NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF NAVAL ARCHITECTURE AND MARINE ENGINEERING
SECTION OF NAVAL AND MARINE HYDRODYNAMICS

# Modeling & Analysis of Hydro/Piezo/Electric Systems

(Diploma Thesis)

by

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*Dedicated to the memory*
*of my grandparents*

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## Synopsis in English

Over the last two decades, two of the trends that can be identified in Renewable Energy Technologies are the use of piezoelectric materials for the construction of energy harvesting devices, and the growing interest in exploitation of the energy contained in sea waves. The original idea of the present thesis is to investigate whether the aforementioned trends could be combined, i.e. if a piezo-electric sea-wave absorption system could be feasible.

The present thesis consists of two parts. The first part deals with the construction of a Variational Principle for the whole hydro/piezo/electric phenomenon by combining established Variational Principles for each of the constituent phenomena (elastodynamics, electrodynamics and hydrodynamics) and addressing the coupling mechanisms between these phenomena when piezoelectric bodies are considered. The aim of the construction of the Variational Principle in order to facilitate a tool for deriving consistent and efficient models of the coupled physical these system(s) in a systematic way.

While the equations of hydro/piezo/electricity obtained in part I are accurate, they are valid only for a conservative system, since the Variational Principle constructed does not take into consideration any mechanism of energy dissipation. Since energy flow is the operating principle of every energy harvesting device, the case of a certain non-conservative hydro/piezo/electric system is examined in the first chapter of part II of the present thesis. More specifically, in this part, the non-conservative element of an external electric circuit is connected to the piezoelectric element, and the net power flow from the sea waves towards this external circuit is calculated. The evaluation of net power flow seen as a percentage of the total wave power showed that such a harvester could be efficient if new materials, that are both flexible and exhibit a strong piezoelectric effect, are constructed. Last, in the final chapter of part II, a variational formulation for a non-conservative lumped piezoelectric system is constructed.

**Keywords:** renewable energy; piezoelectricity; sea wave energy; variational formulation; lumped model

## Σύνοψη στα Ελληνικά (Synopsis in Greek)

Τις τελευταίες δύο δεκαετίες παρατηρούνται μεταξύ άλλων δύο τάσεις στο πεδίο των Ανανεώσιμων Πηγών Ενέργειας: η χρήση πιεζοηλεκτρικών υλικών και οι όλο και περισσότερες προσπάθειες για εκμετάλλευση της ενέργειας των θαλάσσιων κυμάτων. Η αρχική σύλληψη της παρούσας διπλωματικής ήταν να μελετηθεί αν οι δύο αυτές τάσεις θα μπορούσαν να συνδυαστούν σε μια διάταξη εκμετάλλευσης της ενέργειας των θαλάσσιων κυμάτων με την χρήση πιεζοηλεκτρικών υλικών.

Η παρούσα διπλωματική εργασία αποτελείται από δύο μέρη. Το πρώτο μέρος ασχολείται με την διατύπωση ενός Θεωρήματος Μεταβολής για το συνολικό υδρο/πιεζο/ηλεκτρικό πρόβλημα μέσω του συνδυασμού ήδη γνωστών μεταβολικών αρχών για τα επιμέρους φαινόμενα (ελαστοδυναμικό, ηλεκτροδυναμικό, υδροδυναμικό φαινόμενο) και της εισαγωγής των αλληλεπιδράσεων μεταξύ των επιμέρους φαινομένων αυτών για την περίπτωση πιεζοηλεκτρικών σωμάτων. Το Θεώρημα Μεταβολής αυτό κατασκευάζεται ώστε να αποτελέσει ένα εργαλείο για την συνεπή και αποδοτική διατύπωση συνεπών απλοποιημένων μοντέλων για το συζευγμένο υδρο/πιεζο/ηλεκτρικό πρόβλημα κατά έναν συστηματικό τρόπο.

