| x \|_X . \end{aligned}$$

Since the last expression tends to zero as $a, b \to \infty$, the integral converges. It is easy to verify the linearity of R_λ. Moreover, for each $x \in X$, we have:

$$\| R_\lambda x \|_X \leq \int_0^\infty e^{-\lambda t} \| S_t \|_{B(X)} \| x \|_X \; dt \leq \frac{1}{\lambda} \| x \|_X .$$

This means that $R_\lambda \in B(X)$ and $\| R_\lambda \|_{B(X)} \leq \frac{1}{\lambda}$, for each $\lambda > 0$.

Claim: For each $\lambda > 0$, the operator R_λ coincides with $R(\lambda, A)$.

Proof of Claim: We will show that for each $\lambda > 0$, the operator R_λ is the right and the left inverse of the operator $\lambda I - A$. Let $x \in X$, $\lambda > 0$ and $h > 0$. We have:

$$\begin{aligned} \frac{S_h R_\lambda x - R_\lambda x}{h} &= \frac{1}{h} \int_0^\infty e^{-\lambda t} S_{t+h} x \; dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} S_t x \; dt \\ &= \frac{e^{\lambda h}}{h} \int_h^\infty e^{-\lambda t} S_t x \; dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} S_t x \; dt \\ &= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} S_t x \; dt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} S_t x \; dt, \end{aligned}$$

where in the first equality we have used Proposition (A.4.2). Now observe that,

$$\begin{aligned}
\lim_{h\to 0^+}\left\| \frac{\int_0^h e^{-\lambda t} S_t x \; dt}{h} - x \right\| &\leq \lim_{h\to 0^+} \frac{1}{|h|}\int_0^h \left\| e^{-\lambda t} S_t x - x \right\| \; dt \\
&= \lim_{h\to 0^+} \left\| e^{-\lambda h} S_h x - x \right\| = 0,
\end{aligned}$$

since $(S_t)_t$ is strongly continuous. Therefore,

$$\lim_{h\to 0^+} \frac{S_h R_\lambda x - R_\lambda x}{h} = \lambda R_\lambda x - x,$$

from where we deduce that $R_\lambda x \in D(A)$ and $AR_\lambda x = \lambda R_\lambda x - x$, for each $x \in X$. Thus,

$$AR_\lambda = \lambda R_\lambda - I \Leftrightarrow (\lambda I - A)R_\lambda = I.$$

Now, let $x \in D(A)$. We have:

$$\begin{aligned}
R_\lambda A x &= \int_0^\infty e^{-\lambda t} S_t A x \; dt \\
&= \int_0^\infty e^{-\lambda t} (\frac{d}{dt} S_t x) \; dt \\
&= \lim_{t\to\infty} e^{-\lambda t} S_t x - x + \lambda \int_0^\infty e^{-\lambda t} S_t x \; dt \\
&= \lambda R_\lambda x - x,
\end{aligned}$$

where in the second equality we have used Corollary (1.4.1). For the last equality, observe that:

$$\left\| e^{-\lambda t} S_t x \right\|_X \quad \leq \quad e^{-\lambda t} \left\| x \right\|_X \to 0, \;\text{ as t} \to \infty$$

Thus,

$$R_\lambda A = \lambda R_\lambda - I \Leftrightarrow R_\lambda(\lambda I - A) = I.$$

The proof of the necessity is complete.

Definition 2.2.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X, which satisfies conditions (1) and (2) of the Hille-Yosida Theorem (2.2.1). For $\lambda > 0$, we define the operator $A_\lambda : X \to X$, where $A_\lambda = \lambda A R(\lambda, A)$, which is called* ***the Yosida approximation*** *of A.*

Lemma 2.2.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X, which satisfies conditions (1) and (2) of the Hille-Yosida Theorem (2.2.1). Then:*

(i) $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$*, for each $x \in X$.*

(ii) $A_\lambda x = \lambda^2 R(\lambda, A)x - \lambda x$*, for each $x \in X$ and $\lambda > 0$.*

(iii) $\lim_{\lambda\to\infty} A_\lambda x = Ax$*, for each $x \in D(A)$.*

Proof:

(i) Observe that for $x \in D(A)$, we have:

$$\lambda R(\lambda, A)x - x = \lambda R(\lambda, A)x - R(\lambda, A)(\lambda I - A)x = R(\lambda, A)Ax.$$

Therefore, for each $x \in D(A)$ we have:

$$\lim_{\lambda\to\infty} \left\| \lambda R(\lambda, A)x - x \right\| = \lim_{\lambda\to\infty} \left\| R(\lambda, A)Ax \right\| \leq \lim_{\lambda\to\infty} \frac{1}{\lambda} \left\| Ax \right\| = 0.$$

Since $D(A)$ is dense in X and $\left\| \lambda R(\lambda, A) \right\|_{B(X)} \leq 1$, for each $\lambda > 0$, by virtue of Proposition (A.7.1), we deduce that $\lim_{\lambda\to\infty} \left\| \lambda R(\lambda, A)x - x \right\| = 0$, for each $x \in X$, as desired.

(ii) $\lambda^2 R(\lambda, A) - \lambda I = \lambda^2 R(\lambda, A) - \lambda(\lambda I - A)R(\lambda, A) = \lambda A R(\lambda, A) = A_\lambda$.

(iii) For $x \in D(A)$, we have:

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda A R(\lambda, A)x = \lim_{\lambda\to\infty} \lambda R(\lambda, A)Ax = Ax,$$

where in the last equality, we used (i) of this Lemma. For the second equality, arguing as in the proof of (i) we can show that for each $x \in D(A)$ and $\lambda > 0$, $AR(\lambda, A)x = R(\lambda, A)Ax$.

Lemma 2.2.2. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X, which satisfies conditions (1) and (2) of the Hille-Yosida Theorem (2.2.1). Then:*

(i) *For each $\lambda > 0$, the operator A_λ is the infinitesimal generator of the uniformly continuous semigroup $\{e^{tA_\lambda} : t \geq 0\}$ and $\| e^{tA_\lambda} \|_{B(X)} \leq 1$, for each $t \geq 0$.*

(ii) *For each $x \in X$, $t \geq 0$ and $\lambda, \mu > 0$ it holds that:*

$$\| e^{tA_\lambda} x - e^{tA_\mu} x \| \leq t \| A_\lambda x - A_\mu x \| .$$

Proof:

(i) Let $\lambda > 0$. From Lemma (2.2.1) we have that $A_\lambda x = \lambda^2 R(\lambda, A)x - \lambda x$, for each $x \in X$. Thus, $A_\lambda \in B(X)$. From Corollary (1.4.2), A_λ is the generator of the semigroup $\{e^{tA_\lambda} : t \geq 0\}$ which is of course uniformly continuous. Moreover, for each $t \geq 0$:

$$\begin{aligned} \| e^{tA_\lambda} \| &= \left\| e^{t\lambda^2 R(\lambda,A) - t\lambda I} \right\| \\ &\leq e^{t\lambda^2 \| R(\lambda,A) \|} \left\| e^{-t\lambda I} \right\| \leq e^{t\lambda} e^{-t\lambda} = 1. \end{aligned}$$

(ii) It is easy to verify that for each $\lambda, \mu > 0$ the operators $A_\lambda, A_\mu, e^{tA_\lambda}, e^{tA_\mu}$ commute. In order to show that $A_\lambda A_\mu = A_\mu A_\lambda$ use (ii) of Lemma (2.2.1), while for $A_\lambda e^{tA_\mu} = e^{tA_\mu} A_\lambda$, use Remark (A.3.5) and the fact that $A_\lambda {A_\mu}^n = {A_\mu}^n A_\lambda$, for each $n \in \mathbb{N}$ (by induction). Therefore we have:

$$\begin{aligned} \| e^{tA_\lambda} x - e^{tA_\mu} x \| &= \left\| \int_0^1 \frac{d}{ds} (e^{stA_\lambda} e^{(1-s)tA_\mu} x) \, ds \right\| \\ &= \left\| \int_0^1 \frac{d}{ds} (e^{st(A_\lambda - A_\mu)} e^{tA_\mu} x) \, ds \right\| \\ &= \left\| \int_0^1 t(A_\lambda - A_\mu) e^{st(A_\lambda - A_\mu)} e^{tA_\mu} x \, ds \right\| \\ &\leq \int_0^1 t \| A_\lambda x - A_\mu x \| \, \| e^{stA_\lambda} \| \left\| e^{(1-s)tA_\mu} \right\| \, ds \\ &\leq t \| A_\lambda x - A_\mu x \| , \end{aligned}$$

where in the first equality we used Corollary (A.3.2), in the third equality we used Proposition (1.2.6) and in the last inequality (i) of this Lemma.

Proof (Sufficiency): Let $t \geq 0$. From Lemma (2.2.1) and Lemma (2.2.2) it follows that for each $x \in D(A)$,

$$\| e^{tA_\lambda} x - e^{tA_\mu} x \| \leq t \| A_\lambda x - A_\mu x \| \to 0, \text{ as } \lambda, \mu \to \infty. \tag{2.2.1}$$

Thus, by completeness of X, we can define the operator $S_t : D(A) \to X$,

$$S_t x = \lim_{\lambda\to\infty} e^{tA_\lambda} x, \; x \in D(A). \tag{2.2.2}$$

By using (2.2.1), we can show that for $x \in D(A)$ the convergence of the maps $([0,\infty) \ni t \to e^{tA_\lambda}x \in X)_{\lambda>0}$ in (2.2.2) is uniform on compact subsets in $\mathbb{R}^+$. Now, it is easy to verify that S_t is a linear and bounded operator. Observe that from Lemma (2.2.2), we deduce that for each $x \in D(A)$, $\| S_t x \| \le \| x \|$, thus $\| S_t \| \le 1$. Because S_t is a densely defined linear and bounded operator, there exists a unique extension $\tilde{S}_t : X \to X$, such that $\tilde{S}_{t|D(A)} = S_t$, $\tilde{S}_t \in B(X)$ and $\left\| \tilde{S}_t \right\|_{B(X)} \le 1$. Moreover, by Proposition (A.7.1) and (2.2.2) it follows that

$$\tilde{S}_t x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \text{ for each } x \in X, \tag{2.2.3}$$

since $D(A)$ is dense and $\left\| e^{tA_\lambda} \right\| \le 1$, for each $\lambda > 0$. Again the convergence is uniform on compact subsets in $\mathbb{R}^+$. It is an easy task, to verify that $\{\tilde{S}_t : t \ge 0\}$ is a semigroup. To see this, use the semigroup poperty of the exponential and Proposition (A.7.3). In the sequel, we will show that it is strongly continuous. Let $x \in X$, $T > 0$ and $\varepsilon > 0$. Since the convergence in relationship (2.2.3) is uniform on compact subsets in $\mathbb{R}^+$, we can choose a sufficiently large λ_0, such that,

$$\left\| \tilde{S}_t x - e^{tA_{\lambda_0}}x \right\| < \frac{\varepsilon}{2}, \text{ for each } t \in [0,T].$$

Furthermore, $\lim_{t\to 0^+} e^{tA_{\lambda_0}}x = x$, since the semigroup $\{e^{tA_{\lambda_0}} : t \ge 0\}$ is uniformly continuous. Thus, we can choose $\delta > 0$ such that, for each $0 < t < \delta$

$$\| e^{tA_{\lambda_0}}x - x \| < \frac{\varepsilon}{2}$$

Therefore, for each $0 < t < \min\{\delta, T\}$ we have,

$$\left\| \tilde{S}_t x - x \right\| \le \left\| \tilde{S}_t x - e^{tA_{\lambda_0}}x \right\| + \| e^{tA_{\lambda_0}}x - x \| < \varepsilon.$$

This means that $\lim_{t\to 0^+} \tilde{S}_t x = x$, for each $x \in X$ and the strong continuity has been proved. It remains to show that $A : X \supset D(A) \to X$ is the generator of the strongly continuous semigroup of contractions $\{\tilde{S}_t : t \ge 0\}$. To this aim, assume that $B : X \supset D(B) \to X$ is the generator of the semigroup and let $x \in D(A)$. Then, observe that

$$\lim_{\lambda\to\infty} e^{tA_\lambda}A_\lambda x = \tilde{S}_t Ax \tag{2.2.4}$$

and the convergence is uniform on compact subsets of $\mathbb{R}^+$. Indeed,

$$\begin{aligned} \left\| e^{tA_\lambda}A_\lambda x - \tilde{S}_t Ax \right\| &\le \| e^{tA_\lambda}A_\lambda x - e^{tA_\lambda}Ax \| + \left\| e^{tA_\lambda}Ax - \tilde{S}_t Ax \right\| \\ &\le \| A_\lambda x - Ax \| + \left\| e^{tA_\lambda}Ax - \tilde{S}_t Ax \right\|, \end{aligned}$$

which tends to zero, as $\lambda \to \infty$, due to Lemma (2.2.1(iii)) and formula (2.2.3). Thus, for $h > 0$ we have:

$$\begin{aligned} \tilde{S}_h x - x &= \lim_{\lambda\to\infty} e^{hA_\lambda}x - x \\ &= \lim_{\lambda\to\infty} \int_0^h e^{tA_\lambda}A_\lambda x \, dt \\ &= \int_0^h \tilde{S}_t Ax \, dt, \end{aligned}$$

where in the second equality we used Corollary (1.4.1(iii)) and the fact that A_λ is the generator of $\{e^{tA_\lambda} : t \ge 0\}$, for $\lambda > 0$ and in the third equality we used formula (2.2.4) and the uniform convergence theorem (A.3.3). Therefore,

$$\lim_{h\to 0^+} \frac{\tilde{S}_h x - x}{h} = Ax, \text{ for each } x \in D(A).$$

This means that $D(A) \subset D(B)$ and $Bx = Ax$, for each $x \in D(A)$. Finally, we will show that $D(A) = D(B)$. Since B is the generator of a C_0-semigroup of contractions, by the necessity of our theorem, we conclude that $1 \in \rho(B)$. Thus $(I-B)^{-1}(X) = D(B)$. But $(I-B)(D(A)) = (I-A)(D(A)) = X$, i.e $(I-B)^{-1}(X) = D(A)$. Thus, $D(A) = D(B)$.

Corollary 2.2.1. *Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup of contractions on a Banach space X. If $A : X \supset D(A) \to X$ is its infinitesimal generator and A_λ is the Yosida approximation of A, then,*

$$T_t x = \lim_{\lambda\to\infty} e^{tA_\lambda} x, \quad \textit{for each}\, x \in X.$$

Proof: It is a consequence of the proof of the sufficiency of the Hille-Yosida Theorem (2.2.1) and of Proposition (1.4.2).

2.3 The Feller-Miyadera-Phillips Theorem

Theorem 2.3.1 (The Feller-Miyadera-Phillips Theorem)**.** *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is the infinitesimal generator of a strongly continuous semigroup $\{S_t : t \geq 0\} \subset B(X)$ of type (M, ω), (i.e $\parallel S_t \parallel \leq Me^{\omega t}$, for each $t \geq 0$), if and only if:*

(1) *$A(D)$ is dense in X and A is closed.*

(2) *$(\omega, \infty) \subset \rho(A)$ and for each $\lambda > \omega$ and $n \in \mathbb{N}^*$ we have:*

$$\parallel R(\lambda, A)^n \parallel_{B(X)} \leq \frac{M}{(\lambda - \omega)^n}.$$

Remark 2.3.1. For the proof of the Feller-Miyadera-Phillips Theorem (**??**), it is enough to consider the case $\omega = 0$. To see this take for granted the Feller-Miyadera-Phillips Theorem (**??**) for the case $\omega = 0$ and suppose firstly that $A : X \supset D(A) \to X$ is the generator of a C_0-semigroup $(S_t)_t$ of type (M, ω). Then as we have already seen in Example (1.4.2), the strongly continuous semigroup $\{e^{-\omega t} S_t : t \geq 0\}$ is of type $(M, 0)$ and $B : D(A) \to X$, $B = A - \omega I$, is its generator. Moreover, since $\lambda I - A = (\lambda - \omega)I - B$ and $(0, \infty) \subset \rho(B)$, we deduce that $(\omega, \infty) \subset \rho(A)$. In addition $\left\| (\lambda I - A)^{-1} \right\|_{B(X)} = \left\| ((\lambda - \omega)I - B)^{-1} \right\|_{B(X)} \leq \frac{M}{(\lambda-\omega)^n}$, for each $\lambda > \omega$. Conversely, suppose that $A : X \supset D(A) \to X$ satisfies conditions (1) and (2) of Theorem (**??**). Consider $B = A - \omega I$ and arguing as before show that B is the infinitesimal generator of a C_0-semigroup $(T_t)_t$ of type $(M, 0)$. It follows that $\{e^{\omega t} T_t : t \geq 0\}$ is a C_0-semigroup of type (M, ω) and its generator is $B + \omega I = A$.

We begin with the following Lemma.

Lemma 2.3.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X, such that $(0, \infty) \subset \rho(A)$ and $\parallel \lambda^n R(\lambda, A)^n \parallel_{B(X)} \leq M$, for each $n \in \mathbb{N}$ and $\lambda > 0$. Then, there exists a norm $|\ | : X \to [0, \infty)$, with the properties:*

(i) *$\parallel x \parallel \leq |x| \leq M \parallel x \parallel$, for each $x \in X$.*

(ii) *$|\lambda R(\lambda, A)x| \leq |x|$, for each $x \in X$ and $\lambda > 0$*

Proof: For $\mu > 0$, we define $|\ |_\mu : X \to [0, \infty)$, such that,

$$|x|_\mu = \sup_{n\in\mathbb{N}} \parallel \mu^n R(\mu, A)^n x \parallel, \text{ for each}\, x \in X.$$

It is easy to verify that $|\ |_\mu$ is a norm on X and satisfies,

$$\parallel x \parallel \quad \leq \quad |\, x \,|_\mu \leq M \parallel x \parallel, \text{ for each}\, x \in X. \tag{2.3.1}$$

$$|\, \mu R(\mu, A)x \,|_\mu \quad \leq \quad |\, x \,|_\mu, \text{ for each}\, x \in X. \tag{2.3.2}$$

We will show now that,

$$|\, \lambda R(\lambda, A)x \,|_\mu \leq |\, x \,|_\mu, \text{ for each } x \in X \text{ and}\, \lambda \in (0, \mu]. \tag{2.3.3}$$

By virtue of the ResolventEequation (Lemma (2.1.1)), for each $\lambda \in (0, \mu]$ we have:

$$\begin{aligned}
|\, R(\lambda, A)x \,|_\mu &= |\, R(\mu, A)x + (\mu - \lambda)R(\mu, A)R(\lambda, A)x \,|_\mu \\
&= \frac{1}{\mu}\,|\, \mu R(\mu, A)(x + (\mu - \lambda)R(\lambda, A)x) \,|_\mu \\
&\leq \frac{1}{\mu}\,|\, x + (\mu - \lambda)R(\lambda, A)x \,|_\mu \\
&\leq \frac{1}{\mu}\,|\, x \,|_\mu + (1 - \frac{\lambda}{\mu})\,|\, R(\lambda, A)x \,|_\mu ,
\end{aligned}$$

where in the first inequality we used that $\mu R(\mu, A)$ is a contraction on $(X, |\ |_\mu)$, by (2.3.2). Therefore, we have shown (2.3.3). Now , by (2.3.1) and (2.3.3) we get:

$$\|\, \lambda R(\lambda, A)x \,\| \leq |\, \lambda R(\lambda, A)x \,|_\mu \leq |\, x \,|_\mu\,, \text{ for each } x \in X,\ \lambda \in (0, \mu]. \tag{2.3.4}$$

Moreover, by induction we have:

$$\|\, \lambda^n R(\lambda, A)^n x \,\| \leq |\, \lambda^n R(\lambda, A)^n x \,|_\mu \leq |\, x \,|_\mu\,, \text{ for each } x \in X,\ \lambda \in (0, \mu] \text{ and } n \in \mathbb{N},$$

from where we deduce that,

$$|\, x \,|_\lambda \leq |\, x \,|_\mu\,, \text{ for each } \lambda \in (0, \mu],\ x \in X.$$

Therefore we can define the norm $|\ | : X \to [0, \infty)$,

$$|\, x \,| = \lim_{\mu \to \infty} |\, x \,|_\mu\,, \text{ for each } x \in X.$$

Taking the limits in relationships (2.3.1) and (2.3.3), we deduce the desired properties of the norm.

Remark 2.3.2. The key idea of the proof of Feller-Miyadera-Phillips Theorem (**??**), is based on the fact that equivalent metrics on a space, induce the same topology. Thus all topological properties of a set or a function (e.g continuity, closedness e.t.c) with respect to equivalent norms, are the same. This idea, combined with the Hille-Yosida Theorem (2.2.1), leads us to the desired results.

Proof (Necessity): Let $\{S_t : t \geq 0\}$, be a semigroup on X, such that $\|\, S_t \,\| \leq M$, for each $t \geq 0$ and let $A : X \supset D(A) \to X$ be its infinitesimal generator. Define $|||\ ||| : X \to [0, \infty)$,

$$|||x||| = \sup_{t \geq 0} \|\, S_t x \,\| \,.$$

It is easy to verify, that $|||\ |||$ is a norm on X. Moreover,

$$\|\, x \,\| \leq |||x||| \leq M \,\|\, x \,\|\,, \text{ for each } x \in X. \tag{2.3.5}$$

Thus, we have equivalent norms. In addition,

$$|||S_t x||| = \sup_{s \geq 0} \|\, S_s S_t x \,\| \leq \sup_{t \geq 0} \|\, S_t x \,\| = |||x|||, \text{ for each } x \in X \text{ and } t \geq 0.$$

This means, that A is the generator of the semigroup of contractions $\{S_t : t \geq 0\}$ on $(X, |||\ |||)$. Thus, by virtue of the Hille-Yosida Theorem (2.2.1), A is densely defined and closed and $(0, \infty) \subset \rho(A)$, with

$$|||\lambda R(\lambda, A)|||_{B(X)} \leq 1, \text{ for each } \lambda > 0. \tag{2.3.6}$$

So, accordingly to Remark (2.3.2), it remains to show, that $\|\, \lambda^n R(\lambda, A)^n \,\|_{B(X)} \leq M$, for each $\lambda > 0$. To this end, observe that from relationships (2.3.5) and (2.3.6) we deduce that, for each $x \in X$ and $\lambda > 0$, we have:

$$\|\, \lambda^n R(\lambda, A)^n x \,\| \leq |||\lambda^n R(\lambda, A)^n x||| \leq |||x||| \leq M \,\|\, x \,\| \,.$$

Proof (Sufficiency): Assume that conditions (1) and (2) of Theorem (**??**) are satisfied. By virtue of Lemma (2.3.1), there exists a norm $|\ |$ on X, such that $\|\, x \,\| \leq |x| \leq M \,\|\, x \,\|$ and $|\lambda R(\lambda, A)x| \leq |x|$, for each $x \in X$ and $\lambda > 0$. Therefore, A satisfies the conditions of the Hille-Yosida Theorem on $(X, |\ |)$ and so, it is the generator of a C_0-semigroup of contractions $\{S_t : t \geq 0\}$ on $(X, |\ |)$. It remains to show that $\|\, S_t \,\|_{B(X)} \leq M$, for each $t \geq 0$. Indeed, for each $x \in X$ and $t \geq 0$,

$$\|\, S_t x \,\| \leq |S_t x| \leq |x| \leq M \,\|\, x \,\| \,.$$

2.4 Some extra results

Lemma 2.4.1. *Let $\{T_t : t \geq 0\} \subset B(X)$ be a C_0-semigroup on a Banach space X and $A : X \supset D(A) \to X$ its infinitesimal generator. The following assertions hold,*

(i) *For each $\lambda \in \mathbb{C}$, $t > 0$, $x \in X$, $\int_0^t e^{-\lambda s} T_s x \, ds \in D(A)$ and*

$$e^{-\lambda t} T_t x - x = (A - \lambda I) \int_0^t e^{-\lambda s} T_s x \, ds.$$

(ii) *For each $x \in D(A)$ we have:*

$$e^{-\lambda t} T_t x - x = \int_0^t e^{-\lambda s} T_s (A - \lambda I) x \, ds.$$

Proof:

i) For $x \in X$, $t > 0$, $\lambda \in \mathbb{C}$, $h > 0$ we have:

$$\begin{aligned}
\frac{T_h \int_0^t e^{-\lambda s} T_s x \, ds - \int_0^t e^{-\lambda s} T_s x \, ds}{h} &= \frac{\int_0^t e^{-\lambda s} T_{s+h} x \, ds - \int_0^t e^{-\lambda s} T_s x \, ds}{h} \\
&= \frac{e^{\lambda h} \int_h^{t+h} e^{-\lambda s} T_s x \, ds - \int_0^t e^{-\lambda s} T_s x \, ds}{h} \\
&= \frac{1}{h}(e^{\lambda h} - 1) \int_0^t e^{-\lambda s} T_s x \, ds + \frac{1}{h} e^{\lambda h} \int_t^{t+h} e^{-\lambda s} T_s x \, ds \\
&- \frac{1}{h} \int_0^h e^{-\lambda s} T_s x \, ds \\
&\to \lambda \int_0^t e^{-\lambda s} T_s x \, ds + e^{-\lambda t} T_t x - x, \text{ as } h \to 0^+,
\end{aligned}$$

from where we deduce that (i) holds.

(ii) For each $x \in D(A)$ we have:

$$\begin{aligned}
\int_0^t e^{-\lambda s} T_s A x \, ds &= \int_0^t e^{-\lambda s} \left(\frac{d}{ds} T_s x\right) ds \\
&= [e^{-\lambda t} T_t x - x] + \lambda \int_0^t e^{-\lambda s} T_s x \, ds \\
&= [e^{-\lambda t} T_t x - x] + \int_0^t e^{-\lambda s} T_s \lambda x \, ds.
\end{aligned}$$

Theorem 2.4.1. *Let $\{S_t : t \geq 0\} \subset B(X)$ be a C_0- semigroup of type (M, ω), (i.e $\| S_t \| \leq M e^{\omega t}$, for each $t \geq 0$), on a Banach space X. Then, for its infinitesimal generator $A : X \supset D(A) \to X$, the following properties hold:*

(i) *If $\lambda \in \mathbb{C}$ such that the improper integral*

$$R_\lambda x := \int_0^\infty e^{-\lambda t} S_t x \, dt$$

exists in X for all $x \in X$, then $\lambda \in \rho(A)$ and $R_\lambda = R(\lambda, A)$.

(ii) *If $Re\lambda > \omega$, then $\lambda \in \rho(A)$ and the resolvent is given by the integral expression in (i).*

(iii) $\| R(\lambda, A) \| \leq \frac{M}{Re\lambda - \omega}$, *for each $Re\lambda > \omega$.*

Proof:

(i Argue as in the Proof of the necessity of The Hille-Yosida Theorem (2.2.1). (See the proof of the claim).

(ii) For each $a, b \geq 0$, $a \leq b$, we have:

$$\begin{aligned}\left\| \int_a^b e^{-\lambda t} S_t x \, dt \right\|_X &\leq \int_a^b |e^{-\lambda t}| \, \| S_t \|_{B(X)} \| x \|_X \, dt \\ &\leq M \| x \|_X \int_a^b e^{(-Re\lambda + \omega)t} \, dt \to 0, \quad \text{as } a, b \to \infty.\end{aligned}$$

Consequently, $\int_0^s e^{\lambda t} S_t x \, dt$, converges (absolutely) in X as $s \to \infty$, for all $x \in X$.

(iii) Moreover, for each $x \in X$, we have:

$$\| R_\lambda x \|_X \leq \int_0^\infty M e^{(-Re\lambda + \omega)t} \| x \|_X \, dt \leq \frac{M}{Re\lambda - \omega} \| x \|_X .$$

This means that $R_\lambda \in B(X)$ and $\| R_\lambda \|_{B(X)} \leq \frac{M}{Re\lambda - \omega}$, for each $Re\lambda > \omega$.

Proposition 2.4.1. *Let $\{S_t : t \geq 0\} \subset B(X)$ be a C_0- semigroup of type (M, ω), (i.e $\| S_t \| \leq M e^{\omega t}$, for each $t \geq 0$), on a Banach space X and let $A : X \supset D(A) \to X$ be its infinitesimal generator. Then,*

$$\| R(\lambda, A)^n \| \leq \frac{M}{(Re\lambda - \omega)^n},$$

for each $n \in \mathbb{N}$ and $Re\lambda > \omega$.

Proof: Note that for each $Re\lambda > \omega$ and $n \in \mathbb{N}$ it holds that

$$\begin{aligned}R(\lambda, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)x \\ &= \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} S_t x \, dt,\end{aligned}$$

for all $x \in X$. To see the second equality, observe that by virtue of Theorem (2.4.1), we have

$$\begin{aligned}\frac{d}{d\lambda} R(\lambda, A)x &= \frac{d}{d\lambda} \int_0^\infty e^{-\lambda t} S_t x \, dt \\ &= -\lambda \int_0^\infty t e^{-\lambda t} S_t x \, dt,\end{aligned}$$

for each $Re\lambda > \omega$ and $x \in X$. Continue with induction. Therefore it follows that,

$$\begin{aligned}\| R(\lambda, A)^n x \| &= \frac{1}{(n-1)!} \left\| \int_0^\infty t^{n-1} e^{-\lambda t} S_t x \, dt \right\| \\ &\leq \frac{M \| x \|}{(n-1)!} \int_0^\infty t^{n-1} e^{(\omega - Re\lambda)t} \, dt \\ &= \frac{M}{(Re\lambda - \omega)^n} \| x \| .\end{aligned}$$

for all $x \in X$.

Corollary 2.4.1. *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is the infinitesimal generator of a C_0-semigroup of contractions if and only if :*

(1) *$D(A)$ is dense and A is closed.*

(2) *For each $\lambda \in \mathbb{C} : Re\lambda > 0$, one has $\lambda \in \rho(A)$ and it holds that:*

$$\| R(\lambda, A) \|_{B(X)} \leq \frac{1}{Re\lambda}.$$

Proof: It is a consequence of the Hille-Yosida Theorem (2.2.1) and Theorem (2.4.1).

Corollary 2.4.2. *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is the infinitesimal generator of a strongly continuous semigroup $\{S_t : t \geq 0\} \subset B(X)$ of type (M, ω), (i.e $\| S_t \| \leq Me^{\omega t}$, for each $t \geq 0$), if and only if:*

(1) *$A(D)$ is dense in X and A is closed.*

(2) *For each $\lambda \in \mathbb{C} : Re\lambda > \omega$, one has $\lambda \in \rho(A)$ and also:*

$$\| R(\lambda, A)^n \|_{B(X)} \leq \frac{M}{(Re\lambda - \omega)^n},$$

for each $n \in \mathbb{N}$, $Re\lambda > \omega$.

Proof: It is a consequence of the Feller-Miyadera-Phillips Theorem (**??**), Theorem (2.4.1) and Proposition (2.4.1).

2.5 The Lumer-Phillips Theorems

Let X be a Banach space and $x^* \in X^*$ a linear bounded functional. $x \in X$,

$$x^*(x) \ := \ \langle x, x^* \rangle \ = \ \langle x^*, x \rangle.$$

Definition 2.5.1. *Let $(X, \| \|_X)$ be a Banach space. We define the **duality map**,*

$$F : X \to \mathcal{P}(X^*), \quad F(x) \,=\, \{x^* \in X^* : \ \langle x^*, x \rangle \,=\, \| x \|^2 \,=\, \| x^* \|^2\}.$$

Remark 2.5.1. The Hahn-Banach Theorem, ensures us that $F(x) \neq \emptyset$, for each $x \in X$.

Definition 2.5.2. *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is said to be **dissipative** if-f for each $x \in D(A)$, there exists $x^* \in F(x)$, such that $Re\langle Ax, x^* \rangle \leq 0$.*

Theorem 2.5.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X. The following statements are equivalent:*

(i) *A is dissipative*

(ii) *For each $x \in D(A)$ and $\lambda > 0$, we have:*

$$\lambda \| x \| \leq \| (\lambda I - A)x \| .$$

Proof: Assume that A is dissipative. Let $x \in D(A)$ and $\lambda > 0$. Choose $x^* \in X^*$, such that $\langle x, x^* \rangle = \| x \|^2 = \| x^* \|^2$ and $Re\langle Ax, x^* \rangle \leq 0$. Then we have:

$$\begin{aligned}
\lambda \| x \|^2 &= Re\langle \lambda x, x^* \rangle \ \leq \ Re\langle \lambda x, x^* \rangle \ - \ Re\langle Ax, x^* \rangle \\
&= Re\langle \lambda x - Ax, x^* \rangle \\
&\leq |\langle \lambda x - Ax, x^* \rangle| \\
&\leq \| x^* \| \, \| \lambda x - Ax \| \ = \ \| x \| \, \| \lambda x - Ax \| .
\end{aligned}$$

Conversely, (LATER)

Theorem 2.5.2 (Lumer-Phillips Theorem I)**.** *Let $A : X \supset D(A) \to X$ be a densely defined linear operator on a Banach space X. Then, A is the infinitesimal generator of a C_0-semigroup of contractions on X, if and only if:*

((1) *A is dissipative*

(2) *There exists $\lambda > 0$ such that $R(\lambda I - A) = X$, i.e $\lambda I - A$ is surjective.*

Proof (Necessity): Assume, that A is the infinitesimal generator of a C_0-semigroup of contractions $\{S_t : t \geq 0\}$. By virtue of the Hille-Yosida Theorem (2.2.1), we deduce that $(0, \infty) \subset \rho(A)$. Thus, for each $\lambda > 0$, $R(\lambda I - A) = X$. Now, let $x \in D(A)$ and $x^* \in F(x)$. Then, for each $t > 0$, we have:

$$|\langle S_t x, x^* \rangle| \leq \| x^* \| \| S_t x \| \leq \| x \|^2 .$$

Thus,

$$Re\langle S_t x - x, x^* \rangle = Re\langle S_t x, x^* \rangle - Re\langle x, x^* \rangle \leq \| x \|^2 - \| x \|^2 = 0.$$

Therefore,

$$Re\langle Ax, x^* \rangle = \lim_{t \to 0^+} \frac{Re\langle S_t x - x, x^* \rangle}{t} \leq 0, \text{ for each } x \in D(A) \text{ and } x^* \in F(x).$$

Proof (Sufficiency): Since A is dissipative, by Theorem (2.5.1), we deduce, that for each $x \in D(A)$ and $\lambda > 0$ we have:

$$\| \lambda x \| \leq \| (\lambda I - A)x \| . \tag{2.5.1}$$

Observe, that relationship (2.5.1), implies that $\lambda I - A$ is one to one for each $\lambda > 0$. Consider a $\lambda_0 > 0$, such that $\lambda_0 I - A$ is surjective. Then, $(\lambda_0 I - A)^{-1}$ exists and from (2.5.1) we get that $\left\| (\lambda_0 I - A)^{-1} y \right\| \leq \frac{\| y \|}{\lambda_0}$, for each $y \in X$. Thus $(\lambda_0 I - A)^{-1} \in B(X)$ and $\lambda_0 \in \rho(A)$. This implies that $(\lambda_0 I - A)^{-1}$ is closed, thus $\lambda_0 I - A$ and A are also closed.
Claim: In order to complete the proof, it suffices to show that for each $\lambda > 0$, $\lambda I - A$ is surjective.
Proof of Claim: Indeed, in this case, relationship (2.5.1) implies that $\lambda \in \rho(A)$ and $\| R(\lambda, A) \| \leq \frac{1}{\lambda}$. Since, A is densely defined and closed, by the Hille-Yosida Theorem, we get the desired result.
We will show now, that for each $\lambda > 0$, $R(\lambda I - A) = X$. To this end, define the set

$$\Lambda = \{\lambda > 0 : \; R(\lambda I - A) = X\}.$$

Let $\lambda \in \Lambda$. Then, $\lambda \in \rho(A)$ and because $\rho(A)$ is open in $\mathbb{C}$, there exists an open neighborhood $\mathcal{U}$ of λ that is contained in $\rho(A)$. Observe that $\mathcal{U} \cap \mathbb{R}^+ \backslash \{0\}$ is an open interval and is contained in Λ. Thus Λ is an open subset in $(0, \infty)$. Next, we will show that Λ is closed. To this aim, consider $\{\lambda_n\}_{n=1}^{\infty} \subset \Lambda$, such that $\lim_{n\to\infty} \lambda_n = \lambda$. It is clear that $\lambda > 0$. Moreover, if $y \in X$, then for each $n \in \mathbb{N}$, we can choose $x_n \in D(A)$, such that

$$\lambda_n x_n - A x_n = y \tag{2.5.2}$$

By (2.5.1), we deduce that for each $n \in \mathbb{N}$,

$$\| x_n \| \leq \frac{\| y \|}{\lambda_n} \leq c,$$

for some positive c, since $(\lambda_n)_n$ is positive and convergent. In addition, from (2.5.1), we deduce that for each positive integers $m > n$, we have,

$$\begin{aligned}
\lambda_m \| x_n - x_m \| &\leq \| \lambda_m (x_n - x_m) - A(x_n - x_m) \| \\
&= \| \lambda_m x_n - A x_n - y \| \\
&= \| \lambda_m x_n - A x_n - \lambda_n x_n + A x_n \| \\
&= |\lambda_n - \lambda_m| \, \| x_n \| \\
&\leq c\, |\lambda_n - \lambda_m|.
\end{aligned}$$

Thus, $(x_n)_n$ is Cauchy and so, by completeness of X, there exists $x \in X$: $\lim_{n\to\infty} x_n = x$. Then, by (2.5.2) $\lim_{n\to\infty} A x_n = \lambda x - y$. By closedness of A, we conclude that $x \in D(A)$ and $y = \lambda x - Ax$. This means that $\lambda \in \Lambda$. Thus Λ is a closed and an open subset in $(0, \infty)$. As a result $\Lambda = \emptyset$ or $\Lambda = (0, \infty)$. But by assumption, $\Lambda \neq \emptyset$. Thus, $\Lambda = (0, \infty)$, as desired.

Remark 2.5.2. From the above proof, we deduce that if A is the generator of a C_0-semigroup of contractions, then

$$R(\lambda I - A) = X, \; \text{ for each } \lambda > 0$$

and

$$Re < Ax, x^* > \leq 0, \; \text{ for each } x \in D(A) \text{ and } x^* \in F(x).$$

Corollary 2.5.1. *Let $A : X \supset D(A) \to X$ be a densely defined closed operator on a Banach space X. If A and A^* are dissipative, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

Proof: By virtue of the Lumer-Phillips Theorem (2.5.2), it suffices to show that $I - A$ is surjective.
Claim: $R(I - A)$ is a closed subset in X.
Proof of Claim: Consider $\{y_n\}_{n=1}^{\infty} \subset R(I-A)$, such that $\lim_{n\to\infty} y_n = y \in X$. For each $n \in \mathbb{N}$, there exists $x_n \in D(A)$, such that $y_n = x_n - Ax_n$. Since A is dissipative, for each positive integers $m > n$ we have:

$$\| x_n - x_m \| \leq \| y_n - y_m \| .$$

Thus, $(x_n)_n$ is Cauchy. So, by completeness of X, we deduce that there exists $x \in X$, such that $\lim_{n\to\infty} x_n = x$. But then, $\lim_{n\to\infty} Ax_n = x - y$. By closedness of A, $x \in D(A)$ and $y = Ax - x$. Therefore, $y \in R(I - A)$. Now, assume that $R(I - A) \neq X$. From a corollary of Hahn-Banach Theorem, there exists $x^* \in X^*$, $x^* \neq 0$, such that $\langle x^*, x - Ax\rangle = 0$, for each $x \in D(A)$. This implies, that $x^* - A^*x^* = 0$. Since A^* is dissipative, we conclude that $x^* = 0$, which leads us to a contradiction. Thus, $R(I - A) = X$.

Definition 2.5.3. *A linear operator $A : X \supset D(A) \to X$ on a Banach space X, is said to be **closable** if-f it can be extended to a closed linear operator, i.e there exists a linear operator $B : X \supset D(B) \to X$, such that, $D(A) \subset D(B)$ and $Bx = Ax$, for each $x \in D(A)$.*

Lemma 2.5.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space X. The following assertions are equivalent:*

(i) *A is closable*

(ii) *The closure of $G_A = \{(x, Ax) : x \in D(A)\} \subset X \times X$, is the graph of a closed linear operator.*

(iii) *If $\{x_n\}_{n=1}^{\infty} \subset D(A)$ such that, $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} Ax_n$ exists, then $\lim\limits_{n\to\infty} Ax_n = 0$.*

Proof:

(ii) ⇒ *(i)* It is clear that if B is a closed linear operator such that $G_B = \overline{G}_A$, then $Bx = Ax$, for each $x \in D(A)$. This is direct from the fact that G_B is the graph of a function.

(i) ⇒ *(iii)* Let A be a closable linear operator and $(x_n)_n$ a sequence in $D(A)$, such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Ax_n = y \in X$. Consider a closed linear operator $B : X \supset D(B) \to X$, such that $B_{|D(A)} = A$. Then, by the closedness of B, we conclude that $y = B(0) = 0$, as desired.

(iii) ⇒ *(ii)* We will show that $\overline{G}_A$ is the graph of a linear operator $\overline{A}$. To this end, it suffices to show that if (x, y), $(x, y') \in \overline{G}_A$, then $y = y'$. Observe now, that if (x, y), $(x, y') \in \overline{G}_A$, then there exist sequences $(x_n)_n$ and $(x'_n)_n$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} Ax_n = y$ and $\lim_{n\to\infty} x'_n = x$, $\lim_{n\to\infty} Ax'_n = y'$. Therefore, $\lim_{n\to\infty}(x_n - x'_n) = 0$ and $\lim_{n\to\infty} A(x_n - x'_n) = y - y'$. Thus, from (iii) we get the desired result. Therefore, $\overline{G_A}$ is the graph of an operator $\overline{A} : X \supset D(\overline{A}) \to X$. It remains to show the linearity of $D(\overline{A})$ and $\overline{A}$. Let $x, y \in D(\overline{A})$, then there exist sequences $(x_n)_n$ and $(y_n)_n$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} Ax_n = \overline{A}x$ and $\lim_{n\to\infty} Ay_n = \overline{A}y$. So, $\lim_{n\to\infty}(x_n + y_n) = x + y$ and $\lim_{n\to\infty}(Ax_n + Ay_n) = \overline{A}x + \overline{A}y$. Since $G_{\overline{A}}$ is closed we get $x + y \in D(\overline{A})$ and $\overline{A}(x + y) = \overline{A}x + \overline{A}y$. Similarly we work for the scalar multiplication.

Definition 2.5.4. *Let $A : X \supset D(A) \to X$ be a closable linear operator on a Banach space X. Then, the unique closed operator $\overline{A}$ with the property $G_{\overline{A}} = \overline{G_A}$, is said to be the **closure** of A.*

Theorem 2.5.3. *Let $A : X \supset D(A) \to X$ be a linear dissipative operator on a Banach space X. Then*

(i) *If there exists $\lambda_0 > 0$ such that $rg(\lambda_0 I - A) = X$, then $rg(\lambda I - A) = X$, for each $\lambda > 0$.*

(ii) *If A is closable, then $\overline{A}$ is dissipative.*

(iii) *If $D(A)$ is dense in X, then A is closable.*

Proof:

(i) We have already show this. See the proof of the sufficiency of Lummer-Phillips Theorem (2.5.2).

(ii) Let $x \in D(\overline{A})$. Then, there exists a sequence $(x_n)_n$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = \overline{A}x$. Since A is dissipative for each $\lambda > 0$ and $n \in \mathbb{N}$ we have,

$$\| \lambda x_n \| \leq \| \lambda x_n - Ax_n \|$$

Taking the limits as $n \to \infty$ in the last relationship we conclude that

$$\| \lambda x \| \leq \left\| \lambda x - \overline{A}x \right\|, \text{ for each } x \in D(\overline{A}), \lambda > 0,$$

from where we deduce that $\overline{A}$ is also dissipative.

(iii) Let $(x_n)_n$ be a sequence in $D(A)$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Ax_n = y \in X$. By virtue of (iii) of Lemma (2.5.1), it is enough to show that $y = 0$. Let $x \in D(A)$. Since A is dissipative for each $\lambda > 0$ we have,

$$\left\| \frac{1}{\lambda} x_n + x \right\| \leq \left\| (\frac{1}{\lambda} x_n + x) - \lambda A(\frac{1}{\lambda} x_n + x) \right\|$$

Therefore, taking the limits as $n \to \infty$ and $\lambda \to 0$ successively we deduce that

$$\| -y + x \| \geq \| x \|, \text{ for each } x \in D(A)..$$

Now, let $\varepsilon > 0$. Since $D(A)$ is dense in X, there exists $x_\varepsilon \in D(A)$, such that $\| y - x_\varepsilon \| < \varepsilon$. So, from this last result and (2.5) we have that

$$\| y \| \leq \| y - x_\varepsilon \| + \| x_\varepsilon \| < 2\varepsilon.$$

Since this is true for each $\varepsilon > 0$, we conclude that $\| y \| = 0$, thus $y = 0$.

Theorem 2.5.4 (Lumer-Phillips Theorem II)**.** *Let $A : X \supset D(A) \to X$ be a linear densely defined dissipative operator on a Banach space X. The closure $\overline{A}$ of A is the infinitesimal generator of a C_0-semigroup of contractions if and only if there exists $\lambda_0 > 0$ such that $rg(\lambda_0 I - A)$ is dense in X.*

Proof:

" $\Rightarrow$ " By virtue of Theorem (2.5.3), A is closable. If $\overline{A}$ is the generator of a strongly continuous semigroup of contractions then by the Hille-Yosida Theorem (2.2.1), there exists $\lambda_0 > 0$ such that $rg(\lambda_0 I - \overline{A}) = X$. Let $y \in X$. Then, there exists $x \in D(\overline{A})$ such that $y = \lambda_0 x - \overline{A}x$. Moreover, there exists $(x_n)_n$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = \overline{A}x$. Therefore, $\lim_{n\to\infty}(\lambda_0 x_n - Ax_n) = y$, from where we deduce the desired result.

" $\Leftarrow$ " By virtue of Theorem (2.5.3), A is closable and $\overline{A}$ is dissipative. Obviously $\overline{A}$ is densely defined. Arguing as in the proof of Corollary (2.5.1)we can show that $rg(\lambda_0 I - \overline{A})$ is closed in X. Therefore, $rg(\lambda_0 I - \overline{A}) = \overline{rg(\lambda_0 I - \overline{A})} \supset \overline{rg(\lambda_0 I - A)} = X$. Thus we can use the Lumer-Phillips Theorem I (2.5.2) for $\overline{A}$.

Corollary 2.5.2. *Let $A : X \supset D(A) \to X$ be a densely defined linear operator on a Banach space X. If A and A^* are dissipative, then the closure $\overline{A}$ of A is the ifinitesimal operator of a C_0-semigroup of contractions on X.*

Proof: It is a consequence of Theorem (2.5.3) and Corollary (2.5.1).

Part II

Stochastic analysis in infinite dimensions

Chapter 3

Probability in Banach Spaces

In this chapter we develop the necessary probabilistic tools for the study of stochastic integrals and stochastic evolution equations in infinite dimensional spaces. We begin with a review of basic probability concepts in $\mathbb{R}$ and $\mathbb{R}^n$. Next we are going to introduce the notions of integrability, measurability in a more general Banach space valued setting. In section (3.3) we study Banach space valued random variables, their Fourier transform and several type of convergence of sequences of random variables. Finally in section (3.4) we will be concerned with martingales in Banach spaces. The line followed in this chapter is taken from [DS I], [DS II], [DP I], [DP II], [NR].

3.1 Basic Probability Theory

In this section we review some important notions and facts from measure-theoretic probability. A more detailed and expanded treatment of this material can be found in [SP], [ASH], [BL] and [PR]. Here, we focus on the notions of Borel σ-algebras, probability measures and measurability and we are interested mainly in real random variables and random vectors. The section is only a collection of definitions and propositions which will be taken for granted in the rest of the text. The notion of Lebesgue integrability is not examined at all and is assumed familiarity with it. Some well known books on measure theory in the real line and in eukleidean spaces are [KM], [FLD] and [CP].