Αν και οι εξισώσεις του υδρο/πιεζο/ηλεκτρισμού που διατυπώθηκαν στο πρώτο μέρος είναι ακριβείς, ισχύουν μόνο στις περιπτώσεις συντηρητικών συστημάτων, καθώς η κατασκευασθείσα Μεταβολική Αρχή δεν συμπεριλαμβάνει φαινόμενα απώλειας ενέργειας. Καθώς η μεταφορά ενέργειας είναι η αρχή λειτουργίας όλων των διατάξεων εκμετάλλευσης ενέργειας, στο πρώτο κεφάλαιο του δευτέρου μέρους της διπλωματικής εξετάζεται ένα μη συντηρητικό υδρο/πιεζο/ηλεκτρικό σύστημα. Ειδικότερα, στο κεφάλαιο αυτό μελετάται μια συγκεκριμένη περίπτωση πιεζοηλεκτρικής διάταξης συνδεδεμένη με ένα εξωτερικό ηλεκτρικό κύκλωμα – καταναλωτή, για τη οποία διάταξη και υπολογίζεται η καθαρή ροή ισχύος από την θάλασσα προς εξωτερικό κύκλωμα. Αξιολογώντας την καθαρή ροή ισχύος ως ποσοστό της συνολικής ισχύος που φέρει το θαλάσσιο κύμα, αποφαινόμαστε ότι μια τέτοια διάταξη θα ήταν αποδοτική εάν συντεθούν νέα υλικά που θα είναι εύκαμπτα και με ισχυρές πιεζοηλεκτρικές ιδιότητες. Επίσης, στο τελικό κεφάλαιο του δεύτερου μέρους παρουσιάζεται μια μεταβολική διατύπωση για την περίπτωση ενός μη συντηρητικού πιεζοηλεκτρικού συστήματος, αποτελούμενου από διακριτά (μη κατανεμημένα) στοιχεία.

# Preface

The present thesis is the result of our investigation over both the mathematical description of hydro/piezo/electricity and a possible application in sea-wave energy harvesting. Thus, the present thesis is divided into two parts, with Part I dealing with the a systematic approach to the mathematical description of conservative hydro/piezoelectric systems though Variational Principles, while Part II investigates a specific case of non-conservative hydro/piezo/electric system and assesses its feasibility as an energy harvester.

Since the aims of the two parts differ substantially, we decided to split the introduction that is usually found in the first pages of a book, into two introductory chapters corresponding to the two parts. In each of these introductions, a detailed description of the chapters contained in the respective part is made and some basic remarks are also outlined. Also, due to the diversity in content between the chapters, it was deemed as more appropriate to show the references appearing in each chapter at the end of each chapter, rather than having a section for all references at the end of the thesis. The references considered as central to our work can also be found in the two introductory chapters.

The first four chapters of Part I correspond to the constituent phenomena (*elastodynamics, electrodynamics* and *hydrodynamics*) that appear in hydro/piezo/electricity. While the main scope of each of these chapters is to present a Variational Principle for each phenomenon in order the total Variational Principle for hydro/piezo/electricity to be constructed by the proper combination of them, a quick and targeted derivation and description of the governing equations as well as the conditions on the boundaries is made in each chapter. Since the total hydro/piezo/electric problem includes not only the active piezoelectric volume but also the ambient volumes of sea and air, notation for volumes is systematic throughout these chapters, in order to avoid any ambiguities in the description of the total problem. The total problem is described in Ch.5 of Part I where both coupling mechanisms and a Variational Principle are expressed. Chapters 6 and 7 of Part I bring the previously derived mathematical description closer to a possible application; in Ch.6 the quasi-static approximation is adopted, which is a standard simplification in order to obtain solutions while in Ch.7 the Voigt notation with respect to indices is performed in a systematic way, expressing thus the equations of piezoelectricity in the form commonly found in related bibliography.

The first chapter of Part II is our paper "Modeling and analysis of a cliff-mounted piezoelectric sea-wave energy absorption system", published in Coupled Systems Mechanics. In this paper, almost analytic solutions are obtained for a certain non-conservative piezoelectric system, and the feasibility of this system as a sea-wave harvester is evaluated. Furthermore, there is a useful nomenclature section at the end of the paper which, while being partial since referring only to the paper, captures most of notation features used throughout the whole thesis. In the second chapter of the same part, the equivalence between the distributed system used in our paper and the lumped systems commonly appearing in bibliography is shown.

The present work is supplemented with Appendix A, where the equivalence between the various alternative constitutive relations of piezoelectricity is shown.

# CONTENTS

# PART I.

## THE CONSERVATIVE HYDRO/PIEZO/ELECTRIC SYSTEM

# 1. Introduction

In the first part of the present thesis, the governing equations of a conservative hydro/piezo/electric system will be derived. The way of approaching the whole coupled system is outlined as follows.

In the first three chapters following this introductory section, the constituent phenomena of the hydro/piezo/electricity, *elastodynamics, electrodynamics* and *hydrodynamics,* are analyzed. The analysis of each (uncoupled) phenomenon consists of i) the fields and material properties appearing, ii) the set of governing equations, followed by a discussion on their derivation and on the balance between the number of equations and unknowns, iii) the formulation of a variational principle for each of the phenomena.

In the fifth chapter, the whole hydro/piezo/electric problem will be considered, introducing the coupling between elastic-electric and hydrodynamic-elastic fields. The electroelastic coupling will be performed using the piezoelectric constitutive relations which will be stated and discussed. After imposing these constitutive relations, a rearrangement in the energy forms appearing in elastodynamics and electrodynamics is conducted. This energy rearrangement gives rise to energy forms and thus to a variational principle for piezoelectricity that is verified by similar works where such a principle is constructed using thermodynamic arguments (Lee 1991). On the other hand, the hydrodynamic-elastic coupling is performed as in the standard variational principle for hydroelastic systems (Athanassoulis 1982) by matching the fluid velocity and pressure with the respective elastic fields, on the fluid – piezoelectric body interface.

In the sixth chapter, the quasi-static approach is adopted. Under this approach, the magnetic field is negligible since sea-wave frequency is much lower than the frequencies that excite magnetic phenomena.

In the seventh chapter, the reduction of the indices known as Voigt notation is performed to the equations obtained for quasi-static piezoelectricity. The expression of equations under Voigt notation is helpful since the values of material property tensors are given under this notation.

**References**

Athanassoulis G. A. (1982), *Study of non-steady, free-surface flows using variational principles (Ph.D. thesis),* NTUA (in Greek).

Lee P. C. Y (1991), "A variational principle for the equations of piezoelectromagnetism in elastic dielectric crystals", *J. Appl. Phys.* **69**(11), 7470 – 7473.

# 2. Linear Elastodynamics

## 2.1 Fields and Equations

*General References: Athanassoulis (2007), Dym & Shames (1973) Ch. 1, Landau & Lifshitz (1970) Ch. 1, Wang & Truesdell (1973) Ch. 2.*

The following fields are involved in the equations of linear elastodynamics:

- The *displacement vector* $\boldsymbol{u}(\boldsymbol{x};t)$ which relates each point $\boldsymbol{x}$ of the solid volume $\Omega^{(0)}$ to its total displacement in the time interval $[0,T]$.
  From displacement field $\boldsymbol{u}(\boldsymbol{x};t)$, two other fields can be easily derived:

  The *velocity vector* $\dot{\boldsymbol{u}}(\boldsymbol{x};t)$ which is the first temporal derivate of displacement field.

  The *linear momentum vector*

  $$\boldsymbol{p}(\boldsymbol{x};t) = \rho_b\, \dot{\boldsymbol{u}}(\boldsymbol{x};t), \tag{2.1}$$

  where $\rho_b$ is the mass density of the elastic body and $\dot{\boldsymbol{u}}(\boldsymbol{x};t)$ is the velocity vector previously defined.

  Velocity $\dot{\boldsymbol{u}}(\boldsymbol{x};t)$ and linear momentum $\boldsymbol{p}(\boldsymbol{x};t)$ vectors are explicitly defined here as they will be used for expressing kinetic energy in the next paragraph.

- The *Cauchy stress tensor* (2nd rank) $\boldsymbol{\sigma}(\boldsymbol{x};t)$ which is defined so that the stress vector $\boldsymbol{T}(\boldsymbol{x};t;\boldsymbol{n})$ of a surface element with normal vector $\boldsymbol{n}$ to be given as

  $$T_i = \sigma_{ji}\, n_j \tag{2.2}$$

  By Eq. (2.2), the physical meaning of the first and second subscripts in the elements $\sigma_{ij}$ of the stress tensor is explained as follows

  First subscript $i$ denotes the direction of the normal vector of surface element.

  Second subscript $j$ denotes the direction of the applied mechanical load (force).

  And so, we can define each element $\sigma_{ij}$ of the stress tensor as the force in $j$-direction applied on an element perpendicular to $i$-direction.