Definition 3.1.1. *Let $\Omega \neq \emptyset$ be a set. A family $\mathfrak{F} \subset \mathcal{P}(\Omega)$ is said to be a σ-* ***algebra of subsets of*** *Ω if-f*

(i) $\emptyset \in \mathfrak{F}$

(ii) *If $A \in \mathfrak{F}$, then $A^c \in \mathfrak{F}$*

(iii) *If $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathfrak{F}$.*

In this case, the pair $(\Omega, \mathfrak{F})$ is called a ***measurable space****.*

Definition 3.1.2. *Let $\Omega \neq \emptyset$ be a set and $\mathfrak{C} \subset \mathcal{P}(\Omega)$ a class of subsets of Ω. The smallest $\sigma-$algebra of subsets of Ω that contains $\mathfrak{C}$ is said to be* ***the σ-algebra generated by*** *$\mathfrak{C}$ and is denoted by $\sigma(\mathfrak{C})$. In other words,*

$$\sigma(\mathfrak{C}) := \bigcap \{\mathfrak{F} : \mathfrak{F} \text{ is a } \sigma\text{-algebra and } \mathfrak{C} \subset \mathfrak{F}\}.$$

Remark 3.1.1. *Note that the existence of the above σ-algebra is clear from the fact that the intersection of an arbitrary family of σ-algebras is also a $\sigma-$algebra of subsets of Ω.*

Definition 3.1.3. *Let $(X, \mathfrak{T})$ be a topological space. Then the $\sigma-$algebra*

$$\mathfrak{B}(X) := \sigma(\mathfrak{T}) = \sigma(\{U \subset X : U \text{ is } \mathfrak{T}\text{-open}\})$$

is called the ***Borel σ-algebra*** *of subsets of X.*

Proposition 3.1.1. *For the Borel $\sigma-$algebra $\mathfrak{B}(\mathbb{R})$ of $\mathbb{R}$ the following assertions hold,*

(i) *It is generated by the class of closed subsets of $\mathbb{R}$.*

(ii) *It is generated by the class of open intervals of* $\mathbb{R}$, $\{(a,b) : a,b \in \mathbb{R}, a < b\}$.

(iii) *It is generated by the class of closed intervals of* $\mathbb{R}$, $\{[a,b] : a,b \in \mathbb{R}, a < b\}$.

(iv) *It is generated by the class of half open intervals of* $\mathbb{R}$, $\{(a,b] : a,b \in \mathbb{R}, a < b\}$ *or* $\{[a,b) : a,b \in \mathbb{R}, a < b\}$.

(v) *It is generated by the class* $\{[a,\infty) : a \in \mathbb{R}\}$ *or* $\{(-\infty,a] : a \in \mathbb{R}\}$ *or* $\{(a,\infty) : a \in \mathbb{R}\}$ *or* $\{(-\infty,a) : a \in \mathbb{R}\}$.

Proposition 3.1.2. *Let* $(\Omega,\mathfrak{F})$ *be a measurable space and suppose that* $\mathfrak{F} = \sigma(\mathfrak{C})$. *If* $\Omega_0 \subset \Omega$, *then* $\mathfrak{F}_0 := \mathfrak{F} \cap \Omega_0 := \{A \cap \Omega_0 : A \in \mathfrak{F}\}$ *is a* $\sigma-$*algebra of subsets of* Ω_0 *and* $\mathfrak{F}_0 = \sigma(\mathfrak{C}_0)$, *where* $\mathfrak{C}_0 := \mathfrak{C} \cap \Omega_0 := \{C \cap \Omega_0 : C \in \mathfrak{C}\}$. *Moreover, if* $\Omega_0 \in \mathfrak{F}$, *then* $\mathfrak{F}_0 = \{A \subset \Omega_0 : A \in \mathfrak{F}\}$.

Definition 3.1.4. *Let* $(\Omega,\mathfrak{F})$ *be a measurable space. A function* $\mu : \mathfrak{F} \to [0,\infty]$ *is said to be a* ***measure*** *on* $(\Omega,\mathfrak{F})$ *if-f*

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \geq 0$, *for each* $A \in \mathfrak{F}$

(iii) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, *for each* $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ *such that* $A_n \cap A_m = \emptyset$, *for* $m \neq n$.

The triple $(\Omega,\mathfrak{F},\mu)$ *is called* ***measure space***. *Moreover,*

- *If there exists a sequence* $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ *such that* $\mu(A_n) < \infty$, *for each* $n \in \mathbb{N}$ *and* $\Omega = \cup_{n=1}^{\infty} A_n$, *then* μ *is a* σ***-finite measure***.
- *If* $\mu(\Omega) < \infty$, *then then* μ *is a* ***finite measure***.
- *If* $\mu(\Omega) = 1$, *then then* μ *is a* ***probability measure***.

Remark 3.1.2. *It is direct from the definition of a sigma-algebra that it is closed under at most countably infinite operations between sets. Moreover, it can be easily verified that for a probability space* $(\Omega,\mathfrak{F},\mathbb{P})$, *the well known relationships from elementary probability stay true, e.g* $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B), \mathbb{P}(A - B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$*e.t.c.*

Definition 3.1.5. *Let* $(F,\mathfrak{F})$ *and* $(E,\mathfrak{E})$ *be two measurable spaces. A function* $f : F \to E$ *is called* $(\mathfrak{F},\mathfrak{E})$***-measurable*** *if-f*

$$f^{-1}(\mathfrak{E}) \subset \mathfrak{F},$$

which means that

$$f^{-1}(B) \in \mathfrak{F}, \textit{ for each } B \in \mathfrak{E}.$$

Furthermore, if $(\Omega,\mathfrak{F},\mathbb{P})$ *is a p.s then we say that a* $X : \Omega \to E$ *is a* ***random variable*** *if-f it is measurable. Usually we take* $E = \mathbb{R}^n$ *and* $\mathfrak{E} = \mathfrak{B}(\mathbb{R}^n)$.

Proposition 3.1.3. *If* $\sigma(\mathfrak{C}) = \mathfrak{E}$, *then* X *is a random variable if and only if* $X^{-1}(\mathfrak{C}) \subset \mathfrak{F}$.

Corollary 3.1.1. *A function* $X : (\Omega,\mathfrak{F},\mathbb{P}) \to \mathbb{R}$ *is a random variable if and only if* $X^{-1}((-\infty,a]) \in \mathfrak{F}$, *for all* $a \in \mathbb{R}$.

Proposition 3.1.4. *A vector function* $X : \Omega \to \mathbb{R}^n$, $X = (X_1, X_2, \ldots, X_n)$, *is a random vector if and only if* X_i *is a random variable, for each* $i = 1, \ldots, n$.

Proposition 3.1.5. *Let* $(\Omega,\mathfrak{F},\mathbb{P})$ *be a p.s,* $(E,\mathfrak{E})$ *a measurable space and* $X : \Omega \to E$ *a random variable. Then the space* $(E,\mathfrak{E},\mathbb{P}_X)$, *where*

$$\mathbb{P}_X(B) = \mathbb{P}[X \in B], \;\; B \in \mathfrak{E},$$

is a probability space. Moreover the probability measure $\mathbb{P}_X$ *is called the* ***distribution*** *of* X.

When $E = \mathbb{R}^n$ the distribution P_X of a random variable X is uniquely determined by its **distribution function** $F_X : \mathbb{R}^n \to [0,1]$ which is defined by

$$F_X(x_1, x_2, \ldots, x_n) := \mathbb{P}_X(\prod_{i=1}^{n}(-\infty, x_i]).$$

In the case $n = 1$ a distribution function is always a non decreasing, right continuous function and

$$\lim_{x\to\infty} F(x) = 1 \;\text{ and }\; \lim_{x\to-\infty} F(x) = 0.$$

Remark 3.1.3. *Note that the set of real valued measurable functions is closed under addition. multiplication, subtraction, and division. Furthermore, the supremum, infimum, limit superior and limit inferior of a sequence of real random variables is also a measurable function with values in $\overline{R} := \mathbb{R} \cup \{-\infty, \infty\}$. Thus, if the limit of a sequence of real random variables exists, then it is a measurable function.*

Definition 3.1.6. *A real random variable X on a probability space $\mathbb{P}$ is called:*

(i) ***discrete*** *if-f the set $X(\Omega) = rg(X)$ is countable*

(ii) ***continuous*** *if-f its distribution function is everywhere continuous, equivalently if-f $\mathbb{P}[X = x] = 0$, for all $x \in \mathbb{R}$.*

(iii) ***absolutely continuous*** *if-f its distribution $\mathbb{P}_X$ is absolutely continuous with respect to the Lebesgue measure. In this case, from the Radon-Nikodym Theorem* (e.g see [CP] ch.7) *we deduce that there exists a non negative integrable function $f_X : \mathbb{R} \to [0, \infty]$ such that*

$$\mathbb{P}[X \in B] = \int_B f_X(x)\ dm(x).$$

*We call this function the **density** of X. In particular for its distribution function we have that,*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dm(t), \quad \textit{for each } x \in \mathbb{R}.$$

and that it is differentiable wherever the density is continuous, with derivative

$$\frac{d}{dx} F_X(x) = f_X(x).$$

Remark 3.1.4. *We can generalize the previous definition for a random vector.*

3.2 Integration of Banach space valued functions II.

As we have already seen, when we have a continuous Banach space valued function defined on a real interval, the Riemann integration is a choice with quite good properties for our purposes. In contrast, if a Banach space valued function is defined on an arbitrary measurable space $(A, \mathcal{A})$ then the notions of continuity and Riemann integration have no meaning at all. Therefore, the scope of this section is an introduction to the basic definitions and properties of the so called Bochner integration, which will be very useful in the sequel.

3.2.1 The Pettis Measurability Theorem

Consider a separable Banach space X, equipped with its Borel $\sigma-$algebra $\mathfrak{B}(X)$ and a measurable space $(A, \mathfrak{A})$. We remind you that the operations of addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{K} \times X \to X$ are continuous, thus Borel functions. Furthermore, since X is separable we have that $\mathfrak{B}(X \times X) = \mathfrak{B}(X) \otimes \mathfrak{B}(X)$ and $\mathfrak{B}(\mathbb{K} \times X) = \mathfrak{B}(\mathcal{K}) \otimes \mathfrak{B}(X)$. Thus, if $f, g : A \to X$ are two measurable functions and $\lambda : A \to \mathbb{K}$ is also measurable, then $(f, g) : A \to X \times X$ and $(\lambda, f) : A \to \mathbb{K} \times X$ are also measurable. Thus, we can conclude that $f + g$ and λf are measurable as compositions of Borel functions with measurable functions. However, when X is not separable, it is not true in general that $\mathfrak{B}(X \times X) = \mathfrak{B}(X) \otimes \mathfrak{B}(X)$ and $\mathfrak{B}(\mathbb{K} \times X) = \mathfrak{B}(\mathcal{K}) \otimes \mathfrak{B}(X)$, so the above reasoning fails to guarantee the measurability of $f + g$ and λf. So we can see that separability plays a very important role in the Banach space valued setting. This will be even more clear after the introduction of new measurability notions, the so called strong and weak measurability which are also necessary for the generalization of the Lebesgue integral to the Banach space valued setting. The main result of this subsection is the Pettis measurability theorems which guarantee that a function is strongly measurable if and only if it is almost separably valued and weakly measurable.

Definition 3.2.1. *Let X be a Banach space, $F \subset X^*$ a subspace of X^* and $S \subset X$ a subset of X. Then,*

- *F is said to be **norming for** S if-f, for each $x \in S$ it holds that*

$$\| x \| = \sup\{| < x^*, x > | : x^* \in F, \| x^* \|_{X^*} \leq 1\}. \tag{3.2.1}$$

- *We say that F **separates the points of** S if-f for each $x, y \in S$, with $x \neq y$, there exists $x^* \in F$, such that $< x^\star, x > \neq < x^\star, y >$.*

Remark 3.2.1. It is direct that if F is norming for S, then it separates the points of S. The converse is not always true.

Lemma 3.2.1. *Let X be a Banach space, S a separable subspace of X and F a subspace of X^*. Then, the following assertions hold:*

(i) *If F is norming for S, then there exists a sequence of unit vectors $\{x_n^*\}_{n=1}^\infty \subset F$ which is norming for S.*

(ii) *If F separates the points of S, then there exists a sequence $\{x_n^*\}_{n=1}^\infty \subset F$ which separates the points of S.*

Proof:

(i) Let $(x_n)_n$ be a dense sequence in S and $\{\varepsilon_n\}_{n=1}^\infty \subset (0, 1]$ a sequence which converges to zero. According to (3.2.1) and the property of the supremum, we can choose a sequence $\{{x^*}_n\}_{n=1}^\infty \subset F$, with $\| x_n^* \| = 1$ such that

$$| < x_n^*, x_n > | > (1 - \varepsilon_n) \| x_n \| , \quad \text{for each} \quad n \in \mathbb{N}.$$

We claim that the sequence $\{{x^*}_n\}_{n=1}^\infty$ is norming for S. Indeed, for a fixed $x \in S$, since $\{| < x_n^*, x > | : n \in \mathbb{N}\} \subset \{| < x^*, x > | : x^* \in F, \| x^* \| \leq 1\}$, we derive that $\| x \| \geq \sup_{n \in \mathbb{N}} | < x_n^*, x > |$. Now, consider an arbitrary $\delta > 0$. For this $\delta > 0$ we can choose $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0} < \delta$ and $\| x_{n_0} - x \| < \delta$. Therefore,

$$\begin{aligned}
(1 - \delta) \| x \| &\leq (1 - \varepsilon_{n_0}) \| x \| \\
&\leq (1 - \varepsilon_{n_0}) \| x - x_{n_0} \| + (1 - \varepsilon_{n_0}) \| x_{n_0} \| \\
&\leq \delta + | < x_{n_0}^*, x_{n_0} > | \\
&\leq \delta + \| x_{n_0}^* \| \| x_{n_0} - x \| + | < x_{n_0}^*, x > | \\
&\leq | < x_{n_0}^*, x > | + 2\delta \leq \sup_{n \in \mathbb{N}} | < x_n^*, x > | + 2\delta
\end{aligned}$$

Therefore, $\| x \| \leq \sup_{n \in \mathbb{N}} | < x_n^*, x > |$.

(ii) Since F separates the points of S, then for each $x \in S \backslash \{0\}$, there exists $f_x \in F$ such that $f_x(x) \neq 0$. So for fixed $x \in X$, we define

$$V_x := \{y \in S \backslash \{0\} : f_x(y) \neq 0\}.$$

Observe that $\{x\} \subset V_x$ and V_x is open in $S \backslash \{0\}$. This means that $(V_x)_{x \in S \backslash \{0\}}$ is an open cover of $S \backslash \{0\}$. But since $S \backslash \{0\}$ is separable, we conclude that there exists a sequence $\{x_n\}_{n=1}^\infty \subset S \backslash \{0\}$, such that $S \backslash \{0\} \subset \cup_{n=1}^\infty V_{x_n}$. This is because each open cover of a separable metric space has a countable subcover(for details see [TS] ch.9). Now it is easy to see that $\{f_{x_n}\}_{n \in \mathbb{N}} \subset F$ separates the points of S. Indeed, for each $x, y \in S$ with $x \neq y$, since $x - y \in S \backslash \{0\}$, there exists $n_0 \in \mathbb{N}$ such that $x - y \in V_{x_{n_0}}$. Therefore, $f_{x_{n_0}}(x) \neq f_{x_{n_0}}(y)$.

Definition 3.2.2. *A function $f : A \to X$ is said to be $\mathfrak{A}$**-simple** if-f it has the form*

$$f = \sum_{i=1}^n \mathbb{I}_{A_i} x_i,$$

where $n \in \mathbb{N}$, $\{A_i\}_{i=1}^n \subset \mathfrak{A}$ and $\{x_i\}_{i=1}^n \subset X$.

Definition 3.2.3. *Consider a function $f : A \to X$, where $(A, \mathfrak{A})$ is a measurable space and X is Banach space. Then,*

- *f is called **stongly $\mathfrak{A}$-measurable** if-f there exits a sequence $(f_n)_{n=1}^\infty$ of $\mathfrak{A}$-simple functions $f_n : A \to X$, $n \in \mathbb{N}$, which converges pointwise to f.*

- *f is called **separably valued** if-f there exists a closed separable subspace $E \subset X$ such that $rg(f) \subset E$.*

- f *is called* ***weakly*** $\mathfrak{A}$***-measurable*** *if-f the scalar valued function* $x^* \circ f$ *is* $\mathfrak{A}$*-measurable for each* $x^* \in X^*$.

Remark 3.2.2. Obviously if f is $\mathfrak{A}$-measurable, then it is also weakly $\mathfrak{A}$-measurable, since x^* is a Borel function, i.e $(\mathfrak{B}(X), \mathfrak{B}(\mathbb{K}))$-measurable

We are now ready to state and prove the first version if the Pettis measurability theorem due to Pettis in his celebrated paper [PT].

Theorem 3.2.1 (The Pettis measurability Theorem I)**.** *Consider a function* $f : A \to X$, *where* $(A, \mathfrak{A})$ *is a measurable space and* X *is Banach space and let* $F \subset X^*$ *be a norming subspace. Then, the following assertions are equivalent:*

(1) f *is strongly* $\mathfrak{A}$*-measurable.*

(2) f *is separably valued and weakly* $\mathfrak{A}$*-measurable.*

(3) f *is separably valued and the scalar function* $x^* \circ f$ *is* $\mathfrak{A}$*-measurable for each* $x^* \in F$.

Proof:

(1) $\to$ (2) Since f is strongly $\mathfrak{A}$-measurable there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$-simple functions which converges pointwise to f. Set

$$E := \overline{< \cup_{n=1}^{\infty} rg(f_n) >} =: \overline{< \{\varepsilon_n : n \in \mathbb{N}\} >}.$$

Observe that the set $M = \{\sum_{i=1}^{n} r_i \varepsilon_i : n \in \mathbb{N}, \{r_i\}_{i=1}^{n} \subset D\}$, where D is countable and dense in $\mathbb{K}$, is countable and dense in E. Therefore E is a closed separable subspace of X. Moreover, $rg(f) \subset E$. Finally, for a fixed $x^* \in X^*$, we have that $(x^* \circ f_n)_{n\in\mathbb{N}}$ is a sequence of scalar $\mathfrak{A}$-measurable functions which converges pointwise to $x^* \circ f$. Therefore, $x^* \circ f$ is $\mathfrak{A}$-measurable.

(2) $\to$ (3) It is obvious.

(3) $\to$ (1) Let E be a closed and separable subspace of X, such that $rg(f) \subset E$. We want to construct a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$-simple functions $f_n : A \to X$, $n \in \mathbb{N}$, which converges pointwise to f. To this aim, consider a sequence $(x_n)_{n\in\mathbb{N}}$ which is dense in E and for each $n \in \mathbb{N}$ define the function $s_n : E \to \{x_1, x_2, \ldots, x_n\}$ as follows: For each $y \in E$, consider the minimum $k(n, y) \in \{1, \ldots, n\}$ with the property

$$\| x_{k(n,y)} - y \| = \min_{1\leq i\leq n} \| x_i - y \|$$

and set $s_n(y) := x_{k(n,y)}$. Moreover, observe that

$$\lim_{n\to\infty} \| s_n(y) - y \| = 0,$$

Indeed, let $\varepsilon > 0$ then there exists $n_0 \in \mathbb{N}$ such that $\| x_{n_0} - y \| < \varepsilon$. Therefore, for each $n \geq n_0$ we have that

$$\| s_n(y) - y \| \leq \| x_{n_0} - y \| < \varepsilon.$$

For each $n \in \mathbb{N}$ define the function $f_n : A \to X$, $f_n(a) = s_n(f(a))$, $a \in A$. Obviously, for each $n \in \mathbb{N}$, f_n takes values in $\{x_1, \ldots, x_n\}$ and according to the last observation

$$\lim_{n\to\infty} f_n(a) = f(a), \text{ for each } a \in A.$$

Furthermore, for each $k \in \{1, \ldots, n\}$ we have

$$\{a \in A : f_n(a) = x_k\} = \{a \in A : s_n(f(a) = x_k\}$$
$$= \{a \in A : \| f(a) - x_k \| = \min_{1\leq i\leq n} \| f(a) - x_i \|\} \cap \{a \in A : \| f(a) - x_i \| > \| f(a) - x_k \|, \forall i = 1, \ldots, k-1\}.$$

It remains to show that $\{a \in A : f_n(a) = x_k\} \in \mathfrak{A}$. To this end it suffices to show the following claim.
Claim: For each $x \in E$ the scalar function $A \ni a \to \| f(a) - x \|$ is $\mathfrak{A}$-measurable.
Proof of Claim: Since F is norming and E is a separable subspace, by virtue of Lemma (3.2.1) we

can choose a sequence of unit vectors $\{x_n^*\}_{n=1}^{\infty} \subset F$ which is norming for E. This means that for a fixed $x \in E$ we have

$$\| f(a) - x \| = \sup_{n\in\mathbb{N}} | < x_n^*, f(a) > - < x_n^*, x > |, \text{ for each } a \in A.$$

Since, by assumption $(x_n^* \circ f)_n$ is a sequence of scalar $\mathfrak{A}$-measurable functions we get the desired result. The proof now is complete.

Corollary 3.2.1. *Let $f_n : A \to X$, $n \in \mathbb{N}$,(where $(A, \mathfrak{A})$ is a measurable space and X is a Banach space), be strongly $\mathfrak{A}$-measurable functions such that $\lim_{n\to\infty} f_n = f$ pointwise. Then, f is also strongly $\mathfrak{A}$-measurable.*

Proof: For each $n \in \mathbb{N}$, consider a closed separable subspace $E_n \subset X$, such that $rg(f_n) \subset E_n$. Then, $\cup_{n=1}^{\infty} E_n$ is also separable, thus $< \cup_{n=1}^{\infty} E_n >$ is separable and finally E:=$\overline{< \cup_{n=1}^{\infty} E_n >}$ is a separable and closed subspace of X. Moreover $rg(f) \subset E$. By virtue of Theorem (3.2.1), for fixed $x^* \in X^*$, $(x^* \circ f_n)_n$ is a sequence of scalar $\mathfrak{A}$-measurable functions which converges to $x^* \circ f$ pointwise. Therefore, $x^* \circ f$ is $\mathfrak{A}$-measurable.

Corollary 3.2.2. *Let X, Y be two Banach spaces and $(A, \mathfrak{A})$ a measurable space. If $f : A \to X$ is a strongly $\mathfrak{A}$-measurable function and $g : X \to Y$ a continuous function, then $g \circ f : A \to Y$ is strongly $\mathfrak{A}$-measurable.*

Proof: Let $(f_n)_n$ be a sequence of $\mathfrak{A}$- simple functions which converges to f pointwise. Then $(g \circ f_n)_n$ is a sequence of $\mathfrak{A}$- simple functions which converges to $g \circ f$ pointwise.

Proposition 3.2.1. *For a function $f : A \to X$, where $(A, \mathfrak{A})$ is a measurable space and X is a Banach space the following assertions are equivalent:*

(1) *f is strongly $\mathfrak{A}$-measurable.*

(2) *f is separably valued and $\mathfrak{A}$-measurable.*

Proof:

(1) ⇒ (2) By virtue of Theorem (3.2.1), f is separably valued. In order to show that f is $\mathfrak{A}$-measurable, it is enough to show that $f^{-1}(U) \in \mathfrak{A}$, for each $U \subset X$ open. Consider a sequence $(f_n)_n$ of $\mathfrak{A}$- simple functions which converges pointwise to f. Since U is open we have,

$$\begin{aligned} a \in f^{-1}(U) &\Rightarrow \lim_{n\to\infty} f_n(a) \in U \\ &\Rightarrow \exists n_0 \in \mathbb{N} : \forall n \geq n_0,\ f_n(a) \in U \\ &\Rightarrow \exists n_0 \in \mathbb{N} : \forall n \geq n_0\ \exists m \in \mathbb{N} \text{ such that } B(f_n(a), \frac{1}{m}) \subset U \\ &\Rightarrow \exists n_0 \in \mathbb{N} : \forall n \geq n_0\ \exists m \in \mathbb{N}, \text{ such that } f_n(a) \in U \text{ and } d(f_n(a), U^c) > \frac{1}{m}. \end{aligned}$$

Therefore, if for $r > 0$ we set $U_r := \{x \in U : d(x, U^c) > r\}$, we have that

$$f^{-1}(U) = \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \bigcup_{m=1}^{\infty} f_n^{-1}(U_{\frac{1}{m}}).$$

Since f_n is $\mathfrak{A}$- simple, $f_n^{-1}(U_r) \in \mathfrak{A}$, for each $n \in \mathbb{N}$ and $r > 0$. Therefore, $f^{-1}(U) \in \mathfrak{A}$.

(2) ⇒ (1) Since f is $\mathfrak{A}$-measurable, it is also weakly $\mathfrak{A}$-measurable, hence the result follows from Theorem (3.2.1).

Corollary 3.2.3. *Let X be a separable Banach space, $(A, \mathfrak{A})$ a measurable space and consider a function $f : A \to X$. Then the following assertions are equivalent:*

(1) *f is strongly $\mathfrak{A}$-measurable*

(2) *f is weakly $\mathfrak{A}$-measurable*

(3) *f is $\mathfrak{A}$-measurable*

Definition 3.2.4. *Let $(A, \mathfrak{A}, \mu)$ be a σ-finite measure space, X a Banach space and $f : A \to X$ a function.*

- *f is said to be **μ-simple** if-f it is has the form $f = \sum_{i=1}^{N} \mathbb{I}_{A_i} x_i$, where $N \in \mathbb{N}$, $A_i \in \mathfrak{A}$, $\mu(A_i) < \infty$, for all $i = 1, \ldots, N$ and $\{x_i\}_{i=1}^{N} \subset X$.*
- *f is said to be **strongly μ-measurable** if-f there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of μ-simple functions which converges to f μ-almost everywhere.*
- *a function $\tilde{f} : A \to X$ is said to be a **μ-version** of f, if-f $f = \tilde{f}$ μ-almost everywhere.*
- *f is called **weakly μ-measurable** if-f the scalar function $x^* \circ f$ is μ-measurable, for each $x^* \in X^*$.*
- *f is called **μ-separably valued** if-f there exists a closed separable subspace $E \subset X$ such that $f(a) \in E$, for μ-almost all $a \in A$.*

Proposition 3.2.2. *If f is strongly $\mathfrak{A}$-measurable, then it is also strongly μ-measurable.*

Proof: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of $\mathfrak{A}$-simple functions which converges pointwise to f and $(A_n)_n$ a sequence in $\mathfrak{A}$ such that $A_n \subset A_{n+1}$, $\mu(A_n) < \infty$ and $A = \cup_{n=1}^{\infty} A_n$, then $(\mathbb{I}_{A_n} f_n)_n$ is a sequence of μ-simple functions which converges to f pointwise .

Proposition 3.2.3. *For a function $f : A \to X$ the following assertions are equivalent*

(1) *f is strongly μ-measurable.*

(2) *f has a μ-version which is strongly $\mathfrak{A}$-measurable.*

Proof:

(1) $\Rightarrow$ (2) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of μ-simple functions such that $\lim_{n\to\infty} f_n(a) = f(a)$, for each $a \in A \backslash N$, where $N \in \mathfrak{A}$ and $\mu(N) = 0$, then $(\mathbb{I}_{N^c} f_n)_n$ is a sequence of $\mathfrak{A}$- simple functions which converges pointwise to $\tilde{f} = \mathbb{I}_{N^c} f$. Furthermore $\tilde{f}$ is a μ-version of f.

(2) $\Rightarrow$ (1) Let $\tilde{f}$ be a stronlgy $\mathfrak{A}$-measurable μ-version of f and $(\tilde{f}_n)_n$ a sequence of $\mathfrak{A}$-simple functions which converges pointwise to $\tilde{f}$. If $(A_n)_n$ is an increasing sequence in $\mathfrak{A}$ such that $\mu(A_n) < \infty$ and $A = \cup_{n=1}^{\infty} A_n$, then $(\mathbb{I}_{A_n} \tilde{f}_n)_{\tilde{n}}$ is a sequence of μ-simple functions which converges to f μ-almost everywhere (since it converges to $\tilde{f}$ pointwise).

We are now ready to state and prove the second and most important version of the Pettis measurability theorem which can be found again in [PT].

Theorem 3.2.2 (The Pettis measurability Theorem II)**.** *Let $f : A \to X$ be a function, where $(A, \mathfrak{A}, \mu)$ is a σ-finite measure space and X is a Banach space, and let $F \subset X^*$ be a norming subspace. Then, the following assertions are equivalent:*

(1) *f is strongly μ-measurable*

(2) *f is μ-separably valued and weakly μ-measurable.*

(3) *f is μ-separably valued and $x^* \circ f$ is μ-measurable, for each $x^* \in F$.*

Proof:

(1) $\Rightarrow$ (2) By Proposition (3.2.3), f has a μ-version $\tilde{f}$ which is strongly $\mathfrak{A}$-measurable. By the Pettis measurability Theorem I (3.2.1) $\tilde{f}$ is separably valued, thus f is μ-separably valued. Again by Theorem (3.2.1) $x^* \circ \tilde{f}$ is $\mathfrak{A}$-measurable, for $x^* \in X^*$. Therefore if $B \in \mathfrak{B}(\mathbb{K})$, $\mathfrak{A}_\mu$ is the completion of $\mathfrak{A}$ and $f(a) = \tilde{f}(a)$, for each $a \in A \backslash N$, where $N \in \mathfrak{A}$ with $\mu(N) = 0$, we have

$$[(x^* \circ f) \in B] = ([(x^* \circ \tilde{f}) \in B] \cap N^c) \bigcup ([(x^* \circ f) \in B] \cap N) \ \in \mathfrak{A}_\mu.$$

This means that $x^* \circ f$ is $\mathfrak{A}_\mu$-measurable, for each $x^* \in X^*$.

(2) $\Rightarrow$ (3) It is obvious

(3) $\Rightarrow$ (1) Let $E \subset X$ be a closed and separable subspace, such that $f(a) \in E$, for all $a \in A \backslash N$, where $N \in \mathfrak{A}$ and $\mu(N) = 0$. Consider a dense sequence $(x_n)_n$ in E. For each $n \in \mathbb{N}$ we define the function $s_n : E \to \{x_1, \ldots, x_n\}$, as in the proof of Theorem (3.2.1) and the function $\tilde{f}_n : A \to X$,

$$\tilde{f}_n(a) = \begin{cases} s_n(f(a)) & \text{when } a \in N^c, \\ 0 & \text{when } a \in N. \end{cases}$$

Arguing again as in the proof of Theorem (3.2.1), we conclude that

$$\lim_{n\to\infty} \tilde{f}_n(a) = f(a), \text{ for each } a \in A\backslash N.$$

and $\tilde{f}_n$ is $\mathfrak{A}_\mu$-simple, for each $n \in \mathbb{N}$. Easily, for each $n \in \mathbb{N}$ we can construct a μ-version $\hat{f}_n$ of $\tilde{f}_n$, which is $\mathfrak{A}$-simple. For more details on this see [KM] pg 72. Let $(A_n)_n$ be an increasing sequence in $\mathfrak{A}$ such that $\mu(A_n) < \infty$ and $A = \cup_{n=1}^{\infty} A_n$. Then, $(\mathbb{I}_{A_n}\hat{f}_n)_n$ is a sequence of μ-simple functions which converges to f μ-almost everywhere.

Both versions of the Pettis measurability theorem remain true if in the statements we replace the norming subspace $F \subset X^*$ with a subspace which separates the point of X. The proof is more complicated in this case. For more details on this see [DS I].

Corollary 3.2.4. *The almost everywhere limit of a sequence of strongly μ-measurable functions is strongly μ-measurable*

Corollary 3.2.5. *Let X, Y be two Banach spaces and $(A, \mathfrak{A}, \mu)$ a σ- finite measure space. If $f : A \to X$ is strongly μ-measurable and $g : X \to Y$ continuous, then $g \circ f$ is strongly μ-measurable.*

For the proof of the above two corollaries combine Proposition (3.2.3), Corollary (3.2.1) and Corollary (3.2.2)

Corollary 3.2.6. *If $f, g : A \to X$ are two strongly μ-measurable functions such that $x^* \circ f = x^* \circ g$ μ-almost everywhere, for each $x^* \in F$, where $F \subset X^*$ is a subspace which separates the points of X, then $f = g$, μ-almost everywhere*

Proof: If f takes almost all its values in a separable closed subspace $E_1 \subset X$ and g takes almost all its values in a separable closed subspace $E_2 \subset X$, then both f, g take almost all their values (outside a μ-null set $N \in \mathfrak{A}$) in the separable closed subspace $E := \overline{< E_1 \cup E_2 >} \subset X$. Since F separates the points of E, by virtue of Lemma (3.2.1), we can choose a sequence $\{x_n^*\}_{n=1}^{\infty} \subset F$, which separates the points of E. But by our assumption, for each $n \in \mathbb{N}$, there exists a μ-null set $N_n \in \mathfrak{A}$, such that $(x_n^* \circ f)(a) = (x_n^* \circ g)(a)$, for each $a \in A\backslash N_n$. Therefore, $x_n^*(f(a)) = x_n^*(g(a))$, for each $n \in \mathbb{N}$ and $a \in A\backslash(N \bigcup \cup_{n=1}^{\infty} N_n)$. Therefore, $f(a) = g(a)$, for each $a \in A\backslash(N \bigcup \cup_{n=1}^{\infty} N_n)$.

3.2.2 The Bochner Integral

In this subsection we turn our attention to the so called Bochner integral, the natural generalization of the Lebesgue integral to the Banach space valued setting

Definition 3.2.5. *A function $f : A \to X$, where $(A, \mathfrak{A}, \mu)$ is a σ-finite measure space and X is Banach, is said to be μ-**Bochner integrable** if-f there exists a sequence $(f_n)_n$ of μ-simple functions such that*

(1) $\lim_{n\to\infty} f_n = f$ *μ-almost everywhere.*

(2) $\lim_{n\to\infty} \int_A \| f_n - f \| \ d\mu = 0$.

Remark 3.2.3.

1. If f is μ-Bochner integrable then it is strongly μ-measurable. The same is true for the functions $f_n - f$, $n \in \mathbb{N}$. So, by the continuity of the norm and Corollary (3.2.5), we deduce that $\| f_n - f \|$ is strongly μ-measurable, so $\| f_n - f \|$ is μ-measurable, for each $n \in \mathbb{N}$.

2. A μ-simple function is trivially μ-Bochner integrable.

Definition 3.2.6. *Let $f : A \to X$ be a μ-simple function, as in Definition (3.2.4), which has the form $f = \sum_{i=1}^{N} \mathbb{I}_{A_i} x_i$. Then we define* ***the Bochner integral of f with respect to*** *μ as follows:*

$$\int_A f \; d\mu = \sum_{i=1}^{N} \mu(A_i) x_i \quad \in X.$$

Definition 3.2.7. *Let $f : A \to X$ be a μ-Bochner integrable function as in Definition (3.2.5). Then the limit*

$$\int_A f \; d\mu := \lim_{n\to\infty} \int_A f_n \; d\mu$$

exists in X and is called ***the Bochner integral*** *of f with respect to μ.*

Remark 3.2.4. In definition (3.2.6) the integral is independent of the representation of the μ-simple function f. In definition (3.2.7) the integral is independent of the approximating sequence.

Proposition 3.2.4. *If f is μ-Bochner integrable and g is a μ-version of f, then g is also μ-Bochner integrable and $\int_A f \; d\mu = \int_A g \; d\mu$*

Proof: It is direct from Definitions (3.2.5) and (3.2.7).

Proposition 3.2.5. *Let $f : A \to X$,(where $(A, \mathfrak{A}, \mu)$ is a σ-finite measure space and X is Banach), be a μ-Bochner integrable function. Then, for each $x^* \in X^*$ it holds that*

$$x^*(\int_A f \; d\mu) = \int_A (x^* \circ f) \; d\mu.$$

Proof: Make use of the standard machinery, i.e prove the above identity for μ-simple functions and then the general case follows by approximation.

A practical necessary and sufficient condition for Bochner-integrability is given by the following proposition:

Proposition 3.2.6. *Let $(A, \mathfrak{A}, \mu)$ be a σ-finite measure space and X a Banach space. A strongly μ-measurable function $f : A \to X$ is μ-Bochner integrable if and only if:*

$$\int_A \| f \| \; d\mu < \infty.$$

Proof:

" $\Rightarrow$ " Let f be μ-Bochner integrable and consider a sequence $(f_n)_n$ of μ-simple functions which satisfies the two conditions of Definition (3.2.5). Obviously, for each $n \in \mathbb{N}$, we have that $\int_A \| f_n \| \; d\mu < \infty$. In addition, for a fixed $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$, such that $\int_A \| f_n - f \| \; d\mu < \varepsilon$, for each $n \geq n_0$. So we have,

$$\int_A \| f \| \; d\mu \leq \int_A \| f_{n_0} - f \| \; d\mu + \int_A \| f_{n_0} \| \; d\mu < \infty.$$

" $\Leftarrow$ " Let $(f_n)_n$ be a sequence of μ-simple functions such that $\lim_{n\to\infty} f_n(a) = f(a)$, for each $a \in A \backslash N$, where $N \in \mathfrak{A}$, $\mu(N) = 0$. For each $n \in \mathbb{N}$, define the function

$$g_n := \mathbb{I}_{[\| f_n \| \leq 2 \| f \|]} f_n.$$

Then, $(g_n)_n$ is a sequence of μ-simple functions which converges to f μ-almost everywhere. Indeed, for a fixed $a \in A \backslash N$, there exists $n_0 \in \mathbb{N}$, such that

$$\| f_n(a) \| - \| f(a) \| \leq \| f_n(a) - f(a) \| \leq \| f(a) \| , \quad \text{for each } n \geq n_0.$$

Therefore, for each $n \geq n_0$, we have $g_n(a) = f_n(a)$ and the result follows.
On the other hand, for each $n \in \mathbb{N}$ it holds that

$$\| g_n - f \| \leq \| g_n \| + \| f \| \leq 2 \| f \| + \| f \| = 3 \| f \| .$$

Therefore, by the scalar Dominated Convergence Theorem we get that

$$\lim_{n\to\infty} \int_A \| g_n - f \| \; d\mu = 0.$$

Proposition 3.2.7. *If $f : A \to X$ is μ-Bochner integrable then it holds that $\left\| \int_A f \, d\mu \right\| \leq \int_A \| f \| \, d\mu$.*

Proof: Use the standard machinery. The above inequality is true for μ-simple functions. The general case follows by approximation.

Remark 3.2.5. If $f : A \to X$ is μ-Bochner integrable and $B \in \mathfrak{A}$, then the function $\mathbb{I}_B f$ is μ-Bochner integrable and the function $f|_B$ is $\mu|_B$-Bochner integrable and it holds that $\int_A \mathbb{I}_B f \, d\mu = \int_B f|_B \, d\mu|_B$.

Definition 3.2.8. *If $f : A \to X$ is a μ-Bochner integrable function and $B \in \mathfrak{A}$, then we define*

$$\int_B f \, d\mu := \int_A \mathbb{I}_B f \, d\mu = \int_B f|_B \, d\mu|_B.$$

Proposition 3.2.8. *If $f : A \to X$ is a μ-Bochner integrable function and $\mu(A) = 1$, then*

$$\int_A f \, d\mu \in \overline{conv\{f(a) : a \in A\}}.$$

Proof: Suppose that $\int_A f \, d\mu \notin \overline{conv\{f(a) : \ a \in A\}}$. Then, by the Hahn-Banach Separation Theorem ([BR] Th. I.7) we can choose $x^* \in X^*$ and $\delta \in \mathbb{R}$ such that:

$$Re < x^*, \int_A f \, d\mu > < \delta \leq Re < x^*, f(a) >, \text{ for each } a \in A.$$

Equivalently, by Proposition (3.2.5)

$$\int_A Re < x^*, f > \, d\mu \ < \delta \leq Re < x^*, f(a) >, \text{ for each } a \in A.$$

By integration, since $\mu(A) = 1$, we get

$$\int_A Re < x^*, f > \, d\mu \ < \delta \leq \int_A Re < x^*, f > \, d\mu,$$

which leads us to a contradiction.

Proposition 3.2.9 (Dominated Convergence Theorem)**.** *Let $(f_n)_n$ be a sequence of μ-Bochner integrable functions from a σ-finite measure space $(A, \mathfrak{A}, \mu)$ to a Banach space X such that*

1. $\lim_{n\to\infty} f_n = f$, *$\mu$-almost everywhere.*
2. *there exists a μ-Bochner integrable function $g : A \to \mathbb{K}$, such that $\| f_n \| \leq |g|$, μ-almost everywhere.*

Then, f is μ-Bochner integrable and

$$\lim_{n\to\infty} \int_A \| f_n - f \| \, d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$

Proof: Because $\| f \| \leq |g|$ μ-almost everywhere we deduce from Proposition (3.2.6), that f is $\mu-$Bochner integrable. In addition, since $\| f_n - f \| \leq 2|g|$, μ-almost everywhere, we can apply the scale Dominated convergence theorem to get the first identity. Moreover,

$$\begin{aligned} \left\| \int_A f_n \, d\mu - \int_A f \, d\mu \right\| &= \left\| \int_A (f_n - f) \, d\mu \right\| \\ &\leq \int_A \| f_n - f \| \, d\mu \to 0, \text{ as } n \to \infty. \end{aligned}$$

Proposition 3.2.10. *Let $(A, \mathfrak{A}, \mu)$ be a σ-finite measure space and X, Y two Banach spaces. If $f : A \to X$ is a μ-Bochner integrable function and $T \in B(X, Y)$ is a linear and bounded operator, then $Tf : A \to Y$ is μ-Bochner integrable and it holds that*

$$T \int_A f \, d\mu = \int_A Tf \, d\mu.$$

Proof: It is a direct consequence of Definition (3.2.5) and the continuity of T.

Theorem 3.2.3 (Hille). *Let $f : A \to X$ be a μ-Bochner integrable function and $T : X \supset D(T) \to Y$, where Y is Banach, a closed linear operator. If f takes μ-almost all its values in $D(T)$ and the μ-almost everywhere defined function $Tf : A \to Y$, is μ-Bochner integrable, then*

$$\int_A f \, d\mu \in D(T)$$

and

$$T \int_A f \, d\mu = \int_A Tf \, d\mu.$$

Proof: Consider the Banach space $X \times Y$, equipped with the norm $\| (x, y) \|_{X \times Y} = \| x \|_X + \| y \|_Y$. Since, T is a closed operator the graph G_T of T is a closed subset in $X \times Y$. Consider the μ-almost everywhere defined function $g : A \to G_T \subset X \times Y$, $g = (f, Tf)$. Then, it is easy to see that g is strongly μ-measurable and since

$$\int_A \| g \| \, d\mu = \int_A \| f \| \, d\mu + \int_A \| Tf \| \, d\mu < \infty,$$

we conclude, by Proposition (3.2.6) that g is μ-Bochner integrable. Furthermore, since G_T is closed, we have that $\int_A g \, d\mu \in G_T$. But applying Proposition (3.2.10), for the two projections of $X \times Y$ to X and Y respectively, we get

$$\int_A g \, d\mu = (\int_A f \, d\mu, \int_A Tf \, d\mu) \in G_T.$$

Therefore, $\int_A f \, d\mu \in D(T)$ and $T \int_A f \, d\mu = \int_A Tf \, d\mu$.

Theorem 3.2.4 (Pettis). *Let $(A, \mathfrak{A}, \mu)$ be a finite measure space and consider a fixed $1 < p < \infty$. If $f : A \to X$ is a strongly μ-measurable function such that $< x^*, f > \in L^p(A)$, for each $x^* \in X^*$, then there exists a unique $x_f \in X$ such that*

$$< x^*, x_f > = \int_A < x^*, f > \, d\mu, \text{ for all } x^* \in X^*.$$

In this case, the element x_f is said to be the ***Pettis integral*** *of f with respect to μ.*

Proof: Without loss of generality (Pr. (3.2.3)), we assume that f is strongly $\mathfrak{A}$-measurable, thus $\mathfrak{A}$-measurable (Pr. (3.2.1)). For each $n \in \mathbb{N}$, set $A_n = [\| f \| \leq n] \in \mathfrak{A}$. Then, by Proposition (3.2.6) the Bochner integral $\int_{A_n} f \, d\mu$ exists in X, for each $n \in \mathbb{N}$. Moreover, the linear operator $S : X^* \to L^p(A)$, $Sx^* = < x^*, f >$ is closed, so by the closed graph theorem is bounded. We will show that the limit $\lim_{n\to\infty} \int_{A_n} f \, d\mu$ exists in X. Since X is Banach, it is enough to show $(\int_{A_n} f \, d\mu)_n$ is Cauchy. To this aim, by using Hölder's inequality, for each, $x^* \in X^*$ and positive integers $m > n$ we have,

$$\begin{aligned} \left| < x^*, \int_{A_m \setminus A_n} f \, d\mu > \right| &\leq \mu(A_m \setminus A_n)^{\frac{1}{q}} (\int_A | < x^*, f > |^p \, d\mu)^{\frac{1}{p}} \\ &= \mu(A_m \setminus A_n)^{\frac{1}{q}} \| Sx^* \|_{L^p} \\ &\leq \mu(A_m \setminus A_n)^{\frac{1}{q}} \| S \| \| x^* \| , \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore,

$$\left\| \int_{A_m \setminus A_n} f \, d\mu \right\| \leq \mu(A_m \setminus A_n)^{\frac{1}{q}} \| S \| \to 0, \text{ as } n, m \to \infty.$$

So, the limit

$$x_f := \lim_{n\to\infty} \int_{A_n} f \; d\mu \in X.$$

Moreover, for each $x^* \in X^*$

$$< x^*, x_f >= \lim_{n\to\infty} \int_{A_n} < x^*, f > \; d\mu = \int_A < x^*, f > \; d\mu.$$

Finally the uniqueness of x_f is obvious from the Hahn-Banach Theorem.

Definition 3.2.9. *Let $(A, \mathfrak{A}, \mu)$ be a σ-finite measure space, X a Banach space and $1 \leq p < \infty$. We define the space $\mathbf{L^p(A, X)}$ as the linear space of (equivalence classes of) strongly μ-measurable functions $f : A \to X$, which satisfy*

$$\int_A \| f \|^p \; d\mu < \infty.$$

This space equipped with the norm

$$\| f \|_{L^p(A,X)} = (\int_A \| f \|^p \; d\mu)^{\frac{1}{p}}$$

is a Banach space.

Definition 3.2.10. *Let $(A, \mathfrak{A}, \mu)$ be a σ-finite measure space, X a Banach space. We define the space $\mathbf{L^\infty(A, X)}$ as the linear space of (equivalence classes of) strongly μ-measurable functions $f : A \to X$, which satisfy the condition*

$$\text{there exists } r \geq 0 \text{ such that } \mu\{\| f \| > r\} = 0.$$

This space equipped with the norm

$$\| f \|_{L^\infty(A,X)} = \inf\{r \geq 0 : \; \mu\{\| f \| > r\} = 0\}$$

is a Banach space.

For further information on the above spaces of functions see [DS I].

3.3 Random variables in Banach spaces

In this section we study random variables with values in Banach spaces. We begin with the definition of a Banach valued random variable X on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ as a strongly $\mathbb{P}$-measurable function. By virtue of Proposition (3.2.3), this automatically allows us to assume that X is $\mathfrak{F}$-measurable choosing an $\mathfrak{F}$-measurable $\mathbb{P}$-version $\hat{X}$ of X when necessary. This leads us naturally to the extension of the definition of the distribution of a random variable to the Banach space valued setting. We continue with the notions of the independency and the Fourier transform of a probability measure and of a random variable, which are very important for the forthcoming material. The section ends with the celebrated Itô-Nisio theorem which gives equivalent types of convergence of sums of independent and symmetric random variables.
From now on, we consider that all the vector spaces are real.

3.3.1 Random variables

Definition 3.3.1. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and E a Banach space. A function $X : \Omega \to E$ is said to be a **random variable** if-f it is strongly $\mathbb{P}$-measurable.*

Definition 3.3.2. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and E a Banach space. For a $\mathbb{P}$-Bochner integrable random variable $X : \Omega \to E$ we define the **mean value** or **expectation** of X as the integral*

$$\mathbb{E}[X] := \int_\Omega X \; d\mathbb{P}.$$

Remark 3.3.1. It is easy to check the following aseertions. If $X : \Omega \to E$, $Y : \Omega \to E$ are two E-valued random variables and $\lambda \in \mathbb{R}$, then $X + Y$ and λX are E-valued random variables and (X, Y) is an $E \times E$-valued random variable. Furthermore, if X, Y are Bochner integrable, then the same is true for (X, Y), $X + Y$ and λX and $\mathbb{E}(X, Y) = (\mathbb{E}X, \mathbb{E}Y)$, $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ and $\mathbb{E}[\lambda X] = \lambda\mathbb{E}[X]$. Moreover, if ξ is a real random variable on $(\Omega, \mathfrak{F}, \mathbb{P})$ and $x \in E$, then $\xi \cdot x$ is an E-valued random variable. If in addition ξ is integrable, then $\xi \cdot x$ is also integrable and $\mathbb{E}[\xi] \cdot x = \mathbb{E}[\xi \cdot x]$.