- The *strain tensor* (2nd rank) $\boldsymbol{e}(\boldsymbol{x};t)$ whose elements are normalized measures of deformation representing the displacement between particles in the solid volume relative to a reference length.
  More specifically, the strain tensor in linear elastodynamics is introduced by the following definition relation **(system of 6 scalar equations)**

  $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.3}$$

  By Eq. (2.3), we conclude that:
  The strain tensor is symmetric, $e_{ij} = e_{ji}$.
  The diagonal elements, $e_{11}$, $e_{22}$, $e_{33}$, (normal strains) express variations in length.
  The doubled off-diagonal elements, $2e_{12}$, $2e_{23}$, $2e_{31}$, (shear strains) express variations in angle.

The dynamical equations of motion of an elastic solid volume are *Newton's Second Law* expressed for the case of a continuous medium. More specifically:

- *The balance of linear momentum*, which, in the linear case examined, is formulated as the following **system of 3 scalar equations**

  $$\rho_b \ddot{u}_i = \sigma_{ji,j} + \hat{f}_i, \tag{2.4}$$

  where $\rho_b$ is the mass density of the elastic body and $\hat{f}_i$ are the (given) external forces applied on the elastic body.

- *The balance of angular momentum*, which is expressed as a **symmetry property** of the stress tensor $\boldsymbol{\sigma}$

  $$\sigma_{ij} = \sigma_{ji} \tag{2.5}$$

**Balance between number of equations and unknowns in (2.4) system**

| |
|---|
| Number of scalar equations: 3<br><br>Number of unknown scalar fields: 9<br>3 components of the displacement vector<br>9 components of the strain tensor, reduced to 6 using symmetry property (2.5). |

Thus we have **9 unknown scalar fields to be determined by 3 scalar equations.**

We will try to remedy this unbalanced system of equations by introducing the strain tensor as an additional unknown field and strain definition relation (2.3) as six additional scalar equations.

**Balance between number of equations and unknowns in (2.4)+(2.3) system**

| |
|---|
| Number of scalar equations: 9<br>3 scalar equations from Eq.(2.4)<br>6 scalar equations from Eq.(2.3)<br><br>Number of unknown scalar fields: 15<br>3 components of the displacement vector<br>9 components of the strain tensor, reduced to 6 using symmetry property (2.5).<br>9 components of stress tensor, reduced to 6 using symmetry property from Eq.(2.3). |

Thus we have **15 unknown scalar fields to be determined by 9 scalar equations.**

Clearly, the previous step of involving strains into the system had no contribution to the balance of the system since it resulted in 6 new unknowns and 6 new equations. But with this step, a way of balancing the system of equations is revealed:

- Newton's Law in Eq.(2.4) relates **displacements** $u_i$ **to stresses** $\sigma_{ij}$.
- Strain definition relation in Eq.(2.3) relates **displacements** $u_i$ **to strains** $e_{ij}$.

What we need an additional set of 6 equations, to relate s**tresses** $\sigma_{ij}$ **to strains** $e_{ij}$ which, in case of linear elastic materials is the *generalized Hooke's Law*:

$$\sigma_{ij} = c_{ijk\ell}\, e_{k\ell}, \tag{2.6}$$

where $\boldsymbol{c}$ is the elastic stiffness property tensor (4th rank) of the elastic material.
Elastic stiffness tensor $\boldsymbol{c}$ inherits the following set of symmetries from the $\sigma_{ij}$ and $e_{k\ell}$ symmetries

Minor Symmetries: $c_{ijk\ell} = c_{jik\ell}$ and $c_{ijk\ell} = c_{ij\ell k}$
Major Symmetry: $c_{ijk\ell} = c_{k\ell ij}$

It has to be noted that generalized Hooke's law is a constitutive relation, meaning that this relation does not capture a fundamental natural law, but models the behavior of a certain class of materials under a certain type of excitation. In the present case, generalized Hooke's law models the elastic response $\boldsymbol{e}$ of the class of linear elastic media, under the elastic excitation $\boldsymbol{\sigma}$. If the class of materials is changed, another constitutive relation between $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ shall be considered. In contrast with the physical laws which are deduced from the application of general principles (e.g. the conservation of energy), constitutive relations are defined experimentally.