Remark 3.3.2. Let $X : \Omega \to E$ be a random variable. Then by virtue of Proposition (3.2.3) there exists a $\mathbb{P}$-version $\tilde{X}$ of X, which is strongly $\mathfrak{F}$-measurable. By Proposition (3.2.1) it follows that $\tilde{X}$ is also $\mathfrak{F}$-measurable. Moreover, it is easy to verify that for two strongly $\mathfrak{F}$-measurable versions $\tilde{X}, \tilde{Y}$ of X it holds that

$$\mathbb{P}[\tilde{X} \in B] = \mathbb{P}[\tilde{Y} \in B], \quad \text{for each} \quad B \in \mathfrak{B}(E).$$

Definition 3.3.3. *Let $X : \Omega \to E$ be a random variable and $\tilde{X}$ a strongly $\mathfrak{F}$-measurable version of X. We define the set function $\mathbb{P}_X : \mathfrak{B}(E) \to [0, 1]$,*

$$\mathbb{P}_X(B) := \mathbb{P}[X \in B] := \mathbb{P}[\tilde{X} \in B].$$

Then $(E, \mathfrak{B}(E), \mathbb{P}_X)$ is a probability space and the Borel probability measure $\mathbb{P}_X$ is called the ***distribution*** *of X.*

Definition 3.3.4. *Two E-valued random variables which are not necessarily defined in the same probability space but they have the same distribution are called* ***identically distributed****.*

Proposition 3.3.1 (Change of Variable)**.** *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, E, F two Banach spaces, $X : \Omega \to E$ a random variable and $g : E \to F$ a continuous function. Then, it holds that*

$$\int_\Omega g(X)\ d\mathbb{P} = \int_E g(x)\ d\mathbb{P}_X(x).$$

whenever at least one of the above integrals is defined.

Proof: Use the standard machinery, i.e prove the above for simple functions and then the general case follows by approximation.

Corollary 3.3.1. *Let X, Y be two E-valued random variables that are identically distributed and $p \geq 1$. Then,*

$$\mathbb{E} \parallel X \parallel^p = \mathbb{E} \parallel Y \parallel^p ,$$

Proposition 3.3.2. *Let $X : \Omega \to E$ be a random variable. Then, it is* ***tight****, i.e for each $\varepsilon > 0$, there exists a compact subset $K \subset E$, such that $\mathbb{P}[X \notin K] < \varepsilon$.*

Proof: Without loss of generality we assume that E is separable. Consider a dense sequence $\{x_n\}_{n=1}^\infty$ in E. Then, for each $k \in \mathbb{N}$ we have that $E = \bigcup_{n=1}^{\infty} \overline{B}(x_n, \frac{1}{k})$. For each $n \in \mathbb{N}$, set $B_n := \bigcup_{i=1}^{n} \overline{B}(x_i, \frac{1}{k})$. Then, it holds that

$$1 = \mathbb{P}[\bigcup_{n=1}^{\infty} [X \in B_n]] = \lim_{n\to\infty} \mathbb{P}[X \in B_n].$$

Therefore, for a fixed $\varepsilon > 0$, there exists $N_k \in \mathbb{N}$, such that

$$|\mathbb{P}[X \in B_{N_k}] - 1| < \frac{\varepsilon}{2^k},$$

which implies that

$$P[X \in \bigcup_{n=1}^{N_k} \overline{B}(x_n, \frac{1}{k})] > 1 - \frac{\varepsilon}{2^k}.$$

Observe now that the set $K := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{N_k} \overline{B}(x_n, \frac{1}{k})$ is closed and totally bounded, thus compact, since E is complete. Moreover,

$$\mathbb{P}[X \notin K] \leq \sum_{k=1}^{\infty} \mathbb{P}[X \notin \bigcup_{n=1}^{N_k} \overline{B}(x_n, \frac{1}{k})] < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Definition 3.3.5. *A family $\mathfrak{X}$ of E-valued random variables on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is said to be **uniformly tight** if-f for each $\varepsilon > 0$, there exists a compact $K \subset E$ such that $\mathbb{P}[X \notin K] < \varepsilon$, for each $X \in \mathfrak{X}$.*

Lemma 3.3.1. *If $\mathfrak{X}$ is a uniformly tight family, then $\mathfrak{X} - \mathfrak{X} = \{X_1 - X_2 : \ X_1, X_2 \in \mathfrak{X}\}$ is uniformly tight.*

Proof: Let $\varepsilon > 0$. Consider a compact set $K \subset E$ such that $\mathbb{P}[X \notin K] < \frac{\varepsilon}{2}$, for each $X \in \mathfrak{X}$. Then, the set $L = \{x_1 - x_2 : \ x_1, x_2 \in K\}$ is compact as the image of the continuous function $K \times K \ni (x, y) \to x - y \in K$. Moreover, for each $X_1, X_2 \in \mathfrak{X}$, we have

$$\mathbb{P}[(X_1 - X_2) \notin L] \leq \mathbb{P}[X_1 \notin K] + \mathbb{P}[X_2 \notin K] < \varepsilon.$$

3.3.2 Convergence in Probability

Definition 3.3.6. *Let $(X_n)_n$ be a sequence of E-valued random variables, (where E is Banach), on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We say that*

- *$(X_n)_n$ **converges in Probability** to the random variable X, if-f for each $r > 0$:*

$$\lim_{n\to\infty} \mathbb{P}[\| X_n - X \| > r] = 0.$$

- *$(X_n)_n$ **converges in** L^p to the random variable X, (for a fixed $1 \leq p < \infty$), if-f $\mathbb{E} \| X_n \|^p < \infty$, $\mathbb{E} \| X \|^p < \infty$ and*

$$\lim_{n\to\infty} \mathbb{E} \| X_n - X \|^p = 0.$$

Lemma 3.3.2 (Chebyshev's Inequality)**.** *For a random variable $X : \Omega \to E$, $X \in L^p$, for some fixed $1 \leq p < \infty$, it holds that, for each $r > 0$*

$$\mathbb{P}[\| X \| \geq r] \leq \frac{1}{r^p} \mathbb{E} \| X \|^p .$$

Proof:

$$\mathbb{E}(\| X \|^p) \geq \int_{[\| X \|^p \geq r^p]} \| X \|^p \ d\mathbb{P} \geq \int_{[\| X \|^p \geq r^p]} r^p \ d\mathbb{P} = r^p \mathbb{P}[\| X \| \geq r].$$

Corollary 3.3.2. *If $\lim_{n\to\infty} X_n = X$, in L^p, for some $1 \leq p \leq \infty$, then $\lim_{n\to\infty} X_n = X$, in probability.*

Proof: Apply the Chebyshev's inequality.

Proposition 3.3.3. *If $X_n : (\Omega, \mathfrak{F}, \mathbb{P}) \to E$, $n \in \mathbb{N}$ converges in probability, then there exists an almost sure convergent subsequence,*

Proof: Assume that $\lim_{n\to\infty} X_n = X$ in probability. Then, we can choose an increasing sequence of indices $n_1 < n_2 < \ldots$, such that

$$\mathbb{P}[\| X_{n_k} - X \| > \frac{1}{k}] < \frac{1}{2^k}, \text{ for each } k \in \mathbb{N}.$$

Set

$$A_k := [\| X_{n_k} - X \| > \frac{1}{k}], \ k \in \mathbb{N},$$

Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, by the first Borel-Cantelli Lemma (e.g see [SP] pg 9) we have that for

$$N := \{\omega \in \Omega : \omega \in A_k \ \ i.o\} \in \mathfrak{F},$$

it is $\mathbb{P}(N) = 0$. Observe now that for each $\omega \notin N$, there exists $k_0 \in \mathbb{N}$, such that

$$\| X_{n_k}(\omega) - X(\omega) \| \leq \frac{1}{k}, \quad \text{for each } k \geq k_0.$$

Therefore, for each $\omega \notin N$, we have $\lim_{k\to\infty} X_{n_k}(\omega) = X(\omega)$. Thus, $\lim_{k\to\infty} X_{n_k} = X$, almost surely.

3.3.3 Independency and Fourier Transforms

Definition 3.3.7. *Let E be a Banach space and μ a Borel probability measure on E. The* ***Fourier transform*** *of μ, is the function $\widehat{\mu}: E^* \to \mathbb{C}$*

$$\widehat{\mu}(x^*) := \int_E e^{i<x^*,x>} \, d\mu(x).$$

Definition 3.3.8. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, and E a Banach space. The* ***Fourier transform of a random variable*** *$X : \Omega \to E$ is the Fourier transform of its distribution $\mathbb{P}_X : \mathfrak{B}(E) \to [0,1]$. This is*

$$\widehat{X}(x^*) := \int_E e^{i<x^*,x>} \, d\mathbb{P}_X(x) = \int_\Omega e^{i<x^*,X>} \, d\mathbb{P} = \mathbb{E}[e^{i<x^*,X>}].$$

Theorem 3.3.1 (Uniqueness of the Fourier Transform)**.** *Let X_1, X_2 be two E-valued random variables such that $\widehat{X_1}(x^*) = \widehat{X_2}(x^*)$, for each $x^* \in E^*$. Then, they are identically distributed. In other words if $\widehat{\mathbb{P}}_X = \widehat{\mathbb{P}}_Y$, then $\mathbb{P}_X = \mathbb{P}_Y$.*

For a proof of this result see [NR] pg 19.

Definition 3.3.9. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, I a parameter set and $(X_i)_{i\in I}$ a family of random variables on $(\Omega, \mathfrak{F}, \mathbb{P})$ such that, for each $i \in I$, the r.v X_i takes its values in the Banach space E_i. The family $(X_i)_{i\in I}$ is said to be* ***independent*** *if-f for each $N \in \mathbb{N}$ and all choices of distinct indices $i_1, i_2, \ldots, i_N$ and Borel sets $B_1 \in \mathfrak{B}(E_{i_1}), B_2 \in \mathfrak{B}(E_{i_2}), \ldots, B_N \in \mathfrak{B}(E_{i_N})$, it holds that*

$$\mathbb{P}(\bigcap_{s=1}^{N} [X_{i_s} \in B_s]) = \prod_{s=1}^{N} \mathbb{P}[X_{i_s} \in B_s].$$

Proposition 3.3.4. *Let $(X_1, X_2, \ldots, X_N)$ be random variables on $(\Omega, \mathfrak{F}, \mathbb{P})$ with values in the Banach spaces $E_1, E_2, \ldots, E_N$ respectively. Then, $(X_1, X_2, \ldots, X_N)$ is independent if and only if*

$$\mathbb{P}_{(X_1,X_2,\ldots,X_N)} = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2} \otimes \ldots \otimes \mathbb{P}_{X_N}.$$

Proof: If the random variables are independent, then the above probability measures are equal in the class of measurable rectangles $\mathfrak{C} = \{B_1 \times B_2 \times \ldots \times B_N : B_i \in \mathfrak{B}(E_i), i = 1, \ldots, N\}$, which is a π-system. Therefore, by Dynkin's Lemma we conclude that the two probability measures are equal in $\sigma(\mathfrak{C}) = \mathfrak{B}(E_1) \otimes \mathfrak{B}(E_2) \otimes \ldots \otimes \mathfrak{B}(E_N)$, as desired.

For a detailed discussion of π-systems, λ-systems and Dynkin's Lemma see [BL] ch.1 section 3, while for a beautiful presentation of product measure spaces and σ-algebras see [KM] ch. 9.
In the following Proposition the random variables are defined on $(\Omega, \mathfrak{F}, \mathbb{P})$ and are E-valued.

Proposition 3.3.5. *If $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$ in probability and for each $n \in \mathbb{N}$ X_n is independent of Y_n, then X, Y are independent.*

Proof: By virtue of Proposition (3.3.3), without loss of generality we assume that $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$ $\mathbb{P}$-a.s. Then, for each $x^* \in E^*$ and $y^* \in E^*$ we have

$$\begin{aligned}
\widehat{\mathbb{P}}_{(X,Y)}(x^*, y^*) &= \mathbb{E}\left\{e^{i(<x^*,X>+<y^*,Y>)}\right\} \\
&= \lim_{n\to\infty} \mathbb{E}\left\{e^{i<x^*,X_n>} e^{i<y^*,Y_n>}\right\} \\
&= \lim_{n\to\infty} \mathbb{E}\left\{e^{i<x^*,X_n>}\right\} \mathbb{E}\left\{e^{i<y^*,Y_n>}\right\} \\
&= \mathbb{E}\left\{e^{i<x^*,X>}\right\} \mathbb{E}\left\{e^{i<y^*,Y>}\right\} \\
&= \widehat{\mathbb{P}}_X(x^*) \cdot \widehat{\mathbb{P}}_Y(y^*) \;=\; \widehat{\mathbb{P}_X \otimes \mathbb{P}_Y}(x^*, y^*),
\end{aligned}$$

where in the second and fourth line we used the scalar Dominated Convergence Theorem, in the third line we used the independency of (X_n, Y_n) and in the last line the Fubini's Theorem. So, we have concluded that $\widehat{\mathbb{P}}_{(X,Y)} = \widehat{\mathbb{P}_X \otimes \mathbb{P}_Y}$. Therefore, by the uniqueness of the Fourier transform (Th. (3.3.1)) we get that $\mathbb{P}_{(X,Y)} = \mathbb{P}_X \otimes \mathbb{P}_Y$. By virtue of Proposition (3.3.4), the random variables X, Y, are independent.

Definition 3.3.10. *A random variable $X : \Omega \to E$ is said to be* ***symmetric*** *if-f the r.v X and $-X$ are identically distributed.*

Remark 3.3.3. If X is an E-valued symmetric random variable and $\lambda \in \mathbb{R}$, then λX is also an E-valued symmetric random variable. Moreover, if ξ is a real symmetric random variable and $x \in E$, then $\xi \cdot x$ is an E-valued symmetric random variable. Indeed, for $B \in \mathfrak{B}(E)$

$$\mathbb{P}(\{\omega \in \Omega : \ \xi(\omega)x \in B\}) = \mathbb{P}(\{\omega \in \Omega : \ \xi(\omega) \in f^{-1}(B)\}) = \mathbb{P}(\{\omega \in \Omega : \ -\xi(\omega) \in f^{-1}(B)\}) =$$
$$= \mathbb{P}(\{\omega \in \Omega : \ -\xi(\omega)x \in B\}),$$

where $f : \mathbb{R} \to E$, $f(a) = a \cdot x$ is Borel measurable.

Proposition 3.3.6. *Let $X, Y : \Omega \to E$, be two random variables. If X is symmetric and independent of Y, then for each $1 \le p < \infty$, we have:*

$$\mathbb{E} \parallel X \parallel^p \le \mathbb{E} \parallel X + Y \parallel^p$$

Proof: The random variables $X + Y$ and $-X + Y$ are identically distributed. This is because the random vectors (X, Y) and $(-X, Y)$ are identically distributed, since their distributions are equal in the π-system of measurable rectangles. Therefore we have,

$$\begin{aligned}
(\mathbb{E} \parallel X \parallel^p)^{\frac{1}{p}} &= \frac{1}{2}(\mathbb{E} \parallel (X+Y) + (X-Y) \parallel^p)^{\frac{1}{p}} \\
&\le \frac{1}{2}(\mathbb{E} \parallel X+Y \parallel^p)^{\frac{1}{p}} + \frac{1}{2}(\mathbb{E} \parallel X-Y \parallel^p)^{\frac{1}{p}} \\
&= (\mathbb{E} \parallel X+Y \parallel^p)^{\frac{1}{p}}
\end{aligned}$$

Lemma 3.3.3 (Lévy's Inequality). *Let $X_1, \ldots, X_n$ be independent and symmetric random variables defined on $(\Omega, \mathfrak{F}, \mathbb{P})$ and with values in E and set $S_k = \sum_{i=1}^k X_i$, $k = 1, \ldots, n$. Then, for each $r \ge 0$, we have:*

$$\mathbb{P}\left\{\max_{1 \le k \le n} \parallel S_k \parallel > r\right\} \le 2\mathbb{P}\left\{\parallel S_k \parallel > r\right\}.$$

Proof: For each $k = 1, \ldots, n$, we set

$$A_k := \{\parallel S_1 \parallel \le r, \ldots, \parallel S_{k-1} \parallel \le r, \parallel S_k \parallel > r\}.$$

Observe that $A_l \cap A_m = \emptyset$, for each $l \ne m$ and $\cup_{k=1}^n A_k = \{\max_{1 \le k \le n} \parallel S_k \parallel > r\}$. Furthermore, for each $1 \le k \le n$ we have,

$$S_k = \frac{1}{2}(S_n + (2S_k - S_n)).$$

Therefore,

$$\{\parallel S_k \parallel > r\} \subset \{\parallel S_n \parallel > r\} \cup \{\parallel 2S_k - S_n \parallel > r\}.$$

Moreover, for a fixed $k \in \{1, \ldots, n\}$ it is easy to verify that $(X_1, \ldots, X_n)$ and $(X_1, \ldots, X_k, -X_{k+1}, \ldots, -X_n)$ are identically distributed. To see this use the symmetry of each X_i and Proposition (3.3.4). Furthermore, since

$$S_n = S_k + X_{k+1} + \ldots + X_n \quad \text{and } 2S_k - S_n = S_k - X_{k+1} - \ldots X_n,$$

we can conclude that $(X_1, \ldots, X_k, S_n)$ and $(X_1, \ldots, X_k, 2S_k - S_n)$ are identically distributed. To see the last statement, observe first of all that since $(X_k, X_{k+1}, \ldots, X_n)$ and $(-X_k, -X_{k+1}, \ldots, -X_n)$ are identically distributed, then $f((X_k, X_{k+1}, \ldots, X_n))$ and $f(-X_k, -X_{k+1}, \ldots, -X_n)$, where $f : E^n \to E$ is the continuous function $f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$, are identically distributed. But these are $S_n - S_k$ and $2S_k - S_n - S_k$. Now, arguing as before we get that $(X_1, \ldots, X_k, S_n - S_k)$ and $(X_1, \ldots, X_k, 2S_k - S_n - S_k)$ are identically distributed and the desired result follows by making a continuous transformation. After all that, we have,

$$\mathbb{P}(A_k) \le \mathbb{P}(A_k \cap \{\parallel S_n \parallel > r\}) + \mathbb{P}(A_k \cap \{\parallel 2S_k - S_n \parallel > r\}) = 2\mathbb{P}(A_k \cap \{\parallel S_n \parallel > r\}).$$

Therefore,

$$\mathbb{P}(A) = \sum_{k=1}^n \mathbb{P}(A_k) \le 2\sum_{k=1}^n \mathbb{P}(A_k \cap \{\parallel S_n \parallel > r\}) = 2\mathbb{P}\{\parallel S_n \parallel > r\}.$$

We end the section with the celebrated Itô-Nisio theorem which was proved in their paper [I-N].

Theorem 3.3.2 (Itô-Nisio Theorem). *Let $X_n : \Omega \to E, n \in \mathbb{N}$ be independent and symmetric random variables and set $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$. Then, the following statements are equivalent:*

(1) $\lim_{n\to\infty} < x^*, S_n > = < x^*, S > \mathbb{P}$*-a.s, for each* $x^* \in E^*$.

(2) $\lim_{n\to\infty} < x^*, S_n > = < x^*, S >$ *in probability, for each* $x^* \in E^*$.

(3) $\lim_{n\to\infty} S_n = S$ $\mathbb{P}$*-a.s*

(4) $\lim_{n\to\infty} S_n = S$ *in probability.*

where $S : \Omega \to E$ is a random variable.
Furthermore, if the following conditions are true and $\mathbb{E} \parallel S \parallel^p < \infty$, for some $1 \le p < \infty$, then

$$\lim_{n\to\infty} \mathbb{E} \parallel S_n - S \parallel^p = 0.$$

3.4 Gaussian measures in Hilbert spaces

Let $(U, <,>)$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $\mathfrak{B}(U)$ its Borel σ-algebra. Let μ be a probability measure on $(U, \mathfrak{B}(U))$. By a **real random variable** on the probability space $(U, \mathfrak{B}(U), \mu)$ we understand a measurable function $X : (U, \mathfrak{B}(U)) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$. For each $v \in U$ define $f_v \in U^*$ by $f_v(u) = < u, v >_U$, $u \in U$.

Definition 3.4.1. *A probability measure μ on $(U, \mathfrak{B}(U))$ is said to be **Gaussian measure** if-f for each $v \in U$, the element $f_v \in U^*$ has Gaussian law, as a real random variable on $(U, \mathfrak{B}(U), \mu)$. That is, for each $v \in U$ there exist $\sigma_v \ge 0$ and $m_v \in \mathbb{R}$ such that, if $\sigma_v > 0$*

$$\mu_v(B) := \mu_{f_v}(B) = \mu(\{u \in U : f_v(u) \in B\}) = \frac{1}{\sqrt{2\pi{\sigma_v}^2}} \int_B e^{-\frac{(t-m_v)^2}{2{\sigma_v}^2}}\, dt, \quad \forall B \in \mathfrak{B}(\mathbb{R}).$$

If $\sigma_v = 0$, then $\mu_v = \delta_{m_v}$, where δ_{m_v} is the Dirac measure concentrated at m_v.

In order to give a characterization of Gaussian measures on Hilbert spaces we need the following Lemma

Lemma 3.4.1. *Let μ be a probability measure on $(U, \mathfrak{B}(U))$ and $k \in \mathbb{N}$ such that*

$$\int_U | < z, x > |^k \, d\mu(x) < \infty, \quad \text{for all } z \in U.$$

Then, there exists a constant $C(\mu, k) > 0$ such that for all $h_1, h_2, \ldots, h_k \in U$

$$\int_U | < h_1, x > \ldots < h_k, x > | \, d\mu(x) \le C(\mu, k) \parallel h_1 \parallel \ldots \parallel h_k \parallel .$$

This means that the symmetric k-form

$$(h_1, \ldots, h_k) \to \int_U < h_1, x > \ldots < h_k, x > \, d\mu(x)$$

is continuous.

Proof: For each $n \in \mathbb{N}$ set

$$U_n = \{z \in U : \int_U | < z, x > |^k \, d\mu(x) \le n\}.$$

Obviously $U = \cup_{n=1}^{\infty} U_n$. Moreover, for each $n \in \mathbb{N}$, U_n is closed. Indeed if $\{z_j\}_{j=1}^{\infty} \subset U_n$ such that $\lim_{j\to\infty} z_j = z$. Then, $\lim_{j\to\infty} | < z_j, x > |^k = | < z, x > |^k$. Thus, by Fatou's Lemma

$$\int_U | < z, x > |^k \, d\mu(x) \le \liminf_{j\to\infty} \int_U | < z_j, x > |^k \, d\mu(x) \le n.$$

Thus, $z \in U_n$. Now, since U is complete by virtue of Baire's Category Theorem, there exists $n_0 \in \mathbb{N}$ such that U_{n_0} is not nowhere dense. This means, that there exists $r_0 > 0$ and $z_0 \in U_{n_0}$ such that $B(z_0, r_0) \subset \overline{U}_{n_0} \Rightarrow \overline{B}(z_0, r_0) \subset U_{n_0}$. Therefore, for each $y \in \overline{B}(0, r_0)$ we have $z_0, z_0 + y \in \overline{B}(z_0, r_0) \subset U_{n_0}$. Thus,

$$\int_U | < z_0 + y, x > |^k \, d\mu(x) \leq n_0$$

And for each $y \in \overline{B}(0, r_0)$ we have,

$$\int_U | < y, x > |^k \, d\mu(x) = \int_U | < z_0 + y, x > - < z_0, x > |^k \, d\mu(x)$$

$$\leq 2^k \int_U | < z_0 + y, x > |^k \, d\mu(x) + 2^k \int_U | < z_0, x > |^k \, d\mu(x) \leq 2^{k+1} n_0.$$

So, if $z \in U$ and $\| z \| = 1$, $y = r_0 z \in \overline{B}(0, r_0)$. Thus,

$$\int_U | < z, x > |^k \, d\mu(x) = r_0^{-k} \int_U | < y, x > |^k \, d\mu(x) \leq 2^{k+1} n_0 {r_0}^{-k}.$$

Therefore, for all $h_1, \ldots, h_k \in U \backslash \{0\}$, by Hölder's inequality we have,

$$\int_U \left| < \frac{h_1}{\| h_1 \|}, x > \ldots < \frac{h_k}{\| h_k \|}, x > \right| \, d\mu(x)$$

$$\leq \left(\int_U | < \frac{h_1}{\| h_1 \|}, x > |^k \, d\mu(x) \right)^{\frac{1}{k}} \ldots \left(\int_U | < \frac{h_k}{\| h_k \|}, x > |^k \, d\mu(x) \right)^{\frac{1}{k}}$$
$$\leq 2^{k+1} n_0 {r_0}^{-k}.$$

Theorem 3.4.1 (Characterization of Gaussian measures)**.** *A finite measure μ on $(U, \mathfrak{B}(U))$ is Gaussian if and only if*

$$\hat{\mu}(u) = \int_U e^{i<u,x>_U} \, d\mu(x) = e^{i<m,u>_U - \frac{1}{2}<Qu,u>_U}, \qquad u \in U,$$

where $m \in U$ and $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$. In this case we write $\mu = N(m, Q)$, m is called the ***mean*** *of μ and Q is called the* ***covariance operator*** *of μ. The probability measure μ is uniquely characterized by m and Q.*

Proof: (Sufficiency) Assume that

$$\hat{\mu}(u) = \int_U e^{i<u,x>_U} \, d\mu(x) = e^{i<m,u>_U - \frac{1}{2}<Qu,u>_U}, \qquad u \in U.$$

We will show that μ is Gaussian, i.e for each $v \in U$ the real-valued random variable $f_v \in U^*$ has Gaussian law. But, for each $t \in \mathbb{R}$

$$\hat{\mu}_{f_v}(t) = \int_U e^{itf_v(x)} \, d\mu(x) = \int_U e^{it<x,v>_U} \, d\mu(x)$$
$$= \int_U e^{i<tv,x>_U} \, d\mu(x) = \hat{\mu}(tv).$$

Thus,

$$\hat{\mu}_{f_v}(t) = e^{it<m,v> - \frac{1}{2}t^2 <Qv,v>}.$$

By the uniqueness of the Fourier transform of Borel probability measures (Theorem (3.3.1)),this means that f_v is a Gaussian real valued random variable with $m_v =< m, v > \in \mathbb{R}$ and ${\sigma_v}^2 =< Qv, v > \geq 0$.
(Necessity) Conversely, now assume that μ is a Gaussian measure on $(U, \mathfrak{B}(U))$. Since, for each $v \in U$, $f_v \in U^*$ is real valued Gaussian random variable, arguing as before we have

$$\hat{\mu}(v) = \hat{\mu}_{f_v}(1) = e^{im_v - \frac{1}{2}{\sigma_v}^2}.$$

So, it is enough to show that there exists $m \in U$ such that $m_v =< m, v >$, for each $v \in U$ and there exists $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$ such that ${\sigma_v}^2 =< Qv, v >$, for all $v \in U$. Note that

$$\int_U | < x, v > | \, d\mu(x) = \int_U |f_v| \, d\mu = \int_{\mathbb{R}} |t| \, d\mu_{f_v}(t) < \infty, \quad \text{for all} \quad v \in U$$

Thus, form Lemma (3.4.1) we have that the map

$$U \ni v \to \int_U < x, v > \; d\mu(x) \in \mathbb{R}.$$

is a linear and bounded functional on U. Thus, by Riesz Representation Theorem, there exist a $m \in U$ such that

$$< m, v >= \int_U < x, v > \; d\mu(x) = \mathbb{E}[f_v] = m_v, \quad \text{for all} \quad v \in U.$$

Moreover we have

$$\int_U | < x, v > |^2 \; d\mu(x) = \int_U |f_v|^2 \; d\mu = \int_{\mathbb{R}} t^2 \; d\mu_{f_v}(t) < \infty, \quad \text{for all } v \in U.$$

Therefore, again by Lemma (3.4.1), the symmetric bilinear form

$$(h_1, h_2) \to \int_U < x, h_1 >< x, h_2 > \; d\mu(x) - < m, h_1 >< m, h_2 >$$

is continuous. Thus there exists a symmetric $Q \in B(U)$ such that

$$< Qh_1, h_2 >= \int_U < x, h_1 >< x, h_2 > \; d\mu(x) - < m, h_1 >< m, h_2 >, \quad \text{for all} \quad h_1, h_2 \in U.$$

Therefore,

$$< Qv, v >= \int_U < x, v >^2 \; d\mu(x) - < m, v >^2 = \mathbb{E}[{f_v}^2] - \mathbb{E}[f_v]^2 = \sigma_v^2 \geq 0, \quad \text{for all} \quad v \in U.$$

It remains to show that $Tr(Q) < \infty$. Without loss of generality we may assume that $m = 0$. This is because the translated measure $\tilde{\mu}(A) = \mu(A + m)$, $A \in \mathfrak{B}(U)$ has zero mean and the same covariance operator with μ. Indeed it is an easy task to show that for an integrable random variable $f : U \to \mathbb{R}$ it holds that $\int_U f(x) \; d\tilde{\mu}(x) = \int_U f(x - m) \; d\mu(x)$. Therefore, for each $u \in U$ we have $\hat{\tilde{\mu}}(u) = e^{-\frac{1}{2} < Qu, u >}$.
So, we assume that $m = 0$ and thus we have

$$e^{-\frac{1}{2} < Qh, h >} = \int_U e^{i < h, x >} \; d\mu(x) = \int_U \cos < h, x > \; d\mu(x).$$

By using the inequality $1 - \cos x \leq \frac{1}{2} x^2$, we get that for a fixed $c > 0$

$$1 - e^{-\frac{1}{2} < Qh, h >} = \int_U (1 - \cos < h, x >) \; d\mu(x) \leq \frac{1}{2} \int_{\{\| x \| \leq c\}} | < h, x > |^2 \; d\mu(x) + 2\mu(\{x : \| x \| > c\}).$$

We define $Q_c \in B(U)$, $Q_c \geq 0$ by

$$< Q_c h_1, h_2 >= \int_{\{\| x \| \leq c\}} < h_1, x >< h_2, x > \; d\mu(x), \qquad h_1, h_2 \in U.$$

Observe that $Tr(Q_c) < \infty$. Indeed,

$$Tr(Q_c) = \sum_{k=1}^{\infty} < Q_c e_k, e_k >= \sum_{k=1}^{\infty} \int_{\{\| x \| \leq c\}} < e_k, x >^2 \; d\mu(x)$$

$$= \int_{\{\| x \| \leq c\}} \sum_{k=1}^{\infty} < e_k, x >^2 \; d\mu(x) = \int_{\{\| x \| \leq c\}} \| x \|^2 \; d\mu(x) \leq c^2 < \infty.$$

We will show that there exist $c > 0$ such that

$$< Qh, h > \leq \; 2\log 4 \; < Q_c h, h > , \quad \text{for all} \quad h \in U, \tag{3.4.1}$$

which implies that $Tr(Q) \leq 2\log 4 \; Tr(Q_c) < \infty$. Since $\lim_{c\to\infty} \mu(\{x \in U : \| x \| > c\}) = 0$, we can choose $c > 0$ such that $\mu(\{x \in U : \| x \| > c\}) \leq \frac{1}{8}$. And for this $c > 0$ choose $h \in U$ such that $< Q_c h, h > \leq 1$. Thus, we have $1 - e^{-\frac{1}{2}<Qh,h>} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, which implies that

$$< Qh, h > \leq 2 \; \log 4. \tag{3.4.2}$$

Now, for an arbitrary $h \in U$ such that $< Q_c h, h > \neq 0$ we replace h with $\frac{h}{\sqrt{<Q_c h,h>}}$ in (3.4.2) and we derive (3.4.1). On the other hand, if $< Q_c h, h >= 0$, then for each $n \in \mathbb{N}$ $< Q_c nh, nh >= 0 \leq 1$ and thus

$$< Qh, h > \leq n^{-2} \, 2 \; \log 4.$$

So $< Qh, h >= 0$, which means that in this case (3.4.1) is true.

Corollary 3.4.1. *Let μ be a Gaussian measure on $(U, \mathfrak{B}(U))$ with mean m and covariance operator Q. Then for each $u, v \in U$ we have*

(1) $< m, u >_U = m_u = \mathbb{E}[f_u] = \int_U < x, u > \; d\mu(x)$.

(2) $< Qu, v >_U = \int_U < x - m, u >_U < x - m, v >_U \; d\mu(x)$.

(3) $Tr(Q) = \int_U \| x - m \|_U^2 \; d\mu(x)$.

Proof:

(1) It is direct from the proof of the previous Theorem.

(2) Use (1) and do the computations to derive that $\int_U < x - m, u >_U < x - m, v >_U \; d\mu(x) = \int_U < x, u >< x, v > \; d\mu(x) - < m, u >< m, v >=< Qu, v >$.

(3) Use the Monotone convergence Theorem to interchange the sum and the integral and Parseval's equality.

Definition 3.4.2. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a p.s. A random variable $X : \Omega \to U$ is an $(\mathfrak{F}, \mathfrak{B}(U))$ measurable mappping. We say that X is* ***Gaussian random variable*** *if-f the distribution $\mathbb{P}_X$ is a Gaussian measure on $(U, \mathfrak{B}(U))$, i.e $\mathbb{P}_X = N(m, Q)$, for some $m \in U$ and $Q \in B(U)$, $Q \geq 0$, $Tr(Q) < \infty$. In this case m is called the* ***mean*** *of X and Q is called the* ***covariance operator*** *of X.*

Proposition 3.4.1. *If X is a U-valued Gaussian random variable with mean m and covariance operator Q, then for each $u, v \in U$*

(1) $\mathbb{E}[< X, u >_U] =< m, u >_U$

(2) $\mathbb{E}(< X - m, u >_U < X - m, v >_U) =< Qu, v >_U$

(3) $\mathbb{E}(\| X - m \|_U^2) = Tr(Q)$

(4) $\mathbb{E}[X] = m$ *and* $Qu = \mathbb{E}(< X - m, u > (X - m))$, *for all* $u \in U$

Proof: (1), (2), (3) follows from Corollary (3.4.1) by a change of variables. By Proposition (3.2.5) for $x^* = f_u$, observe that for each $u \in U$, $< \mathbb{E}[X], u >= \mathbb{E}[< X, u >] =< m, u >$ and thus $\mathbb{E}[X] = m$. Moreover for all $u, v \in U$ we have

$$< Qu, v >= \mathbb{E}\left(< X - m, u >< X - m, v >\right) = \mathbb{E}\left(<< X - m, u > \cdot (X - m), v >\right) =< \mathbb{E}\left(< X - m, u > (X - m)\right), v > .$$

Therefore, $Qu = \mathbb{E}(< X - m, u > (X - m))$, for all $u \in U$.

Remark 3.4.1. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and U a separable Hilbert space. A U-valued random variable $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to U$ is Gaussian if and only if for each $u \in U$, $< u, X >$ is an $\mathbb{R}$-valued Gaussian random variable. This is direct from Definition (3.4.2) and Definition (3.4.1).

Proposition 3.4.2. *Let $m \in U$, $Q \in B(U)$, $Q \geq 0$ with $Tr(Q) < \infty$. A U-valued random variable $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to U$ is Gaussian with $\mathbb{P}_X = N(m, Q)$ if and only if X can be represented as a $\mathbb{P}$-a.s convergent series*

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k e_k, \tag{3.4.3}$$

where (λ_k, e_k) are the eigenpairs of Q, and $\{\beta_k\}_{k=1}^{\infty}$ are independent real random variables with $\mathbb{P}_{\beta_k} = N(0,1)$, when $\lambda_k > 0$ and $\beta_k = 0$, when $\lambda_k = 0$. Moreover, the series in (3.4.3) is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)$.

Proof: Assume that X is a Gaussian U-valued random variable. Then by remark (3.4.1) we have that for each $u \in U$, $< u, X >$ is a real Gaussian random variable. Moreover by Proposition (3.4.1) we have

$$\mathbb{E}[< X, e_k >] =< m, e_k > \tag{3.4.4}$$

and

$$\mathbb{E}[< X - m, e_k >< X - m, e_l >] =< Qe_k, e_l >= \lambda_k \delta_{kl}. \tag{3.4.5}$$

Observe that when $\lambda_k = 0$, by (3.4.5) we deduce that $< X - m, e_k >= 0$ $\mathbb{P}$-a.s. Therefore we have

$$\begin{aligned} X(\omega) &= \sum_{k=1}^{\infty} < X(\omega), e_k) > e_k \\ &= m + \sum_{k=1}^{\infty} < X(\omega) - m, e_k > e_k \\ &= m + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(\omega) e_k, \quad \mathbb{P}\text{-a.s} \end{aligned}$$

where

$$\beta_k = \begin{cases} \frac{1}{\sqrt{\lambda_k}} < X - m, e_k > & \text{when } \lambda_k > 0, \\ 0 & \text{when } \lambda_k = 0. \end{cases}$$

By (3.4.4) and (3.4.5) is direct that when $\lambda_k > 0$ the real random variable β_k has distribution $\mathbb{P}_{\beta_k} = N(0,1)$. It remains to show that $\{\beta_k\}_{k=1}^{\infty}$ are independent. In order to show this, we will use the well known fact from probability theory (see Th. 16.4 [PR]), which says that if $Y = (Y_1, Y_2, \ldots, Y_n)$ is an $\mathbb{R}^n$-valued Gaussian random variable, then the family $\{Y_k\}_{k=1}^{n}$ of real random variables is independent if and only if for each $k \neq l$, $Cov[Y_k, Y_l] = 0$. In our case, for a fixed $n \in \mathbb{N}$, $\beta = (\beta_1, \ldots, \beta_n)$ is an $\mathbb{R}^n$-valued Gaussian random variable. Indeed, for each $\nu \in \mathbb{R}^n$ we have,

$$\begin{aligned} < \beta, \nu >_{\mathbb{R}^n} = \sum_{k=1}^{n} \nu_k \beta_k &= \sum_{\lambda_k > 0} \nu_k \frac{1}{\sqrt{\lambda_k}} < X - m, e_k >_U \\ &=< X, \sum_{\lambda_k > 0} \frac{\nu_k}{\sqrt{\lambda_k}} e_k > + C, \end{aligned}$$

which is an $\mathbb{R}$-valued Gaussian random variable. And by (3.4.5), we deduce that for $k \neq l$, $\mathbb{E}[\beta_k \beta_l] = 0$. This shows the desired independency.

Finally we will show that the series in (3.4.3) is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)$. Indeed we have $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$ and thus

$$\begin{aligned} \left\| \sum_{k=n}^{m} \sqrt{\lambda_k}\beta_k e_k \right\|^2_{L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)} &= \mathbb{E}\left(\left\| \sum_{k=n}^{m} \sqrt{\lambda_k}\beta_k e_k \right\|^2_U \right) \\ = \mathbb{E}\left(\sum_{k=n}^{m} \lambda_k {\beta_k}^2 \right) = \sum_{k=n}^{m} \lambda_k \mathbb{E}[{\beta_k}^2] &= \sum_{k=n}^{m} \lambda_k \to 0 \;\; n, m \to \infty. \end{aligned}$$

Conversely, let β_k, e_k and λ_k be as assumed and define

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k e_k.$$

By the above computation the series is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)$ and thus by lemma (3.3.2) is convergent in probability. So by the Itô-Nisio Theorem (3.3.2) the series is convergent $\mathbb{P}$-almost surely. We want to show that X is Gaussian with mean m and covariance operator Q. Let $u \in U$. We have

$$< m + \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k e_k, u >=< m, u > + \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k < e_k, u > \tag{3.4.6}$$

is a real Gaussian random variable, since $\{\beta_i\}_{i=1}^{n}$ are independent real Gaussian random variables. Moreover the sequence of partial sums in (3.4.6) is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{R})$. Indeed, since $Y_k = \sqrt{\lambda_k}\beta_k < e_k, u >$, $k = 1, \ldots, m$ are independent real Gaussian random variables with mean zero we have

$$Var[\sum_{k=n}^{m} Y_k] = \mathbb{E}[\left\{\sum_{k=n}^{m} Y_k\right\}^2] = \sum_{k=n}^{m} \mathbb{E}[{Y_k}^2] = \sum_{k=n}^{m} Var[Y_k].$$

Therefore,

$$\left\| \sum_{k=n}^{m} \sqrt{\lambda_k}\beta_k < e_k, u > \right\|^2_{L^2(\Omega,\mathbb{R})} = \mathbb{E}\left| \sum_{k=n}^{m} \sqrt{\lambda_k}\beta_k < e_k, u > \right|^2$$

$$= \sum_{k=n}^{m} \mathbb{E}[\lambda_k{\beta_k}^2 < e_k, u >^2] \leq \| u \|^2 \sum_{k=n}^{m} \lambda_k \mathbb{E}[{\beta_k}^2] = \| u \|^2 \sum_{k=n}^{m} \lambda_k \to 0, \quad n, m \to \infty.$$

Since the series is $L^2((\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{R})$ convergent, we get (see Th. 4.1.5 [BW]) that the limit $< X, u >$ is a Gaussian real random variable with $\mathbb{E}[< X, u >] =< m, u >$. Moreover, for each $u, v \in U$ we have

$$\mathbb{E}[< X - m, u >< X - m, v >] = \mathbb{E}[< \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k e_k, u >< \sum_{l=1}^{\infty} \sqrt{\lambda_l}\beta_l e_l, v >]$$

$$= \mathbb{E}\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sqrt{\lambda_k}\sqrt{\lambda_l}\beta_k\beta_l < e_k, u >< e_l, v > \right)$$

$$= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sqrt{\lambda_k}\sqrt{\lambda_l}\mathbb{E}[\beta_k\beta_l] < e_k, u >< e_l, v >$$

$$= \sum_{k=1}^{\infty} \lambda_k < e_k, u >< e_k, v >=< Qu, v > .$$

The interchange of the integral and the infinite sum in the above computations is due to Beppo Levy's theorem, since

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{E}\left| \sqrt{\lambda_k}\sqrt{\lambda_l}\beta_k\beta_l < e_k, u >< e_l, v > \right|$$

$$= \sum_{k=1}^{\infty} \lambda_k | < e_k, u >< e_k, v > | \leq \| u \| \| v \| \sum_{k=1}^{\infty} \lambda_k < \infty.$$

Corollary 3.4.2 (Existence of Gaussian measures)**.** *For each $m \in U$, $Q \in B(U)$, $Q \geq 0$, $Tr(Q)$ there exists a Gaussian measure on $(U, \mathfrak{B}(U))$ such that $\mu = N(m, Q)$.*

Proof: We assume that there exists a probability space with a countably infinite family of independent real Gaussian random variables. This is a non-trivial fact from probability theory (see [KAL]). For a given $m \in U$ and $Q \geq 0$, $Tr(Q) < \infty$ construct a Gaussian random variable X according to Proposition (3.4.2) and take $\mu = \mathbb{P}_X$.

3.5 Infinite dimensional Wiener Processes

Definition 3.5.1. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and $I \subset \mathbb{R}$ an interval. A U-valued* ***stochastic process*** *$\{X(t)\}_{t\in I}$ is a family of U-valued random variables $X(t)$ on $(\Omega, \mathfrak{F}, \mathbb{P})$ indexed by I.*

Definition 3.5.2. *Two U-valued stochastic processes $(X(t))_{t\in I}$ and $(Y(t))_{t\in I}$ on a probability spaces $(\Omega, \mathfrak{F}, \mathbb{P})$ are said to be* ***versions*** *or* ***modifications*** *of each other if-f*

$$\mathbb{P}([X(t) \neq Y(t)]) = 0, \qquad \textit{for each } t \in I.$$

In addition, we say that they are ***indistinguishable*** *if-f*

$$\mathbb{P}(\bigcup_{t\in I}[X(t) \neq Y(t)]) = 0$$

Definition 3.5.3. *Let $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$. A U-valued stochastic process $\{W(t)\}_{t\geq 0}$ is said to be* ***nuclear Q-Wiener process*** *if-f*

(1) $W(0) = 0$

(2) $(W(t))_{t\geq 0}$ *has continuous paths. That is for all $\omega \in \Omega$ the mapping $t \to W(t, \omega)$ is continuous.*

(3) *For each $0 \leq s < t$, we have $W(t) - W(s) \sim N(0, (t-s)Q)$.*

(4) $(W(t))_{t\geq 0}$ *has independent increments. That is, for all $0 = t_0 < t_1, \ldots < t_n < \infty$ the random variables $\{W(t_j) - W(t_{j-1})\}_{j=1}^n$ are independent.*

Proposition 3.5.1 (Representation of Q-Wiener processes). *Let $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$. A U-valued stochastic process $(W(t))_{t\geq 0}$ is a U-valued Q-Wiener process if and only if for each $t \geq 0$, $W(t)$ can be represented as a $\mathbb{P}$-a.s convergent series*

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \tag{3.5.1}$$

where (λ_k, e_k) are the eigenpairs of Q and $\{\beta_k(t)\}_{t\geq 0}$ are independent real valued standard Brownian motions on $(\Omega, \mathfrak{F}, \mathbb{P})$. Moreover, for each $T > 0$ the series in (3.5.1) converges in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), C([0,T], U))$. In particular for every $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$, there exists a U-valued Q-Wiener process.

Proof: Assume that $(W(t))_{t\geq 0}$ is a U-valued Q-Wiener process. Then it is direct by Definition (3.5.3) that for each $t \geq 0$, $W(t) \sim N(0, tQ)$. Arguing as in the Proof of Proposition (3.4.2) we derive that for a fixed $t \geq 0$

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k \qquad \mathbb{P} - a.s,$$

where

$$\beta_k(t) = \begin{cases} \frac{1}{\sqrt{\lambda_k}} < W(t), e_k > & \text{when } \lambda_k > 0, \\ 0 & \text{when } \lambda_k = 0. \end{cases}$$

Moreover for a fixed $t \geq 0$, we have that $\{\beta_k(t)\}_{k=1}^{\infty}$ are independent real Gaussian random variables, with $\beta_k(t) \sim N(0, t)$, for all $\lambda_k > 0$. In addition the series in (3.5.1) is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)$.
We will show that for each fixed $k \in \mathbb{N}$ such that $\lambda_k > 0$ the stochastic process $(\beta_k(t))_{t\geq 0}$ is an $\mathbb{R}$-valued standard Brownian motion. Indeed it is direct that $\beta_k(0) = 0$ and since $(W(t))_{t\geq 0}$ has continuous paths, by the continuity of the inner product the same is true for $(\beta_k(t))_{t\geq 0}$. In addition, for $0 = t_0 < t_1 < \ldots < t_n$

$$\beta_k(t_j) - \beta_k(t_{j-1}) = \frac{1}{\sqrt{\lambda_k}} < W(t_j) - W(t_{j-1}), e_k > .$$

So, by the independency of $\{W(t_j) - W(t_{j-1})\}_{j=1}^n$, we deduce (see Pr.13.13 [KM]) the independency of $\{\beta_k(t_j) - \beta_k(t_{j-1})\}_{j=1}^n$. Now, by assumption for each $0 \leq s < t$, $W(t) - W(s) \sim N(0, (t-s)Q)$, so by proposition (3.4.1) we have $\mathbb{E}[< W(t) - W(s), e_k >] =< 0, e_k >= 0$ and $\mathbb{E}[< W(t) - W(s), e_k >^2] = (t-s)\lambda_k$. Thus, $\beta_k(t) - \beta_k(s) \sim N(0, t-s)$.

Next, we will show that $(\beta_k())_{k=1}^{\infty}$ is a family of independent stochastic processes. So, for distinct $\{k_i\}_{i=1}^{n}$ and $0 = t_0 < t_1 < \ldots < t_m$ we have to show that

$$(\beta_{k_1}(t_1), \ldots, \beta_{k_1}(t_m)), \ldots (\beta_{k_n}(t_1), \ldots \beta_{k_n}(t_m))$$

are independent. For each $i = 1, \ldots n$, the $\mathbb{R}^m$-valued random variable $(\beta_{k_i}(t_1), \ldots \beta_{k_i}(t_m))$ is Gaussian with zero mean. So, in order to show the desired independency it is enough to show that for $\{a_j\}_{j=1}^{m}, \{a'_j\}_{j=1}^{m} \subset \mathbb{R}^m$

$$\mathbb{E}[\sum_{j=1}^{m} a_j \beta_{k_i}(t_j) \sum_{j=1}^{m} a'_j \beta_{k_{i'}}(t_j)] = 0, \quad \text{for } \ k_i \neq k_{i'}.$$

This is direct from the following claim.
Claim: For $t > s > 0$ and $i \neq j$ such that $\lambda_i, \lambda_j > 0$ we have $\mathbb{E}[\beta_i(t)\beta_j(s)] = 0$.
Proof of the Claim:

$$\mathbb{E}[\beta_i(t)\beta_j(s)] = \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} \mathbb{E}[< W(t), e_i >< W(s), e_j >]$$

$$= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} \{\mathbb{E}[< W(t) - W(s), e_i >< W(s), e_j >] + \mathbb{E}[< W(s), e_i >< W(s), e_j >]\}$$

$$= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} s < Qe_i, e_j >= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} s\lambda_i < e_i, e_j >= 0.$$

Conversely, let $(\beta_k())_{k\in\mathbb{N}}$ and Q be given as in the statement and define

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k.$$

We will give a sketch of the basic steps. The very technical points are omitted. Note again that the series in the above formula is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), U)$ due to the fact that $Tr(Q) < \infty$. It is direct that $W(0) = 0$. Arguing again as in the proof of Proposition (3.4.2) one can shows that $W(t) - W(s) \sim N(0, (t-s)Q)$, for $0 \leq s < t$ and the independency of the increments. The continuity of the paths is a result of the convergence of the series in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), C([0,T], U))$.Here, we consider the $C([0,T], U)$-valued random variables $(\xi_j)_{j=1}^{\infty}$, defined by

$$\xi_j(t) = \sqrt{\lambda_j} \beta_j(t) e_j, \quad t \in [0, T].$$

In order to show that the sequence of partial sums $S_N = \sum_{j=1}^{N} \xi_j$ is convergent in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), C([0,T], U))$, we will use the Doob's maximal inequality (Th 4.1 [BRZ]), which states that for a real valued p-integrable martingale $\{M(t)\}_{t\geq 0}$ it holds that

$$\left(\mathbb{E}[\sup_{0\leq t\leq T} |M(t)|^p]\right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\mathbb{E}(|M(T)|^p)\right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$

We know that a real-valued Brownian motion $(B(t))_{t\geq 0}$ is an $\mathfrak{F}_t^B$-martingale, where $\mathfrak{F}_t^B = \sigma(\{B_s : \ 0 \leq s \leq t\})$. Therefore,

$$\left\| \sum_{k=n}^{m} \sqrt{\lambda_k} \beta_k(t) e_k \right\|^2_{L^2((\Omega,\mathfrak{F},\mathbb{P}),C([0,T],U))} = \mathbb{E}\left(\sup_{0\leq t\leq T} \left\| \sum_{k=n}^{m} \sqrt{\lambda_k} \beta_k(t) e_k \right\|_U^2\right)$$

$$= \mathbb{E}\left(\sup_{0\leq t\leq T} \sum_{k=n}^{m} \lambda_k \beta_k(t)^2\right) \leq \sum_{k=n}^{m} \lambda_k \mathbb{E}\left(\sup_{0\leq t\leq T} \beta_k(t)^2\right)$$

$$\leq 4 \sum_{k=n}^{m} \lambda_k \mathbb{E}(\beta_k(T)^2) = 4T \sum_{k=n}^{m} \lambda_k \to 0, \qquad n, m \to \infty,$$

since $Tr(Q) < \infty$. By the completeness of $L^2((\Omega, \mathfrak{F}, \mathbb{P}), C([0,T], U))$ we get the desired result.
For the existence of a U-valued Q-Wiener process it is enough to consider a probability space with a countably infinite set of independent Brownian motions (the existence of such a space is a non-trivial fact from probability theory, see [KAL]) and construct a random process as described in this Proposition.

Definition 3.5.4. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A* ***filtration*** *is a family of sub-sigma-algebras $\{\mathfrak{F}_t\}_{t\geq 0}$ such that $\mathfrak{F}_t \subset \mathfrak{F}_s \subset \mathfrak{F}$, for all $t \leq s$.*
A filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ is said to be ***normal*** *if-f*

(1) *$\mathfrak{F}_0$ contains all sets $A \in \mathfrak{F}$ such that $P(A) = 0$.*

(2) *$\mathfrak{F}_t = \mathfrak{F}_{t^+} := \cap_{s>t}\mathfrak{F}_s$, for all $t \geq 0$.*

Definition 3.5.5. *A Q-Wiener process $(W(t))_{t\geq 0}$ is called* ***Q-Wiener process with respect to the filtration*** *$\{\mathfrak{F}_t\}_{t\geq 0}$ if-f*

(1) *$\{W(t)\}_{t\geq 0}$ is* ***adapted*** *to $\{\mathfrak{F}_t\}_{t\geq 0}$, that is $W(t)$ is $\mathfrak{F}_t$-measurable, for all $t \geq 0$.*

(2) *For each $0 \leq s < t$, the random variable $W(t) - W(s)$ is independent of $\mathfrak{F}_s$.*

For a given Q-Wiener process $(W(t))_{t\geq 0}$, there is always a normal filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ such that $\{W(t)\}_{t\geq 0}$ becomes a Q-Wiener process with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$. To see this, define

$$\mathcal{N} := \{A \in \mathfrak{F} : \ P(A) = 0\}, \quad \tilde{\mathfrak{F}}_s := \sigma(W(r) : 0 \leq r \leq s) \quad \tilde{\mathfrak{F}}_s^0 := \sigma(\mathcal{N} \cup \tilde{\mathfrak{F}}_s)$$

and

$$\mathfrak{F}_s := \cap_{r>s}\tilde{\mathfrak{F}}_r^0. \tag{3.5.2}$$

Proposition 3.5.2. *If $\{W(t)\}_{t\geq 0}$ is a U-valued Q-Wiener process on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, then $\{W(t)\}_{t\geq 0}$ is a Q-Wiener process with respect to the normal filtration defined in (3.5.2).*

Proof: Since $\mathfrak{F}_s \supset \tilde{\mathfrak{F}}_s$, we have that $(W(t))_{t\geq 0}$ is $\mathfrak{F}_t$-adapted. So we have to show that for fixed $0 \leq s < t$, the random variable $W(t) - W(s)$ is independent of $\mathfrak{F}_s$. Observe that for all the choices of $0 \leq t_1 < t_2 < \ldots < t_n \leq s$ we have

$$\sigma(W(t_1), \ldots, W(t_n)) = \sigma(W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1}))$$

which is independent of $W(t) - W(s)$, since $(W(t))_t$ has independent increments. By this observation we deduce that $W(t) - W(s)$ is independent of $\tilde{\mathfrak{F}}_s$ and therefore of $\tilde{\mathfrak{F}}_s^0$ as well. Finally by the continuity of the paths,

$$W(t) - W(s) = \lim_{n\to\infty}(W(t) - W(s + \frac{1}{n})).$$

If n is large enough such that $s + \frac{1}{n} \leq t$, then $W(t) - W(s + \frac{1}{n})$ is independent of $\tilde{\mathfrak{F}}^0_{s+\frac{1}{n}} \supset \mathfrak{F}_s$ and hence of $\mathfrak{F}_s$. This shows that $W(t) - W(s)$ is independent of $\mathfrak{F}_s$.

3.6 Martingales in Banach spaces

Let E be a separable Banach space and $\mathfrak{B}(E)$ be its Borel σ-algebra. When we say that $X : (\Omega, \mathfrak{F}, \mathbb{P}) \to E$ is a random variable we mean that it is $(\mathfrak{F}, \mathfrak{B}(E))$-measurable. The following Proposition is a generalization of the existence of the conditional expectation of an integrable random variable to the Banach space-valued setting. For the proof see [KV] pg.28-30.

Proposition 3.6.1. *Let E be a real separable Banach space and X an E-valued Bochner integrable random variable on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then there exists a unique (up to a set of $\mathbb{P}$- measure zero) random variable $Z \in L^1((\Omega, \mathcal{G}, \mathbb{P}), E)$ such that*

(1) *Z is $\mathcal{G}$-measurable*

(2) *$\int_A X \, d\mathbb{P} = \int_A Z \, d\mathbb{P}$, for all $A \in \mathcal{G}$.*

The random variable Z is called the ***conditional expectation*** *of X given $\mathcal{G}$ and is denoted by $\mathbb{E}[\mathbf{X}|\mathcal{G}]$. Furthermore,*

$$\| \mathbb{E}[X|\mathcal{G}] \| \leq \mathbb{E}[\| X \| |\mathcal{G}] \quad \mathbb{P} - a.s.$$

Lemma 3.6.1 (Law of double expectation)**.** *Let E be a real separable Banach space and X an E-valued Bochner integrable random variable on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.*

Proof: In (2) of Proposition (3.6.1), put $A = \Omega$.

Lemma 3.6.2. *Let E be a real separable Banach space and X an E-valued Bochner integrable random variable on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then for all $x^* \in E^*$ we have*

$$x^*(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(x^*(X)|\mathcal{G}) \quad \mathbb{P} - a.s.$$

Proof: First of all, $\mathbb{E}(|x^*(X)) \leq \| x^* \| \mathbb{E}(\| X \|) < \infty$. Thus $\mathbb{E}(x^*(X)|\mathcal{G})$ is well defined. Of course $x^*(\mathbb{E}(X|\mathcal{G}))$ is $\mathcal{G}$-measurable and for each $A \in \mathcal{G}$ we have

$$\int_A \mathbb{E}(X|\mathcal{G})\ d\mathbb{P} = \int_A X\ d\mathbb{P}$$

$$\Rightarrow x^*(\int_A \mathbb{E}(X|\mathcal{G})\ d\mathbb{P}) = x^*(\int_A X\ d\mathbb{P})$$

$$\Rightarrow \int_A x^*(\mathbb{E}(X|\mathcal{G}))\ d\mathbb{P} = \int_A x^*(X)\ d\mathbb{P}.$$

Therefore, $x^*(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(x^*(X)|\mathcal{G}) \quad \mathbb{P} - a.s.$, for all $x^* \in E^*$.

Corollary 3.6.1. *Let E be a real separable Banach space and X an E-valued Bochner integrable random variable on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. If X is independent of $\mathcal{G}$, then $x^*(X)$ is also independent of $\mathcal{G}$, for each $x^* \in E^*$ and*

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X).$$

Proof: Let $x^* \in E^*$. Since X is independent of $\mathcal{G}$, we have for each $A \in \mathfrak{B}(\mathbb{R})$ and $B \in \mathcal{G}$

$$\mathbb{P}([x^*(X) \in A] \cap B) = \mathbb{P}([X \in {x^*}^{-1}(A)] \cap B) = \mathbb{P}([X \in {x^*}^{-1}(A)])\ \mathbb{P}(B) = \mathbb{P}([x^*(X) \in A])\ \mathbb{P}(B),$$

since ${x^*}^{-1}(A) \in \mathfrak{B}(E)$. Therefore, $x^*(X)$ is independent of $\mathcal{G}$. Now, by virtue of Lemma (3.6.2), for each $x^* \in E^*$ we have

$$x^*(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(x^*(X)|\mathcal{G}) = \mathbb{E}(x^*(X)) = x^*(\mathbb{E}(X)) \qquad \mathbb{P} - a.s.$$

Now by Corollary (3.2.6), we get $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ $\mathbb{P}$-a.s.

Definition 3.6.1. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, E a separable Banach space, $(M(t))_{t\geq 0}$ an E-valued stochastic process and $(\mathfrak{F}(t))_{t\geq 0}$ a filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$. The stochastic process $(M(t))_{t\geq 0}$ is said to be* ***$\mathfrak{F}_t$-martingale*** *if-f*

(1) $\mathbb{E}(\| M(t) \|) < \infty$, *for all* $t \geq 0$.

(2) $(M(t))_t$ *is* $\mathfrak{F}_t$*-adapted.*

(3) $\mathbb{E}(M(t)|\mathfrak{F}_s) = M(s)$, *for all* $0 \leq s < t$.

Theorem 3.6.1. *Let E be a separable Banach space and $(M(t))_{t\geq 0}$ an E-valued stochastic process and $(\mathfrak{F}(t))_{t\geq 0}$ a filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$. If $(M(t))_{t\geq 0}$ is an $\mathfrak{F}_t$-martingale, then for all $x^* \in E^*$, $(x^*(M(t)))_{t\geq 0}$ is a real-valued $\mathfrak{F}_t$-martingale. Conversely, if $\mathbb{E}(\| M(t) \|) < \infty$, for all $t \geq 0$ and for all $x^* \in E^*$, $(x^*(M(t)))_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathfrak{F}_t$-martingale, then $(M(t))_{t\geq 0}$ is an E-valued $\mathfrak{F}_t$-martingale.*

Proof: Assume that $(M(t))_{t\geq 0}$ is an $\mathfrak{F}_t$-martingale and fix a $x^* \in E^*$. Then, we have

$$\mathbb{E}(|x^*(M(t))|) \leq \| x^* \| \mathbb{E}(\| M(t) \|) < \infty.$$

for all $t \geq 0$. Moreover, $M(t)$ is $\mathfrak{F}_t$-measurable, thus $x^*(M(t))$ is also $\mathfrak{F}_t$-measurable, for all $t \geq 0$. Finally, for $0 \leq s < t$ we have

$$\mathbb{E}(x^*(M(t))|\mathfrak{F}_s) = x^*(\mathbb{E}(M(t)|\mathfrak{F}_s)) = x^*(M(s)), \quad \mathbb{P} - a.s.$$

Conversely, assume that $\mathbb{E}(\| M(t) \|) < \infty$ for all $t \geq 0$ and that $(x^*(M(t)))_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathfrak{F}_t$-martingale for all $x^* \in E^*$. Since $x^*(M(t))$ is $\mathfrak{F}_t$-measurable for all $x^* \in E^*$, by the Pettis measurability Theorem (3.2.1) we deduce that $M(t)$ is $\mathfrak{F}_t$-measurable for all $t \geq 0$. Moreover, for $0 \leq s < t$ and for all $x^* \in E^*$ we have

$$x^*(\mathbb{E}(M(t)|\mathfrak{F}_s)) = \mathbb{E}(x^*(M(t))|\mathfrak{F}_s) = x^*(M(s)), \quad \mathbb{P} - a.s.$$

Therefore by Corollary (3.2.6) we get $\mathbb{E}(M(t)|\mathfrak{F}_s) = M(s)$, $\mathbb{P}$-a.s.

Remark 3.6.1. The assumption $\mathbb{E}(\| M(t) \|) < \infty$ for all $t \geq 0$ in the converse statement is necessary. It is possible $x^*(Z) \in L^1((\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{R})$, for all $x^* \in E^*$ but Z not in $L^1((\Omega, \mathfrak{F}, \mathbb{P}), E)$. For more details on this see [KV] pg 34.

Theorem 3.6.2 (Doob's Maximal Inequality). *Let E be a separable Banach space and $(M(t))_{t\geq 0}$ an E-valued $\mathfrak{F}_t$-martingale. If $M(t) \in L^p((\Omega, \mathfrak{F}, \mathbb{P}), E)$ for all $t \geq 0$, for some $p \geq 1$, then $(\| M(t) \|^p)_{t\geq 0}$ is a non-negative real-valued $\mathfrak{F}_t$-sub-martingale. That is*

$$\mathbb{E}(\| M(t) \|^p | \mathfrak{F}_s) \geq \| M(s) \|^p, \quad \textit{for } \ 0 \leq s < t. \tag{3.6.1}$$

Moreover if $p > 1$, $T \geq 0$, then

$$\mathbb{E}(\sup_{0\leq t\leq T} \| M(t) \|^p) \leq (\frac{p}{p-1})^p \ \mathbb{E}(\| M(T) \|^p) \tag{3.6.2}$$

Proof: For each $0 \leq s < t$, we have

$$\mathbb{E}(\| M(t) \|^p | \mathfrak{F}_s) \geq \{\mathbb{E}(\| M(t) \| | \mathfrak{F}_s)\}^p \geq \| \mathbb{E}(M(t)|\mathfrak{F}_s) \|^p = \| M(s) \|^p .$$

Where in the first inequality we used Jensen's inequality for $\psi(x) = x^p, x \geq 0$. The next of the proof is consequence of Doob's maximal inequality for positive real-valued sub-martingales.

Definition 3.6.2. *Let E be a separable Banach space, $(\Omega, \mathfrak{F}, \mathbb{P})$ a probability space and $(\mathfrak{F}(t))_{t\geq 0}$ a filtration. For $T > 0$ we define the linear space $\mathbf{M^2_T(E)}$ of continuous square integrable E-valued $\{\mathfrak{F}_t\}_{t\in[0,T]}$-martingales $(M(t))_{t\in[0,T]} =: M$. We equip the space $M^2_T(E)$ with the norm*

$$\| M \|_{M^2_T(E)} := \sup_{t\in[0,T]} (\mathbb{E}(\| M(t) \|^2))^{\frac{1}{2}}.$$

Observe that from (3.6.1) by taking the expectations we get

$$\mathbb{E}(\| M(s) \|^2) \leq \mathbb{E}(\| M(T) \|^2), \quad \forall 0 \leq s < T.$$

Therefore, $\| M \|_{M^2_T(E)} = \mathbb{E}(\| M(T) \|^2)^{\frac{1}{2}}$.

Proposition 3.6.2. *The space $M^2_T(E)$ is a Banach space and for each $M \in M^2_T(E)$ we have*

$$\| M \|_{M^2_T(E)} \leq (\mathbb{E}(\sup_{t\in[0,T]} \| M(t) \|^2))^{\frac{1}{2}} \leq 2 \| M \|_{M^2_T(E)} . \tag{3.6.3}$$

Proof: The first inequality is obvious and the second one is derived by (3.6.2) for $p = 2$. Let $(M_n)_n$ be a Cauchy sequence in $M^2_T(E)$. Then by (3.6.3), $(M_n)_n$ is also Cauchy in $L^2((\Omega, \mathfrak{F}, \mathbb{P}), (C([0, T]), E))$ which is Banach. Thus, there exists $M \in L^2((\Omega, \mathfrak{F}, \mathbb{P}), (C([0, T]), E))$ such that

$$\| M_n - M \|_{L^2((\Omega,\mathfrak{F},\mathbb{P}),(C([0,T]),E))} \to 0, \quad n \to \infty.$$

Therefore by (3.6.3) we have that

$$\| M_n - M \|_{M^2_T(E)} \to 0, \quad n \to \infty.$$

It remains to show that M is an $\mathfrak{F}_t$-martingale. Fix $0 \leq s < t$. For each $t \in [0, T]$ we have $\| M_n(t) - M(t) \|_{L^2(\Omega,E)} \to 0$, as $n \to \infty$. Therefore $\mathbb{E}[M_n(t)|\mathfrak{F}_s] \to \mathbb{E}(M(t)|\mathfrak{F}_s)$ in $L^2(\Omega, E)$. Indeed, $\| \mathbb{E}(M_n(t) - M(t)|\mathfrak{F}_s) \|^2 \leq \mathbb{E}(\| M_n(t) - M(t) \|^2 |\mathfrak{F}_s)$. Thus, $\mathbb{E}(\| \mathbb{E}(M_n(t) - M(t)|\mathfrak{F}_s) \|^2) \leq \mathbb{E}(\mathbb{E}(\| M_n(t) - M(t) \|^2 |\mathfrak{F}_s)) = \mathbb{E} \| M_n(t) - M(t) \|^2 \to 0$, as $n \to \infty$. On the other hand, $\mathbb{E}(M_n(t)|\mathfrak{F}_s) = M_n(s)$, a.s, from where we deduce that $\mathbb{E}(M_n(t)|\mathfrak{F}_s) \to M(s)$ in $L^2(\Omega, E)$. From the uniqueness of the limit we conclude that $\mathbb{E}(M(t)|\mathfrak{F}_s) = M(s)$ a.s.

Proposition 3.6.3. *Let $(W(t))_{t\geq 0}$ be a U-valued Q-Wiener process with respect to the normal filtration $(\mathfrak{F}_t)_{t\geq 0}$ on $(\Omega, \mathfrak{F}, \mathbb{P})$. Then $W \in M^2_T(E)$ for all $T > 0$.*

Proof: $(W(t))_{t\geq 0}$ has continuous trajectories. Moreover $\mathbb{E}(\| W(t) \|^2) = t\ Tr(Q) \leq T\ Tr(Q) < \infty$, for all $t \in [0, T]$ and $(W(t))_t$ is $\mathfrak{F}_t$-adapted. Finally for $0 \leq s < t$ we have

$$\mathbb{E}(W(t)|\mathfrak{F}_s) = \mathbb{E}(W(t) - W(s)|\mathfrak{F}_s) + \mathbb{E}(W(s)|\mathfrak{F}_s) = \mathbb{E}(W(t) - W(s)) + W(s) = W(s), \quad a.s,$$

since $W(t) - W(s)$ is independent of $\mathfrak{F}_s$ and $W(s)$ is $\mathfrak{F}_s$-measurable.

Chapter 4

Stochastic integration in Hilbert spaces

4.1 The stochastic integral for nuclear Wiener processes

4.1.1 Measurability of operator valued random variables

Consider the space $B(U,H)$ of linear and bounded operators $L : U \to H$, where U, H are separable Hilbert spaces. It is well known that $B(U,H)$ is a Banach space, when it is endowed with the norm

$$\| T \|_{B(U,H)} = \sup\{\| Tx \| : \ x \in U, \| x \| \leq 1\}.$$

We define the **uniform Borel σ-algebra $\mathcal{B}_{\mathbf{uni}}(\mathbf{B}(\mathbf{U},\mathbf{H}))$** as the smallest σ-algebra which contains the open balls

$$B_r(T) = \{L \in B(U,H) : \ \| L - T \| < r\}, \quad r > 0 \ \ T \in B(U,H).$$

Generally the space $B(U,H)$ is not separable (see [KV] page 37) thus it has too many open balls and as a result the class of $B(U,H)$-valued measurable functions with respect to $\mathcal{B}_{uni}(B(U,H))$ is very small. Instead of the uniform Borel σ-algebra we consider the **strong Borel σ-algebra** $\mathcal{B}_{str}(B(U,H))$ on $B(U,H)$ defined as the smallest σ-algebra which contains the sets of the form

$$\{T \in B(U,H) : \ Tx \in A, \ \forall x \in U\}, \quad A \in \mathfrak{B}(H).$$

Definition 4.1.1. *Let $(\Omega, \mathfrak{F})$ be a measurable space and $\mathcal{G} \subset \mathfrak{F}$ a sub-σ-algebra. A mapping $L : \Omega \to B(U,H)$ is said to be* ***strongly $\mathcal{G}$-measurable****if-f it is $\mathfrak{G}$-measurable, when we endow $B(U,H)$ with the strong Borel σ-algebra $\mathcal{B}_{str}(B(U,H))$. This means that $L : \Omega \to B(U,H)$ is strongly $\mathcal{G}$-measurable if and only if for each $x \in U$, $Lx : (\Omega, \mathfrak{F}) \to (H, \mathfrak{B}(H))$ is $\mathcal{G}$-measurable.*

Generally, it holds that $\mathcal{B}_{str}(B(U,H)) \subset \mathcal{B}_{uni}(B(U,H))$. Moreover it can be shown that $\mathfrak{B}(B_2(U,H)) \subset \mathcal{B}_{str}(B(U,H))$. In particular $B_2(U,H)$ is a strongly measurable subset of $B(U,H)$. For more details see [KV].

Lemma 4.1.1. *If L is a $B(U,H)$-valued strongly measurable mapping and ξ a U-valued measurable mapping on a measurable space $(\Omega, \mathfrak{F})$, then $L\xi : \Omega \to H$ is an H-valued measurable mapping on $(\Omega, \mathfrak{F})$.*

Proof: Since H is a separable Hilbert space, by virtue of the Pettis measurability Theorem (3.2.1), $L\xi$ is measurable if and only if for each $x \in H$, $< x, L\xi >$ is $\mathbb{R}$-valued $\mathfrak{F}$-measurable. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for U. Then,

$$< L\xi, x >=< \xi, L^* x >= \sum_{k=1}^{\infty} < \xi, e_k >< Le_k, x > .$$

Since Le_k is $(\mathfrak{F}, \mathfrak{B}(H))$-measurable and ξ is $(\mathfrak{F}, \mathfrak{B}(U))$-measurable we deduce that $< \xi, e_k >< Le_k, x >$ is $(\mathfrak{F}, \mathfrak{B}(\mathbb{R}))$-measurable and so the everywhere convergent sum $< L\xi, x >$ is $\mathfrak{F}$-measurable as well.

4.1.2 The stochastic integral for elementary random processes.

In this section we consider that $(U, <\,,\,>_U)$ and $(H, <\,,\,>_H)$ are separable Hilbert spaces, $T > 0$ fixed and $(W(t))_{t\in[0,T]}$ is a U-valued Q-Wiener process on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with respect to the normal filtration $(\mathfrak{F}(t))_{t\in[0,T]}$.

Definition 4.1.2. *A $B(U,H)$-valued random process $(\Phi(t))_{t\in[0,T]}$ is said to be* ***elementary process*** *if-f there exists a partition $0 = t_0 < t_1 < \ldots < t_N = T$, $N \in \mathbb{N}$ such that*

$$\Phi(t) = \sum_{m=0}^{N-1} \Phi_m \mathbb{I}_{(t_m, t_{m+1}]}(t), \quad t \in [0,T] \tag{4.1.1}$$

where

- *$\Phi_m : \Omega \to B(U,H)$ is strongly $\mathfrak{F}_{t_m}$-measurable.*
- *Φ_m takes only a finite number of values in $B(U,H)$. That is,*

$$\Phi_m(\omega) = \sum_{j=1}^{k_m} \mathbb{I}_{\Omega_j^m}(\omega) L_j^m,$$

  where $L_j^m \in B(U,H)$ and $\Omega = \cup_{j=1}^{k_m} \Omega_j^m$ with the union being disjoint. The linear space of elementary stochastic processes is denoted by $\mathcal{E}$.

Definition 4.1.3. *For each $\Phi \in \mathcal{E}$ having the representation in (4.1.1) we define*

$$I_t(\Phi) :=: \int_0^t \Phi \, dW := \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)), \quad t \in [0,T],$$

where $\Delta W_n(t) = W(t_{n+1} \wedge t) - W(t_n \wedge t)$ and $t \wedge s = \min\{t, s\}$.

Proposition 4.1.1. *For each $\Phi \in \mathcal{E}$, $(\int_0^t \Phi \, dW)_{t\in[0,T]}$ is a continuous square integrable H-valued $\mathfrak{F}_t$-martingale. In other words, $(I_t(\Phi))_{t\in[0,T]} \in M_T^2(H)$.*

Proof: Let $\Phi \in \mathcal{E}$ having the representation in (4.1.1). Define $M(t) = \int_0^t \Phi \, dW$, $t \in [0,T]$.

- $(M(t))_{t\in[0,T]}$ has continuous paths.
  For a fixed $\omega \in \Omega$ the mapping $[0,T] \ni t \to \sum_{n=0}^{N-1} \Phi_n(\omega)(\Delta W_n(t)(\omega))$ is continuous. This is because $t \to \Delta W_n(t)(\omega)$ is continuous and $\Phi_n(\omega)$ is continuous for all $n \in \mathbb{N}$.

- $(M(t))_{t\in[0,T]}$ is square integrable.
  For each $t \in [0,T]$ we have

$$\begin{aligned}
\mathbb{E}(\| M(t) \|^2) &= \mathbb{E}(\left\| \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)) \right\|^2) \\
&\leq N \sum_{n=0}^{N-1} \mathbb{E}\left(\| \Phi_n(\Delta W_n(t)) \|^2 \right) \\
&\leq N \sum_{n=0}^{N-1} \mathbb{E}\left(\| \Phi_n \|_{B(U,H)}^2 \| \Delta W_n(t) \|_U^2 \right) \\
&\leq N \sum_{n=0}^{N-1} \mathbb{E}\left((\sum_{j=0}^{k_n} \| L_j^n \|_{B(U,H)}^2) \| \Delta W_n(t) \|_U^2 \right) \\
&\leq N \max_{n=0\ldots N-1} (\sum_{j=0}^{k_n} \| L_j^n \|_{B(U,H)}^2) \sum_{n=0}^{N-1} \mathbb{E}(\| \Delta W_n(t) \|_U^2) < \infty.
\end{aligned}$$

- $(M(t))_{t\in[0,T]}$ is an $\mathfrak{F}_t$-martingale.
  - $\mathbb{E}(\| M(t) \|) < \infty$, for all $t \in [0,T]$ since each $M(t)$ is square integrable.
  - For each $n = 0, \ldots, N-1$, $\Phi_n(\Delta W_n(t))$ is $\mathfrak{F}_t$-measurable H-valued random variable and thus $\Phi(t)$ is $\mathfrak{F}_t$-measurable H-valued random variable as well. Indeed, first of all observe that $\Delta W_n(t)$ is U-valued and $\mathfrak{F}_t$-measurable. In particular for $t \in (t_k, t_{k+1}]$ we have

$$\Delta W_n(t) = \begin{cases} W(t_{n+1}) - W(t_n), & \text{when } t_n < t_k, \\ W(t) - W(t_k), & \text{when } t_n = t_k. \\ 0, & \text{when } t_n > t_k. \end{cases}$$

Therefore, for each $t_n > t_k$, we have $\Phi_n(\Delta W_n(t)) = 0$ which is $\mathfrak{F}_t$-measurable. In any other case $\mathfrak{F}_{t_n} \subset \mathfrak{F}_t$ and thus Φ_n is strongly $\mathfrak{F}_t$-measurable. The conclusion now follows from Lemma (4.1.1).
  - For each $0 \le s < t$ it holds that $\int_0^s \Phi\ dW = \mathbb{E}(\int_0^s \Phi\ dW | \mathfrak{F}_s)$.
Indeed, assume that $0 = t_0 < t_1 < \ldots < t_l < s \le t_{l+1} < \ldots < t_k < t \le t_{k+1} < \ldots < t_N = T$. Then,

$$\int_0^t \Phi\ dW = \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)) =$$

$$= \sum_{n=0}^{l-1} \Phi_n(\Delta W_n(t)) + \Phi_l(W(s) - W(t_l)) + \Phi_l(W(t_{l+1} \wedge t) - W(s)) + \sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)) =$$

$$= \sum_{n=0}^{l} \Phi_n(\Delta W_n(s)) + \Phi_l(W(t_{l+1} \wedge t) - W(s)) + \sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)) =$$

$$= \int_0^s \Phi\ dW + \Phi_l(W(t_{l+1} \wedge t) - W(s)) + \sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)).$$

Therefore,

$$\mathbb{E}\left(\int_0^t \Phi\ dW | \mathfrak{F}_s\right) = \mathbb{E}\left(\int_0^s \Phi\ dW | \mathfrak{F}_s\right) + \mathbb{E}\left(\Phi_l(W(t_{l+1} \wedge t) - W(s)) | \mathfrak{F}_s\right)$$

$$+ \mathbb{E}\left(\sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)) | \mathfrak{F}_s\right).$$

Since $\int_0^s \Phi\ dW$ is $\mathfrak{F}_s$-measurable, we have $\mathbb{E}\left(\int_0^s \Phi\ dW | \mathfrak{F}_s\right) = \int_0^s \Phi\ dW$.
For the second term we will make use of the following well-known result for real-valued martingales.

Lemma 4.1.2. *Let X, Y be real valued martingales on $(\Omega, \mathfrak{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathfrak{F}$ be a sub-σ-algebra. If X is $\mathcal{G}$-measurable and $Y, XY \in L^1((\Omega, \mathfrak{F}, \mathbb{P}), \mathbb{R})$, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$.*

Returning now to our proof, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for U. Then, for each $x \in H$, we have

$$< \mathbb{E}\left(\Phi_l(W(t_{l+1} \wedge t) - W(s) | \mathfrak{F}_s), x\right) >=$$

$$=< \mathbb{E}(\Phi_l(\sum_{k=1}^{\infty} < W(t_{l+1} \wedge t) - W(s), e_k > e_k) | \mathfrak{F}_s), x >$$

$$= \sum_{k=1}^{\infty} \mathbb{E}\left(< W(t_{l+1} \wedge t) - W(s), e_k >< \Phi_l e_k, x > | \mathfrak{F}_s\right)$$

$$= \sum_{k=1}^{\infty} < \Phi_l e_k, x > \ \mathbb{E}\left(< W(t_{l+1} \wedge t) - W(s), e_k > | \mathfrak{F}_s\right)$$

$$= \sum_{k=1}^{\infty} < \Phi_l e_k, x > \ \mathbb{E}\left(< W(t_{l+1} \wedge t) - W(s), e_k >\right) = 0,$$

where in the fourth equality we used that $< \Phi_l e_k, x >$ is real-valued $\mathfrak{F}_s$-measurable and Lemma (4.1.2) and in the last inequality that $W(t_{l+1} \wedge t) - W(s)$ is independent of $\mathfrak{F}_s$. Eventually we have shown that $\mathbb{E}\left(\Phi_l(W(t_{l+1} \wedge t) - W(s))|\mathfrak{F}_s\right) = 0$.
The rest of the terms are of the form $\mathbb{E}\left(\Phi_m(W(t_{m+1} \wedge t) - W(t_m))|\mathfrak{F}_s\right)$, where $s \leq t_m \leq t$. For all $A \in \mathfrak{F}_s$ we have,

$$\begin{aligned}\int_A \Phi_m(W(t_{m+1} \wedge t) - W(t_m))\, d\mathbb{P} &= \int_A \sum_{j=1}^{k_m} \mathbb{I}_{\Omega_j^m} L_j^m (W(t_{m+1} \wedge t) - W(t_m))\, d\mathbb{P} \\ &= \sum_{j=1}^{k_m} L_j^m \int_{A \cap \Omega_j^m} (W(t_{m+1} \wedge t) - W(t_m))\, d\mathbb{P}.\end{aligned}$$

But $A \in \mathfrak{F}_s \subset \mathfrak{F}_{t_m}$ and $\Omega_j^m \in \mathfrak{F}_{t_m}$, so $A \cap \Omega_j^m \in \mathfrak{F}_{t_m}$, for all $j = 1, \ldots, k_m$. Thus,

$$\begin{aligned}&\sum_{j=1}^{k_m} L_j^m \int_{A \cap \Omega_j^m} (W(t_{m+1} \wedge t) - W(t_m))\, d\mathbb{P} \\ &= \sum_{j=1}^{k_m} L_j^m \int_{A \cap \Omega_j^m} \mathbb{E}(W(t_{m+1} \wedge t) - W(t_m)|\mathfrak{F}_{t_m})\, d\mathbb{P} \\ &= \sum_{j=1}^{k_m} L_j^m \int_{A \cap \Omega_j^m} \mathbb{E}((W(t_{m+1} \wedge t) - W(t_m))\, d\mathbb{P} = 0.\end{aligned}$$

So by the definition of conditional expectation we conclude that

$$\mathbb{E}\left(\Phi_m(W(t_{m+1} \wedge t) - W(t_m))|\mathfrak{F}_s\right) = 0.$$

Remark 4.1.1. *Since $M(t) = \int_0^t \Phi\, dW$ is a martingale we have*

$$\mathbb{E}(\int_0^t \Phi\, dW) = \mathbb{E}(M(0)) = 0.$$

Definition 4.1.4. *For $\Phi \in \mathcal{E}$ we define*

$$\| \Phi \|_T := (\mathbb{E}(\int_0^T \left\| \Phi(s) Q^{1/2} \right\|^2_{B_2(U,H)} ds))^{\frac{1}{2}} = (\mathbb{E}(\int_0^T \| \Phi(s) \|^2_{L_2^0} ds))^{\frac{1}{2}}.$$

The following identity, called the Itô isometry, will be crucial when we extend the stochastic integral to a larger class of integrands.

Proposition 4.1.2 (Itô isometry). *If $\Phi \in \mathcal{E}$, then*

$$\mathbb{E}\left(\left\| \int_0^T \Phi\, dW \right\|^2\right) = \mathbb{E}\left(\int_0^T \left\| \Phi(s) Q^{1/2} \right\|^2_{B_2(U,H)} ds\right) \tag{4.1.2}$$

or equivalently,

$$\left\| \int_0^{\cdot} \Phi\, dW \right\|_{M_T^2(H)} = \| \Phi \|_T .$$

Proof: Let $\Phi \in \mathcal{E}$. First of all,

$$\int_0^T \Phi\, dW = \sum_{n=0}^{N-1} \Phi_n(\Delta W_n),$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$. So,

$$\mathbb{E}\Big(\Big\| \int_0^T \Phi\, dW \Big\|^2\Big) = \mathbb{E}\Big(\Big\langle \sum_{n=0}^{N-1} \Phi_n \Delta W_n, \sum_{m=0}^{N-1} \Phi_m \Delta W_m \Big\rangle\Big)$$

$$= \mathbb{E}\Big(\sum_{n=0}^{N-1} \|\Phi_n \Delta W_n\|^2\Big) + 2\mathbb{E}\Big(\sum_{m<n} \langle \Phi_n \Delta W_n, \Phi_m \Delta W_m\rangle\Big)$$
$$= T_1 + T_2.$$

We will show that $T_1 = \|\Phi\|_T^2$ and that $T_2 = 0$. Let $\{f_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of H and $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of U such that $Qe_k = \lambda_k e_k$. Then,

$$\mathbb{E}\Big(\|\Phi_n \Delta W_n\|^2\Big) = \mathbb{E}\Big(\sum_l \langle \Phi_n \Delta W_n, f_l\rangle^2\Big) = \sum_l \mathbb{E}\Big(\langle \Phi_n \Delta W_n, f_l\rangle^2\Big)$$
$$= \sum_l \mathbb{E}\Big(\mathbb{E}\big(\langle \Phi_n \Delta W_n, f_l\rangle^2 \big| \mathcal{F}_{t_n}\big)\Big) = \sum_l \mathbb{E}\Big(\mathbb{E}\big(\langle \Delta W_n, \Phi_n^* f_l\rangle^2 \big| \mathcal{F}_{t_n}\big)\Big).$$

But,

$$\langle \Delta W_n, \Phi_n^* f_l\rangle^2 = \Big(\sum_k \langle \Delta W_n, e_k\rangle \langle \Phi_n^* f_l, e_k\rangle\Big)^2$$
$$= \Big(\sum_k \underbrace{\langle f_l, \Phi_n e_k\rangle}_{=a_k} \underbrace{\langle \Delta W_n, e_k\rangle}_{=b_k}\Big)^2 = \Big(\sum_k a_k b_k\Big)^2 = \sum_{k,j} a_k a_j b_k b_j.$$

Thus,

$$\begin{aligned}
\mathbb{E}\Big(\langle \Delta W_n, \Phi_n^* f_l\rangle^2 \Big| \mathcal{F}_{t_n}\Big) &= \mathbb{E}\Big(\sum_{k,j} a_k a_j b_k b_j \Big| \mathcal{F}_{t_n}\Big) \\
&= \sum_{k,j} \mathbb{E}\Big(a_k a_j b_k b_j \Big| \mathcal{F}_{t_n}\Big) && (4.1.3) \\
&= \sum_{k,j} a_k a_j \mathbb{E}\big(b_k b_j \big| \mathcal{F}_{t_n}\big) = \sum_{k,j} a_k a_j \mathbb{E}\big(b_k b_j\big) && (4.1.4) \\
&= \sum_{k,j} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_j\rangle \mathbb{E}(\langle \Delta W_n, e_k\rangle \langle \Delta W_n, e_j\rangle) \\
&= \sum_{k,j} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_j\rangle \Delta t_n \langle Qe_k, e_j\rangle && (4.1.5) \\
&= \sum_{k,j} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_j\rangle \Delta t_n \lambda_k \langle e_k, e_j\rangle \\
&= \sum_{k=1}^{\infty} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_k\rangle \Delta t_n \lambda_k \\
&= \Delta t_n \sum_{k=1}^{\infty} \langle \Phi_n^* f_l, Q^{1/2} e_k\rangle^2 \\
&= \Delta t_n \sum_{k=1}^{\infty} \langle Q^{1/2} \Phi_n^* f_l, e_k\rangle^2 \\
&= \Delta t_n \Big\| Q^{1/2} \Phi_n^* f_l \Big\|^2 \quad \mathbb{P} - \ a.s
\end{aligned}$$

We used the Beppo Levy Theorem in 4.1.3, properties of conditional expectation in 4.1.4, and the assumption on the increments of a Q-Wiener process in 4.1.6. Hence, using property (1) in Remark **??**,

$$\begin{aligned}
T_1 &= \sum_{n=0}^{N-1} \sum_{l=1}^{\infty} \mathbb{E}\left(\Delta t_n \|Q^{1/2} \Phi_n^* f_l\|^2\right) = \mathbb{E}\Big(\sum_{n=0}^{N-1} \Delta t_n \|Q^{1/2} \Phi_n^*\|^2_{B_2(H,U)}\Big) \\
&= \mathbb{E}\Big(\sum_{n=0}^{N-1} \Delta t_n \|\Phi_n Q^{1/2}\|_{B_2(U,H)}\Big) = \mathbb{E}\Big(\int_0^T \|\Phi(s) Q^{1/2}\|_{B_2(U,H)}\, ds\Big).
\end{aligned}$$

Following the similar reasoning for the second term we conclude easily that $T_2 = 0$.

Corollary 4.1.1. *If $\Phi_1, \Phi_2 \in \mathcal{E}$, then*

$$\mathbb{E}\Big(\Big\langle \int_0^T \Phi_1\, dW, \int_0^T \Phi_2\, dW \Big\rangle_H\Big) = \mathbb{E}\Big(\int_0^T \langle \Phi_1(s)Q^{1/2}, \Phi_2(s)Q^{1/2}\rangle_{B_2(U,H)}\, ds\Big).$$

Remark 4.1.2. The functional $\|\cdot\|_T$ is only a seminorm on $\mathcal{E}$. Indeed, if $\Phi \in \mathcal{E}$ and

$$\|\Phi\|_T^2 \quad = \quad \mathbb{E}\Big(\int_0^T \|\Phi(s)Q^{1/2}\|^2_{B_2(U,H)}\, ds\Big) = 0.$$

then $\Phi(s)Q^{1/2} = 0$, $\mathbb{P}_T := m \times \mathbb{P}$-a.s.. Therefore, $\Phi = 0$ on $Q^{1/2}(U)$, $m \times \mathbb{P}$-a.s. Let

$$\mathcal{E}_0 := \Big\{\Phi \in \mathcal{E} : \Phi = 0 \text{ on } Q^{1/2}(U),\ \mathbb{P}_T\text{-a.s.}\Big\}.$$

We re-define $\mathcal{E}$ to be the quotient space $\mathcal{E} := \mathcal{E}/\mathcal{E}_0$. Then $\|\cdot\|_T$ is a norm on $\mathcal{E}$.

4.2 Extension of the stochastic integral to more general processes

We have already seen that the map

$$Int : \big(\mathcal{E}, \|\cdot\|_T\big) \to \big(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}\big)$$

is isometric. Since, the space $\big(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}\big)$ is Hilbert , Int extends uniquely to an isometric mapping to the abstract completion $\overline{\mathcal{E}}$ of $\mathcal{E}$, by the obvious way. In this section we will give a characterization of $\overline{\mathcal{E}}$. Here it is convenient to treat the processes as random variables from $\Omega_T := [0,T] \times \Omega$ to $B(U,H)$, where the product space Ω_T is equipped with the product σ-algebra $\mathfrak{B}([0,T]) \otimes \mathfrak{F}$ and the product measure $\mathbb{P}_T := m \otimes \mathbb{P}$. The σ-field just introduced does not take into acount the adaptivity of the considered process. To this aim we introduce the following σ-algebra

$$\mathcal{P}_T = \sigma\Big(\Big\{(s,t] \times F : 0 \le s < t \le T,\ F \in \mathfrak{F}_s\Big\} \cup \Big\{\{0\} \times F : F \in \mathfrak{F}_0\Big\}\Big).$$

and introduce the notion of predictable process.

Definition 4.2.1. *If $\tilde{H}$ is a separable Hilbert space and $Y : (\Omega_T, \mathbb{P}_T) \to (\tilde{H}, \mathfrak{B}(\tilde{H}))$ is measurable, then Y is called $\tilde{H}$-**predictable**.*

The next proposition shows that the class of predictable processes is rich.

Proposition 4.2.1. *If H is a separable Hilbert space, then the following σ-algebras coincide.*

1. $\mathcal{P}_1 = \sigma($*adapted continuous processes*$)$
2. $\mathcal{P}_2 = \sigma($*adapted left continuous processes with right hand limits*$)$
3. $\mathcal{P}_3 = \sigma($*adapted left continuous processes*$)$
4. $\mathcal{P}_T$

We are now in position to characterize the proper class of integrands.

Theorem 4.2.1. *There is an explicit characterization of $\overline{\mathcal{E}}$ given by*

$$\begin{aligned}\mathcal{N}_W^2 &= \mathcal{N}_W^2(0,T;H)\\ &= \Big\{\Phi : [0,T] \times \Omega \to L_2^0 : \Phi \text{ is } L_2^0\text{-predictable and } \|\Phi\|_T < \infty\Big\}\\ &= L^2\big([0,T] \times \Omega, \mathcal{P}_T, m \times P; L_2^0\big).\end{aligned}$$

In fact

1. *If a mapping Φ from Ω_T in to $B(U,H)$ is $B(U,H)$-predictable, then Φ is also L_2^0-predictable. In particular elementary processes are L_2^0-predictable.*
2. *If Φ is an L_2^0- predictable process with $\| \Phi \|_T < \infty$ then there exists a sequence $\Phi_n \subset \mathcal{E}$ with $\lim_{n\to\infty} \| \Phi_n - \Phi \|_T = 0$.*

In the last case we define $Int(\Phi) = \lim_{n\to\infty} \Phi_n$, where the limit is in $(M_T^2(H), \| \ \|_{M_T^2(H)})$. It is direct that this limit exists and is independent of the choice of the approximating sequence. Both Itô's Isometry and Corollary 4.1.1 still hold for $\Phi \in \mathcal{N}_W^2$

4.3 Stochastic integral for cylindrical Wiener processes

In order to extend the construction of stochastic integrals to the case where the covariance operator Q is only bounded but not necessarily of finite trace, one needs to extend the notion of a Q- Wiener process. We would like to consider a Wiener process $\{W(t)\}_{t\geq 0}$ with covariance operator Q such that $Tr(Q)=\infty$, for example, $Q=I$.If $Tr(Q)=\infty$, then the sum

$$W(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k$$

does not even converge in $L^2(\Omega,U)$, since

$$\mathbb{E}\left(\left\|\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k\right\|\right)=t\;Tr(Q)=\infty.$$

The above sum does converge in a suitable bigger space $\tilde{H}$ where it defines an $\tilde{H}$-valued nuclear Wiener process. Neverthless the formal sum (3.5.1) is called the **cylindrical Wiener process**.
We begin with the remark that when $Tr(Q)<\infty$ the inclusion

$$J:(U_0,\langle\cdot\,,\cdot\rangle_0)\to(U,\langle\cdot\,,\cdot\rangle),\quad \text{with } x\mapsto Jx=x,$$

is Hilbert-Schmidt. Indeed, if $\{e_k\}$ is an ortonormal basis for U_0, then $f_k=Q^{-1/2}e_k$, $k\in\mathbb{N}$ is an orthonormal basis for $Ker(Q)^{\perp}$, thus

$$\begin{aligned}\|J\|^2_{B_2(U_0,U)}&=\sum_k\langle Je_k,Je_k\rangle_U=\sum_k\langle e_k,e_k\rangle_U\\&=\sum_k\langle Q^{1/2}Q^{-1/2}e_k,Q^{1/2}Q^{-1/2}e_k\rangle_U\\&=\sum_k\langle Q^{1/2}f_k,Q^{1/2}f_k\rangle_U=Tr(Q)<\infty,\end{aligned}$$

Therefore, if $Tr(Q)=\infty$, then we need to consider another Hilbert space $(\tilde{U},[\cdot,\cdot],[\![\cdot]\!])$ such that there is an embedding $J:U_0\to\tilde{U}$ which is Hilbert-Schmidt. This can always be done (see [KV])

Proposition 4.3.1 (Cylindrical Wiener process). *Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space and let $Q\in B(U)$, $Q\geq 0$. If $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for the Cameron Martin space $U_0=Q^{1/2}(U)$ and $\{\beta_k\}_{k\in\mathbf{N}}$ is a family of independent real-valued Brownian motions and $(\tilde{U},[\cdot,\cdot],[\![\cdot]\!])$ is a separable Hilbert space such that there is an embedding $J:U_0\to\tilde{U}$ is Hilbert-Schmidt, then $\tilde{Q}:\tilde{U}\to\tilde{U}$ defined by $\tilde{Q}:=JJ^*$ is bounded, $\tilde{Q}\geq 0$, $Tr(\tilde{Q})<\infty$, and the series*

$$\tilde{W}(t)=\sum_{k=1}^{\infty}\beta_k(t)Je_k,\quad t\in[0,T],\tag{4.3.1}$$

converges in $\mathcal{M}^2_T(\tilde{U})$ and defines a $\tilde{Q}$-Wiener process on $\tilde{U}$. Moreover,

$$\tilde{U_0}:=\tilde{Q}^{1/2}(\tilde{U})=J(U_0)$$

and, for all $u\in U_0$,

$$\|u\|_0=[\![\tilde{Q}^{-1/2}Ju]\!]:=[\![Ju]\!]_0.$$

That is, $J:U_0\to\tilde{U_0}$ is an isometric isomorphism.