**Balance between number of equations and unknowns in (2.4)+(2.3)+(2.6) system**

| |
|---|
| Number of scalar equations: 15<br>3 scalar equations from *Newton's Law* of Eq.(2.4)<br>6 scalar equations from *strain definition relation* of Eq.(2.3) |

| |
|---|
| 6 scalar equations from *generalized Hooke's Law* of Eq.(2.6)<br><br>Number of unknown scalar fields: 15<br>3 components of the *displacement vector*<br>9 components of the *strain tensor*, reduced to 6 using symmetry property (2.5).<br>9 components of *stress tensor*, reduced to 6 using symmetry property from Eq.(2.3). |

Thus we have **15 unknown scalar fields to be determined by 15 scalar equations and the system of equations is <u>well-balanced</u>.**

### 2.2 Variational Formulation of Elastodynamics: Hamilton's Principle

*Related References: Athanassoulis (2007), Dym & Shames (1973) Ch. 3, Goldstein et al. (2000) Ch. 2, Love (1944) Ch. VII, Sokolnikoff & Specht (1946) Ch. V.*

The variational principle to be formulated in the present paragraph will have as only independent argument the displacement vector field $\boldsymbol{u}(\boldsymbol{x};t)$ plus a new auxiliary (considered as independent) boundary vector field $\boldsymbol{\lambda}(\boldsymbol{x};t)$ whose meaning will become evident after the variational formulation.
Since the only independent argument is the displacement vector $\boldsymbol{u}(\boldsymbol{x};t)$, we should express the energy quantities (kinetic and elastic) appearing in elastic volume $\Omega^{(0)}$ in terms of field $\boldsymbol{u}(\boldsymbol{x};t)$. However, as the observant reader will notice, elastic energy quantity $U_{\text{elastic}}$ will be expressed in terms of the strain field $\boldsymbol{e}(\boldsymbol{x};t)$. Field $\boldsymbol{e}(\boldsymbol{x};t)$ *is considered as auxiliary and not independent*, since its components stand merely as an aggregated notation for the respective components of the displacement field using strain definition relation (2.3). The use of field $\boldsymbol{e}(\boldsymbol{x};t)$ will help us perform the following Gâteaux derivation in more contracted expressions until the final step of derivation, where the independent field $\boldsymbol{u}(\boldsymbol{x};t)$ will appear via Eq.(2.3).

<u>Kinetic Energy</u>: $$U_{\text{kinetic}}(\boldsymbol{u}) = \frac{1}{2}\iiint_{\Omega^{(0)}} p_i\,\dot{u}_i\;dV \tag{2.7a}$$

Substituting linear momentum definition relation (2.1) into Eq.(2.7a) we obtain

$$U_{\text{kinetic}}(\boldsymbol{u}) = \frac{1}{2}\rho_b\iiint_{\Omega^{(0)}} \dot{u}_i\,\dot{u}_i\;dV \tag{2.7b}$$

<u>Elastic Energy</u>: $$U_{\text{elastic}}(\boldsymbol{u}) = \frac{1}{2}\iiint_{\Omega^{(0)}} \sigma_{ij}\,e_{ij}\;dV \tag{2.8a}$$

Substituting Hooke's Law of Eq.(2.6) into Eq.(2.8a) we obtain

$$U_{\text{elastic}}(\boldsymbol{u}) = \frac{1}{2} c_{ijk\ell} \iiint\limits_{\Omega^{(0)}} e_{ij} e_{k\ell} \, dV \tag{2.8b}$$

Thus, Hamilton's Principle for Linear Elastodynamics is expressed as:

$$\delta \mathscr{H}[\boldsymbol{u}; \boldsymbol{\lambda}: \delta\boldsymbol{u}, \delta\boldsymbol{\lambda}] = 0 \tag{2.9}$$

where $\mathscr{H}[\boldsymbol{u}; \boldsymbol{\lambda}]$ is the functional of Eq.(2.10) and $\delta$ denotes the total Gâteaux functional derivative.

$$\begin{aligned}
\mathscr{H}[\boldsymbol{u}; \boldsymbol{\lambda}] &= L[\boldsymbol{u}] + W_{\Omega^{(0)}}^{\substack{\text{external}\\ \text{force}}}[\boldsymbol{u}] + I_{\partial\Omega_u^{(0)}}^{\text{given } u_i}[\boldsymbol{u}; \boldsymbol{\lambda}] + I_{\partial\Omega_T^{(0)}}^{\text{given } T_i}[\boldsymbol{u}] = \\
&= \int_{t_0}^{t_1} \left[ U_{\text{kinetic}}(\boldsymbol{u}) - U_{\text{elastic}}(\boldsymbol{u}) \right] dt + \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \hat{f}_i u_i \, dV \, dt + \\
&+ \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_u^{(0)}} \left( u_i - \hat{u}_i \right) \lambda_i \, dS \, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_T^{(0)}} \hat{T}_i u_i \, dS \, dt
\end{aligned} \tag{2.10}$$