Proof: First of all we wiil prove th properties of $\tilde{Q}$. Obviously $\tilde{Q}\in B(\tilde{U})$ and $\tilde{Q}$ is selfadjoint since $\tilde{Q}^*=(JJ^*)^*=JJ^*=\tilde{Q}$. Moreover, for each $u\in\tilde{U}$ we have

$$[\tilde{Q}u,u]_{\tilde{U}}=[JJ^*u,u]_{\tilde{U}}=||J^*u||^2_{U_0}\geq 0.$$

Furthermore, if $(\phi_k)_k$ is an orthonormal basis for $\tilde{U}$ then ,

$$Tr(\tilde{Q})=\sum_{k=1}^{\infty}[\tilde{Q}\phi_k,\phi_k]_{\tilde{U}}=\|\,J^*\,\|^2_{B_2(\tilde{U},U_0)}=\|\,J\,\|^2_{B_2(U_0,\tilde{U})}<\infty.$$

Next, we wil show first that $\{\tilde{W}(t)\}_{t\in[0,T]}$ is a $\tilde{Q}$-Wiener process on $\tilde{U}$. For $j\in\mathbb{N}$ set $\xi_j(t)=\beta_j(t)Je_j$, $t\in[0,T]$ and define

$$\mathcal{G}_t := \sigma\Big(\bigcup_{j=1}^{\infty}\sigma(\{\beta_j(s):\ \ ,0\le s\le t\})\Big),\quad t\in[0,T].$$

Then for each fixed $j\in\mathbb{N}$ the process $\{\xi_j(t)\}_{t\in[0,T]}$ is a continuous $\tilde{U}$-valued square integrable martingale with respect to $\{\mathcal{G}_t\}_{t\ge 0}$. For the martingale property we will use the following Lemma from real valued martingales.

Lemma 4.3.1. *Let $X\in L^1(\Omega,\mathfrak{F},\mathbb{P};\mathbb{R})$ be a random variable and $\mathcal{G}_1,\mathcal{G}_2\subset\mathfrak{F}$ be σ-algebras. If $\mathcal{G}_1$ is independent of $\sigma(\sigma(X)\cup\mathcal{G}_2)$, then*

$$\mathbb{E}\big(X\big|\sigma(\mathcal{G}_1\cup\mathcal{G}_2)\big)=\mathbb{E}\big(X\big|\mathcal{G}_2\big).$$

Indeed, take $0\le s\le t\le T$ and then

$$\mathbb{E}\Big(\beta_j(t)\Big|\mathcal{G}_s\Big)=\mathbb{E}\Big(\beta_j(t)\Big|\sigma(\{\beta_j(u)\}_{u\le s}\Big)=\beta_j(s),$$

This follows from Lemma 4.3.1 with $X=\beta_j(t)$, $\mathcal{G}_2=\sigma(\{\beta_j(u)\}_{u\le s})$, and $\mathcal{G}_1=\sigma\Big(\bigcup_{k\ne j}\{\beta_k(u)\}_{u\le s}\Big)$. The independency of $\sigma(\sigma(X)\cup\mathcal{G}_2)$ and $\mathcal{G}_1$ follows by the independency of $(\beta_k)_{k=1}^{\infty}$. Therefore

$$\tilde{W}_n(t):=\sum_{j=1}^{n}\beta_j(t)Je_j,\quad t\in[0,T]$$

is also in $M_T^2(\tilde{U})$. We will now show that $((\tilde{W}_n(t))_t)_{n=1}^{\infty}$ is Cauchy in $M_T^2(\tilde{U})$. Indeed for $m>n$,

$$\begin{aligned}
&\|\tilde{W}_m-\tilde{W}_n\|^2_{\mathcal{M}_T^2(\tilde{U})}\\
&=\mathbb{E}\Big(\|\tilde{W}_m(T)-\tilde{W}_n(T)\|^2\Big)=\mathbb{E}\Big(\Big\|\sum_{j=n+1}^{m}\beta_j(T)Je_j\Big\|^2\Big)\\
&=T\sum_{j=n+1}^{m}\|Je_j\|^2\to 0,\quad n\to\infty.
\end{aligned}$$

since $J:U_0\to\tilde{U}$ is a Hilbert-Schmidt operator, Therefore, $\tilde{W}_n$ converges in $\mathcal{M}_T^2(\tilde{U})$ and its limit $\tilde{W}\in\mathcal{M}_T^2(\tilde{U})$ is a continuous process. It follows easily that $\tilde{W}(0)=0$ and that the increments are independent. It remains to show that for $0\le s<t$, $\tilde{W}(t)-\tilde{W}(s)\ N(0,(t-s)\tilde{Q})$. For all $u\in\tilde{U}$,

$$[\tilde{W}(t)-\tilde{W}(s),u]=\sum_{j=1}^{\infty}(\beta_j(t)-\beta_j(s))[Je_j,u].$$

In addition,

$$\mathbb{E}\left(\left|\sum_{j=n+1}^{m}\{\beta_j(t)-\beta_j(s)\}[Je_j,u]\right|^2\right)=$$

$$=\sum_{j=n+1}^{m}\mathbb{E}\left((\beta_j(t)-\beta_j(s))^2[Je_j,u]^2\right)\le\|u\|^2(t-s)\sum_{j=n+1}^{m}\|Je_j\|$$

which converges to zero as $n,m\to\infty$ since J is Hilbert-Schmidt. Thus $[\tilde{W}(t)-\tilde{W}(s),u]$ is Gaussian being an $L^2(\Omega,\mathfrak{F},\mathbb{P};\mathbb{R})$ limit of Gaussian random variables. To compute the mean we have

$$\mathbb{E}([\tilde{W}(t)-\tilde{W}(s),u])=\sum_{n=1}^{\infty}\mathbb{E}(\beta_n(t)-\beta_n(s))[Je_n,u]=<0,u>.$$

Thus $m = 0$. To compute the covariance operator of the increments, take $u, v \in \tilde{U}$, and

$$\mathbb{E}\big([\tilde{W}(t) - \tilde{W}(s), u] \cdot [\tilde{W}(t) - \tilde{W}(s), v]\big) = \sum_{k=1}^{\infty}(t-s)[Je_k, u][Je_k, v]$$
$$= \sum_{k=1}^{\infty}(t-s)\langle e_k, J^*u\rangle_0 \langle e_k, J^*v\rangle_0 = (t-s)\langle J^*u, J^*u\rangle_0 = (t-s)[JJ^*u, v],$$

where we used that $\mathbb{E}\big(\beta_j(t) - \beta_j(s))(\beta_k(t) - \beta_k(s)\big) = \delta_{jk}$. Thus, $\tilde{Q} = JJ^*$. For the remain of the proof we use a result from operator theory. For details see [KV].
Now we are ready to define the stochastic integral with respect to a cylindrical Wiener process. Since $Tr(\tilde{Q}) < \infty$, we can integrate processes $\{\Phi(t)\}_{t\in[0,T]}$ which are in $L^2(([0,T] \times \Omega, \mathcal{P}_T, m \otimes \mathbb{P}); B_2(\tilde{U_0}, H))$. But we are aiming at integrating processes with values in $B_2(U_0, H)$. Since $J : U_0 \to \tilde{U_0}$ is isometrically isomorphsmi we have that if $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for U_0, then $\{Je_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for $\tilde{U_0}$. This leads us to the very important remark that,

$$\Phi \in B_2(U_0, H) \quad \Leftrightarrow \quad \Phi J^{-1} \in B_2(\tilde{U_0}, H).$$

Indeed,

$$\|\Phi\|^2_{B_2(U_0,H)} = \sum_{k=1}^{\infty}\langle \Phi e_k, \Phi e_k\rangle = \sum_{k=1}^{\infty}\langle \Phi J^{-1}Je_k, \Phi J^{-1}Je_k\rangle$$
$$= \|\Phi J^{-1}\|^2_{B_2(\tilde{U}_0,H)}.$$

Note that an $B_2(U_0, H)$-valued process is $\{\Phi(t)\}_{t\in[0,T]}$ is $B_2(U_0, H)$-predictable if and only if $\{\Phi(t)J^{-1}\}_{t\in[0,T]}$ is $L_2(\tilde{U_0}, H)$-predictable.

Definition 4.3.1 (Integral with respect to a cylindrical Wiener process)**.** *Let $\{W(t)\}_{t\in[0,T]}$ be a cylindrical Wiener process. For processes $\{\Phi(t)\}_{t\in[0,T]} \in \mathcal{N}^2_W := L^2(([0,T] \times \Omega, \mathcal{P}_T, m \otimes \mathbb{P}); B_2(U_0, H))$ we define the stochastic integral by*

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) := \int_0^t \Phi(s)J^{-1}\,\mathrm{d}\tilde{W}(s), \quad t \in [0,T],$$

Remark 4.3.1. *The cylindrical Wiener process $\{\tilde{W}(t)\}_{t\in[0,T]}$ constructed in Proposition 4.3.1 depends on J but $\int_0^t \Phi\,\mathrm{d}W$ does not.*

Part III

Stochastic evolution equations with additive noise

Chapter 5

The deterministic abstract Cauchy problem

In this chapter we will study how the variation of constants formula

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)\ ds$$

which is the solution to the inhomogeneous problem

$$\begin{cases} u'(t) &= Au(t) + f(t), \\ u(0) &= x \in X, \end{cases} \quad .$$

where X is Banach and $A \in B(X)$ can be generalised to give a solution formula for the more general case where A is a generator of a strongly continuous semigroup. The notion of a classical solution to the inhomogeneous abstract Cauchy problem (see section (5.3)) leads us to the so-called problem of maximal regularity(e.g see [PZ]). In order to overcome this we introduce two alternative notions of solutions in terms of the integrated equation, the so-called strong and weak solutions. The main result of this chapter is Theorem (5.3.1) which is due to Ball [BAL].

5.1 Overview of Weak and Weak* topologies

In this section we overview the very basics of weak and weak* topologies. For more details see [BR] chapter III.

Definition 5.1.1. *Let $X \neq \emptyset$ be a non empty set, $\mathfrak{X} = \{(Y_a, \mathcal{S}_a) : a \in J\}$, a collection of topological spaces indexed by J and $\mathfrak{F} = \{f_a : X \to Y_a : a \in J\}$ a family of maps. The weakest topology with respect to which the functions $f \in \mathfrak{F}$ are continuous is called the $\sigma(\mathbf{X}, \mathfrak{F})$ topology on X,.*

Remark 5.1.1. Note that the existence of the $\sigma(\mathbf{X}, \mathfrak{F})$ topology is clear by the fact that the intersection of a family of topologies on X is also a topology. Moreover,

$$\sigma(X, \mathfrak{F}) = \bigcap \{\mathcal{T} : \mathcal{T} \text{ topology on } X \text{ and } \mathcal{T} \supset \cup_{a\in J}\{f_a^{-1}(V) : V \in \mathcal{S}_a\}\}.$$

Theorem 5.1.1. *Let $X \neq \emptyset$ be a non empty set, $\mathfrak{X} = \{(Y_a, \mathcal{S}_a) : a \in J\}$, a collection of topological spaces indexed by J and $\mathfrak{F} = \{f_a : X \to Y_a : a \in J\}$ a family of maps. If for each $a \in J$, the topological space $(Y_a, \mathcal{S}_a)$ is Hausdorff and if $\mathfrak{F}$ separates the points of X, then $\sigma(X, \mathfrak{F})$ is Hausdorff.*

Proof: Let $x, y \in X$ with $x \neq y$. Then, there exists $a \in J$ such that $f_a(x) \neq f_a(y)$ and since $(Y_a, \mathcal{S}_a)$ is Hausdorff, there exist $U, V \in \mathcal{S}_a$ such that $f_a(x) \in U$, $f_a(y) \in V$ and $U \cap V = \emptyset$. But then $f_a^{-1}(U), f_a^{-1}(V) \in \sigma(X, \mathfrak{F})$ with $f_a^{-1}(U) \cap f_a^{-1}(V) = \emptyset$ and $x \in f_a^{-1}(U)$, $y \in f_a^{-1}(V)$. Therefore, $\sigma(X, \mathfrak{F})$ is Hausdorff.

Definition 5.1.2. *Let $(X, \mathcal{T})$ be a topological space. A class of open sets $\mathcal{B} \subset \mathcal{T}$ is called **base** for the topology $\mathcal{T}$ if-f each element of $\mathcal{T}$ can be written as union of elements of $\mathcal{B}$.*

Proposition 5.1.1. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$ a collection of open sets. Then the following statements are equivalent*

1. *$\mathcal{B}$ is a base for $\mathcal{T}$.*
2. *For each $\emptyset \neq U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Proof:

(1) $\rightarrow$ (2) Let $\emptyset \neq U \in \mathcal{T}$ and $x \in U$. Then, $U = \cup_{i \in I} B_i$, where I is an index set and $\{B_i\}_{i \in I} \subset \mathcal{B}$. So, there exists $i \in I$, such that $x \in B_i \subset U$

(2) $\rightarrow$ (3) If $U \in \mathcal{T}$ and $x \in U$, then by assumption there exists $V_x \in \mathcal{B}$, such that $\{x\} \subset V_x \subset U$. Therefore, $U = \cup_{x \in U} V_x$.

Proposition 5.1.2. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$ a base for $\mathcal{T}$. Then,*

$$\mathcal{T} = \{G \subset X : \forall x \in G, \exists B \in \mathcal{B} : x \in B \subset G\}.$$

Proof: Use similar arguments as in the proof of Proposition (5.1.1)

Definition 5.1.3. *Let $(X,\mathcal{T})$ be a topological space. A collection of open sets $\mathcal{S} \subset \mathcal{T}$ is said to be a* ***sub-base*** *for $\mathcal{T}$ if-f the collection of intersections of finite families of members of $\mathcal{S}$ is a base for $\mathcal{T}$.*

Proposition 5.1.3. *Let $X \neq \emptyset$ be a non empty set and $\mathcal{S}$ a collection of subsets of X which covers X. This is, for each $x \in X$, there exists $A \in \mathcal{S}$ such that $x \in A$. Denote by $\mathcal{B}$ the collection of intersections of finite families of elements of $\mathcal{S}$. Then,*

$$\mathcal{T} = \{\cup_{i \in I} B_i : \{B_i\}_{\imath \in I} \subset \mathcal{B}\} \cup \{\emptyset\}$$

is a topology on X and is the weakest topology on X which contains the family $\mathcal{S}$. Moreover, $\mathcal{S}$ is a sub-base for $\mathcal{T}$ and $\mathcal{B}$ is a base for $\mathcal{T}$.

Proof: It is obvious that $\emptyset, X \in \mathcal{T}$ and that $\mathcal{T}$ is closed under arbitrary unions. It remains to show, that if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$. If A or B is empty then the result is trivial. Suppose that $A, B \neq \emptyset$. Then, $A = \cup_\alpha A_\alpha$ and $B = \cup_\beta B_\beta$, where $\{A_\alpha\}_\alpha, \{B_\beta\}_\beta \subset \mathcal{B}$. Therefore,

$$A \cap B = \cup_{\alpha,\beta}(A_\alpha \cap B_\beta).$$

But since each A_α and B_β is a finite intersection of elements of $\mathcal{S}$, the same is true for all $A_\alpha \cap B_\beta$. This means that $A_\alpha \cap B_\beta \in \mathcal{B}$. Therefore, $A \cap B \in \mathcal{T}$. So we have shown that $\mathcal{T}$ is a topology on X. Let $\mathcal{T}'$ be a topology on X such that $\mathcal{S} \subset \mathcal{T}'$. Then, $\mathcal{B} \subset \mathcal{T}'$ and finally $\mathcal{T} \subset \mathcal{T}'$.

Corollary 5.1.1. *Let $X \neq \emptyset$ be a non empty set, $\mathfrak{X} = \{(Y_a, \mathcal{S}_a) : a \in J\}$, a collection of topological spaces indexed by J and $\mathfrak{F} = \{f_a : X \to Y_a : a \in J\}$ a family of maps. Then*

$$\mathcal{S} = \{f_a^{-1}(V) : a \in J, V \in \mathcal{S}_a\}$$

is a sub-base for the $\sigma(X,\mathcal{F})$ topology.

Definition 5.1.4. *Let $(X, \| \, \|_X)$ be a normed space. The $\mathcal{T}_w := \sigma(X, X^*)$ topology on X is called* ***the weak topology*** *on X.*

Proposition 5.1.4. *Let $(X, \| \, \|_X)$ be a normed space. A non empty set $G \neq \emptyset$ is* ***weakly open*** *if and only if for each $a \in G$, there exist $n \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$ such that*

$$\mathcal{N}(a, x_1^*, \ldots, x_n^*, \varepsilon) = \{x \in X : |x_j^*(x) - x_j^*(a)| < \varepsilon, \, j = 1, \ldots, n\} \subset G.$$

Proof: As we have already discussed in Corollary (5.1.1), $\mathcal{S} = \{(x^*)^{-1}(U) : x^* \in X^*, U \subset \mathbb{C} \,\text{open}\}$ is a sub-base for $\mathcal{T}_w$. In other words

$$\mathcal{B} := \{\cap_{i=1}^k S_i : k \in \mathbb{N}, \, \{S_i\}_{i=1}^k \subset \mathcal{S}\}$$

is a base for $\mathcal{T}_w$. Therefore by Proposition (5.1.3) a $\emptyset \neq G \subset X$ is weakly open if and only if for each $a \in G$, there exists B such that $a \in B \subset G$, where B has the form

$$B = (x_1^*)^{-1}(U_1) \cap (x_2^*)^{-1}(U_2) \cap \ldots \cap (x_n^*)^{-1}(U_n),$$

for some $n \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $U_1, \ldots, U_n$ open subsets in $\mathbb{C}$. But for a fixed $j \in \{1, \ldots, n\}$, $a \in (x_j^*)^{-1}(U_j) \Leftrightarrow x_j^*(a) \in U_j$ and since U_j is open there exists $\varepsilon_j > 0$ such that $B(x_j^*(a), \varepsilon_j) \subset U_j$. So we can easily derive that G is weakly open if and only if for each $a \in G$, there exists B such that $a \in B \subset G$, where B has the form

$$B = (x_1^*)^{-1}(B(x_1^*(a), \varepsilon_1)) \cap \ldots \cap (x_n^*)^{-1}(B(x_n^*(a), \varepsilon_n)),$$

for some $n \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$. By taking $\varepsilon = \min\{\varepsilon_i\}_{i=1}^n$, we conclude that G is weakly open if and only if for each $a \in G$, there exists B such that $a \in B \subset G$, where B has the form

$$B = (x_1^*)^{-1}(B(x_1^*(a), \varepsilon)) \cap \ldots \cap (x_n^*)^{-1}(B(x_n^*(a), \varepsilon)) = \{x \in X : |x_j^*(x) - x_j^*(a)| < \varepsilon, \, j = 1, \ldots, n\},$$

for some $n \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$.

Remark 5.1.2. Each $\mathcal{N}(a, x_1^*, \ldots, x_n^*, \varepsilon)$ is weakly open and contains a. Therefore, an equivalent definition of a weakly open set $G \neq \emptyset$ is*: G is weakly open if and only if it can be written as a union of elements which have the form* $\mathcal{N}(a, x_1^*, \ldots, x_n^*, \varepsilon)$, $a \in G$, $n \in \mathbb{N}$, $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ *and* $\varepsilon > 0$.

Definition 5.1.5. *Let* $(X, \mathcal{T})$ *be a topological space and* $x \in X$. *A collection* $\mathcal{N}_x$ *of neighborhoods of* x *is said to be a* ***neighborhood base*** *at* x *if-f for all* $G \in \mathcal{T}$ *such that* $x \in G$, *there exists* $N \in \mathcal{N}_x$ *such that* $x \in N$ *and* $N \subset G$.

Corollary 5.1.2. *Let* $(X, \| \|_X)$ *be a normed space and* $a \in X$. *The family* $\mathcal{N}_\alpha = \{\mathcal{N}(a, x_1^*, \ldots, x_n^*, \varepsilon) : n \in \mathbb{N}, \varepsilon > 0, x_1^*, x_2^*, \ldots, x_n^* \in X^*\}$ *is an open neighborhood base at* a *for the weak topology on* X.

Proposition 5.1.5. *Let* $(X, \| \|_X)$ *be a normed space. Then* $(X, \mathcal{T}_w)$ *is Hausdorff.*

Proof: The space $(\mathbb{C}, \mathcal{T}_{||})$ is Hausdorff, as a metric space and by the Hahn-Banach theorem X^* separates the points of X. Therefore the result follows by Theorem (5.1.1).

Definition 5.1.6. *Let* $(X, \| \|_X)$ *be a normed space. The* ***weak* topology*** *on* X^* *is the* $\sigma(X^*, \hat{X})$ *topology, where*

$$\hat{X} = \{\hat{x} : X^* \to \mathbb{C}, \; \hat{x}(x^*) = x^*(x) : x \in X\}.$$

Proposition 5.1.6. *The topological space* $(X^*, \mathcal{T}_{w^*})$ *is Hausdorff.*

Proof: The space $(\mathbb{C}, \mathcal{T}_{||})$ is Hausdorff, as a metric space and $\hat{X}$ separated the points of X^*. Indeed if $x^*, y^* \in X^*$ with $x^* \neq y^*$, then there exists $x \in X$ such that $x^*(x) \neq y^*(x) \Leftrightarrow \hat{x}(x^* \neq \hat{x}(y^*))$. Therefore the result follows by Theorem (5.1.1).

Remark 5.1.3. It is well known that $\hat{X} \subset X^{**}$. Therefore, the weak* topology on X^* is weaker than the weak topology on X^*, i.e the $\sigma(X^*, X^{**})$ topology. Of course, if X is reflexive, then the two topologies coincide. In fact, the inverse statement is also true, i.e X is reflexive if and only if the weak topology and the weak* topology on X^* coincide.

Proposition 5.1.7. *Let* $(X, \| \|_X)$ *be a normed space. A non empty subset* $G \subset X^*$ *is weakly* open if and only if for each* $l \in G$, *there exist* $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ *and* $\varepsilon > 0$ *such that*

$$\mathcal{N}(l, x_1, x_2, \ldots, x_n, \varepsilon) = \{x^* \in X^* : |x^*(x_i) - l(x_i)| < \varepsilon, \, i = 1, \ldots, n\} \subset G.$$

Proof: Use similar arguments as in the proof of Proposition (5.1.4).

Corollary 5.1.3. *Let* $(X, \| \|_X)$ *be a normed space and* $l \in X^*$. *The family* $\mathcal{N}_l = \{\mathcal{N}(l, x_1, \ldots, x_n, \varepsilon) : n \in \mathbb{N}, \varepsilon > 0, x_1, x_2, \ldots, x_n \in X\}$ *is an open neighborhood base at* l *for the weak* topology on* X^*.

5.2 Duality of densely defined linear operators

Definition 5.2.1. *Let X, Y be two Banach spaces and $A : X \supset D(A) \to Y$ a densely defined linear operator. The **adjoint** of A is the operator $A^* : Y^* \supset D(A^*) \to X^*$, where*

$$D(A^*) = \{y^* \in Y^* : \exists x^* \in X^* : < x^*, x >=< y^*, Ax >, \ \forall x \in D(A).\} \ \textit{and} \ A^* y^* = x^*.$$

Remark 5.2.1. Note that in the definition of $D(A)$, the element x^* is unique, since $D(A)$ is dense in X, therefore A^* is well defined. Moreover it is easy to check that $D(A^*)$ is a linear subspace of Y^* and A^* is a linear operator.

Lemma 5.2.1. *Let X be a Banach space and consider a nonempty subset $\emptyset \neq V \subset X$. Then, the **annihilator***

$$V^\perp = \{x^* \in X^* : < x^*, v >= 0 \quad \forall v \in V \}$$

is weakly closed in X^*.*

Proof: Let $y^* \in X^* \backslash V^\perp$. Then, there exists $v \in V$ such that $< y^*, v > \neq 0$. Observe now, that the set

$$U = \mathcal{N}(y^*, v, \frac{| < y^*, v > |}{2}) = \{x^* \in X^* : \ |y^*(v) - x^*(v)| < \frac{| < y^*, v > |}{2}\}$$

is weakly* open, contains y^* and is contained in $X^* \backslash V^\perp$, since $U \cap V^\perp = \emptyset$. Therefore $X^* \backslash V^\perp$ is weakly* open.

Proposition 5.2.1. *Let X be a Banach space. Then, a linear subspace $F \subset X^*$ is weakly* dense if and only if it separates the points of X.*

Proposition 5.2.2. *Let X, Y be two Banach space and $A : X \supset D(A) \to Y$ a densely defined linear operator. Then, the following assertions are valid*

(i) *The adjoint $A^* : Y^* \supset D(A^*) \to X^*$ is weakly* closed, i.e the graph G_{A^*} of A^* is weakly* closed in $Y^* \times X^*$.*

(ii) *If in addition A is closed, then A^* is densely defined, i.e $D(A^*)$ is weakly* dense in Y^*.*

Proof: First of all, let us note that $Y^* \times X^*$ coincides with $(Y \times X)^*$ via the mapping $Y^* \times X^* \ni (y^*, x^*) \to (y \otimes x)^* \in (Y \times X)^*$, where

$$< (y \otimes x)^*, (y, x) >:=:< (y^*, x^*), (y, x)) >=< y^*, y > + < x^*, x >, \quad \text{for all } x \in X, \ y \in Y.$$

(i) We have the following equivalences

$$(y^*, x^*) \in G_{A^*} \Leftrightarrow < y^*, Ax > = < x^*, x >, \ \forall x \in D(A) \Leftrightarrow < (y^*, x^*), (-Ax, x) > = 0, \ \forall x \in D(A).$$

Therefore, $G_{A^*} = (\rho(G_A))^\perp$, where $\rho : X \times Y \to Y \times X$ with $\rho(x, y) = (-y, x)$. Thus, by Lemma (5.2.1) we conclude that G_{A^*} is weakly* closed.

(ii) It is enough to show that $D(A^*)$ separates the points of Y. To this aim consider $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Then, $(0_X, y_1 - y_2) \in X \times Y \backslash \{0_{X \times Y}\}$, therefore $(0, y_1 - y_2) \ni G_A$. But since G_A is closed in $X \times Y$, by virtue of the Hahn-Banach Theorem we can choose a $(x^*, y^*) \in X^* \times Y^*$, such that $(x^*, y^*) \in (G_A)^\perp$ and $< (x^*, y^*), (0, y_1 - y_2) > \neq 0 \Leftrightarrow y^*(y_1) \neq y^*(y_2)$. So, it remains to show that $y^* \in D(A^*)$. This follows by the fact that for each $x \in D(A)$ we have $(x, Ax) \in G_A$. Therefore, $< (x^*, y^*), (x, Ax) >= 0 \ \Leftrightarrow < -x^*, x > = < y^*, Ax >$. This means that $y^* \in D(A^*)$ and $A^* y^* = -x^*$.

Proposition 5.2.3. *Let X, Y be two Banach spaces and $A : X \supset D(A) \to Y$ be a closed linear and densely defined operator. If for some $x \in X$, $y \in Y$ it holds that*

$$< y^*, y > = < A^* y^*, x >, \quad \textit{for each} \quad y^* \in D(A^*),$$

then $x \in D(A)$ and $Ax = y$.

Proof: We want to show that $(x, y) \in G_A$. Since G_A is closed in $X \times Y$, it is enough to show that $< (x^*, y^*), (x, y) >= 0$, for each $(x^*, y^*) \in (G_A)^\perp$. To this aim, fix a $(x^*, y^*) \in (G_A)^\perp$. With the same arguments as in the proof of Proposition (5.2.2) we obtain that $y^* \in D(A^*)$ and $A^* y^* = -x^*$. Combining this result with the assumption we get

$$< (x^*, y^*), (x, y) > = < -A^* y^*, x > + < y^*, y > = 0.$$

5.3 Weak solutions and the variation of constants formula

Let X be a Banach space and $A : X \supset D(A) \to X$ the infinitesimal generator of a C_0-semigroup $\{T_t : t \geq 0\}$ on X. In this section, we consider the **Inhomogeneous Abstract Cauchy Problem (IACP)**

$$\begin{cases} u'(t) \; = \; Au(t) + f(t), & t \in [0,T]; \\ u(0) \; = \; x, & . \end{cases}$$

where $x \in X$, $T \geq 0$ and $f \in L^1([0,T], X)$.

Definition 5.3.1. *A function $u : [0,T] \to X$ is called a **classical solution of the (IACP)** if-f it satisfies the following conditions*

- *u is continuous on $[0,T]$*
- *u is continuously differentiable on $(0,T]$*
- *$u(t) \in D(A)$, for each $t \in (0,T]$*
- *$u(0) = x$ and $u'(t) = Au(t) + f(t)$, for all $t \in (0,T]$.*

Proposition 5.3.1. *If $f = 0$ and $x \in D(A)$, then the **homogeneous abstract Cauchy Problem***

$$\begin{cases} u'(t) \; = \; Au(t), & t \in [0,T]; \\ u(0) \; = \; x, & . \end{cases}$$

admits a unique classical solution which is given by $u(t) = T_t x$, $t \in [0,T]$.

Proof: The existence is a direct consequence of Theorem (1.4.1(1)) and Corollary (1.4.1(i)). For the uniqueness, use exactly the same reasoning as in the proof of Proposition (1.4.2).

Definition 5.3.2. *A function $u : [0,T] \to X$ is called a **strong solution of the (IACP)** if-f it satisfies the following conditions*

- *$u \in L^1([0,T], X)$.*
- *for each $t \in [0,T]$, $(\int_0^t u(s)\ ds) \in D(A)$.*
- *for each $t \in [0,T]$, $u(t) = x + A \int_0^t u(s)\ ds + \int_0^t f(s)\ ds$.*

Definition 5.3.3. *A function $u : [0,T] \to X$ is called a **weak solution of the (IACP)** if-f it satisfies the following conditions*

- *$u \in L^1([0,T], X)$.*
- *for each $t \in [0,T]$ and for all $x^* \in D(A^*)$ we have*

$$< x^*, u(t) >=< x^*, x > + \int_0^t < A^*x^*, u(s) > \ ds \ + \ \int_0^t < x^*, f(s) > \ ds.$$

Remark 5.3.1. We can trivially check that every strong solution of the (IACP) is also a weak solution.

Proposition 5.3.2. *Each weak solution of the (IACP) is a strong solution*

Proof: Observe that for all $t \in [0,T]$ and $x^* \in D(A^*)$ we have

$$< x^*, u(t) - x - \int_0^t f(s)\ ds > = < A^*x^*, \int_0^t u(s)\ ds > .$$

The result now follows by Proposition (5.2.3).

Theorem 5.3.1. *For each $x \in X$ and $f \in L^1([0,T],X)$ the (IACP) admits a unique strong solution which is given by the* ***variation of constants formula***

$$u(t) = T_t x + \int_0^t T_{t-s} f(s)\ ds. \tag{5.3.1}$$

If $f \in L^p(0,T;X)$, for $1 \le p < \infty$, then $u \in L^p(0,T;X)$.

Proof: First of all, note that due to the properties of convolutions of functions and the fact that $t \to T_t$ is continuous (thus integrable) and $f \in L^1$, we can easily conclude that u is well defined and in L^1 and moreover if $f \in L^p$, then $u \in L^p$. By virtue of Proposition (5.3.2), in order to prove that the function u which is given by the variation formula (5.3.1) is a strong solution of the (IACP), it is enough to show that u is a weak solution. For each $x^* \in D(A^*)$, the function $[0,T] \ni t \to < x^*, T_t x >$ is differentiable for each $x \in X$ with

$$\frac{d}{dt} < x^*, T_t x > = < A^* x^*, T_t x >, \quad \text{for all } t \in [0,T].$$

Indeed, if $x \in D(A)$ the above formula follows by Theorem (1.4.1). For an arbitrary $x \in X$, we can use Proposition (A.7.1) since $D(A)$ is dense in X and $\sup_{t\in[0,T]} \| T_t \| < \infty$ because of the strong continuity of the semigroup. After these observations, for each $t \in [0,T]$ and $x^* \in D(A^*)$ we have

$$\begin{aligned}
\int_0^t < A^*x^*, u(s) > \ ds &= \int_0^t < A^*x^*, \int_0^s T_{s-r} f(r)\ dr > \ ds \ + \ \int_0^t < A^*x^*, T_s x > \ ds \\
&= \int_0^t \int_0^s < A^*x^*, T_{s-r} f(r) > \ dr\ ds \ + \ \int_0^t < A^*x^*, T_s x > \ ds \\
&= \int_0^t \int_r^t < A^*x^*, T_{s-r} f(r) > \ ds\ dr \ + \ \int_0^t < A^*x^*, T_s x > \ ds \\
&= \int_0^t \int_r^t \frac{d}{ds} < x^*, T_{s-r} f(r) > \ ds\ dr \ + \ \int_0^t \frac{d}{ds} < x^*, T_s x > \ ds \\
&= \int_0^t (< x^*, T_{t-r} f(r) > - < x^*, f(r) >)\ dr \ + \ < x^*, T_t x > - < x^*, x > \\
&= < x^*, u(t) > - < x^*, x > - \int_0^t < x^*, f(r) > \ dr.
\end{aligned}$$

In order to show the uniqueness, suppose that $\bar{u}$ is another strong solution of the (IACP) and set $w = u - \bar{u}$. Then, it is direct that w is integrable $\int_0^t w(s)\ ds \in D(A)$ and

$$w(t) = A \int_0^t w(s)\ ds, \ \text{ for all } t \in [0,T]$$

Now set

$$z(t) = \int_0^t \int_0^s w(r)\ dr\ ds.$$

By using Theorem (A.3.2) and Proposition (A.4.2) we obtain that $z(t) \in D(A)$ and

$$z'(t) = \int_0^t w(s)\ ds = \int_0^t A \int_0^s w(r)\ dr\ ds = Az(t)$$

for all $t \in [0,T]$. But this means that $z = 0$. Indeed, for a fixed $t \in [0,T]$, consider the differentiable function $g : [0,t] \to X$, $g(s) = T_{t-s} z(s)$ with derivative

$$g'(s) = -AT_{t-s} z(s) + T_{t-s} z'(s) = 0.$$

So g is constant, therefore

$$z(t) = g(t) = g(0) = T_t z(0) = 0.$$

Since $z = 0$ we deduce that $u = \bar{u}$ almost everywhere and the proof is complete.

Chapter 6

The stochastic abstract Cauchy problem with additive noise

Let $\{W(t)\}_{t\in[0,T]}$ be an U-valued Q-Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We consider equations written formally as

$$\begin{aligned} \mathrm{d}X(t) &= (AX(t) + f(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ X(0) &= \xi, \end{aligned} \tag{6.0.1}$$

where we make the following assumptions.

(A1) $A : \mathcal{D}(A) \subset H \to H$ is linear operator, generating a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{S(t)\}_{t\geq 0}$.

(A2) $B \in B(U, H)$;

(A3) $\{f(t)\}_{t\in[0,T]}$ a predictable H-valued process with Bochner integrable trajectories, that is, $t \mapsto f(\omega, t)$ is Bochner integrable on $[0, T]$ for P-almost all $\omega \in \Omega$;

(A4) ξ is an $\mathcal{F}_0$-measurable H-valued random variable.

Under assumption (A1) the deterministic evolution problem (abstract Cauchy problem)

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad t > 0, \\ u(0) &= x, \end{aligned}$$

is well-posed (under some weak assumptions on f) and its unique (mild) solution is given by the variation of constants formula

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,\mathrm{d}s.$$

Remark 6.0.2. *Since H is, in particular, a reflexive Banach space it follows that $\{S(t)^*\}_{t\geq 0}$ is also a C_0-semigroup on H with generator given by A^*, the adjoint of A. In non-reflexive Banach spaces this is not true in general.*

Next we discuss what we mean by the solution of the formal equation (6.0.1). In this section we always assume (A1)–(A4).

Definition 6.0.4 (Strong solution)**.** *An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a strong solution of* (6.0.1) *if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $X(t, \omega) \in \mathcal{D}(A)$ P_T-almost surely, $\int_0^T \|AX(t)\|\,\mathrm{d}t < \infty$ P-almost surely, and, for all $t \in [0, T]$,*

$$X(t) = \xi + \int_0^t \big(AX(s) + f(s)\big)\,\mathrm{d}s + \int_0^t B\,\mathrm{d}W(s), \quad P\text{-a.s.}$$

Recall that the integral $\int_0^t B\,\mathrm{d}W(s)$ is defined if and only if $\|\, B^2 \,\|_{L_2^0} = Tr(BQB^*) < \infty$.

For $\eta \in H$, we define

$$l_\eta : H \to \mathbf{R}, \quad l_\eta(h) := \langle h, \eta\rangle,\ h \in H. \tag{6.0.2}$$

Definition 6.0.5 (Weak solution)**.** *An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a weak solution of* (6.0.1) *if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories P-almost surely and*

$$\langle X(t),\eta\rangle = \langle \xi,\eta\rangle + \int_0^t \big(\langle X(s),A^*\eta\rangle + \langle f(s),\eta\rangle\big)\,\mathrm{d}s \\ + \int_0^t l_\eta B\,\mathrm{d}W(s), \quad P\text{-a.s.},\ \forall \eta\in\mathcal{D}(A),\ t\in[0,T].$$

Note that the stochastic integral may be written formally as

$$\int_0^t l_\eta B\,\mathrm{d}W(s) = \int_0^t \langle B\,\mathrm{d}W(s),\eta\rangle.$$

We will show that the unique weak solution of (6.0.1) is given by the variation of constants formula

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(s)\,\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s).$$

We will need the following lemma about interchanging the stochastic integral with closed operators.

Lemma 6.0.1. *Let E be a separable Hilbert space. Let $\Phi\in\mathcal{N}_W^2$, $A:\mathcal{D}(A)\subset H\to E$ be a closed, linear operator with $\mathcal{D}(A)$ being a Borel subset of H. If $\Phi(t)u\in\mathcal{D}(A)$ P-almost surely for all $t\in[0,T]$ and $u\in U$ and $A\Phi\in\mathcal{N}_W^2$, then*

$$P\Big(\int_0^T \Phi(s)\,\mathrm{d}W(s)\in\mathcal{D}(A)\Big) = 1$$

and

$$A\Big(\int_0^T \Phi(s)\,\mathrm{d}W(s)\Big) = \int_0^T A\Phi(s)\,\mathrm{d}W(s), \quad P\text{-a.s.} \tag{6.0.3}$$

Note that if $A\in L(H,E)$, then (6.0.3) holds for all $\Phi\in\mathcal{N}_W^2$. We define the stochastic convolution

$$W_A(t) := \int_0^t S(t-s)B\,\mathrm{d}W(s)$$

and the operator

$$Q_t = \int_0^t S(s)BQB^*S(s)^*\,\mathrm{d}s,$$

where the integral is a strong Bochner integral. The following theorem provides the basic properties of the stochastic convolution.

Theorem 6.0.2. *If for some $T>0$,*

$$\int_0^T \|S(t)B\|^2_{L_2^0}\,\mathrm{d}s = \int_0^T Tr(S(t)BQB^*S(t)^*)\,\mathrm{d}t = Tr(Q_T) < \infty,$$

then

1. *$W_A\in C\big([0,T],L_2(\Omega,\mathfrak{F},\mathbb{P};H)\big)$ and W_A has an H-predictable version;*
2. *$\{W_A(t)\}_{t\in[0,T]}$ is a Gaussian process and*

$$Cov(W_A(t)) = \int_0^t S(s)BQB^*S(s)^*\,\mathrm{d}s = Q_t.$$

Proof: Let $0\le s\le t\le T$ and define

$$\Phi(r) = S(t-r)B, \quad M_t(s) = \int_0^s \Phi\,\mathrm{d}W = \int_0^s S(t-r)B\,\mathrm{d}W(r).$$

Then

$$\mathbb{E}\int_0^t \|\Phi\|^2_{L_2^0}\,\mathrm{d}r = \int_0^t \|S(t-r)B\|^2_{L_2^0}\,\mathrm{d}r = \int_0^t \|S(r)B\|^2_{L_2^0}\,\mathrm{d}r$$
$$\leq \int_0^T \|S(r)B\|^2_{L_2^0}\,\mathrm{d}r < \infty.$$

Thus, $M_t(s)$ is well defined, in particular, for $s=t$ it follows that $M_t(t)=W_A(t)$ is well defined. To show mean square continuity, let $0\leq s\leq t\leq T$. Then

$$\begin{aligned} W_A(t)-W_A(s) &= \int_0^t S(t-r)B\,\mathrm{d}W(r) - \int_0^s S(s-r)B\,\mathrm{d}W(r) \\ &= \int_0^s \big(S(t-r)-S(s-r)\big)B\,\mathrm{d}W(r) \\ &\quad + \int_0^t 1_{(s,t]}S(t-r)B\,\mathrm{d}W(r) = X+Y. \end{aligned} \tag{6.0.4}$$

The random variables X and Y are independent with zero mean and therefore, using also Itô's isometry,

$$\begin{aligned} \mathbb{E}(\|W_A(t)-W_A(s)\|^2) &= \mathbb{E}\Big(\Big\|\int_0^s \big(S(t-s)-I\big)S(s-r)B\,\mathrm{d}W(r)\Big\|^2\Big) \\ &\quad + \mathbb{E}\Big(\Big\|\int_0^t 1_{(s,t]}S(t-r)B\,\mathrm{d}W(r)\Big\|^2\Big) \\ &= \int_0^s \|(S(t-s)-I)S(r)BQ^{1/2}\|^2_{B_2(U,H)}\,\mathrm{d}r \\ &\quad + \int_0^{t-s} \|S(r)BQ^{1/2}\|^2_{B_2(U,H)}\,\mathrm{d}r \to 0 \quad \text{as } s\to t. \end{aligned}$$

The second integral converges 0 by the Dominated Convergence Theorem. For the first one, we have

$$1_{(0,s]}(r)\|(S(t-s)-I)S(r)BQ^{1/2}\|^2_{B_2(U,H)} \leq 2\max_{0\leq s\leq T}\|S(s)\|^2_{B(H)}\|S(r)BQ^{1/2}\|^2_{B_2(U,H)},$$

and therefore we may use again dominated convergence together with the fact that $S(t-s)-I\to 0$ strongly as $t-s\to 0$.

For the existence of a predictable version of $\{W_A(t)\}_{t\in[0,T]}$ note that if $\{X(t)\}_{t\in[0,T]}$ is mean square continuous, then it is uniformly stochastically continuous[1] on $[0,T]$. This follows from the observation that the mean square continuity of $\{X(t)\}_{t\in[0,T]}$ means that $X(\cdot)$ is continuous as a function $[0,T]\to L_2(\Omega,\mathcal{F},P;H)$. Since $[0,T]$ is compact $\{X(t)\}_{t\in[0,T]}$ is uniformly mean square continuous on $[0,T]$. We have that

$$P(\|X(t)-X(s)\|^2\geq\epsilon^2)\leq\frac{1}{\epsilon^2}\mathbb{E}(\|X(t)-X(s)\|^2)$$

and hence $\{X(t)\}_{t\in[0,T]}$ is uniformly stochastically continuous on $[0,T]$. By [**?**, Proposition 3.6], $\{X(t)\}_{t\in[0,T]}$ has a predictable version since it is clearly adapted and stochastically continuous.

For t fixed, the random variable $W_A(t)$ is Gaussian. This follows from the construction of the integral and the fact that for elementary deterministic processes the stochastic integral is a Gaussian random variable. An easy calculation shows, similar to the one in (6.0.4), that for all $u_1,u_2,\ldots,u_n\in U$, $(\langle W_A(t_1),u_1\rangle,\ldots,\langle W_A(t_n),u_n\rangle)$ is an $\mathbf{R}^n$-valued Gaussian random variable using also Lemma 1.4.1 for $A=l_{u_i},\ i=1,\ldots,n$. Finally, the covariance operator Q_t of $W_A(t)$ can be computed in a straightforward fashion using Lemma 1.4.1, Corollary 4.1.1 and Parseval's formula.

Before proving the existence and uniqueness of weak solutions of (6.0.1) we need a few preparatory results which we state with only a reference to the proofs. Consider the following assumptions.

1. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space and $\{\mathfrak{F}_t\}_{t\geq 0}$ a filtration. Let $\Phi\in\mathcal{N}^2_W[0,T]$, ϕ be an H-valued predictable process, Bochner integrable on $[0,T]$ $\mathbb{P}$-almost surely, and $X(0)$ be an $\mathfrak{F}_0$-measurable H-valued random variable.

[1]A process $\{X(t)\}_{t\in[0,T]}$ is uniformly stochastically continuous on $[0,T]$ if $\forall\varepsilon>0,\ \forall\delta>0,\ \exists\gamma>0,$ such that $P(\|X(t)-X(s)\|\geq\varepsilon)\leq\delta,\ |t-s|<\gamma,\ t,s\in[0,T]$.

2. Let $F : [0,T] \times H \to \mathbf{R}$ and assume that the Fréchet derivatives $F_t(t,x)$, $F_x(t,x)$, and $F_{xx}(t,x)$ are uniformly continuous as functions of (t,x) on bounded subsets of $[0,T] \times H$. Note that, for fixed t, $F_x(t,x) \in L(H,\mathbf{R})$ and we consider $F_{xx}(t,x)$ as an element of $B(H)$.

Theorem 6.0.3 (Itô's formula)**.** *Under assumptions (1) and (2) above, let*

$$X(t) = X(0) + \int_0^t \phi(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s), \quad t \in [0,T].$$

Then, $\mathbb{P}$-almost surely and for all $t \in [0,T]$,

$$\begin{aligned} F(t,X(t)) = F(0,X(0)) &+ \int_0^t F_x(s,X(s))\Phi(s)\,\mathrm{d}W(s) \\ &+ \int_0^t \Big(F_t(s,X(s)) + F_x(s,X(s))(\phi(s)) \\ &\quad + \tfrac{1}{2} Tr\big((F_{xx}(s,X(s)))(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^* \big) \Big)\,\mathrm{d}s. \end{aligned}$$

The next result is the stochastic version of Fubini's Theorem. Consider the following.

(3) Let $(E,\mathcal{E})$ be a measurable space and

$$\Phi : (\Omega_T \times E, \mathcal{P}_T \times \mathcal{E}) \to (L_2^0, \mathcal{B}(L_2^0))$$

be a measurable mapping.

(4) Let μ be a finite positive measure on $(E,\mathcal{E})$.

(5) Assume that $\int_E \|\Phi(\cdot,\cdot,x)\|_T\,\mathrm{d}\mu(x) < \infty$.

Note, that, in particular, for fixed $x \in E$, the process $\Phi(\cdot,\cdot,x)$ is L_2^0-predictable and $\Phi(\cdot,\cdot,x) \in \mathcal{N}_W^2[0,T]$.

Theorem 6.0.4 (Stochastic Fubini's Theorem)**.** *Assuming (3)–(5) above, we have P-almost surely,*

$$\int_E \int_0^T \Phi(t,x)\,\mathrm{d}W(t)\,\mathrm{d}\mu(x) = \int_0^T \int_E \Phi(t,x)\,\mathrm{d}\mu(x)\,\mathrm{d}W(t). \tag{6.0.5}$$

Note that the inner integral on the right hand side of (6.0.5) is an L_2^0-valued Bochner integral. Now we can the prove existence of weak solutions of (6.0.1). Let

$$X(t) := S(t)\xi + \int_0^t S(t-s)f(s)\,\mathrm{d}s + W_A(t) = Y(t) + W_A(t). \tag{6.0.6}$$

Theorem 6.0.5 (Existence of weak solutions)**.** *Assume (A1)–(A4) and*

$$\int_0^T \|S(r)B\|_{L_2^0}^2\,\mathrm{d}r < \infty.$$

Then $\{X(t)\}_{t\in[0,T]}$ defined in (6.0.6) *has a version which is a weak solution of* (6.0.1).

Proof: The process $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories and an H-predictable version by Theorem 6.0.2. Since $\{Y(t)\}_{t\in[0,T]}$ is the (unique) weak solution of

$$\begin{aligned} &Y'(t) = AY(t) + f(t), \quad t > 0, \\ &Y(0) = \xi, \end{aligned}$$

it follows that $\{X(t)\}$ is a weak solution of (6.0.3) if and only if $W_A(t) = X(t) - Y(t)$ is a weak solution of

$$\begin{aligned} &\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = 0. \end{aligned} \tag{6.0.7}$$

Therefore, without loss of generality, we may set $\xi = 0$, $f = 0$ and show that $W_A(t)$ is a weak solution of (6.0.7). If $t \in [0,T]$ and $\eta \in \mathcal{D}(A^*)$, then

$$\int_0^t \langle A^*\eta, W_A(s)\rangle \,\mathrm{d}s = \int_0^t \Big\langle A^*, \int_0^t 1_{[0,s]}(r)S(s-r)B\,\mathrm{d}W(r)\Big\rangle \,\mathrm{d}s.$$

Following (6.0.2), we set $l_{A^*\eta}(u) := \langle A^*\eta, u\rangle$. Then, by Lemma 1.4.1 and Theorem 6.0.4,

$$\begin{aligned}\int_0^t \langle A^*\eta, W_A(t)\rangle \,\mathrm{d}s &= \int_0^t l_{A^*\eta}\Big(\int_0^t 1_{[0,s]}(r)S(s-r)B\,\mathrm{d}W(r)\Big)\,\mathrm{d}s \\ &= \int_0^t \int_0^t 1_{[0,s]}(r) l_{A^*\eta} S(s-r)B\,\mathrm{d}W(r)\,\mathrm{d}s \\ &= \int_0^t \int_0^t 1_{[0,s]}(r) l_{A^*\eta} S(s-r)B\,\mathrm{d}s\,\mathrm{d}W(r) \\ &= \int_0^t \int_r^t l_{A^*\eta} S(s-r)B\,\mathrm{d}s\,\mathrm{d}W(r).\end{aligned}$$

For all $u \in U$,

$$l_{A^*\eta}S(s-r)Bu = \langle A^*\eta, S(s-r)Bu\rangle = \langle S(s-r)^*A^*\eta, Bu\rangle,$$

and hence, using that $\eta \in \mathcal{D}(A^*)$,

$$\begin{aligned}\int_r^t l_{A^*\eta}S(s-r)Bu\,\mathrm{d}s &= \int_r^t \langle S(s-r)^*A^*\eta, Bu\rangle\,\mathrm{d}s \\ &= \int_r^t \langle A^*S(s-r)^*\eta, Bu\rangle\,\mathrm{d}s \\ &= \int_r^t \frac{\mathrm{d}}{\mathrm{d}s}\langle S(s-r)^*\eta, Bu\rangle\,\mathrm{d}s \\ &= \langle \eta, S(s-r)Bu\rangle - \langle \eta, Bu\rangle.\end{aligned}$$

Finally, by Lemma 1.4.1,

$$\begin{aligned}\int_0^t \int_0^t 1_{[0,s]}(r) l_{A^*\eta} S(s-r)B\,\mathrm{d}s\,\mathrm{d}W(r) &= \int_0^t l_\eta S(t-r)B\,\mathrm{d}W(s) \\ &\quad - \int_0^t l_\eta B\,\mathrm{d}W(s) = \langle \eta, W_A(t)\rangle - \int_0^t l_\eta B\,\mathrm{d}W(s),\ P\text{-a.s.}\end{aligned}$$

To prove uniqueness of weak solutions of (6.0.1) we need the following two results.

Lemma 6.0.2. *Let $(C, \mathcal{D}(C))$ be the generator of a C_0-semigroup on the separable Hilbert space H. Then, the vector space $\mathcal{D}(C)$ endowed with inner product $\langle x,y\rangle_C := \langle x,y\rangle_H + \langle Cx, Cy\rangle_H$ and norm $\|x\|_C := \langle x,x\rangle_C^{1/2}$ is a separable Hilbert space.*

Proposition 6.0.3. *Let $\{X(t)\}_{t\geq 0}$ be a weak solution of (6.0.1) with $f = 0$ and $\xi = 0$. Then, for all $\rho \in C^1([0,T], \mathcal{D}(A^*))$ and $t \in [0,T]$,*

$$\langle X(t), \rho(t)\rangle = \int_0^t \langle X(s), \rho'(s) + A^*\rho(s)\rangle\,\mathrm{d}s + \int_0^t l_{\rho(s)}B\,\mathrm{d}W(s).$$

Proof: First, let $\rho(s) := \rho_0\phi(s)$, $\rho_0 \in \mathcal{D}(A^*)$, $\phi \in C^1([0,T], \mathbf{R})$ and define

$$Y_{\rho_0}(t) := \int_0^t \langle X(s), A^*\rho_0\rangle\,\mathrm{d}s + \int_0^t l_{\rho_0}B\,\mathrm{d}W(s).$$

Note that if $\{X(t)\}_{t\in[0,T]}$ is a weak solution with $f = 0$ and $\xi = 0$, then

$$\langle X(t), \rho_0\rangle = Y_{\rho_0}(t), \quad t \in [0,T]. \tag{6.0.8}$$

If $F(t,x) := \phi(t)x$, $x \in \mathbf{R}$, $t \in [0,T]$, then

$$F_t(t,x) = x\phi'(t), \quad F_x(t,x) = \phi(t), \quad F_{xx}(t,x) = 0,$$

and hence, by Theorem 6.0.3 and (6.0.8),

$$\begin{aligned}\langle X(t), \rho(t)\rangle &= \phi(t)\langle X(t), \rho_0\rangle = \phi(t)Y_{\rho_0}(t) = F(t, Y_{\rho_0}(t)) \\ &= \int_0^t \phi(s) l_{\rho_0} B \,\mathrm{d}W(s) + \int_0^t \big(Y_{\rho_0}(s)\phi'(s) + \phi(s)\langle X(s), A^*\rho_0\rangle\big)\,\mathrm{d}s \\ &= \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) + \int_0^t \langle X(s), \rho'(s) + A^*\rho(s)\rangle \,\mathrm{d}s.\end{aligned}$$

Next consider a general $\rho \in C^1([0,T], \mathcal{D}(A^*))$. By Remark 6.0.2 the operator A^* is the generator of the C_0-semigroup $\{S(t)^*\}_{t\geq 0}$ and hence, by Lemma 6.0.2, $\mathcal{D}(A^*)$ becomes a separable Hilbert space with inner product $\langle x, y\rangle_{A^*} := \langle x, y\rangle_H + \langle A^*x, A^*y\rangle_H$ and norm $\|x\|_{A^*} := \langle x, x\rangle_{A^*}^{1/2}$. Let $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for $(\mathcal{D}(A^*), \|\cdot\|_{A^*})$ and consider the orthogonal expansions

$$\rho(t) = \sum_{k=1}^{\infty} \langle \rho(t), e_k\rangle_{A^*} e_k \quad \text{and} \quad \rho'(t) = \sum_{k=1}^{\infty} \langle \rho'(t), e_k\rangle_{A^*} e_k.$$

For $N \in \mathbf{N}$, define

$$\rho_N(t) := \sum_{k=1}^{N} \langle \rho(t), e_k\rangle_{A^*} e_k, \quad \rho_N'(t) = \sum_{k=1}^{N} \langle \rho'(t), e_k\rangle_{A^*} e_k.$$

Then, by the first part of the proof and linearity,

$$\begin{aligned}\langle X(t), \rho_N(t)\rangle_H = &\int_0^t \big(\langle X(s), \rho_N'(s)\rangle_H + \langle X(s), A^*\rho_N(s)\rangle_H\big)\,\mathrm{d}t \\ &+ \int_0^t l_{\rho_N(s)} B \,\mathrm{d}W(s).\end{aligned} \tag{6.0.9}$$

For the second integral on the right hand side of (6.0.9) we have, using Itô's isometry, that

$$\mathbb{E}\Big\| \int_0^t l_{\rho_N(s)} B \,\mathrm{d}W(s) - \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) \Big\|^2 \to 0,$$

since, by the Dominated Convergence Theorem,

$$\int_0^t \|l_{\rho_N(s)} - l_{\rho(s)}) BQ^{1/2}\|^2_{L^2(U,\mathbf{R})}\,\mathrm{d}s = \int_0^t \|Q^{1/2}B^*(\rho_N(s) - \rho(s))\|_U^2\,\mathrm{d}s \to 0.$$

Finally, we may select a subsequence $\{\rho_{N_k}\}$ such that

$$\int_0^t l_{\rho_{N_k}(s)} B \,\mathrm{d}W(s) \to \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) \quad P\text{-almost surely, as } k \to \infty.$$

For the sake of simplicity we denote the sequence $\{\rho_{N_k}\}$ by $\{\rho_N\}$ again. To deal with the first integral on the right hand side of (6.0.9), we note that $\rho_N(t)$ and $\rho_N'(t)$ converge in the $\|\cdot\|_{A^*}$-norm to $\rho(t)$ and $\rho'(t)$, respectively. Hence, it follows that

$$\langle X(t), \rho_N(t)\rangle_H \to \langle X(t), \rho(t)\rangle_H,$$

$$\langle X(s), \rho_N'(s)\rangle_H \to \langle X(s), \rho'(s)\rangle_H,$$

and

$$\langle X(s), A^*\rho_N(s)\rangle_H \to \langle X(s), A^*\rho(s)\rangle_H$$

as $N \to \infty$. We also have

$$\begin{aligned}|\langle X(s), \rho_N'(s)\rangle|^2 &\leq \|X(s)\|_H^2 \|\rho_N'(s)\|_H^2 \leq \|X(s)\|_H^2 \|\rho_N'(s)\|_{A^*}^2 \\ &\leq \|X(s)\|_H^2 \|\rho'(s)\|_{A^*}^2 \leq K\|X(s)\|_H^2,\end{aligned}$$

and thus,

$$|\langle X(s), \rho_N'(s)\rangle| \leq K\|X(s)\|_H. \tag{6.0.10}$$

Similarly, for the other term,

$$|\langle X(s), A^*\rho_N(s)\rangle| \leq \cdots \leq \|X(s)\|_H \|\rho(s)\|_{A^*} \leq K\|X(s)\|_H. \tag{6.0.11}$$

Since $\{X(t)\}_{t\in[0,T]}$ is a weak solution of (6.0.1) it has Bochner integrable trajectories P-almost surely and hence, by (6.0.10), (6.0.10), and the Dominated Convergence Theorem, we may pass to the limit in (6.0.9) inside the first integral on the right hand side P-almost surely and the proof is complete.

Theorem 6.0.6 (Uniqueness). *If $\{X(t)\}_{t\in[0,T]}$ is a weak solution of* (6.0.1)*, then $X(t)$ is given by* (6.0.6) *P-almost surely, that is, $\{X(t)\}_{t\in[0,T]}$ is a version of* (6.0.6).

Proof: As in the proof of existence of weak solutions of (6.0.1) it suffices to consider the case when $f = 0$ and $\xi = 0$. Let

$$\rho(s) := S(t-s)^*\rho_0, \quad s \in [0,T], \quad \rho_0 \in \mathcal{D}((A^*)^2).$$

Then $\rho'(s) = -A^*S(t-s)^*\rho_0 = -A^*\rho(s)$ and by Lemma 6.0.3,

$$\langle X(t), \rho_0\rangle = \langle X(t), \rho(t)\rangle = \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s).$$

Furthermore,

$$(l_{\rho(s)}B)(u) = \langle S(t-s)^*\rho_0, Bu\rangle = (l_{\rho_0} S(t-s)B)(u)$$

and hence, by Lemma 1.4.1,

$$\begin{aligned}\langle X(t), \rho_0\rangle &= \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) = \int_0^t l_{\rho_0} S(t-s) B \,\mathrm{d}W(s)\\ &= l_{\rho_0}\Big(\int_0^t S(t-s)B\,\mathrm{d}W(s)\Big) = \langle W_A(t), \rho_0\rangle.\end{aligned}$$

Finally, using the fact from semigroup theory that $\mathcal{D}((A^*)^2)$ is dense in H, we conclude that $X(t) = W_A(t)$ P-almost surely.

Chapter 7

Applications to stochastic partial differential equations

7.1 Overview of Sobolev spaces

In this section we overview the basic properties of Sobolev spaces. The Sobolev spaces $W^{k,p}(\Omega)$ are $L^p(\Omega)$ spaces that control the regularity of the derivatives. Their structure and properties make them particularly suitable for the formulation of partial differential equations in a functional analytic setting. When we study a partial differential equation, we understand-in the classical sense- that a solution must be differentiable at least as many times as the order of the equation and that it must satisfy the equation everywhere in the space and time. However such a point of view is very restrictive and several interesting equations which model physical phenomena will fail to possess such solutions and thus we will prevented from studying mathematically such physical situations. As a first towards this, the notion of differentiable functions must be generalized via the notion of Sobolev spaces.

7.1.1 Domain boundary and its regularity

Definition 7.1.1. *A subset $\Omega \subset \mathbb{R}^d$ is said to be a **domain** if-f it is nonempty open and connected.*

Definition 7.1.2. *A subset $\Omega \subset \mathbb{R}^d$ is said to be **connected** if-f for each two points in Ω are connected by a continuous curve which lies in Ω.*

Consider an infinite sequence $(\Omega_n)_n$ of bounded domains such that $\Omega_n \subset \Omega_{n+1}$, where Ω_0 is a symmetric equilateral hexagon with unit edge length. For every $n \in \mathbb{N}$ the domain Ω_{n+1} is obtained from Ω_n as follows: Each edge of Ω_n is split into three equally long parts e_{left}, e_{mid} and e_{right}. An open equilateral triangle of the edge-length $|e_{mid}|$ is attached from outside to e_{mid}. Consider the limit set Ω. Then it can be proven that the unit normal vector to the boundary $\partial\Omega$ is defined nowhere to on $\partial\Omega$. In order to avoid similar unpleasant situations we introduce the notion of a **Lipschitz continuous boundary** $\partial\Omega$. For an exact definition see Adams [AD]. Roughly speaking the boundary $\partial\Omega$ is said to be Lipschitz continuous if-f there exists a finite covering of $\partial\Omega$ of open d-dimensional rectangles such that in each rectangle $\partial\Omega$ can be expressed as a Lipschitz continuous function of $d-1$ variables.When $\partial\Omega$ is Lipschitz continuous then a unique unit outer normal vector is defined almost everywhere on $\partial\Omega$.

7.1.2 Weak derivatives

Definition 7.1.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set. We call **multi-index** a vector $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$. The **length** of the multi-index is given by $|a| = \sum_{i=1}^d a_i$. Let $f : \Omega \to \mathbb{R}$ be a m-times continuously differentiable function. For $|a| \leq m$, we define the* $\mathbf{a^{th}}$ ***partial derivative of*** *f by*

$$D^a f = \frac{\partial^{|a|} f}{\partial {x_1}^{a_1}, \partial {x_2}^{a_2}, \ldots, \partial {x_d}^{a_d}}.$$

Example 7.1.1. If $a = (0, 0, \ldots, 0)$, then $D^a f = f$.
If $a = (1, 0, \ldots, 0)$, then $D^a f = \frac{\partial f}{\partial x_1} =: \partial_i f$.
If $a = (1, 1, \ldots, 1)$, then $D^a f = \frac{\partial^d f}{\partial x_1 \partial x_2 \ldots \partial x_d} =: \partial^d_{1,\ldots,d} f$.

Definition 7.1.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set. We define the space of **test functions***

$$C_c^\infty(\Omega) := \{\phi \in C^\infty(\Omega) : supp(\phi) \subset \Omega \text{ and } supp(\phi) \text{ is compact}\},$$

*where the **support** of a function ϕ is the set*

$$supp(\phi) := \overline{\{\mathbf{x} \in \Omega : \phi(\mathbf{x}) \neq 0\}}.$$

Example 7.1.2. Consider the bounded domain $\Omega = (-1, 1) \subset \mathbb{R}$ and the functions

$$\phi(x) = \cos(\pi x) + 1 \qquad\qquad \psi(x) = e^{-\frac{1}{1-x^2}}.$$

Neither ϕ nor ψ is a test function, since $supp(\phi) = supp(\psi) = [-1, 1]$ is not contained in Ω. However, ψ can be extended by zero to be a distribution in the interval $\tilde{\Omega} = (-1 - \varepsilon, 1 + \varepsilon)$, for some $\varepsilon > 0$. This is not possible for the function ϕ, since its second derivative would be discontinuous in $\tilde{\Omega}$

Remark 7.1.1. For each $\phi \in C_c^\infty(\Omega)$, there exists at least a thin belt along the boundary $\partial\Omega$, where ϕ vanishes. This is straightforward by the fact that $supp(\phi)$ is compact, so it is contained to a closed ball that is a pure subset of the open set Ω.

In the sequel we will review basic facts from multidimensional calculus which are needed for the definition of the weak derivatives. For more details see Spivak [**?**].

Theorem 7.1.1 (Green's Formula)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. For every $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$, we have*

$$\int_\Omega \frac{\partial u}{\partial x_i}\, v\, d\mathbf{x} = -\int_\Omega u\, \frac{\partial v}{\partial x_i}\, d\mathbf{x} + \int_{\partial\Omega} u\, v\, \nu_i\, dS,$$

where $\nu(\mathbf{x}) = (\nu_1, \ldots, \nu_d)^T(\mathbf{x})$ is the unit normal vector to the boundary $\partial\Omega$. (which is defined almost everywhere).

Theorem 7.1.2 (Gauss's Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Every smooth vector field $\mathbf{w} \in [C^1(\Omega) \cap C(\overline{\Omega})]^d$ satisfies*

$$\int_\Omega \nabla \cdot \mathbf{w}(\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{w}(\mathbf{x}) \cdot \nu(\mathbf{x})\, dS$$

where $\nu(\mathbf{x}) = (\nu_1, \ldots, \nu_d)^T(\mathbf{x})$ is the unit normal vector to the boundary $\partial\Omega$

Proof: For each $i = 1, \ldots, d$ apply Green's Formula (thm (7.1.1)) for $u = w_i$ and $v = 1$. By addition of the derived equations, we get the desired result.

Corollary 7.1.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $f \in C^m(\Omega)$ and a a multi-index such that $|a| \leq m$. Then,*

$$\int_\Omega D^a f(\mathbf{x})\, \phi(\mathbf{x})\, d\mathbf{x} = (-1)^{|a|} \int_\Omega f(\mathbf{x})\, D^a \phi(\mathbf{x})\, d\mathbf{x}, \text{ for all } \phi \in C_c^\infty(\Omega).$$

Proof: Apply Green's formula (thm(7.1.1))

Definition 7.1.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set. A measurable function $f : \Omega \to \mathbb{R}$ is said to be **locally integrable** if-f $\int_K |f|\, d\mathbf{x} < \infty$, for each $K \subset \Omega$ compact. The space of locally integrable functions is denoted by $L^1_{loc}(\Omega)$.*

Remark 7.1.2. Observe that $L^p(\Omega) \subset L^1_{loc}(\Omega)$ for each open $\Omega \subset \mathbb{R}^d$ and $1 \leq p < \infty$. Actually, the space $L^1_{loc}(\Omega)$ is very "large". For example, the function $\frac{1}{x}$ does not belong to $L^p(0, \infty)$ for any $1 \leq p < \infty$ but it belongs to $L^1_{loc}(0, \infty)$.

Lemma 7.1.1 (Generalized Variational Lemma). *Let $\Omega \subset \mathbb{R}^d$ be an open set and $f \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega f(\mathbf{x})\phi(\mathbf{x})\ d\mathbf{x} = 0, \quad \textit{for each} \ \ \phi \in C_c^\infty(\Omega)$$

then $f = 0$ almost everywhere in Ω.

Proof: First case $f \in L^1(\Omega)$ and $|\Omega| < \infty$. Let $\varepsilon > 0$. In this case, there exists $f_1 \in C_c^\infty(\Omega)$ such that $\| f - f_1 \|_1 < \varepsilon$. Therefore, for each $g \in C_c^\infty(\Omega)$ we have

$$\left| \int_\Omega f_1\, g\ d\mathbf{x} \right| = \left| \int_\Omega (f_1 - f)\, g\ d\mathbf{x} \right| \le \| g \|_\infty \| f_1 - f \|_1 < \varepsilon \| g \|_\infty . \tag{7.1.1}$$

Set $K_1 = \{\mathbf{x} \in \Omega : \ f_1(\mathbf{x}) \ge \varepsilon\}$ and $K_2 = \{\mathbf{x} \in \Omega : \ f_1(\mathbf{x}) \le -\varepsilon\}$. We have $K_1 \cap K_2 = \emptyset$ and for each $i = 1, 2$, K_i is a closed (from continuity of f_1) subset *supp*(f_1), which is compact. So each K_i is compact. By virtue of Urysohn's Lemma, there exists a function $h \in C_c^\infty(\Omega)$ such that $h = 1$ on K_1, $h = -1$ on K_2 and $|h(\mathbf{x})| \le 1$, for all $\mathbf{x} \in \Omega$. Set $K = K_1 \cup K_2$. Then by 7.1.1, we have

$$\begin{aligned} \int_K |f_1|\ d\mathbf{x} &= \int_K f_1\, h\ d\mathbf{x} \le \left| \int_K f_1\, h\ d\mathbf{x} \right| \le \left| \int_\Omega f_1\, h\ d\mathbf{x} - \int_{\Omega - K} f_1\, h\ d\mathbf{x} \right| \\ &\le \left| \int_\Omega f_1\, h\ d\mathbf{x} \right| + \left| \int_{\Omega - K} f_1\, h\ d\mathbf{x} \right| \le \varepsilon + \int_{\Omega - K} |f_1|\ d\mathbf{x} \end{aligned}$$

Therefore,

$$\| f_1 \|_1 \le \varepsilon + 2 \int_{\Omega - K} |f_1|\ d\mathbf{x} \le \varepsilon + 2\varepsilon|\Omega|,$$

since $|f_1| \le \varepsilon$ on $\Omega - K$. So we conclude that $\| f \|_1 \le \| f - f_1 \|_1 + \| f_1 \|_1 \le 2\varepsilon + 2\varepsilon|\Omega|$. Since this last inequality is true for an arbitrary $\varepsilon > 0$, we get that $\| f \|_1 = 0$, thus $f = 0$ a.e in Ω.
General case $f \in L^1_{loc}(\Omega)$. We write $\Omega = \cup_{n=1}^\infty \Omega_n$, where for each $n \in \mathbb{N}$ $\Omega_n = \Omega \cap B(\mathbf{0}, n)$. For each $n \in \mathbb{N}$, Ω_n is bounded and relatively compact. Therefore, $\int_{\Omega_n} |f|\ d\mathbf{x} \le \infty$. So, by the previous case, for each $n \in \mathbb{N}$ $f = 0$ a.e in Ω_n. Thus, $f = 0$ a.e in Ω.

Definition 7.1.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $f \in L^1_{loc}(\Omega)$ and a a multi-index. A function $D_w^a f \in L^1_{loc}(\Omega)$ is called the **weak** a^{th} **derivative** of f if-f*

$$\int_\Omega D_w^a f\ \ \phi\ d\mathbf{x} = (-1)^{|a|} \int_\Omega f\, D^a \phi\ d\mathbf{x}, \quad \textit{for all} \ \phi \in C_c^\infty(\Omega).$$

Lemma 7.1.2 (Uniqueness of the weak derivative). *Let $\Omega \subset \mathbb{R}^d$ be an open set, $f \in L^1_{loc}(\Omega)$ and a a multi-index. The weak a^{th} derivative $D_w^a f \in L^1_{loc}(\Omega)$ (if it exists) is unique up to a zero-measure subset of Ω.*

Proof: Assume $g_1, g_2 \in L^1_{loc}(\Omega)$ are the weak a^{th} derivatives of f. Then,

$$\int_\Omega (g_1 - g_2)\phi\ d\mathbf{x} = 0 \ \text{ for all } \ \phi \in C_c^\infty(\Omega).$$

Therefore, by Lemma (7.1.1) we conclude that $g_1 = g_2$ a.e in Ω.

Lemma 7.1.3 (Compatibility of weak and classical derivatives). *Let $\Omega \subset \mathbb{R}^d$ be an open set, $f \in C^m(\Omega)$ and a multi-index $|a| \le m$. Then the classical a^{th} derivative $D^a f$ is identical to the weak a^{th} derivative $D_w^a f$.*

Proof: It is direct from Corollary (7.1.1) and Lemma (7.1.2).

Remark 7.1.3. Classical derivatives are defined pointwise as limits of difference quotients. Weak derivatives are defined only in an integral sense up to a set of measure zero. By arbitarily changing the function f on a set of measure zero we do not affect its weak derivative in any way.

Example 7.1.3. Consider the function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x, & x > 0, \end{cases}$$

which is continuous and piecewise-smooth. We will show that in this case the weak derivative exists and it coincides with the classical derivative which is defined almost everywhere. For all $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} f\,\phi'\;dx = \int_0^\infty x\phi'(x)\;dx = -\int_0^\infty \phi(x)\;dx = -\int_{\mathbb{R}} H(x)\phi(x)\;dx,$$

where

$$H(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0. \end{cases}$$

Therefore, the Heaviside function H is the weak derivative of f.

The next example raises an important interesting point. If a locally integrable function has a classical derivative almost everywhere, which is also locally integrable, then the weak derivative is not necessarily the same with the classical derivative.

Example 7.1.4. The Heaviside function H is locally integrable. We claim that H does not have a weak derivative. Indeed, for all $\phi \in C_c^\infty(\mathbb{R})$

$$-\int_{\mathbb{R}} H(x)\phi'(x)\;dx = -\int_0^\infty \phi'(x)\;dx = \phi(0).$$

Assume that there exists $g \in L^1_{loc}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} g(x)\phi(x)\;dx = \phi(0), \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}).$$

By the Dominated Convergence Theorem we have

$$\lim_{h\to 0^+} \int_{-h}^{h} |g(x)|\;dx = 0.$$

Therefore, we can choose a $\delta > 0$ such that $\int_{-\delta}^{\delta} |g(x)|\;dx \le \frac{1}{2}$. Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function with $\phi(0) = 1$ and with support contained in the interval $[-\delta, \delta]$. Then we have

$$1 = \phi(0) = \int_{\mathbb{R}} g(x)\phi(x)\;dx \le \|\,\phi\,\|_\infty \int_{-\delta}^{\delta} |g(x)|\;dx \le \frac{1}{2},$$

which is a contraction.

Example 7.1.5. Consider the function

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 2 + \sin x, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

Observe that f is nowhere continuous. For if $x \in \mathbb{R}\backslash\mathbb{Q}$ and $\{x_n\}_{n=1}^\infty \subset \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$, then if f was continuous, it should be $f(x_n) \to f(x)$, thus $0 = f(x) = 2 + \sin x \ne 0$. On the other hand if $x \in \mathbb{Q}$, then for all $\delta > 0$ we can choose a $x_1 \in \mathbb{R}\backslash\mathbb{Q}$ with $|x - x_1| < \delta$. But $|f(x) - f(x_1)| = |2 + \cos x_1| > 1$. However it is easy to see that the function $g(x) = \cos x$ is a weak derivative of f. Indeed, since the Lebesgue measure of $\mathbb{Q}$ is zero, for each $\phi \in C_c^\infty(\Omega)$ we have

$$-\int_{\mathbb{R}} f(x)\phi'(x)\;dx = -\int_{\mathbb{R}} (2 + \sin x)\phi'(x)\;dx = \int \cos x\phi(x)\;dx.$$

7.1.3 The Sobolev spaces $W^{k,p}(\Omega)$ and $H^k(\Omega)$.

Definition 7.1.7. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \ge 1$ an integer and $p \in [1, \infty]$. We define the space*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) :\; D_w^a f \text{ exists and lies in } L^p(\Omega), \text{ for all } |a| \le k\}.$$

For all $1 \leq p < \infty$ we define the norm $\| \, \|_{k,p}$ on $W^{k,p}(\Omega)$ by

$$\| f \|_{k,p} = \left(\sum_{|a| \leq k=} \| D_w^a f \|_p^p \right)^{\frac{1}{p}} .$$

Especially, for the space $W^{1,p}(\Omega)$ we have

$$\| f \|_{1,p} = \left(\| f \|_p^p + \| \nabla f \|_p^p \right)^{\frac{1}{p}}$$

For $p = +\infty$ we define the norm $\| \, \|_{k,\infty}$ on $W^{k,\infty}(\Omega)$ by

$$\| f \|_{k,\infty} = \max_{|a| \leq k} \| D_w^a f \|_\infty .$$

In the important special case $p = 2$ we abbreviate

$$H^k(\Omega) := W^{k,p}(\Omega)$$

Furthermore we define the seminorms on $W^{k,p}(\Omega)$ by

$$|f|_{k,p} = \left(\sum_{|a| = k} \| D_w^a f \|_p^p \right)^{\frac{1}{p}}, \quad \textit{for } 1 \leq p < \infty.$$

$$|f|_{k,\infty} = \max_{|a| = k} \| D_w^a f \|_\infty .$$

Especially, for the space $W^{1,p}(\Omega)$ we have

$$|f|_{1,p} = \| \nabla f \|_p .$$

Theorem 7.1.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \geq 1$ an integer and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ is Banach.*

Proof: We will prove this result for the Sobolev space $W^{1,p}(a,b)$ where $(a,b) \subset \mathbb{R}$ an open interval. Let $(f_n)_n$ be a $\| \, \|_{1,p}$-Cauchy sequence in $W^{1,p}(a,b)$. Then $(f_n)_n$ and $(f'_n)_n$ are $\| \, \|_p$-Cauchy sequences in the Banach space $L^p(a,b)$. Therefore, there exist $f, g \in L^p(a,b)$ such that $\lim_{n\to\infty} \| f_n - f \|_p = 0$ and $\lim_{n\to\infty} \| f'_n - g \|_p = 0$. On the other hand, for all $n \in \mathbb{N}$ and $\phi \in C_c^\infty(a,b)$, we have $\int_a^b f_n \phi' \, dx = -\int_a^b f'_n \phi \, dx$. But it holds that $\lim_{n\to\infty} \int_a^b f_n \phi' \, dx = \int_a^b f \phi' \, dx$. Indeed, $\left| \int_a^b f_n \phi \, dx - \int_a^b f \phi \, dx \right| \leq \| \phi \|_\infty \int_a^b |f_n - f| \, dx \leq \| \phi \|_\infty \| f_n - f \|_p (b-a)^{\frac{1}{p^*}} \to 0$, when $n \to \infty$. Similarly, $\lim_{n\to\infty} \int_a^b f'_n \phi \, dx = \int_a^b g \phi \, dx$. Therefore, we conclude that $\int_a^b f \phi' \, dx = -\int_a^b g \phi \, dx$. This means that $f \in W^{1,p}(a,b)$, $f' = g$ and $\lim_{n\to\infty} \| f_n - f \|_{1,p} = 0$.

Proposition 7.1.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $k \geq 1$ an integer. If $1 < p < \infty$, then the Sobolev space $W^{k,p}(\Omega)$ is reflexive.*

Proof: We will prove this result for the space $W^{1,p}(a,b)$. Since the space $L^p(a,b)$ is reflexive for $1 < p < \infty$, the same is true for the space $L^p(a,b) \times L^p(a,b)$. But the space $W^{1,p}(a,b)$ is isometrical isomorphic to a subspace of $L^p(a,b) \times L^p(a,b)$, via the mapping $W^{1,p}(a,b) \ni f \to (f, f') \in L^p(a,b) \times L^p(a,b)$. But the space $W^{1,p}(a,b)$ is closed, since it is Banach. Therefore the above subspace is also a closed subset of the reflexive $L^p(a,b) \times L^p(a,b)$, thus it is also reflexive. Therefore $W^{1,p}(a,b)$ is reflexive.

Theorem 7.1.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $k \geq 1$ an integer. The space $H^k(\Omega)$ endowed with the inner product*

$$< f, g >_{k,2} = \sum_{|a| \leq k} < D_w^a f, D_w^a g >_{L^2(\Omega)}$$

is Hilbert.

Proof: By virtue of Theorem (7.1.3) it is enough to show that $<,>_{k,2}$ is an inner product.

Remark 7.1.4. We denote by $C^\infty(\overline{\Omega})$ the space of all functions which are infinitely differentiable in the open set Ω and such that the functions and all their derivatives possess continuous extensions to $\overline{\Omega}$.

Theorem 7.1.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary, $k \geq 1$ an integer and $1 \leq p < \infty$. Then, the space $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(a,b)$.*

Proof: We will prove this result for the space $W^{1,p}(a,b)$. In the first two steps we prove two useful Lemmas. **Claim** Let $\phi \in C_c^\infty(a,b)$. There exists $\psi \in C_c^\infty(a,b)$ such that $\phi = \psi'$ if and only if $\int_a^b \phi(t)\ dt = 0$
Proof of claim: Assume that there exists $\phi \in C_c^\infty(a,b)$ such that $\phi = \psi'$. Then, $\int_a^b \phi\ dt = \psi(b) - \psi(a) = 0$. Conversely, assume that $\int_a^b \phi\ dt = 0$ and $supp(\phi) \subset [c,d]$. Define the function $\psi(t) = \int_a^t \phi(s)\ ds$. Since $\psi'(t) = \phi(t)$, we have that $\psi \in C^\infty(a,b)$. Furthermore, it is direct that ψ vanishes in (a,c) and (c,d). Therefore, $supp(\psi) \subset [c,d]$, so $\psi \in C_c^\infty(a,b)$.
Claim: Let $f \in L^p(a,b)$, $1 \leq p < \infty$. If for all $\phi \in C_c^\infty(a,b)$ it holds that $\int_a^b f\phi'\ dx = 0$, then f is equal to a constant almost everywhere in (a,b). Therefore, if $f \in W^{1,p}(a,b)$ and $f' = 0$, then f is constant almost everywhere in (a,b)
Proof of claim: Let $\phi_0 \in C_c^\infty(a,b)$ such that $\int_a^b \phi_0\ dx = 1$. Consider an arbitrary $\phi \in C_c^\infty(a,b)$. Set

$$w = \phi - \left(\int_a^b \phi\ dx \right) \phi_0 \quad \in C_c^\infty(a,b). \tag{7.1.2}$$

Then, $\int_a^b w\ dx = 0$ and thus from the previous claim there exists $\psi \in C_c^\infty(a,b)$ such that $w = \psi'$. So, by our assumption we get that $\int_a^b f\, w\ dx = 0$. By substituting w from (7.1.2) we have that $\int_a^b f\phi\ dx = \int_a^b \phi\ dt \cdot \int_a^b f\, \phi_0\ dt$. If we set $c = \int_a^b f\phi_0\ dt$, then $\int_a^b (f-c)\phi\ dt = 0$. Since the last relationship is true for all $\phi \in C_c^\infty(a,b)$, we derive that $f = c$ a.e.

We continue now with the main proof. Obviously $C^\infty[a,b] \subset W^{1,p}(a,b)$ and the weak and classical derivatives coincide. Let $f \in W^{1,p}(a,b)$. Then $f' \in L^p(a,b)$ and thus by density there exists a sequence $(\phi_n)_n$ in $C_c^\infty(a,b)$ such that $\lim_{n\to\infty} \| \phi_n - f' \|_p = 0$. For each $n \in \mathbb{N}$ set $\psi_n = \int_a^t \phi_n\ dx$. Then $\{\psi_n\}_{n=1}^\infty \subset C^\infty[a,b]$. Moreover, for positive integers m, n we have

$$|\psi_n - \psi_m| \leq \int_a^b |\phi_n - \phi_m|\ dx \leq \| \phi_n - \phi_m \|_p (b-a)^{\frac{1}{p*}}$$

Therefore,

$$\| \psi_n - \psi_m \|_p \leq \| \phi_n - \phi_m \|_p (b-a),$$

from where we deduce that $(\psi_n)_n$ is Cauchy in $L^p(a,b)$ and thus there exists $h \in L^p(a,b)$ such that $\lim_{n\to\infty} \| \psi_n - h \|_p = 0$. Moreover since $\psi_n' = \phi_n$, for each $n \in \mathbb{N}$ and $\phi \in C_c^\infty(a,b)$ we have

$$\int_a^b \psi_n \phi'\ dx = -\int_a^b \phi_n \phi\ dx.$$

Taking the limits as $n \to \infty$ we get $\int_a^b h\phi'\ dx = -\int_a^b f'\phi\ dx$, which means that $h \in W^{1,p}(a,b)$ and $h' = f'$. So by the second claim $h - f = c$ almost everywhere in (a,b). Consider now the sequence $x_n = \psi_n - c$. It is $(x_n)_{n=1}^\infty \subset C^\infty[a,b]$ and $\lim_{n\to\infty} \| x_n - f \|_{1,p} = 0$.

Definition 7.1.8. *A function $f : [a,b] \to \mathbb{R}$ is said to be **absolutely continuous** if-f for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite sequence $\{(x_i, x_i')\}_{i=1}^n$ of disjoint open intervals contained in (a,b), with $\sum_{i=1}^n (x_i' - x_i) < \delta$, it holds that $\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon$.*

Remark 7.1.5. For every absolutely continuous function the following assertions are true:

1. Every absolutely continuous function is uniformly continuous.

2. Every absolutely continuous function is differentiable almost everywhere and its derivative is an integrable function.

3. A function f is absolutely continuous if and only if it can be written as an integral of an integrable function. In this case,
$$f(x) = f(a) + \int_a^x f'(t)\ dt$$

4. If $f, g : [a, b] \to \mathbb{R}$ are two absolutely continuous functions, then we can integrate by parts, i.e
$$\int_a^b fg'\ dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)\ dt$$

5. From (3) each $\phi \in C_c^\infty(a, b)$ is absolutely continuous. Therefore, from (4) each absolutely continuous function f is an element of $W^{1,p}(a, b)$ and its weak derivative coincides with the classical derivative (which is defined almost everywhere).

Proposition 7.1.2. *Let $1 \leq p < \infty$ and $f \in W^{1,p}(a, b)$. Then, f is equal to an absolutely continuous function almost everywhere.*

Proof: Let $f \in W^{1,p}(a, b)$. Then $f' \in L^p(a, b)$. Set $g(x) = \int_a^x f'(t)\ dt$. Then g is absolutely continuous, thus $g \in W^{1,p}(a, b)$ and $g' = f'$. So $f = g + c$ almost everywhere and of course $\tilde{f} := g + c$ is absolutely continuous.

7.1.4 The spaces $W_0^{k,p}(\Omega)$ and $H_0^k(\Omega)$.

Definition 7.1.9. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \geq 1$ an integer and $p \in [1, +\infty]$. We define the subspace $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Moreover, we denote $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.*

Remark 7.1.6. One can think the closed subspace $W_0^{k,p}(\Omega)$ as
$$W_0^{k,p}(\Omega) = \{f \in W^{k,p}(\Omega) : \ D_w^a f = 0 \ \text{ on } \partial\Omega, \ \text{ for all } \ |a| < k\}.$$
Especially,
$$W_0^{1,p}(\Omega) = \{f \in W^{1,p}(\Omega) : \ f = 0 \ \text{ on } \ \partial\Omega\}.$$
In the above interpretation there is an important detail that needs to be dealt with caution. In general as the d-dimensional Lebesgue measure of $\partial\Omega$ is zero, it is not meaningful a priori to talk about the value of a function $f \in W^{k,p}(\Omega) \subset L^p(\Omega)$ on $\partial\Omega$, unless say f is at least continuous. The object of **trace theory** is to give a meaning to $f_{|\partial\Omega}$ called the **trace of** f. So the interpretation of $W_0^{1,p}(\Omega)$ as the space of Sobolev functions that vanish on the boundary is made more precise in the trace Theorem which shows the existence of a trace map T that maps a Sobolev function to its boundary values and states that the functions in $W_0^{1,p}(\Omega)$ are those whose trace is equal to zero. This study is beyond the scope of these notes. For more details see sections 2.6 and 2.7 in Kesavan's book [KS I]. In the sequel we will give a proof for the case $W_0^{1,p}(a, b)$.

Proposition 7.1.3. *Let $f \in W^{1,p}(a, b)$ and $\check{f}$ its absolutely continuous representative. Then $f \in W_0^{1,p}(a, b)$ if and only if $\check{f}(a) = \check{f}(b) = 0$.*

Proof: Let $f \in W_0^{1,p}(a, b)$. Then there exists a sequence $(\phi_n)_n$ in $C_c^\infty(a, b)$ such that $\lim_{n\to\infty} \| \phi_n - \check{f} \|_{1,p} = 0$. Since the inclusion map from $W^{k,p}(\Omega)$ into $C[a, b]$ is continuous (see theorem(6.4.3) in [KS II]), we conclude that $\phi_n \to \check{f}$ uniformly. Thus $\check{f}(a) = \lim_{n\to\infty} \phi_n(a) = 0$. Similarly $\check{f}(b) = 0$.
Conversely, assume that $\check{f}(a) = \check{f}(b) = 0$. Then, we have that
$$\check{f}(x) = \int_a^x \check{f}'(t)\ dt$$
Therefore, $\int_a^b \check{f}'(t)\ dt$. Consider now a sequence $(\phi_n)_n$ in $C_c^\infty(a, b)$ with $\lim_{n\to\infty} \| \phi_n - \check{f}' \|_p = 0$.. Then,
$$\left| \int_a^b \phi_n\ dt - \int_a^b \check{f}'\ dt \right| \leq \| \phi_n - \check{f}' \|_p (b - a)^{\frac{1}{p^*}} \to 0$$

and so

$$\int_a^b \phi_n \, dt \to 0.$$

Let $\phi_0 \in C_c^\infty(a,b)$ such that $\int_a^b \phi_0 \, dt = 1$. Then if

$$\psi_n = \phi_n - \left(\int_a^b \phi_n \, dt \right) \phi_0$$

we also have that $\left\| \psi_n - \check{f}' \right\|_p \to 0$ and $\int_a^b \psi_n \, dt = 0$. Therefore, $\psi_n = x_n'$ where $(x_n)_n$ in $C_c^\infty(a,b)$ as well. Since

$$x_n = \int_a^x \psi_n \, dt$$

it follows that $x_n \to \check{f}$ uniformly and so $\left\| x_n - \check{f} \right\|_p \to 0$ as well. Thus, $x_n \in C_c^\infty(a,b)$ and $\| x_n - f \|_{1,p} \to 0$. This shows that $f \in W_0^{1,p}(\Omega)$.

Lemma 7.1.4 (Poincare's Inequality)**.** *Let Ω be a bounded open set. Then there exists a constant $C = C(\Omega, p)$ such that*

$$\| f \|_p \leq C|f|_{1,p} = C \, \| \nabla f \|_p \,, \quad \textit{for all } f \in W_0^{1,p}(\Omega) \quad (1 \leq p < \infty).$$

In particular $f \to |f|_{1,p} = \| \nabla f \|_p$ defines a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\| \; \|_{1,p}$. On $H_0^1(\Omega)$ the bilinear form

$$(f,g) \to \int_\Omega \nabla f \nabla g \, d\mathbf{x}$$

is an inner product on $H_0^1(\Omega)$ which induces the norm $| \; |_{1,2}$ equivalent to the norm $\| \; \|_{1,2}$.

Proof: We will prove this result for the space $W_0^{1,p}(a,b)$. In particular, we will show that $\| f \|_p \leq (b-a) \| f' \|_p$, for all $f \in W_0^{1,p}(a,b)$. Let $f \in W_0^{1,p}(a,b)$ and $\tilde{f}$ its absolutely continuous representative. Then by Proposition (7.1.3), $\tilde{f}(x) = \int_a^x \tilde{f}'(t) \, dt$. Therefore, $|\tilde{f}(x)| \leq \left\| \tilde{f}' \right\|_p (b-a)^{\frac{1}{p^*}}$. Thus, $\left\| \tilde{f} \right\|_p \leq \left\| \tilde{f}' \right\|_p (b-a)$.

7.1.5 Generalized integration by parts formulae

Theorem 7.1.6 (Green's formula)**.** *Let Ω be a bounded open set of class C^1 lying on the same side of its boundary $\partial\Omega$. Let $u, v \in H^1(\Omega)$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} \, v \, d\mathbf{x} = -\int_\Omega u \, \frac{\partial v}{\partial x_i} \, d\mathbf{x} + \int_{\partial\Omega} u \, v \, \nu_i \, dS,$$

where $\nu(\mathbf{x}) = (\nu_1, \ldots, \nu_d)^T(\mathbf{x})$ is the unit normal vector to the boundary $\partial\Omega$. (which is defined almost everywhere).

Proof: See Theorem(2.7.5) in [KS I]

Remark 7.1.7. In the above theorem by u on $\partial\Omega$ we mean the trace of u on $\partial\Omega$.

Corollary 7.1.2. *Let Ω be a bounded open set of class C^1, $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$. Then,*

$$\int_\Omega u \Delta v \, d\mathbf{x} = -\int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} + \int_{\partial\Omega} u \nabla v \cdot \nu \, dS$$

where $\nu(\mathbf{x}) = (\nu_1, \ldots, \nu_d)^T(\mathbf{x})$ is the unit normal vector to the boundary $\partial\Omega$. (which is defined almost everywhere).

Proof: It is direct from Theorem (7.1.6).

7.2 Laplacian with Dirichlet boundary condition

In this section we start with a simple but fundamental example, the **heat equation with Dirichlet boundary condition**. Let $\Omega \subset \mathbb{R}^d$ be **a bounded open set of class** C^∞ **with boundary** $\partial\Omega$. Consider the problem of finding a function $u : [0,\infty) \times \overline{\Omega} \to \mathbb{C}$ that satisfies

$$\begin{cases} \frac{\partial u}{\partial t} \;=\; \Delta u, & (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ u \;=\; 0, & (t,\mathbf{x}) \in [0,\infty) \times \partial\Omega \\ u(0,\mathbf{x}) \;=\; u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{7.2.1}$$

where $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial {x_i}^2} := \sum_{i=1}^d \partial_i^2$ is the **Laplacian** in the space variables $\mathbf{x}$, t is the time variable and u_0 is the initial data. The main idea is to view $u(t,\mathbf{x})$ as a function of one variable $[0,\infty) \ni t \to u(t) \in X$,where for each $t \geq 0$, $u(t)$ is the function $\Omega \ni \mathbf{x} \to u(t,\mathbf{x})$ (depending only on the space variables) and for each $t \geq 0$ the function $u(t)$ is an element of a suitably chosen infinite-dimensional Hilbert space X (space of functions). Therefore the above partial differential equation can be rewritten as an **abstract Cauchy problem**, i.e an **ordinary differential equation in a Hilbert space** of the form

$$\begin{cases} u'(t) \;=\; Au(t), & t \geq 0 \\ u(0) \;=\; u_0, & , \end{cases}$$

where $A = \Delta : X \supset D(A) \to X$. We wish to make use of the previous theory, so we have to choose carefully both the space X and the domain $D(A)$ in order that the unbounded linear operator $A : X \supset D(A) \to X$ generates a C_0-semigroup of contractions. We make the following choice:

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset X = L^2(\Omega), \\ Au \;=\; \Delta u \,, & , \end{cases}$$

Observe that the choice $D(A) \subset H_0^1(\Omega)$ imposes a condition, that is $u = 0$ on $\partial\Omega$

Theorem 7.2.1. *The above defined A generates a strongly continuous semigroup of contractions on $X = L^2(\Omega)$.*

Proof: We will apply Lumer-Phillips Theorem (2.5.2) to show that $\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is the infinitesimal generator of a C_0-semigroup. Note that $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \supset C_0^2(\Omega)$ and C_0^2 is dense in $L^2(\Omega)$, therefore A is densely defined.
First step: A is dissipative. Notice that integrating by parts we take, for each $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$< A\phi, \phi >= \int_\Omega \Delta\phi \;\overline{\phi}\; d\mathbf{x} = \sum_{j=1}^d \int_\Omega \partial_j^2 \phi \;\overline{\phi}\; d\mathbf{x} = -\sum_{j=1}^d \int_\Omega \partial_j \phi \;\overline{\partial_j \phi}\; d\mathbf{x} = -\int_\Omega |\nabla\phi|^2 \; d\mathbf{x} \;\leq 0.$$

Second step: $\lambda_0 I - A$ **is surjective for some** $\lambda_0 > 0$**.** We have to find $\lambda_0 > 0$ such that the following fact is true: for each $f \in L^2(\Omega)$, there exists $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $f = \lambda_0 \phi - A\phi$. Precisely we are trying to solve the well known **Dirichlet** problem

$$\begin{cases} \lambda_0 \phi - \Delta\phi \;=\; f\,, & \text{on } \Omega \\ \phi \;=\; 0, & \text{on } \partial\Omega. \end{cases} \tag{7.2.2}$$

There is a standard topological way to solve this equation. Observe that $\phi \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is a solution of (7.2.2) if and only if

$$\int_\Omega (\lambda_0 \phi - \Delta\phi) \;\overline{\psi}\; d\mathbf{x} = \int_\Omega f \;\overline{\psi}\; d\mathbf{x},\ \forall \psi \in C_c^\infty(\Omega). \tag{7.2.3}$$

For the opposite direction use the generalized variational Lemma (7.1.1). Now, since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and both sides of (7.2.3) are continuous in ψ with respect to the $H_0^1(\Omega)$-topology, it follows that

$$\int_\Omega (\lambda_0 \phi - \Delta\phi) \;\overline{\psi}\; d\mathbf{x} = \int_\Omega f \;\overline{\psi}\; d\mathbf{x},\ \forall \psi \in H_0^1(\Omega).$$

Exchanging conjugates and integrating by parts (see Green's formula (Cor(7.1.2)) we have that $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ is a solution of (7.2.2) if and only if

$$\lambda_0 \int_\Omega \psi \;\overline{\phi}\; d\mathbf{x} + \int_\Omega \nabla\psi \cdot \overline{\nabla\phi}\; d\mathbf{x} = \int_\Omega \psi \;\overline{f}\; d\mathbf{x},\ \forall \psi \in H_0^1(\Omega). \tag{7.2.4}$$

A function $\phi \in H_0^1(\Omega)$ satisfying equation (7.2.4) is called **weak solution** of the Dirichlet boundary value problem (7.2.2). Therefore, a weak solution $\phi \in H_0^1(\Omega)$ can formally solve (7.2.4) without being in $H^2(\Omega)$ and therefore without the Laplacian $\Delta\phi$ be defined. In the sequel, we will apply the Riesz representation Theorem to prove the existence of a weak solution $\phi \in H_0^1(\Omega)$ of (7.2.4). To this aim, for $\lambda_0 \geq 0$ define the map $<<\, , >>: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ as follows:

$$<< \psi, \phi >>:= \lambda_0 \int_\Omega \psi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega \nabla\psi \cdot \overline{\nabla\phi}\, d\mathbf{x}, \quad \phi, \psi \in H_0^1(\Omega).$$

Then, by virtue of the Poincaré inequality (Lemma(7.1.4)), it is easy to check that for each $\lambda_0 \geq 0$ (although the case $\lambda_0 = 1$ is enough for our scope), the map $<<\, , >>$ defines an inner product on $H_0^1(\Omega)$, equivalent to the usual one defined by

$$< \psi, \phi >_{1,2}:= \int_\Omega \psi\, \overline{\phi}\, d\mathbf{x} \int_\Omega \nabla\psi \cdot \overline{\nabla\phi}\, d\mathbf{x}, \quad \phi, \psi \in H_0^1(\Omega).$$

Now, observe that for a fixed $f \in L^2(\Omega)$ the linear functional $x^* : H_0^1(\Omega) \to \mathbb{C}$ defined by

$$x^*(\psi) := \int_\Omega \psi \overline{f}\, d\mathbf{x},$$

is bounded with respect to the $\|\;\|_{1,2}$ norm and therefore is also bounded with respect to the norm induced by $<<\, , >>$. Indeed, by using Hölder's inequality we get, for all $\psi \in H_0^1(\Omega)$

$$|x^*(\psi)| \leq \|\, \psi \,\|_{L^2(\Omega)} \|\, f \,\|_{L^2(\Omega)} \leq \|\, \psi \,\|_{1,2} \|\, f \,\|_{L^2(\Omega)}\,.$$

Therefore, by Riesz representation Theorem, there exists a unique $\phi \in H_0^1(\Omega)$ such that

$$x^*(\psi) =<< \psi, \phi >>, \quad \forall\psi \in H_0^1(\Omega).$$

So until now we have shown that there exists a weak solution $\phi \in H_0^1(\Omega)$ of the Dirichlet problem (7.2.2). We want to prove that $\phi \in H^2(\Omega)$, too. If this is true then we have finished. This fact is guaranteed by the following theorem (see Brezis [BR] chapter IX Theorem IX.25):

Theorem 7.2.2 (Regularity of weak solutions of the Dirichlet problem). *Let $\Omega \subset \mathbb{R}^d$ be of class C^2 with bounded boundary $\partial\Omega$ and $\phi \in H_0^1(\Omega)$ a weak solution of the Dirichlet problem (7.2.2). Then $\phi \in H^2(\Omega)$ and there exist a constant $C = C(\Omega) > 0$ such that*

$$\|\, \phi \,\|_{H^2(\Omega)} \leq C \,\|\, \phi \,\|_{L^2(\Omega)}\,.$$

Theorem 7.2.3. *For each $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the problem*

$$\begin{cases} \frac{\partial u}{\partial t} \;=\; \Delta u, & (t, \mathbf{x}) \in [0, \infty) \times \Omega \\ u \;=\; 0, & (t, \mathbf{x}) \in [0, \infty) \times \partial\Omega \\ u(0, \mathbf{x}) \;=\; u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

has a unique solution $u \in C^1([0, \infty), L^2(\Omega)) \cap C([0, \infty), H^2(\Omega) \cap H_0^1(\Omega))$.

Proof: Apply Theorem (7.2.1) and Remark (1.4.3).

Remark 7.2.1. Theorem (7.2.3) can be extended to the case $u_0 \in L^2(\Omega)$ with some changes. In this case the self-adjointness of A plays the crucial role. Note, that the price we pay for the initial data being in $X = L^2(\Omega)$, is the lack of differentiability at $t = 0$. For more details on this see Example (4.1.4), Theorem (4.5.2), Theorem (4.5.3) and Theorem (4.6.1) in the book of Kesavan [KS I]. We only state the final result and omit the proof.

Theorem 7.2.4 ([KS I] Theorem (4.6.1)). *Let $u_0 \in L^2(\Omega)$. Then there exists a unique solution u of the heat equation (7.2.1) such that*

$$u \in C([0, \infty), L^2(\Omega)) \cap C^1((0, \infty), L^2(\Omega)) \cap C((0, \infty), H^2(\Omega) \cap H_0^1(\Omega)).$$

Further, for every $\varepsilon > 0$

$$u \in C^\infty([\varepsilon, \infty) \times \overline{\Omega}).$$

7.3 Diffusion operators on L^2.

In this section we enlarge the discussion of the previous section. Let $\Omega \subset \mathbb{R}^d$ be **a bounded open set of class C^∞ with boundery $\partial\Omega$**. Consider the problem of finding a function $u : [0,\infty) \times \overline{\Omega} \to \mathbb{C}$ that satisfies the following **parabolic pde**

$$\begin{cases} \frac{\partial u}{\partial t} \;=\; Lu, & (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ u \;=\; 0, & (t,\mathbf{x}) \in [0,\infty) \times \partial\Omega \\ u(0,\mathbf{x}) \;=\; u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{7.3.1}$$

where

$$L := \sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\partial_{i,j} + \sum_{j=1}^{d} b_j(\mathbf{x})\partial_j - c(\mathbf{x}), \;\; \mathbf{x} \in \Omega$$

is the **diffusion operator**. We will make the following assumptions

- the **diffusion coefficients** $a_{ij} \in C^1(\overline{\Omega}, \mathbb{C})$ for $1 \le i,j \le d$. Moreover $b_j, c \in C(\overline{\Omega}, \mathbb{C})$.
- **[Uniform ellipticity]** there exists $a > 0$ such that $Re \displaystyle\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\zeta_i\overline{\zeta_i} \ge a|\zeta|$, for all $\zeta \in \mathbb{C}^d, \mathbf{x} \in \Omega$.

Of course for $a_{ij} = \delta_{ij}$ and $b_j = c = 0$ we take the Laplace operator.
The above partial differential equation can be rewritten as an **abstract Cauchy problem**, i.e an **ordinary differential equation in a Hilbert space** of the form

$$\begin{cases} u'(t) \;=\; Au(t), & t \ge 0 \\ u(0) \;=\; u_0, & , \end{cases}$$

where

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset X = L^2(\Omega), \\ Au \;= Lu\,, & , \end{cases}$$

In the sequel we will show that A is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.

Theorem 7.3.1. *Under the assumption of uniform ellipticity we the operator A as defined above is the infinitesimal generator of a C_0-semigroup on $X = L^2(\Omega)$.*

Proof: By virtue of Example (1.4.2), it is enough to show that $A_{\lambda_0} := A - \lambda_0 I$ generates a C_0-semigroup of contractions on X, for some $\lambda_0 > 0$. We will apply the Lumer-Phillips Theorem (2.5.2).
First step: $\underline{A_{\lambda_0} = A - \lambda_0 I \textbf{ is dissipative for some } \lambda_0 > 0}$ For each $\phi \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ from integration by parts (theorem (7.1.6))we have

$$\begin{aligned} < L\phi, \phi >_{L^2(\Omega)} \;&=\; \sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \partial_{ij}\phi\, \overline{\phi}\, d\mathbf{x} + \sum_{j=1}^{d} \int_\Omega b_j\, \partial_j\phi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega c\, \phi\, \overline{\phi}\, d\mathbf{x} \\ &=\; -\sum_{i,j=1}^{d} \int_\Omega \partial_i\phi\, \partial_j(a_{ij}\, \overline{\phi})\, d\mathbf{x} + \sum_{j=1}^{d} \int_\Omega b_j\, \partial_j\phi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega c\, |\phi|^2\, d\mathbf{x}. \end{aligned}$$

Now,

$$\int_\Omega \partial_i\phi\, \partial_j(a_{ij}\, \overline{\phi})\, d\mathbf{x} = \int_\Omega \partial_j a_{ij}\, \partial_i\phi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega a_{ij}\, \partial_i\phi\, \overline{\partial_j\phi}\, d\mathbf{x}.$$

So,

$$< L\phi, \phi >_{L^2(\Omega)} = -\int_\Omega \sum_{i,j=1}^{d} a_{ij}\, \partial_i\phi\, \overline{\partial_j\phi}\, d\mathbf{x} + \int_\Omega \sum_{j=1}^{d} \left(-\sum_{i=1}^{d} \partial_i a_{ji} + b_j \right) \partial_j\phi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega c\, |\phi|^2\, d\mathbf{x}.$$

Now taking real parts we have,

$$Re < L\phi, \phi >_{L^2(\Omega)} = -\int_\Omega Re \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i\phi\, \overline{\partial_j\phi} \right) d\mathbf{x} + R \left(\int_\Omega \sum_{j=1}^{d} \left(-\sum_{i=1}^{d} \partial_i a_{ji} + b_j \right) \partial_j\phi\, \overline{\phi}\, d\mathbf{x} + \int_\Omega c\, |\phi|^2\, d\mathbf{x} \right).$$

By uniform ellipticity we have

$$\int_\Omega Re\left(\sum_{i,j=1}^d a_{ij}\,\partial_i\phi\,\overline{\partial_j\phi}\right)\,d\mathbf{x} \geq a\int_\Omega |\nabla\phi|^2\,d\mathbf{x}.$$

Now set $K_1 := \max_{1\le j\le d}\left\|-\sum_{i=1}^d \partial_i a_{ji}+b_j\right\|_\infty < \infty$ and $K_2 := \|\,c\,\|_\infty < \infty$. By applying Hölder's inequality we get

$$\left|\int_\Omega\sum_{j=1}^d\left(-\sum_{i=1}^d\partial_i a_{ji}+b_j\right)\,\partial_j\phi\;\overline{\phi}\;d\mathbf{x}\right| \le K_1\int_\Omega\sum_{j=1}^d|\partial_j\phi|\;|\phi|\;d\mathbf{x}\le d\,K_1\,\|\,\nabla\phi\,\|_2\,\|\,\phi\,\|_2 =: K_3\,\|\,\nabla\phi\,\|_2\,\|\,\phi\,\|_2\,.$$

and

$$\left|\int_\Omega c\;|\phi|^2\;d\mathbf{x}\right| \le K_2\,\|\,\phi\,\|_2^2\,.$$

Now because $ab\le\frac{(a^2+b^2)}{2}$ applying this with $a=\varepsilon\,\|\,\nabla\phi\,\|_2$ and $b=\frac{1}{\varepsilon}\,\|\,\phi\,\|_2$ (where $\varepsilon>0$)we get

$$\|\,\nabla\phi\,\|_2\,\|\,\phi\,\|_2 \le \frac{\varepsilon^2}{2}\,\|\,\nabla\phi\,\|_2^2+\frac{1}{2\varepsilon^2}\,\|\,\phi\,\|_2^2\,.$$

Finally, combining all the above results we take the estimate

$$Re < L\phi,\phi > \le \left(-a+\frac{K_3\varepsilon^2}{2}\right)\|\,\nabla\phi\,\|_2^2+\left(\frac{K_3}{2\varepsilon^2}+K_2\right)\|\,\phi\,\|_2^2\,.$$

Now choose $\varepsilon>0$ small enough such that $K_4=-a+\frac{K_3\varepsilon^2}{2}<0$. Setting $K_5:=\frac{K_3}{2\varepsilon^2}+K_2$, we deduce that for each $\lambda\ge 0$ and $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$

$$Re < A_\lambda\phi,\phi >= Re < (L-\lambda I)\phi,\phi > \le K_4\,\|\,\nabla\phi\,\|_2^2+(K_5-\lambda)\,\|\,\phi\,\|_2^2\,.$$

This gives dissipativity if $K_5-\lambda\le 0$. So since now we will fix some $\lambda_0\ge K_5$.
Second step: $Rg(\mu I-A_{\lambda_0})=X=L^2(\Omega)$, for some $\mu>0$. Because $\mu I-A_{\lambda_0}=\mu I-(L-\lambda_0 I)=(\mu+\lambda_0)I-L$ we consider recall μ the value $\mu+\lambda_0$. Using similar arguments as in the previous section we deduce that a function $\phi\in D(A)=H^2(\Omega)\cap H_0^1(\Omega)$ is a solution of the **Dirichlet problem**

$$\begin{cases}\mu\phi-L\phi\,=\,f\in L^2(\Omega)\,, & \text{on }\Omega\\ \phi\,=\,0, & \text{on }\partial\Omega.\end{cases} \tag{7.3.2}$$

if and only if

$$\int_\Omega\sum_{i,j=1}^d\overline{a_{ij}}\;\partial_i\psi\;\overline{\partial_j\phi}\;d\mathbf{x}+\int_\Omega\sum_{j=1}^d\overline{\left(\sum_{i=1}^d\partial_i a_{ji}-b_j\right)}\;\psi\;\overline{\partial_j\phi}\;d\mathbf{x}+\int_\Omega(\mu-\overline{c})\;\psi\;\overline{\phi}\;d\mathbf{x}=\int_\Omega\psi\;\overline{f}\;d\mathbf{x},\quad\forall\psi\in H_0^1(\Omega). \tag{7.3.3}$$

A **weak solution** of (7.3.2) is a function $u\in H_0^1(\Omega)$ satisfying (7.3.3). The existence of a weak solution is due to a generalization of the Riesz representation Theorem which is called **Lax-Milgram Lemma**. To prepare the discussion call $\tilde{b}_j=\overline{\sum_{i=1}^d\partial_i a_{ji}-b_j}$ and define the form $B:H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$,

$$B(\psi,\phi):=\int_\Omega\sum_{i,j=1}^d\overline{a_{ij}}\;\partial_i\psi\;\overline{\partial_j\phi}\;d\mathbf{x}+\int_\Omega\sum_{j=1}^d\tilde{b}_j\;\psi\;\overline{\partial_j\phi}\;d\mathbf{x}+\int_\Omega(\mu-\overline{c})\;\psi\;\overline{\phi}\;d\mathbf{x} \tag{7.3.4}$$

Clearly B is a **bilinear form** (i.e linear in the first variable and anti-linear in the second variable) and in some sense is going to take the place of the inner product on $H_0^1(\Omega)$. For this we need a kind of equivalence between $\|\ \|_{1,2}$ and B on $H_0^1(\Omega)$. This is contained in the following

Lemma 7.3.1. *Let B be defined by (7.3.4). Under the uniform ellipticity hypothesis we can choose an appropriate $\mu>0$ such that*

1. *B is $H_0^1(\Omega)\times H_0^1(\Omega)$-**continuous**, that is there exists $L>0$ such that $|B(\psi,\phi)|\le L\,\|\psi\|_{1,2}\,\|\phi\|_{1,2}$, for each $\psi,\phi\in H_0^1(\Omega)$.*

2. *B is $H_0^1(\Omega)$-**elliptic**, that is there exists a $l>0$ such that $|B(\psi,\psi)|\ge l\,\|\psi\|^2_{1,2}$, for each $\psi\in H_0^1(\Omega)$.*

Proof of Lemma:

1. By Hölder's inequality we have

$$|B(\psi,\phi)|\le M_1\,\|\nabla\psi\|_2\,\|\nabla\phi\|_2+M_2\,\|\nabla\phi\|_2\,\|\psi\|_2+M_3\,\|\phi\|_2\,\|\psi\|_2\le(M_1+M_2+M_3)\,\|\psi\|_{1,2}\,\|\phi\|_{1,2}\,,$$

for all $\phi,\psi\in H_0^1(\Omega)$, where $M_1:=d^2\max_{1\le i,j\le d}\|a_{ij}\|_\infty$, $M_2:=d\max_{1\le j\le d}\left\|\tilde{b}_j\right\|_\infty$ and $M_3:=\|\mu-c\|_\infty$.

2. Moreover by uniform ellipticity we have

$$|B(\psi,\psi)|\ge ReB(\psi,\psi)\ge a\,\|\nabla\psi\|_2^2-M_2\,\|\psi\|_2\,\|\nabla\psi\|_2+\mu\,\|\psi\|_2^2-\|c\|_\infty\,\|\psi\|_2^2\,.$$

Now again as before,

$$\|\nabla\psi\|_2\,\|\psi\|_2\le\frac{\varepsilon^2}{2}\,\|\nabla\psi\|_2^2+\frac{1}{2\varepsilon^2}\,\|\psi\|_2^2\,.$$

Therefore

$$|B(\psi,\psi)|\ge\left(a-\frac{M_2\varepsilon^2}{2}\right)\|\nabla\psi\|_2^2+(\mu-\|c\|_\infty-\frac{M_2}{2\varepsilon^2})\,\|\psi\|_2^2\,,$$

for all $\psi\in H_0^1(\Omega)$. Now it is clear that choosing ε small enough in such way that $a-\frac{M_2\varepsilon^2}{2}>0$ and μ big enough in such a way that $\mu-\|c\|_\infty-\frac{M_2}{2\varepsilon^2}>0$ we get the conclusion (by virtue of the Poincaré inequality (Lemma (7.1.4))).

The next result as announced is an extension of the Riesz Lemma

Lemma 7.3.2 (Lax-Milgram)**.** *Let X be a Hilbert space, $B:X\times X\to\mathbb{C}$ a bilinear continuous and X-elliptic form. Then, for each $x^*\in X^*$, there exists a unique $x\in X$ such that $x^*(y)=B(y,x)$, for each $y\in X$.*

Proof of Lemma: See Theorem(3.1.4) in [KS I]
Conclusion of the second step: Fix a $f\in L^2(\Omega)$ as we have already shown in the previous section,

$$x^*:H_0^1(\Omega)\to\mathbb{C},\quad x^*(\psi):=\int_\Omega\psi\overline{f}\;d\mathbf{x}$$

is a linear and bounded functional. Therefore since by Lemma (7.3.1) for a fixed μ large enough, B is a bilinear continuous and $H_0^1(\Omega)$-elliptic form, by virtue of Lax-Milgram Lemma, there exists a unique $\phi\in H_0^1(\Omega)$ such that $x^*(\psi)=B(\psi,\phi)$, for every $\psi\in H_0^1(\Omega)$. This $\phi\in H_0^1(\Omega)$ is a weak solution of the Dirichlet problem (7.3.2). It remains to show the regularity of the solution, that is $\phi\in H^2(\Omega)$. To prove that the weak solution has the right regularity is avery difficult and technical job and is contained in

Theorem 7.3.2 (Regularity of weak solutions of the Dirichlet problem)**.** *Let $\Omega\subset\mathbb{R}^d$ of class C^2 with bounded boundary $\partial\Omega$ and $\phi\in H_0^1(\Omega)$ a weak solution of the Dirichlet problem (7.3.2). Then $\phi\in H^2(\Omega)$ and there exist a constant $C=C(\Omega)>0$ such that*

$$\|\phi\|_{H^2(\Omega)}\le C\,\|\phi\|_{L^2(\Omega)}\,.$$

The following Corollary is direct from Theorem (7.3.1) and Remark (1.4.3).

Corollary 7.3.1. *Let $\Omega\subset\mathbb{R}^d$ be a bounded open set of class C^∞ with boundary $\partial\Omega$. Under the uniform ellipticity hypothesis, for each $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$, the parabolic pde*

$$\begin{cases}\frac{\partial u}{\partial t}\;=\;\sum_{i,j=1}^d a_{ij}(\mathbf{x})\partial_{i,j}\,u+\sum_{j=1}^d b_j(\mathbf{x})\partial_j\,u-c(\mathbf{x})\,u, & (t,\mathbf{x})\in[0,\infty)\times\Omega\\ u\;=\;0, & (t,\mathbf{x})\in[0,\infty)\times\partial\Omega\\ u(0,\mathbf{x})\;=\;u_0(\mathbf{x}), & \mathbf{x}\in\Omega,\end{cases}$$

has a unique solution $u\in C^1([0,\infty),L^2(\Omega))\cap C([0,\infty),H^2(\Omega)\cap H_0^1(\Omega))$.

7.4 The wave equation

In this section we assume tha all the vector spaces are real

Theorem 7.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class C^∞. Let $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in H_0^1(\Omega)$. Then there exists a unique solution u of the problem*

$$\left\{\begin{array}{rcll} \frac{\partial^2 u}{\partial t^2} &=& \Delta u, & (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ u &=& 0, & (t,\mathbf{x}) \in [0,\infty) \times \partial\Omega \\ u(0,\mathbf{x}) &=& f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \frac{\partial u}{\partial t}(0,\mathbf{x}) &=& g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{array}\right. \tag{7.4.1}$$

such that $u \in C([0,\infty), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty), H_0^1(\Omega)) \cap C^2([0,\infty), L^2(\Omega))$.

Proof: Since Ω is bounded by virtue of the Poincaré inequality (Lemma (7.1.4)) we can equip the space $H_0^1(\Omega)$ with the inner product

$$< u, v >_{0,1} := \int_\Omega \nabla u \cdot \nabla v \; d\mathbf{x},$$

which is equivalent to the usual one. Therefore, $(H_0^1(\Omega), < , >_{0,1})$ is Hilbert. Let $X = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the inner product

$$< \mathbf{u}, \mathbf{v} >_X = \int_\Omega \nabla u_1 \cdot \nabla v_1 \; d\mathbf{x} + \int_\Omega u_2 \; v_2 \; d\mathbf{x},$$

where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are in $X = H_0^1(\Omega) \times L^2(\Omega)$. Again, it is direct that $(X, < , >)$ is Hilbert. If we set $v = \frac{\partial u}{\partial t}$, then the wave equation (7.4.1) can be rewritten as a system of partial differential equations for the pair (u, v)

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} &=& v, & (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ \frac{\partial v}{\partial t} &=& \Delta u, & (t,\mathbf{x}) \in [0,\infty) \times \Omega \\ u &=& 0, & (t,\mathbf{x}) \in [0,\infty) \times \partial\Omega \\ u(0,\mathbf{x}) &=& f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ v(0,\mathbf{x}) &=& g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{array}\right.$$

With these in mind, the above partial differential equation can be rewritten as an **abstract Cauchy problem**, i.e an **ordinary differential equation in a Hilbert space** of the form

$$\left\{\begin{array}{rcll} \mathbf{u}'(t) &=& \mathbf{A}\mathbf{u}(t), & t \geq 0 \\ \mathbf{u}(0) &=& (f,g), & , \end{array}\right.$$

where

$$\left\{\begin{array}{l} D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \subset X = H_0^1(\Omega) \times L^2(\Omega), \\ \mathbf{A}\mathbf{u} = (v, \Delta u)\,, \mathbf{u} = (u,v) \in D(A) \end{array}\right. .$$

We will show that the conditions of the Lumer-Phillips Theorem (2.5.2) are satisfied, thus $\mathbf{A}$ is the infinitesimal generator of a semigroup of contractions on X. Finally the desired result will be direct from Remark (1.4.3). It is clear that $\mathbf{A}$ is densely defined. Furthermore,
First step: $\mathbf{A}$ is dissipative. Indeed by Green's formula (Cor (7.1.2)), for each $\mathbf{u} = (u,v) \in D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ we have

$$< \mathbf{A}\mathbf{u}, \mathbf{u} >_X = \int_\Omega \nabla v \cdot \nabla u \; d\mathbf{x} + \int_\Omega \Delta u \cdot v \; d\mathbf{x} = 0.$$

Second step: $\mathbf{I} - \mathbf{A}$ is surjective. Let $\mathbf{h} = (h_1, h_2) \in X = H_0^1(\Omega) \times L^2(\Omega)$. Consider the equation $(\mathbf{I} - \mathbf{A})\mathbf{u} = \mathbf{h}$, i.e

$$u - v = h_1 \qquad v - \Delta u = h_2$$

$$u \in H^2(\Omega) \cap H_0^1(\Omega) \quad v \in H_0^1(\Omega).$$

Adding two equations we get

$$u - \Delta u = h_1 + h_2 \tag{7.4.2}$$

and as $h_1 + h_2 \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying (7.4.2) by the existence and regularity results we studied in the two previous sections. Then $v = u - h_1 \in H_0^1(\Omega)$ exists. Thus we have shown that $\mathbf{I} - \mathbf{A}$ is surjective.
Therefore, by the Lumer-Phillips Theorem we deduce that $\mathbf{A}$ generates a C_0-semigroup of contractions. And by Remark (1.4.3) we deduce that if $\mathbf{u}_0 = (f, g) \in D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, then we get that there exists a unique solution $[0, \infty) \ni t \to \mathbf{u}(t) = (u(t), v(t)) \in X$ to the evolution equation

$$\mathbf{u}'(t) = \mathbf{A}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0$$

and u satisfies $u \in C([0, \infty), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty), H_0^1(\Omega)) \cap C^2([0, \infty), L^2(\Omega))$.

Remark 7.4.1. In fact arguing as before we deduce that $-A$ satisfies the conditions of Lumer-Phillips Theorem (2.5.2) as well. In other words A satisfies the condition of the celebrated Stone's generation Theorem (see Theorem (4.5.4) in Kesavan [KS I]). This implies that A and $-A$ generate a group of isometries. Based on this, we can extend the previous Theorem (see (4.7.1) in Kesavan [KS I]. Specifically, we can show that under the assumptions of Theorem (7.4.1), for the unique solution u we have that for every $\epsilon > 0$,

$$u \in C^\infty([\epsilon, \infty) \times \overline{\Omega}).$$

7.5 Stochastic heat equation with additive space-time white noise

Let $U = H = L^2(V)$, where $V \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary ∂V. We consider the following problem

$$\begin{cases} d_t X(t,\xi) \;=\; \Delta_\xi X(t,\xi)\, dt + dW(t,\xi) & t \geq 0, \;\; \xi \in V \\ X(t,\xi) \;=\; 0 & t \geq 0, \;\; \xi \in \partial V \\ X(0,\xi) \;=\; 0, & \xi \in V \end{cases} \tag{7.5.1}$$

where $\Delta_\xi(t,\xi) = \sum_{i=1}^{\infty} \frac{\partial^2}{\partial \xi_i^2} X(t,\xi)$ is the spatial Laplace operator. We write the problem (7.5.1) in the abstract form

$$\begin{cases} dX(t) \;=\; AX(t)\, dt \,+\, dW(t), & t \geq 0 \\ X(0) \;=\; 0, & , \end{cases}$$

where

$$\begin{cases} D(A) = H^2(V) \cap H_0^1(V) \subset L^2(V), \\ A \;= \Delta_\xi \,, & , \end{cases}$$

$U = H = L^2(V)$, $f = 0$ and $B = I$. Consider the case $Tr(Q) < \infty$. Based on Theorem (6.0.5) in order to conclude that the problem (7.5.1) admits a unique weak solution given by the stochastic convolution $W_{\Delta_\xi}(t) = \int_0^t S(t-s)\, dW(s)$, where $S(\cdot)$ is the C_0- semigroup of contraction generated by $A = \Delta_\xi$, it is enough to show that

$$\int_0^t \| S(s) \|_{L_2^0}^2 \; ds = \int_0^t Tr\{S(s) Q S^*(s)\}\; ds < \infty \tag{7.5.2}$$

Indeed if $(e_n)_n$ is an orthonormal basis for $L^2(V)$, then

$$\int_0^t \| S(s) \|_{L_2^0}^2 \; ds = \int_0^t \left\| S(s) Q^{1/2} \right\|_{B_2(L^2(V))}^2 \; ds = \int_0^t \sum_{n=1}^{\infty} \left\| S(s) Q^{1/2} e_n \right\|_{L^2(V)}^2 \; ds$$

$$\leq \int_0^t \| S(s) \|_{B(L^2(V))}^2 \sum_{n=1}^{\infty} \left\| Q^{1/2} e_n \right\|_{L^2(V)}^2 \; ds \leq \int_0^t \sum_{n=1}^{\infty} \left\| Q^{1/2} e_n \right\|_{L^2(V)}^2 \; ds = t\; Tr(Q) < \infty.$$

Thus, when $Tr(Q) < \infty$, there exists a weak solution in any spatial dimension.
Next we consider the case $Q = I$. It is well known that we can represent the trajectories of the semigroup $S(\cdot)$ which is generated by A by

$$S(t)v = \mathrm{e}^{tA} v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\mu_j} \langle v, \phi_j \rangle \phi_j,$$

where $(\mu_i, \phi_i)_{i=1}^{\infty}$ are the eigenpairs of Δ_ξ. The set of the eigenvectors $(\phi_j)_j$ is an orthonormal basis for $L^2(V)$ and for the eigenvalues it holds that

$$0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_j \leq \cdots, \quad \mu_j \approx j^{2/n} \to \infty \text{ as } j \to \infty. \tag{7.5.3}$$

The semigroup is analytic and, in particular, by a simple calculation using Parseval's identity we have

$$\int_0^T \|(-A)^{1/2} \mathrm{e}^{tA} v\|^2 \, \mathrm{d}t = \int_0^T \sum_j \mu_j \mathrm{e}^{-2t\mu_j} \langle v, \phi_j \rangle^2 \, \mathrm{d}t \leq \frac{1}{2} \|v\|^2. \tag{7.5.4}$$

Thus by (7.5.4) we have

$$\begin{aligned}
\int_0^T \|S(t) Q^{1/2}\|_{B_2(L^2(V))}^2 \, \mathrm{d}t &= \int_0^T \|\mathrm{e}^{tA} Q^{1/2}\|_{B_2(L^2(V))}^2 \, \mathrm{d}t \\
&= \int_0^T \sum_k \|\mathrm{e}^{tA} Q^{1/2} e_k\|^2 \, \mathrm{d}t \\
&= \sum_k \int_0^T \|(-A)^{1/2} \mathrm{e}^{tA} (-A)^{-1/2} Q^{1/2} e_k\|^2 \, \mathrm{d}t \\
&\leq \frac{1}{2} \sum_k \|(-A)^{-1/2} Q^{1/2} e_k\|^2 = \frac{1}{2} \|(-A)^{-1/2} Q^{1/2}\|_{B_2(L^2(V))}^2.
\end{aligned}$$

Thus (7.5.2) holds if

$$\|(-A)^{-1/2} Q^{1/2}\|_{B_2(L^2(V))} < \infty. \tag{7.5.5}$$

Then using (7.5.3) we get

$$\|(-A)^{-1/2} Q^{1/2}\|_{B_2(L^2(V))}^2 = \left\| -(A)^{-1/2} \right\|_{HS}^2 = \sum_{k=1}^{\infty} \mu_k^{-1} \sim \sum_{k=1}^{\infty} k^{-2/n}.$$

This is finite if and only if $n = 1$. Thus white noise is too irregular in higher spatial dimensions.

Appendix A

Basic tools from analysis

A.1 Nuclear and Hilbert-Schmidt Operators

Consider two separable real Hilbert spaces $(U, <,>_U)$, $(H, <,>_H)$. Let $\{e_n\}_{n=1}^{\infty} \subset U$ and $\{f_n\}_{n=1}^{\infty} \subset H$ be orthonormal bases for U and H respectively. In this section we will examine two special spaces of linear operators in $B(U, H)$, the nuclear operators and Hilbert-Schmidt operators.

Definition A.1.1. *For $T \in B(U)$ we say that T is* ***positive*** *if-f T is self-adjoint positive semi-definite, that is $T = T^*$ and $< Tu, u > \geq 0$, for all $u \in U$. In this case we denote $\mathbf{T} \geq \mathbf{0}$.*

Definition A.1.2. *An operator $T \in B(U, H)$ is said to be* ***nuclear operator*** *if-f there exist sequences $\{a_n\}_{n=1}^{\infty} \subset U$, $\{b_n\}_{n=1}^{\infty} \subset H$ such that $\sum_{n=1}^{\infty} \| a_n \| \| b_n \| < \infty$ and*

$$Tf = \sum_{n=1}^{\infty} < f, a_n > b_n, \quad \textit{for all } f \in U.$$

In this case we write $\mathbf{T} \in \mathbf{B_1(U, H)}$.

Proposition A.1.1. *The space of nuclear operators $B_1(U, H)$ equipped with the norm*

$$\| T \|_{B_1(U,H)} = \inf\{\sum_{n=1}^{\infty} \| a_n \| \| b_n \| \ : \ Tf = \sum_{n=1}^{\infty} < f, a_n > b_n, \ \ \forall f \in U.\}$$

is a Banach space.

Remark A.1.1. A nuclear operator $T \in B_1(U, H)$ is compact, since by Definition (A.1.2) it can be approximated by operators of finite rank.

Proposition A.1.2. *Let $U, H.K$ be separable Hilbert spaces. Then the following assertions hold*

(i) *If $T \in B_1(U, H)$ and $S_1 \in B(H, K)$, then $S_1 T \in B_1(U, K)$ and $\| S_1 T \|_1 \leq \| T \|_1 \| S_1 \|$.*

(ii) *If $T \in B(U, H)$ and $S_2 \in B_1(H, K)$, then $S_2 T \in B_1(U, K)$ and $\| S_2 T \|_1 \leq \| T \| \| S_2 \|_1$.*

Proof: The proof is simple and is omitted.

Lemma A.1.1. *Let $T \in B_1(U)$ and let $\{e_n\}_{n=1}^{\infty} \subset U$ be an orthonormal basis for U. Then the* ***trace*** *of T*

$$Tr(T) := \sum_{n=1}^{\infty} < Te_n, e_n >$$

exists and does not depend on the choice of the orthonormal basis.

Proof: Since $T \in B_1(U)$, there exist sequences $\{a_n\}_{n=1}^{\infty} \subset U$ and $\{b_n\}_{n=1}^{\infty} \subset U$ such that $\sum_{n=1}^{\infty} \| a_n \| \| b_n \| < \infty$ and

$$< Te_k, e_k > = \sum_{j=1}^{\infty} < e_k, b_j >< a_j, e_k > .$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} | < Te_k, e_k > | &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} | < e_k, b_j >< a_j, e_k > | \\
&\leq \sum_{j=1}^{\infty}\left(\left(\sum_{k=1}^{\infty} | < e_k, b_j > |^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} | < a_j, e_k > |^2\right)^{\frac{1}{2}}\right) \\
&= \sum_{j=1}^{\infty} \| a_j \| \| b_j \| < \infty.
\end{aligned}$$

Therefore, the series converges absolutely. Moreover, by Fubini's Theorem we have

$$\begin{aligned}
Tr(T) &= \sum_{k=1}^{\infty} < Te_k, e_k >= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} < e_k, b_j >< a_j, e_k > \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} < e_k, b_j >< a_j, e_k >= \sum_{j=1}^{\infty} < a_j, b_j >,
\end{aligned}$$

which is independent of the choice of the orthonormal basis.

Corollary A.1.1. *If $T \in B_1(U)$, then $Tr(T) \leq \| T \|_1$.*

Proposition A.1.3. *If $T \in B_1(U)$ and $S \in B(U)$, then $ST, TS \in B_1(U)$ and $Tr(TS) = Tr(ST) \leq \| T \|_1 \| S \|$.*

Proof: Let $\{a_n\}_{n=1}^{\infty} \subset U$, $\{b_n\}_{n=1}^{\infty} \subset U$ such that $\sum_{n=1}^{\infty} \| a_n \| \| b_n \| < \infty$ and $Tf = \sum_{n=1}^{\infty} < f, a_n > b_n$, for all $f \in U$. Then, $STf = \sum_{n=1}^{\infty} < f, a_n > Sb_n$, for all $f \in U$ and $TSf = \sum_{n=1}^{\infty} < f, S^* a_n > b_n$, for all $f \in U$. Therefore, $Tr(ST) = \sum_{n=1}^{\infty} < a_n, Sb_n >= \sum_{n=1}^{\infty} < S^* a_n, b_n >= Tr(TS)$.

Proposition A.1.4. *Let $T \in B(U)$, $T \geq 0$. Then T is nuclear if and only if for an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in U the trace $Tr(T) = \sum_{n=1}^{\infty} < Te_n, e_n > < \infty$ converges.*
In this case $Tr(T) = \| T \|_{B_1(U)}$.

Proof: See Da Prato [DP I] Proposition C.3.

Definition A.1.3. *An operator $T \in B(U, H)$ is said to be **Hilbert-Schmidt** if-f*

$$\sum_{n=1}^{\infty} \| Te_n \|^2 < \infty,$$

for an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of U. In this case we write $\mathbf{T} \in \mathbf{B_2(U, H)}$.

Remark A.1.2. If $T \in B_2(U, H)$ then the sum $\sum_{n=1}^{\infty} \| Te_n \|^2$ is independent of the choice of the orthonormal basis. Indeed, if $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for H, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \| Te_n \|^2 &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} | < Te_n, f_k > |^2 \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} | < e_n, T^* f_k > |^2 = \sum_{k=1}^{\infty} \| T^* f_k \|^2 .
\end{aligned}$$

Proposition A.1.5. *Let U, H be two separable Hilbert spaces with orthonormal bases $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ respectively. The space $B_2(U, H)$ of Hilbert-Schmidt operators, equipped with the inner product*

$$< T, S >_{B_2(U,H)} := \sum_{n=1}^{\infty} < Te_n, Se_n >_H$$

which induces the norm

$$\| T \|_{B_2(U,H)} := \left(\sum_{n=1}^{\infty} \| Te_n \|_H\right)^{\frac{1}{2}}$$

is a separable Hilbert space. More specifically, the set $\{f_j \otimes e_k\}_{j,k \in \mathbb{N}}$ consists an orthonormal basis for $B_2(U, H)$.

Proof: First of all we will show the completeness. Let $\{T_n\}_{n=1}^{\infty} \subset B_2(U,H)$ be a Cauchy sequence in $B_2(U,H)$. Then, since $\| T \| \leq \| T \|_{B_2(U,H)}$, it is direct that $\{T_n\}_{n=1}^{\infty}$ is also Cauchy in the Banach space $B(U,H)$. Therefore, there exists $T \in B(U,H)$ such that

$$\lim_{n\to\infty} \| T_n - T \|_{B(U,H)} = 0.$$

Fix an $\varepsilon > 0$ and choose a $n_0 \in \mathbb{N}$ such that for all positive integers $m, n \geq n_0$, $\| T_n - T_m \|_{B_2(U,H)}^2 < \varepsilon$. Then by Fatou's lemma for each $n \geq n_0$, we get

$$\begin{aligned} \| T_n - T \|_{B_2(U,H)}^2 &= \sum_{k=1}^{\infty} \| (T_n - T)e_k \|^2 \\ &= \sum_{k=1}^{\infty} \lim_{m\to\infty} \| (T_n - T_m)e_k \|^2 \\ &\leq \liminf_{m\to\infty} \sum_{k=1}^{\infty} \| (T_n - T_m)e_k \|^2 \\ &= \liminf_{m\to\infty} \| T_n - T_m \|_{B_2(U,H)}^2 \leq \varepsilon. \end{aligned}$$

So, we have shown that $\lim_{n\to\infty} \| T_n - T \|_{B_2(U,H)} = 0$ and $T \in B_2(U,H)$. We will continue with the separability of $B_2(U,H)$. To this aim we will show that $\{f_j \otimes e_k\}_{j,k\in\mathbb{N}}$ is countable orthonormal basis for $B_2(U,H)$. It is easy to see that for each $j,k \in \mathbb{N}$ the rank one operator $f_j \otimes e_k$ defined by $(f_j \otimes e_k)(u) = f_j < u, e_k >, u \in U$ belongs to $B_2(U,H)$ and $\| f_j \otimes e_k \|_{B_2(U,H)} = 1$. Moreover, for each $T \in B_2(U,H)$ we have

$$< f_j \otimes e_k, T >_{B_2(U,H)} = \sum_{n=1}^{\infty} < f_j, Te_n >< e_k, e_n > = < f_j, Te_k > .$$

From the last one observation, we deduce that

$$< f_j \otimes e_k, f_l \otimes e_m >_{B_2(U,H)} = 0,$$

when $j \neq l$ or $k \neq m$. Therefore the system $\{f_j \otimes e_k\}_{j,k\in\mathbb{N}}$ is orthonormal. Moreover if $T \in B_2(U,H)$ such that $< f_j \otimes e_k, T >_{B_2(U,H)} = 0$ for all $j,k \in \mathbb{N}$, then $< f_j, Te_k > = 0$, for all $j,k \in \mathbb{N}$. Since $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for H, we get that $Te_k = 0$, for all $k \in \mathbb{N}$, thus $T = 0$.

Proposition A.1.6. *$T \in B_2(U,H)$ if and only if $T^* \in B_2(H,U)$ and in this case $\| T \|_{B_2(U,H)} = \| T^* \|_{B_2(H,U)}$.*

Proposition A.1.7. *If $T \in B_2(U,H)$, then $\| T \|_{B(U,H)} \leq \| T \|_2$.*

Proof: Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for H, then for all $f \in U$ we have

$$\| Tf \|^2 = \sum_{n=1}^{\infty} | < Tf, f_n > |^2 = \sum_{n=1}^{\infty} | < f, T^* f_n > |^2$$

$$\leq \| f \|^2 \sum_{n=1}^{\infty} \| T^* f_n \|^2 = \| f \|^2 \| T^* \|_{B_2(H,U)}^2 = \| f \|^2 \| T \|_{B_2(U,H)}^2 .$$

Therefore, $\| T \|_{B(U,H)} \leq \| T \|_{B_2(U,H)}$.

Proposition A.1.8. *If $T \in B_2(U,H)$ and $S \in B(U)$, then $TS \in B_2(U,H)$ and $\| TS \|_{B_2(U,H)} \leq \| T \|_{B_2(U,H)} \| S \|$. If $T \in B(U,H)$ and $S \in B_2(U)$, then $TS \in B_2(U,H)$ and $\| TS \|_{B_2(U,H)} \leq \| T \| \| S \|_{B_2(U,H)}$.*

Proof: It is a matter of simple computations.

Proposition A.1.9. *Let E, F, G be separable Hilbert spaces. If $T \in B_2(E,F)$ and $S \in B_2(F,G)$, then $ST \in B_1(E,G)$ and $\| ST \|_1 \leq \| S \|_2 \| T \|_2$.*

Proof: Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for F. Then, for each $x \in E$ we have

$$STx = S\left(\sum_{n=1}^{\infty} < Tx, f_n > f_n\right) = \sum_{n=1}^{\infty} < x, T^* f_n > Sf_n.$$

Therefore,

$$\| ST \|_1 \leq \sum_{n=1}^{\infty} \| T^* f_n \| \, \| Sf_n \| \leq \left(\sum_{n=1}^{\infty} \| T^* f_n \|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \| Sf_n \|^2\right)^{\frac{1}{2}}$$

$$= \| T^* \|_2 \| S \|_2 = \| T \|_2 \| S \|_2 .$$

Corollary A.1.2. *$T \in B_2(U, H)$ if and only if $TT^* \in B_1(H)$ if and only if $T^*T \in B_1(U)$. In this case, $\| T \|^2_{B_2(U,H)} = Tr(TT^*) = Tr(T^*T)$.*

Proof: If $T \in B_2(U, H)$, then by Proposition (A.1.6) $T^* \in B_2(H, U)$ and thus by Proposition (A.1.9) $TT^* \in B_1(H)$ and $T^*T \in B_1(U)$. Moreover, if $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis in H, then

$$Tr(TT^*) = \sum_{n=1}^{\infty} < TT^* f_n, f_n >= \sum_{n=1}^{\infty} \| T^* f_n \|^2 = \| T^* \|_2^2 = \| T \|_2^2 .$$

Conversely, if $TT^* \in B_1(H)$, then by Lemma (A.1.1) $Tr(TT^*) < \infty$. Since $Tr(TT^*) = \sum_{n=1}^{\infty} \| T^* f_n \|^2$ we have that both T and T^* are Hilbert-Schmidt operators.

Proposition A.1.10. *If $Q \in B(U)$, $Q \geq 0$ and $Tr(Q) < \infty$, then there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty} \subset U$ such that*

$$Qe_n = \lambda_n e_n$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ and $\lim_{k\to\infty} \lambda_k = 0$ and 0 is the only accumulation point of $(\lambda_k)_k$. Moreover,

$$Qx = \sum_{k=1}^{\infty} \lambda_k < x, e_k > e_k, \quad x \in U.$$

Proof: This is a consequence of the Spectral Theorem for self-adjoint compact operators.

A.2 Pseudo-inverse and the Cameron-Martin space

Let U, H be two separable real Hilbert spaces. We know that an operator $T \in B(U, H)$ is $1-1$ if and only if $Ker(T) = \{0\}$. Therefore, the restriction $T|_{(Ker(T))^\perp}$ is always one-to-one. Moreover, since $U = Ker(T) \oplus (Ker(T))^\perp$, we have that $T((Ker(T))^\perp) = T(U)$. This discussion drives us to the following definition.

Definition A.2.1. *Let $T \in B(U, H)$. We define the **pseudo-inverse** of T, still denoted by T^{-1} by*

$$T^{-1} := \left(T|_{(Ker(T))^\perp}\right)^{-1}.$$

The pseudo-inverse T^{-1} is defined on $T(U)$, so

$$T^{-1} : T(U) \to (Ker(T))^\perp.$$

Definition A.2.2. *Let $Q \in B(U)$, $Q \geq 0$ and let $Q^{\frac{1}{2}}$ be its positive square root, that is $Q^{\frac{1}{2}} \geq 0$ and $Q^{1/2}Q^{1/2} = Q$. We define the **Cameron-Martin space** $U_0 = Q^{1/2}(U)$ with inner product*

$$< u_0, v_0 >_{U_0} := < Q^{-1/2} u_0, Q^{-1/2} v_0 >_U, \quad , u_0, v_0 \in U_0,$$

where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$ in case it is not one-to-one.

Remark A.2.1. The map

$$Q^{1/2} : ((Ker(Q^{1/2}))^{\perp}, <, >_U) \to (U_0, <, >_0)$$

is an **isometric isomorphism**. Therefore, the Cameron-Martin space $(U_0, <, >_0)$ is separable and Hilbert. Moreover, if $\{g_n\}_{n=1}^{\infty}$ is an orthonormal basis of $(Ker(Q^{1/2}))^{\perp}$, then $\{Q^{1/2}g_n\}_{n=1}^{\infty}$ is an orthonormal basis for $(U_0, <, >_0)$.

Definition A.2.3. *Let $Q \in B(U)$, $Q \geq 0$ and let $U_0 = Q^{1/2}(U)$ be the Cameron-Martin space endowed with the inner product $<, >_0$. We define the space*

$$\mathbf{L_2^0} := \mathbf{B_2(U_0, H)}$$

and

$$\mathbf{B(U,H)_0} := \{T|_{U_0} : T \in B(U,H)\}.$$

Lemma A.2.1. *Let $Q \in B(U)$, $Q \geq 0$ with $Tr(Q) < \infty$. There exists an orthonormal basis for $L_2^0 = B_2(U_0, H)$ consisting of elements of $B(U,H)_0$. Furthermore, $B(U,H)_0 \subset L_2^0$ and thus it is dense in L_2^0.*

Proof: Consider an orthonormal basis of U, consisting by the eigenvectors of Q. Such a basis exists by (A.1.10). In particular this basis is the union of an orthonormal basis $\{h_k\}_{k=1}^{\infty}$ for $Ker(Q)$, corresponding to the zero eigenvalues of Q and of an orthonormal basis $\{g_k\}_{k=1}^{\infty}$ for $(Ker(Q))^{\perp}$ corresponding to the positive eigenvalues of Q. Note that $Ker(Q) = Ker(Q^{1/2})$. Indeed it is direct that $Ker(Q^{1/2}) \subset Ker(Q)$. For the inverse inclusion, observe that if $x \in Ker(Q)$, then $Q^{1/2}x \in Ker(Q^{1/2}) \cap (Ker(Q^{1/2}))^{\perp}$, so $Q^{1/2}x = 0$. By remark (A.2.1), since $\{g_n\}_{n=1}^{\infty}$ is an orthonormal basis for $(Ker(Q^{1/2}))^{\perp}$, then $\{Q^{1/2}g_n\}_{n=1}^{\infty}$ is an orthonormal basis for U_0. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Then by Proposition (A.1.5), $\{f_j \otimes Q^{1/2}g_k\}_{j,k\in\mathbb{N}}$ is an orthonormal basis for $L_2^0 = B_2(U_0, H)$. It is easy to prove that these operators are elements of $B(U,H)_0$ as well. Indeed, for all $u \in U_0$ we have

$$\left\| (f_j \otimes Q^{1/2}g_k)(u) \right\|_H = \left\| f_j < Q^{1/2}g_k, u >_{U_0} \right\|_H$$

$$= \left\| f_j < g_k, Q^{-1/2}u >_U \right\|_H = \left\| f_j < g_k, \sum_{\lambda_k > 0} \frac{1}{\sqrt{\lambda_k}} < u, g_k > g_k >_U \right\|_H$$

$$\leq \frac{1}{\sqrt{\lambda_k}} \| u \|_U .$$

This proves the first statement. For the second statement, since $Tr(Q) < \infty$, by proposition (A.1.4) we have that $Q \in B_1(U)$. Thus, by Corollary (A.1.2) $Q^{1/2} \in B_2(U)$. Now, if $T \in B(U,H)_0$, by Proposition (A.1.8) $TQ^{1/2} \in B_2(U,H)$. Combining all these, we get

$$\| T \|_{L_2^0}^2 = \sum_{k=1}^{\infty} \left\| TQ^{1/2}g_k \right\|^2$$

$$= \sum_{k=1}^{\infty} \left\| TQ^{1/2}g_k \right\|^2 + \sum_{k=1}^{\infty} \left\| TQ^{1/2}h_k \right\|^2 = \left\| TQ^{1/2} \right\|_{B_2(U,H)}^2 \leq \| T \|_{B(U,H)}^2 \left\| Q^{1/2} \right\|_{B_2(U)}^2 < \infty.$$

Thus, $B(U,H)_0 \subset L_2^0$

A.3 Calculus of Banach space valued functions

A.3.1 Differentiation of Banach space valued functions

Definition A.3.1. *Let $(X, \| \, \|_X)$ be a normed space, $(a,b) \subset \mathbb{R}^+$ an open interval, $f : (a,b) \to X$ a function and $t_0 \in (a,b)$ a point. The function f is said to be **differentiable** at t_0 if-f the limit*

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}$$

*exists in X. In this case, the value of the above limit is denoted by $f'(t_0)$ and is called the **derivative** of f at t_0.*

Remark A.3.1. In the same manner, we can define the **right derivative (left derivative)** of f at a point t_0, assuming that the function is defined in an interval of the form $[t_0, t_0+h)((t_0-h, t_0])$, for some $h>0$.

Remark A.3.2. A lot of standard results in differential calculus of scalar valued functions, remain valid in the case of Banach space valued functions. We will now examine some cases which will be useful in the next sections.

Proposition A.3.1. *Let $(X, \| \|_X)$ be a normed space and $f:(a,b)\to X$ a differentiable function such that $f'(t)=0$, for each $t\in(a,b)$. Then f is constant.*

Proof: Let $x^* \in X^*$ a linear bounded functional. Consider the scalar valued function $g:(a,b)\to\mathbb{R}$(or $\mathbb{C}$), $g(t)=x^*(f(t))$, for each $t\in(a,b)$. By the linearity and continuity of x^*, we have that g is differentiable and $g'(t)=x^*(f'(t))=0$, for each $t\in(a,b)$. Thus, $x^*(f(t))=$ constant, for each $t\in(a,b)$, $x^*\in X^*$. Now, let $t_0\in(a,b)$, then for each $t\in(a,b)$ we have:

$$\|f(t)-f(t_0)\| = \sup\{x^*(f(t)-f(t_0)) : x^*\in X^*, \|x^*\|\le 1\} = 0$$

Proposition A.3.2. *Let $(X, \| \|_X)$ be a normed space and $f:(a,b)\to X$ a differentiable function at a point $t_0\in(a,b)$. Then f is continuous at t_0.*

Remark A.3.3. In contrast to the two previous results, the Mean Value theorem is not true in general for Banach space valued functions. To this end, consider the vector valued function $f:[0,\pi/2]\to\mathbb{R}^2$, $f(t)=(\sin t,\cos t)$. Then, $\| (b-a)f'(t) \|_2 = \pi/2$, for each $t\in(a,b)$, while $\| f(b)-f(a) \|_2 = \sqrt{2}$.

A.3.2 Riemann integral of Banach space valued functions

Definition A.3.2. *Let $(X, \| \|_X)$ be a normed space, $f:[a,b]\to X$ a function,*

$$\mathcal{J} = \{a=t_0<t_1<\ldots<t_k=b\},\ \textit{a partition of } [a,b]$$

$$\Xi = \{\xi_i\}_{i=0}^{k-1}\subset[a,b],\ \textit{such that: } a=t_0\le\xi_0\le t_1\le\ldots\le\xi_{k-1}\le t_k=b.$$

We define the real number:

$$\Delta(\mathcal{J}) = \max\{t_{i+1}-t_i : 0\le i\le k-1\}$$

and the element of X:

$$S(\mathcal{J},\Xi,f) = \sum_{i=0}^{k-1} f(\xi_i)(t_{i+1}-t_i).$$

*The function f is said to be **Riemann integrable** if-f the limit $\lim_{n\to\infty} S(\mathcal{J}_n,\Xi_n,f)$ exists in X, for each sequence of pairs $((\mathcal{J}_n,\Xi_n))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\Delta(\mathcal{J}_n)=0$ and is independent of the choice of the sequence. In this case, the limit $\lim_{n\to\infty} S(\mathcal{J}_n,\Xi_n,f)$ is called the **Riemann integral** of f and is denoted by $\int_a^b f(t)dt$.*

Theorem A.3.1. *Let $(X, \| \|_X)$ be a Banach space and $f:(a,b)\to X$ a function. If f is continuous, then it is also Riemann integrable.*

Proof: Let $((\mathcal{J}_n,\Xi_n))_{n\in\mathbb{N}}$ be a sequence as in Definition (A.3.2), with $\lim_{n\to\infty}\Delta(\mathcal{J}_n)=0$. We will show that the sequence $(S(\mathcal{J}_n,\Xi_n,f))_{n=1}^{\infty}\subset X$ is Cauchy, thus convergent, since X is Banach. Let $\varepsilon>0$. The function f is uniformly continuous in $[a,b]$, since it is continuous and $[a,b]$ is compact. So, there is a $\delta>0$ such that:

$$\|f(s)-f(t)\| < \varepsilon/2(b-a), \text{ for all s,t} \in [a,b], \text{ with } |s-t|<\delta.$$

Consider now the pairs $(\mathcal{J},\Xi)$, $\mathcal{J}=\{t_i\}_{i=0}^{k}$, $\Xi=\{\xi_i\}_{i=0}^{k-1}$, $\Delta(\mathcal{J})<\delta$, where $\mathcal{J},\Xi$ are

as in definition and $(\mathcal{J}',\Xi')$, $\mathcal{J}'=\{t'_i\}_{i=0}^{k'}$, $\Xi'=\{\xi'_i\}_{i=0}^{k'-1}$, $\Delta(\mathcal{J}')<\delta$, where $\mathcal{J}',\Xi'$

are as in definition. Observe that for the pair $(\mathcal{J}'',\Xi'')$, where $\mathcal{J}''=\mathcal{J}\cup\mathcal{J}'$, $\mathcal{J}''=\{t''_i\}_{i=0}^{k''}$, $\Xi''=\{t''_i\}_{i=0}^{k''-1}$, we have that $k''\le k+k'-2$ and $\Delta(\mathcal{J}'')<\delta$. In addition, for $i=0,1,\ldots,k-1$ the interval $[t_i,t_{i+1}]$, either coincides with some $[t''_{j_i}, t''_{j_i+1}]$, or it is a finite union of intervals, i.e $[t_i, t_{i+1}] = [t''_{j_i}, t''_{j_i+1}] \cup \ldots \cup [t''_{j_i+l}, t''_{j_i+l+1}]$, for some positive integer l. Therefore,we have:

$$\begin{aligned} \|f(\xi_i)(t_{i+1} - t_i) - \sum_{m=0}^{l} f(t''_{j_i+m})(t''_{j_i+m+1} - t''_{j_i+m})\| &= \|\sum_{m=0}^{l}[f(\xi_i) - f(t''_{j_i+m})][t''_{j_i+m+1} - t''_{j_i+m}]\| \\ &< \frac{\varepsilon(t_{i+1} - t_i)}{2(b-a)}, \end{aligned}$$

since $\xi_i, t''_{j_i+m} \in [t_i, t_{i+1}]$, for each $m = 0, 1 \ldots, l$, so $|\xi_i - t''_{j_i+m}| < \delta$. Summing over i we get:

$$\|S(\mathcal{J}, \Xi, f) - S(\mathcal{J}'', \Xi'', f)\| < \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon/2.$$

Similarly, we have:

$$\|S(\mathcal{J}', \Xi', f) - S(\mathcal{J}'', \Xi'', f)\| < \varepsilon/2.$$

Therefore:

$$\|S(\mathcal{J}, \Xi, f) - S(\mathcal{J}', \Xi', f)\| < \varepsilon, \tag{A.3.1}$$

for each $(\mathcal{J}, \Xi)$, $(\mathcal{J}', \Xi')$ such that Ξ, Ξ' are as in definition and $\Delta(\mathcal{J})$, $\Delta(\mathcal{J}') < \delta$. This leads us to the observation that $(S(\mathcal{J}_n, \Xi_n, f))_n$ is Cauchy and that the limit $\lim_{n\to\infty} S(\mathcal{J}_n, \Xi_n, f)$ is independent of the choice of the sequence, as desired.

Remark A.3.4. The previous theorem reveals the important role that completeness of Banach spaces plays to the Riemann integration of Banach space valued functions.

Proposition A.3.3. *Let $(X, \| \|_X)$ be a Banach space, $f, g : [a, b] \to X$ Riemann integrable functions and λ, κ scalars. Then, $\lambda f + \kappa g$ is Riemann integrable and*

$$\int_a^b (\lambda f + \kappa g)dx = \lambda \int_a^b f(x)dx + \kappa \int_a^b g(x)dx$$

Proof: For every sequence $((\mathcal{J}_n, \Xi_n))_{n\in\mathbb{N}}$ as in Definition (A.3.2), such that $\lim\limits_{n\to\infty} \Delta(\mathcal{J}_n) = 0$, we have:

$$S(\mathcal{J}_n, \Xi_n, \lambda f + \kappa g) = \lambda S(\mathcal{J}_n, \Xi_n, f) + \kappa S(\mathcal{J}_n, \Xi_n, g), \text{ for each n} \in \mathbb{N}$$

Since f, g are Riemman integrable, it is enough to take the limits as $n \to \infty$ in the last identity.

Proposition A.3.4. *Let $(X, \| \|_X)$ be a Banach space and $f : (a, b) \to X$ a continuous function. Then,*

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \| f(t) \| \, dt$$

Proof: First of all, observe that the real valued function $\|f\|$ is integrable, since it is continuous. Let $((\mathcal{J}_n, \Xi_n))_{n\in\mathbb{N}}$ be a sequence as in Definition (A.3.2), such that $\lim\limits_{n\to\infty} \Delta(\mathcal{J}_n) = 0$. Then, we can easily derive that:

$$\| S(\mathcal{J}_n, \Xi_n, f) \| \leq S(\mathcal{J}_n, \Xi_n, \|f\|), \text{ for each n} \in \mathbb{N}.$$

By integrability of f and $\|f\|$ and continuity of the norm $\| \|$, taking the limits as $n \to \infty$, we have the desired result.

Theorem A.3.2. *Let $(X, \| \|_X)$ be a normed space, $f : [a, b] \to X$ a continuous function and a fixed point $x \in X$. Consider the function $F : [a, b] \to X$,*

$$F(t) = x + \int_a^t f(s)\ ds.$$

Then F is differentiable and $F'(t) = f(t)$, for all $t \in [a, b]$.

Proof: Let $t_0 \in [a,b]$. Without loss of generality, we assume that t_0 is an interior point of $[a,b]$. Let $\varepsilon > 0$. From continuity of f at t_0, there is $\delta > 0$ such that:

$$\|f(t) - f(t_0)\| < \varepsilon, \ \text{for each t} \in [a,b], \ \text{with } |t - t_0| < \delta.$$

Therefore, for each $0 < h < \delta$ we have:

$$\begin{aligned} \left\| \frac{F(t_0+h) - F(t_0)}{h} - f(t_0) \right\| &= \frac{1}{h} \left\| \int_a^{t_0+h} f(s)\ ds - \int_a^{t_0} f(s)\ ds - hf(t_0) \right\| \\ &= \frac{1}{h} \left\| \int_{t_0}^{t_0+h} (f(s) - f(t_0))\ ds \right\| \\ &\le \frac{1}{h} \int_{t_0}^{t_0+h} \|f(s) - f(t_0)\|\ ds < \varepsilon. \end{aligned} \tag{A.3.2}$$

Similarly, we can show that for $-\delta < h < 0$ we have again that:

$$\left\| \frac{F(t_0+h) - F(t_0)}{h} - f(t_0) \right\| < \varepsilon.$$

This means that the last inequality is valid for all $0 < |h| < \delta$, which completes the proof.

Corollary A.3.1. *Let $(X, \| \ \|_X)$ be a normed space and $f : [a,b] \to X$ a function. Then, $f \in C^1([a,b])$ if and only if there exists a continuous function $\phi : [a,b] \to X$, such that:*

$$f(t) = f(a) + \int_a^t \phi(s)\ ds \ \ , \textit{for each } t \in [a,b] \tag{A.3.3}$$

Proof: Suppose that $f \in C^1([a,b])$ and consider the function,

$$z(t) = f(t) - f(a) - \int_a^t f'(s)\ ds, \ \text{t} \in [a,b].$$

By virtue of Theorem (A.3.2), the function z is differentiable in $[a,b]$ and $z'(t) = 0$, for each $t \in [a,b]$. Now, by virtue of Proposition (A.3.1), $z(t) = z(a) = 0$, for each $t \in [a,b]$. The converse is direct from Theorem (A.3.2)

Corollary A.3.2. *Let $(X, \| \ \|_X)$ be a normed space, $f : [a,b] \to X$ a continuously differentiable function in $[a,b]$. Then,*

$$\int_a^b f'(s)\ ds = f(b) - f(a) \tag{A.3.4}$$

Proof: Consider the function, $F(t) = f(t) - \int_a^t f'(s)\ ds$, $t \in [a,b]$. Then, as in the proof of Corollary (A.3.1), we have that F is constant in $[a,b]$. Therefore $F(a) = F(b)$ and the proof is complete.

Theorem A.3.3 (Uniform Convergence Theorem)**.** *Let $(X, \| \ \|_X)$ be a normed space, $f_n : [a,b] \to X$, $n \in \mathbb{N}$, a sequence of continuous functions and $f : [a,b] \to X$ a function. If $f_n \to f$ uniformly as $n \to \infty$, then:*

$$\lim_{n\to\infty} \int_a^b f_n(s)\ ds = \int_a^b f(s)\ ds.$$

Proof: Since the convergence is uniform and $(f_n)_n$ is a sequence of continuous functions, the limit f is also a continuous function, thus Riemann integrable. Moreover, we have:

$$\begin{aligned} \left\| \int_a^b f_n(s)\ ds - \int_a^b f(s)\ ds \right\| &\le \int_a^b \|f_n(s) - f(s)\|\ ds \\ &\le (b-a) \sup_{s\in[a,b]} \|f_n(s) - f(s)\| \to 0, \ \text{ as n} \to \infty \end{aligned}$$

Remark A.3.5. We remind you that if $(X, \| \|_X)$ is a Banach space, then the space $B(X)$ equipped with the operation of composition of two functions is a Banach algebra. Moreover, if $(T_n)_n$, $(S_n)_n$ are sequences in $B(X)$, such that $\lim_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} S_n = S$, then $\lim_{n\to\infty} S_n T_n = ST$, which means that the operation of multiplication is continuous. To see this note that $(\| T_n \|)_n$ is bounded since $(T_n)_n$ is convergent and for each $n \in \mathbb{N}$ we have,

$$\| S_n T_n - ST \| \leq \| S_n T_n - ST_n \| + \| ST_n - ST \| \leq (\sup_{n\in\mathbb{N}} \| T_n \|) \| S_n - S \| + \| S \| \| T_n - T \| .$$

Proposition A.3.5. *Let $(X, \star)$ be a Banach algebra, $f : [a, b] \to X$ a Riemann integrable function and a fixed point $c \in X$. Then $c \star f$ and $f \star c$ are Riemann integrable functions and it holds that:*

$$c \star \int_a^b f(s)\, ds = \int_a^b c \star f(s)\, ds$$

$$\left[\int_a^b f(s)\, ds\right] \star c = \int_a^b f(s) \star c\, ds$$

Proof: For every sequence $((\mathcal{J}_n, \Xi_n))_{n\in\mathbb{N}}$ as in Definition (A.3.2), such that $\lim_{n\to\infty} \Delta(\mathcal{J}_n) = 0$, we have:

$$c \star S(\mathcal{J}_n, \Xi_n, f) = S(\mathcal{J}_n, \Xi_n, c \star f), \text{ for each n} \in \mathbb{N}$$

$$S(\mathcal{J}_n, \Xi_n, f) \star c = S(\mathcal{J}_n, \Xi_n, f \star c), \text{ for each n} \in \mathbb{N}$$

Since f is Riemman integrable and " $\star$ " is a continuous function, it is enough to take the limits as $n \to \infty$.

A.4 Closed Linear Operators

Definition A.4.1. *A linear operator $A : X \supset D(A) \to X$, on a Banach space $(X, \| \|_X)$ (where $D(A)$ is a linear subspace) is said to be **closed** if-f for each sequence $\{x_n\}_{n=1}^{\infty} \subset D(A)$, such that:*

$$\lim_{n\to\infty} x_n = x \in X \quad \textit{and} \quad \lim_{n\to\infty} Ax_n = y \in X,$$

it holds that:

$$x \in D(A) \;\; \textit{and} \;\; y = Ax.$$

Proposition A.4.1. *Let $A : X \supset D(A) \to X$ be a linear operator on a Banach space $(X, \| \|_X)$. Then, A is closed if and only if the graph of A:*

$$G_A = \{(x, y) \in X \times X : x \in D(A),\, y = Ax\}$$

is a closed subset in $X \times X$.

Proof: The proof is direct from the definition of a closed set via sequences.

Remark A.4.1. It is clear that every $T \in B(X)$ is closed. The inverse statement is not always true.

Example A.4.1. Consider the Banach space $(C([0, 1]), \| \|_\infty)$ of continuous complex functions in $[0, 1]$, endowed with the supremum norm and its linear subspace $D(A) = C^1([0, 1]) \subset C([0, 1])$ of continuously differentiable functions. We define the linear operator $A : X \supset D(A) \to X$, $Ax = x'$. We claim that A is not bounded. To this end, consider the sequence $\{x_n\}_{n=1}^{\infty} \subset C^1([0, 1])$, $x_n(t) = t^n$, $0 \leq t \leq 1$, $n \in \mathbb{N}$. Then, $\| x_n \|_\infty = 1$ and $\| Ax_n \|_\infty = n$, for each $n \in \mathbb{N}$. On the other hand, it is easy to verify that A is closed. For this, suppose that $\{x_n\}_{n=1}^{\infty} \subset D(A)$ is a sequence such that: $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Ax_n = y \in X$. Equivalently $x_n \to x$ uniformly and $x_n' \to y$, uniformly. Thus, $x \in C^1([0, 1])$ and $y = x'$, as desired.

Proposition A.4.2. *Let $A : X \supset D(A) \to X$ be a closed linear operator, where X is Banach space and let $[a, b] \ni t \to x_t \in D(A)$ be a Riemann integrable function. If the function $[a, b] \ni t \to Ax_t \in X$ is Riemann integrable, then:*

$$\int_a^b x_s\, ds \in D(A)$$

and

$$A \int_a^b x_s\, ds = \int_a^b Ax_s\, ds$$

Proof: Let $((\mathcal{J}_n, \Xi_n))_{n\in\mathbb{N}}$ be a sequence as in definition (A.3.2), such that $\lim\limits_{n\to\infty} \Delta(\mathcal{J}_n) = 0$. Observe that the the sequence $(S(\mathcal{J}_n, \Xi_n, x))_{n=1}^{\infty}$ is in $D(A)$, since $D(A)$ is a linear space. In addition,

$$A(S(\mathcal{J}_n, \Xi_n, x)) = S(\mathcal{J}_n, \Xi_n, Ax), \text{ for each n} \in \mathbb{N}.$$

Since the functions $t \to x_t$ and $t \to Ax_t$ are Riemann integrable, we have:

$$\lim_{n\to\infty} S(\mathcal{J}_n, \Xi_n, x) = \int_a^b x_s \, ds \text{ and } \lim_{n\to\infty} S(\mathcal{J}_n, \Xi_n, Ax) = \int_a^b Ax_s \, ds.$$

Now, the closedness of A implies the desired results.

Proposition A.4.3. *Let $A : X \supset D(A) \to X$ be a linear operator, where X is a Banach space. We define the norm*

$$\| \, \|_A : D(A) \to [0, \infty), \ \| x \|_A = \| x \|_X + \| Ax \|_X .$$

Then, A is closed if and only if $(D(A), \| \, \|_A)$ is Banach.

Proof:

" $\Rightarrow$ " Assume that A is closed and $(x_n)_n$ is a Cauchy sequence in $(D(A), \| \, \|_A)$. Then, since $\| x \|_X \leq \| x \|_A$ and $\| Ax \|_X \leq \| x \|_A$, for all $x \in D(A)$, we conclude that $(x_n)_n$ and $(Ax_n)_n$ are $\| \, \|_X$-Cauchy. From completeness of $(X, \| \, \|_X)$ and closedness of A, there is $x \in D(A)$ such that: $\lim\limits_{n\to\infty} \| x_n - x \|_X = 0$ and $\lim\limits_{n\to\infty} \| Ax_n - Ax \|_X = 0$. This implies that $\lim\limits_{n\to\infty} \| x_n - x \|_A = 0$.

" $\Leftarrow$ " Consider a sequence $(x_n)_n$ in $D(A)$ such that the sequences $(x_n)_n$ and $(Ax_n)_n$ are $\| \, \|_X$-convergent, so $\| \, \|_X$-Cauchy. Then, $(x_n)_n$ is $\| \, \|_A$-Cauchy. Completeness of $(D(A), \| \, \|_A)$ implies that there is a $x \in D(A)$ such that $\lim\limits_{n\to\infty} \| x_n - x \|_A = 0$. Thus, $\lim\limits_{n\to\infty} \| x_n - x \|_X = 0$ and $\lim\limits_{n\to\infty} \| Ax_n - Ax \|_X = 0$, since $\| x_n - x \|_X \leq \| x_n - x \|_A$ and $\| Ax_n - Ax \|_X \leq \| x_n - x \|_A$.

Proposition A.4.4. *Let $A : X \supset D(A) \to X$ be a closed linear operator, where X is a Banach space. If A is $1-1$, we can define the linear operator:*

$$A^{-1} : R(A) \to D(A), \ \ A^{-1}y = x \Leftrightarrow Ax = y, \ \ \textit{for } x \in D(A), \, y \in R(A).$$

Then, A^{-1} is also closed.

Proof: Let $\{y_n\}_{n=1}^{\infty} \subset rg(A)$ be a sequence such that: $\lim_{n\to\infty} y_n = y \in X$ and $\lim_{n\to\infty} A^{-1}(y_n) = x \in X$. Set $x_n = A^{-1}y_n$, $n \in \mathbb{N}$. Then, $\{x_n\}_{n=1}^{\infty} \subset D(A)$, $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Ax_n = y \in X$. The closedness of A implies that $x \in D(A)$ and $y = Ax$. Equivalently $y \in rg(A)$ and $A^{-1}y = x$. Therefore, A^{-1} is closed.

Corollary A.4.1. *Let $A : X \supset D(A) \to X$ be a linear closed operator, where X is a Banach space. If A is bijective then $A^{-1} \in B(X)$.*

Proof: By virtue of proposition (A.4.4), $A^{-1} : X \to X$ is closed. Now, by virtue of Proposition (A.4.1) and the Closed graph theorem, we conclude that A^{-1} is bounded.

Proposition A.4.5. *Let $A : X \supset D(A) \to X$ be a closed linear operator and $B \in B(X)$, where X is a Banach space. Then, $C : D(A) \to X$, $C = A + B$ is a closed linear operator.*

Proof: Let $\{x_n\}_{n=1}^{\infty} \subset D(A)$ be a sequence such that: $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Cx_n = y \in X$. Then, $\lim_{n\to\infty} Ax_n = y - Bx$, thus by closedness of A, $x \in D(A)$ and $y - Bx = Ax \Leftrightarrow y = Cx$.

Corollary A.4.2. *Let $A : X \supset D(A) \to X$ be a linear operator. Then A is closed if and only if there exists $\lambda \in \mathbb{R}$ such that $\lambda I_X - A$ is closed.*

Proof: This is a consequence of Proposition (A.4.5) and of the equivalence: A is closed if and only if -A is closed.

Definition A.4.2. *Let $A : X \supset D(A) \to X$ be a closed linear operator, where X is a Banach space. A linear subspace $D \subset D(A)$ is said to be a* ***core*** *of A if-f for each $x \in D(A)$, there exists $\{x_n\}_{n=1}^{\infty} \subset D$ such that: $x_n \to x$ and $Ax_n \to Ax$, as $n \to \infty$.*

Remark A.4.2. It is obvious that if D is a core of a closed linear operator A, then D is dense in D(A) and A(D) is dense in $rg(A)$.

Proposition A.4.6. *Let $A : X \supset D(A) \to X$ be a closed linear operator, where X is a Banach space. Assume that there exists $c > 0$ such that:*

$$\| Ax \| \geq c \| x \| , \text{ for each } x \in D(A).$$

If a linear subspace $D \subset D(A)$ has the property: $A(D)$ is dense in $rg(A)$, then D is a core of A.

Proof: $A(D)$ is dense in $rg(A)$. Thus, for each $x \in D(A)$, there exists $(x_n)_n$ in D such that: $\lim_{n\to\infty} Ax_n = Ax$. Now,

$$\| x_n - x \| \leq \frac{1}{c} \| Ax_n - Ax \| \to 0, \text{ as } n \to \infty,$$

which competes the proof.

Remark A.4.3. It is clear that a dense linear subspace is always a core of a bounded linear operator. More generally, the following statement is true.

Proposition A.4.7. *Let $A : X \supset D(A) \to X$ be a closed linear operator, where X is a Banach space. A linear subspace $D \subset D(A)$ is a core of A if and only if D is dense in $(D(A), \| \ \|_A)$.*

A.5 Inverses of Operators

Definition A.5.1. *Let $(X, \| \ \|_X)$,$(Y, \| \ \|_Y)$ be two normed spaces and $T \in B(X, Y)$. We say that T is* ***invertible*** *if-f there is $L \in B(Y, X)$, such that $TL = I_Y$ and $LT = I_X$. Equivalently, $TLy = y$,for each $y \in Y$ and $LTx = x$,for each $x \in X$.*

Remark A.5.1. When the bounded operator L exists, then it coincides with the linear transformation T^{-1}. Indeed, it is easy to verify that if the conditions of Definition (A.5.1) are satisfied, then T is bijective, thus $L = T^{-1}$ and this is the unique operator with these properties. The inverse statement does not always hold. This means, that it is possible the inverse transformation T^{-1} to exist, but not to be bounded. Moreover, in the case where X and Y are both Banach spaces, then the open mapping theorem ensures us that $T \in B(X, Y)$ is invertible if and only if T is bijective.

Theorem A.5.1. *Let $(X, \| \ \|_X)$ be a Banach space and $A \in B(X)$. If $\| A \|_{B(X)} < 1$, then:*

1. *The operator $I - A$ is invertible.*
2. *$\| I - A \| \leq \frac{1}{1 - \| A \|}$*
3. *$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$*

Proof: The sequence of partial sums $(S_n)_n$, where $S_n = \sum_{k=0}^{n} A^k$, $n \in \mathbb{N}$, is Cauchy in the Banach space $B(X)$, thus convergent. Indeed, for positive integers $m > n$ we have:

$$\| S_n - S_m \| = \left\| \sum_{k=n+1}^{m} A^k \right\| \leq \sum_{k=n+1}^{m} \| A \|^k \to 0 \text{ , as n,m} \to \infty,$$

because $\sum_{n=1}^{\infty} \| A \|^n < \infty$, since $\| A \| < 1$.

Moreover, it is to verify that $\sum_{n=0}^{\infty} A^n$ is a linear bounded operator (as expected) and that $\| \sum_{n=0}^{\infty} A^n \|_{B(X)} \leq \frac{1}{1-\| A \|}$. For the last one, observe that for each $x \in X$:

$$\left\| \sum_{n=0}^{\infty} A^n x \right\|_X \leq \sum_{n=0}^{\infty} \| A \|_{B(X)}^n \| x \|_X = \frac{1}{1 - \| A \|_{B(X)}} \| x \|_X$$

In addition, we have:

$$\| (I-A)S_n - I \| = \left\| I - A^{n+1} - I \right\| = \left\| A^{n+1} \right\| \leq \| A \|^{n+1} \to 0, \text{ as n} \to \infty,$$

again because $\| A \| < 1$.

Therefore,

$$(I-A) \sum_{n=0}^{\infty} A^n = \lim_{n \to \infty} (I-A)S_n = I$$

Similarly, we can show that

$$\sum_{n=0}^{\infty} A^n (I-A) = I$$

. This means, that $I - A$ is invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$

A.6 The Banach-Steinhaus Theorem

Lemma A.6.1. *Let X be a Banach space, $(r_n)_n$ a sequence of positive real numbers such that* $\lim_{n\to\infty} r_n = 0$ *and $(x_n)_n$ a sequence of elements of X. If the sequence of sets $(B_n)_n$, where $B_n = \overline{B}(x_n, r_n)$, $n \in \mathbb{N}$ is decreasing, i.e $B_{n+1} \subset B_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$.*

Proof: The sequence $(x_n)_n$ is Cauchy in X. Indeed, for positive integers $m > n$ we have: $\| x_n - x_m \| \leq r_n$. By completeness of X, there exists $x \in X$: $\lim_{n\to\infty} x_n = x$. We will show that $x \in \bigcap_{n=1}^{\infty} B_n$. To this end, observe that for $n \in \mathbb{N}$ it holds that $\{x_N\}_{N \geq n} \subset B_n$ and B_n is closed. Thus $x = \lim_N x_N \in B_n$, for each $n \in \mathbb{N}$.

Definition A.6.1. *Let $(X, \| \|_X)$ be a normed space. A subest $S \subset X$ is said to be **nowhere dense** if-f its closure $\overline{S}$ does not contain any open ball.*

Proposition A.6.1. *Let $(X, \| \|_X)$ be a normed space and $S \subset X$. If S is nowhere dense, then each open ball B in X contains an open ball B' such that: $B' \bigcap S = \emptyset$.*

Proof: Suppose that there exists an open ball B such that: $B' \bigcap S \neq \emptyset$, for each open ball $B' \subset B$. We will show that $B \subset \overline{S}$ which leads us to a contradiction. Let $x \in B$. There exists $n \in \mathbb{N}$ such that: $B(x, \frac{1}{n}) \subset B$, because B is open. By our assumption, for each $N \geq n$: $B(x, \frac{1}{N}) \bigcap S \neq \emptyset$. Thus, for each $N \geq n$ we can choose $x_N \in B(x, \frac{1}{N}) \bigcap S$. Thus, we can conclude that $x = \lim_N x_N \in \overline{S}$.

Theorem A.6.1 (Baire's Category Theorem)**.** *Let $(X, \| \|_X)$ be a Banach space. Then X cannot be represented as a countable union of nowhere dense sets.*

Proof: Suppose that X can be written as: $X = \bigcup_{n=1}^{\infty} S_n$, where $(S_n)_n$ is a sequence of nowhere dense sets. Let $B(0,1)$ be the unitary open ball in X. Since S_1 is nowhere dense, by virtue of Proposition (A.6.1), we can choose an open ball $B_1 = B(x_1, r_1)$ such that:

$$r_1 \leq \frac{1}{2}\ , \overline{B}(x_1, r_1) \subset B(0,1) \text{ and } \overline{B}(x_1, r_1) \bigcap S_1 = \emptyset.$$

Generally having found an open ball B_n, we can choose an open ball $B_{n+1} = B(x_{n+1}, r_{n+1})$ such that:

$$r_{n+1} \leq \frac{1}{n+2}\ , \overline{B}(x_{n+1}, r_{n+1}) \subset B_n \text{ and } \overline{B}(x_{n+1}, r_{n+1}) \bigcap S_{n+1} = \emptyset.$$

Now, the sequences $(r_n)_n$, $(x_n)_n$ and $(\overline{B}_n)_n$, satisfy the conditions of Lemma (A.6.1), so there exists $x \in \bigcap_{n=1}^{\infty} B_n$. But then, $x \in \bigcap_{n=1}^{\infty} (S_n)^c = \emptyset$, which is impossible.

Theorem A.6.2 (Banach-Steinhauss Theorem)**.** *Let $(X, \| \, \|_X)$ be a Banach space $(Y, \| \, \|_Y)$ a normed space and $(A_i)_{i\in I}$ a family of linear bounded operators from X into Y (i.e $\{A_i\}_{i\in I} \subset B(X,Y)$).*
If for each $x \in X$:

$$\sup_{i\in I} \| A_i x \|_Y < \infty,$$

then

$$\sup_{i\in I} \| A_i \|_{B(X,Y)} < \infty$$

Proof: For each $n \in \mathbb{N}$, we set:

$$S_n = \{x \in X : \| A_i x \|_Y \leq n \text{ , for each i} \in I\}.$$

Because of continuity of operators $(A_i)_{i\in I}$, it is easy to verify that S_n is closed, for each $n \in \mathbb{N}$. In addition, by our assumption it follows that

$$X = \bigcup_{n=1}^{\infty} S_n.$$

By virtue of the Baire's Category Theorem (A.6.1), there exists $n_0 \in \mathbb{N}$ such that S_{n_0} is not nowhere dense. This means that there exists an open ball $B(x,r)$ in X such that: $B(x,r) \subset \overline{S_{n_0}} \Rightarrow \overline{B}(x,r) \subset S_{n_0}$. Consider $y \in X$, $y \neq 0_X$ and set $z = x + \frac{r}{\| y \|} y$. Observe that $x, z \in \overline{B}(x,r) \subset S_{n_0}$. Therefore, for each $i \in I$ we have:

$$\begin{aligned} \| A_i y \| &= \left\| \frac{\| y \|}{r} A_i z - \frac{\| y \|}{r} A_i x \right\| \\ &\leq \frac{\| y \|}{r} \| A_i z \| + \frac{\| y \|}{r} \| A_i x \| \\ &\leq \frac{2n_0}{r} \| y \| . \end{aligned}$$

This means that:

$$\| A_i \| \leq \frac{2n_0}{r}, \text{ for each i} \in I.$$

Remark A.6.1. If the conditions of the Banach Steinhauss Theorem are satisfied, then the conclusion is equivalent to the following statement: There exists $K > 0$ such that:

$$\| A_i x \|_Y \leq K \| x \|_X \text{ , for each x} \in X \text{ , i} \in I.$$

Corollary A.6.1. *Let $(X, \| \, \|_X)$ be a Banach space, $(Y, \| \, \|_Y)$ a normed space and $(A_n)_{n\in\mathbb{N}}$ a sequence of linear bounded operators from X into Y. If for each $x \in X$, the limit $\lim_{n\to\infty} A_n x$ exists, then the operator:*

$$A : X \to Y \quad Ax = \lim_{n\to\infty} A_n x,$$

is linear and bounded.

Proof: It is easy to verify the linearity of A. Moreover, for each $x \in X$ the sequence $(\| A_n x \|)_n$ is bounded in $\mathbb{R}$. By virtue of the Banach-Steinhauss theorem (A.6.2) and Remark (A.6.1), there exists $k > 0$ such that:

$$\| A_n x \|_Y \leq k \| x \|_X \text{ , for each x} \in X \text{ , n} \in \mathbb{N}.$$

Now, taking the limits as $n \to \infty$ and because of continuity of the norm, we get:

$$\| Ax \|_Y \leq k \| x \|_X \text{ , for each x} \in X.$$

Thus, $A \in B(X,Y)$.

Remark A.6.2. The above theorem is true only for linear operators. As a counterexample, consider the sequence of continuous functions $x_n(t) = t^n$, $t \in [0,1]$. Then, the sequence converges pointwise on $[0,1]$ to the function:

$$x(t) = \begin{cases} 0 & \text{when } 0 \leq t < 1, \\ 1 & \text{when } t = 1. \end{cases}$$

which is not continuous.

Corollary A.6.2. *Let $(X, \| \, \|_X)$ be a Banach space, $(Y, \| \, \|_Y)$ a normed space and $(A_t)_{t \geq 0}$ a family of linear bounded operators from X into Y. If for each $x \in X$ the limit $\lim_{t \to 0} A_t x$ exists, then there exists $\delta > 0$ such that:*

$$\sup_{0 \leq t \leq \delta} \| A_t \| < \infty.$$

Proof: Suppose that for each $\delta > 0$: $sup_{0 \leq t \leq \delta} \| A_t \| = \infty$. Then, if $\delta > 0$, for each $M \in \mathbb{R}$ there exists $t_0 \in [0, \delta]$ such that: $\| A_{t_0} \| > M$. Therefore, for $n \in \mathbb{N}$, there exists $t_n \in [0, \frac{1}{n}]$ such that: $\| A_{t_n} \| > n$. On the other hand, observe that for each $x \in X$:

$$\lim_{n \to \infty} A_{t_n} x = \lim_{t \to 0} A_t x < \infty.$$

Thus, for each $x \in X$:

$$\sup_{n \in \mathbb{N}} \| A_{t_n} x \| < \infty.$$

By virtue of the Banach-Steinhauss theorem (A.6.2),

$$\sup_{n \in \mathbb{N}} \| A_{t_n} \| < \infty,$$

which leads us to a contradiction.

A.7 Uniform and Strong Operator Topology

Reminder A.7.1. Let $(X, \| \, \|_X)$, $(Y, \| \, \|_Y)$ be two Banach spaces. Consider the vector space $(B(X,Y), \| \, \|_{B(X,Y)})$ of linear bounded operators $T : X \to Y$, equipped with the norm:

$$\| T \|_{B(X,Y)} = \sup\{ \| Tx \|_Y : x \in X, \| x \|_X \leq 1 \}.$$

We already know that since Y is Banach, the same is true for $B(X,Y)$.
In this text, we will be interested mainly in two topologies on the space $B(X,Y)$. The one is the **norm operator or uniform topology on** $B(\mathbf{X}, \mathbf{Y})$. This is the topology induced by the above norm. The second one is called the **strong operator topology on** $B(\mathbf{X}, \mathbf{Y})$. This is the weakest topology on $B(X,Y)$ such that the functions:

$$E_x : B(X,Y) \to Y, \quad E_x(T) = Tx,$$

are continuous, for each $x \in X$. Moreover, the strong operator topology coincides with the **topology of pointwise convergence on** $(\mathbf{X}, \| \, \|_{\mathbf{X}})$, i.e if $(T_n)_n$ are operators in $B(X,Y)$, then T_n converges strongly to T if-f:

$$\lim_{n \to \infty} \| T_n x - T x \|_Y = 0, \text{ for each x} \in X.$$

Note that due to the Banach Steinhauss Theorem (see Corolarry (A.6.1)), if $(X, \| \, \|_X)$ is Banach, a strongly converging sequence of bounded operators is always converging to a linear bounded operator. Finally, if $Y = \mathbb{R} (or\, \mathbb{C})$ the strong operator topology coincides with the weak*- topology on $X^\star$.

Remark A.7.1. It is clear that if a $(T_n)_n$ converges to T w.r.t the uniform topology, i.e

$$\lim_{n \to \infty} \| T_n - T \|_{B(X,Y)} = 0,$$

then it is also converges to T w.r.t the strong operator topology. To this end, observe that for each $x \in X$:

$$\| T_n x - T x \|_Y \leq \| T_n - T \|_{B(X,Y)} \| x \|_X .$$

Therefore:

$$\text{If } \lim_{n\to\infty} \| T_n - T \|_{B(X,Y)} = 0, \text{ then } Tx = (\lim_{n\to\infty} T_n)x = \lim_{n\to\infty} T_n x, \ \forall x \in X$$

In the same manner,

$$\text{If} \quad T = \sum_{n=1}^{\infty} T_n, \text{ then } Tx = (\sum_{n=1}^{\infty} T_n)x = \sum_{n=1}^{\infty} T_n x, \ \forall x \in X.$$

Remark A.7.2. As we have already stated, the strong operator topology coincides with the topology of pointwise convergence on X. Furthermore, on bounded subsets of $B(X)$, the strong operator topology coincides with the topology of pointwise convergence on a dense subset of X.

Proposition A.7.1. *Let $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ be two Banach spaces, $(T_n)_n$ a bounded sequence in $B(X,Y)$ and $D \subset X$ a dense subset. If $(T_n x)_{n\in\mathbb{N}}$ is Cauchy, for each $x \in D$, then T_n converges strongly to some $T \in B(X,Y)$.*

Proof: Set $\sup_{n\in\mathbb{N}} \| T_n \| = M > 0$. Let $x \in X$ and $\varepsilon > 0$. Since D is dense in X we can choose $x_0 \in D$ such that: $\| x - x_0 \| < \frac{\varepsilon}{3M}$. Now, by assumption $(T_n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Therefore, we can choose $n_0 \in \mathbb{N}$ such that: $\| T_n x_0 - T_m x_0 \| < \varepsilon/3$, for each $n, m \geq n_0$. Thus, for each $n, m \geq n_0$:

$$\begin{aligned}
\| T_m x - T_n x \| &\leq \| T_m x - T_m x_0 \| + \| T_m x_0 - T_n x_0 \| + \| T_n x_0 - T_n x \| \\
&\leq M \| x - x_0 \| + \| T_m x_0 - T_n x_0 \| + M \| x - x_0 \| \\
&< \varepsilon.
\end{aligned}$$

This means that $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, for each $x \in X$. Thus, it is also a convergent sequence, since Y is Banach. An application of the Banach Steinhauss Theorem (see Corollary (A.6.1)) ends the proof.

Proposition A.7.2. *Let $(X, \| \ \|_X)$ be a Banach space and $(T_n)_n$ a sequence in $B(X)$. The following statements are equivalent:*

1. *The sequence $(T_n)_n$ is strongly convergent (i.e pointwise convergent on $(X, \| \ \|_X)$)*
2. *The sequence $(T_n)_n$ is uniformly convergent on compact subsets in X.*

Proof: For simplicity, it is enough to consider convergence to 0. If $(T_n)_n$ converges strongly, then by the Banach-Steinhauss Theorem (A.6.2), $M = \sup_{n\in\mathbb{N}} \| T_n \| < \infty$. Let $K \subset X$ be a compact set and $\varepsilon > 0$. Since $K \subset \bigcup_{x\in X} B(x, \varepsilon/2M)$, we can find a finite set $N_\epsilon = \{x_1, \ldots, x_k\} \subset X$, such that $K \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon/2M)$. This means that for each $x \in K$, there is $x_i \in N_\varepsilon$ such that $\| x - x_i \| < \varepsilon/2M$. By our assumption and since N_ε is finite we can find a $n_0 \in \mathbb{N}$ such that $\| T_n x_i \| < \varepsilon/2$, for each $n \geq n_0$ and $i = 1, \ldots, k$. Therefore for $x \in K$,

$$\| T_n x \| \leq \| T_n x_i \| + M \| x - x_i \| < \varepsilon,$$

for each $n \geq n_0$ (and n_0 is the same for all $x \in K$). The converse statement is obvious.

Proposition A.7.3. *Let $(X, \| \ \|_X)$ be a Banach space and $(T_n)_n$, $(S_n)_n$ two sequences in $B(X)$.*

(i) *If T_n converges strongly to T and S_n converges strongly to S, then $T_n S_n$ converges strongly to TS.*

(ii) *If T_n converges strognly to T, then $\lim_{n\to\infty} T_n x_n = Tx$, for each sequence $(x_n)_n \in X$ which converges to x.*

Proof: We will show (i). Since (T_n) converges strongly to T, from the Banach Steinhauss Theorem (A.6.2), $M = \sup_{n\in N} \| T_n \|_{B(X)} < \infty$. So for each $x \in X$ we have,

$$\| T_n S_n x - TSx \| \leq \| T_n S_n x - T_n Sx \| + \| T_n Sx - TSx \| \leq M \| S_n x - Sx \| + \| T_n Sx - TSx \|$$

which tends to zero as $n \to \infty$.

Bibliography

[AD] R. A. Adams and J. Fournier, *Sobolev Spaces*, Elsevier, 2003.

[A-K] F. Albiac and N.J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.

[ARD] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander, *Vectorvalued Laplace Transforms and Cauchy problems*, Monographs in Mathematics, vol. 96, Birkhauser Verlag, Basel, 2001.

[AR I] Argyros S., *Lecture notes on Real analysis*, (in Greek), National Technical University of Athens 2003

[AR II] Argyros S., *Lecture notes on Functional Analysis*, (in Greek), National Technical University of Athens 2004

[ASH] Robert B. Ash, Catherine A.Doléans-Dade, *Probability and Measure theory*, second edition, Academic Press 1999

[BAL] J.M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. 63 (1977), no. 2, 370373.

[BL] P.Billingsley, *Probability and Measure*, John-Wiley and Sons, New York 1995

[BOG] V.I. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[BS] Alberto Bressan, *Lecture notes on functional analysis and linear partial differential equations*, 2011

[BR] H. Brezis *Functional analysis*, Translation in Greek by Kravvaritis Dimitrios, Chrysovergis Ion, NTUA press 1997

[BRZ] Z.Brzezniak, T.Zastawniak ,*Basic stochastic processes*, Springer-Verlag London 1999

[BW] Bobrowski Adam, *Functional analysis for probability and stochastic processes*, Cambridge University Press 2005

[CP] Marek Capiǹski, Ekkehard Kopp, *Measure Integral and Probability*, Springer-Verlag London 2003

[DP I] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[DP II] G. Da Prato, *An Introduction to Infinite-Dimensional Analysis*, Springer, 2006.

[DS I] J. Diestel and J.J. Uhl, Jr., *Vector Measures*, American Mathematical Society, Providence, R.I., 1977.

[DS II] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.

[E-N] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, 2000.

[EV] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1993.

[FLD] Folland, *Real Analysis*, Second Edition, Wiley Interscience 1999,

[FR] X. Fernique, *Intégrabilité des vecteurs gaussiens*, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1698A1699.

[GL] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1985.

[GU] Paolo Guiotto, *Lecture notes on Functional analysis II*, 2010

[I-N] K. Itô and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), 3548.

[KAL] Kallenberg Olav, *Foundations of Modern Probability*, Second Edition Springer 2002

[KS I] S. Kesavan, *Topics in Functional Analysis and Applications.* Wiley, New York, 1989

[KS II] S.Kesavan,*Functional Analysis.* Hindustan Book Agency, India 2009

[KM] Koumoullis G., Negrepontis S., *Measure theory*, (in Greek), Symmetria Athens 2005

[KV] Kovács M.,Larsson S., *Introduction to stochastic partial differential equations*, September 2008

[NR] Jan van Neerven, *Stochastic Evolution Equations*, ISEM Lecture Notes 2007/08

[PR] Jean Jacod, Philip Protter, *Probability Essentials*, Springer 2004

[PT] B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. 44 (1938), no. 2, 277304.

[PZ] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences. Springer Verlag, 1983.

[RR] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, Springer- Verlag, 1993.

[SP] Spiliotis John, *Lecture notes on Probability theory*, (in Greek), National Technical University of Athens 2003.

[SR] Sarantopoulos J., *Lecture notes on measure and integration theory*, National Technical University of Athens 2010.

[TS] Tsamatos Panagiotis, *Topology*, (in Greek), Tziola Thessaloniki 2009

[VR] Ioan I.Vrabie, C_0*-semigroups and applications*, North-Holland Mathematical studies 2003.

[WI] Ivan F.Wilde, *Functional analysis-Gently done*, King's College London

[WII] Ivan F.Wilde, *Stochastic analysis notes*, King's College London 2004