where $L[\boldsymbol{u}]$ is the action functional of the Lagrangian for elastodynamics, $W_{\Omega^{(0)}}^{\substack{\text{external}\\ \text{force}}}[\boldsymbol{u}]$ corresponds to the work of the given external forces $\hat{f}_i$ applied on the whole volume $\Omega^{(0)}$, $I_{\partial\Omega_u^{(0)}}^{\text{given } u_i}[\boldsymbol{u}; \boldsymbol{\lambda}]$ corresponds to the boundary condition on the boundary $\partial\Omega_u^{(0)}$ where displacement is prescribed and $I_{\partial\Omega_T^{(0)}}^{\text{given } T_i}[\boldsymbol{u}]$ corresponds to the boundary condition on the boundary $\partial\Omega_T^{(0)}$ where stress is prescribed.

To validate the above Hamilton's Principle, we perform the variation of Eq.(2.9) and re-obtain the governing equations of linear elastodynamics. For this, the Gâteaux derivatives with regard to $\boldsymbol{u}$ for each part of the functional (2.10) follow:

<u>Kinetic energy part of the Lagrangian</u>

$$L_{\text{kinetic}}[\boldsymbol{u}] = \int_{t_0}^{t_1} U_{\text{kinetic}}(\boldsymbol{u}) \, dt = \frac{1}{2} \rho_b \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \dot{u}_i \dot{u}_i \, dV \, dt \Rightarrow$$

$$\delta_{\boldsymbol{u}} L_{\text{kinetic}}\left[\boldsymbol{u}: \delta \boldsymbol{u}\right] = \frac{1}{2}\rho_b \frac{d}{d\xi}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}}\left(\dot{u}_i + \xi\,\delta\dot{u}_i\right)\left(\dot{u}_i + \xi\,\delta\dot{u}_i\right) dV\ dt\Bigg|_{\xi=0} =$$

$$= \rho_b \int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} \dot{u}_i\,\delta\dot{u}_i\ dV\ dt\ = \left[\text{temporal integration by parts}\right] =$$

$$= \rho_b \cancel{\iiint_{\Omega^{(0)}} \dot{u}_i\,\delta u_i\ dV\Bigg|_{t_0}^{t_1}}^{=0} - \rho_b \int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} \ddot{u}_i\,\delta u_i\ dV\ dt \tag{2.11}$$

<u>Elastic Energy part of the Lagrangian</u>

$$L_{\text{elastic}}\left[\boldsymbol{u}\right] = \int_{t_0}^{t_1} U_{\text{elastic}}\left(\boldsymbol{u}\right) dt = \frac{1}{2} c_{ijk\ell}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} e_{ij}\, e_{k\ell}\ dV\ dt$$

Firstly we perform the Gâteaux derivation with regard to auxiliary field $\boldsymbol{e}$

$$\delta_{\boldsymbol{e}} L_{\text{elastic}}\left[\boldsymbol{u}: \delta \boldsymbol{e}\right] = \frac{1}{2} c_{ijk\ell}\frac{d}{d\xi}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}}\left(e_{ij} + \xi\,\delta e_{ij}\right)\left(e_{k\ell} + \xi\,\delta e_{k\ell}\right) dV\ dt\Bigg|_{\xi=0} =$$

$$= \frac{1}{2} c_{ijk\ell}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} e_{ij}\,\delta e_{k\ell}\ dV\ dt + \frac{1}{2} c_{ijk\ell}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} e_{k\ell}\,\delta e_{ij}\ dV\ dt$$

by interchanging $ij-k\ell$ pairs of indices in the first term and using the $ij-k\ell$ major symmetry of elastic stiffness tensor $\boldsymbol{c}$ we obtain

$$\delta_{\boldsymbol{e}} L_{\text{elastic}}\left[\boldsymbol{u}: \delta \boldsymbol{e}\right] = c_{ijk\ell}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} e_{k\ell}\,\delta e_{ij}\ dV\ dt \tag{2.12}$$

In Eq.(2.12) we perform a change in variable of derivation using

$$\delta e_{ij} = \frac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right),$$

which is a relation obtained by Gâteaux derivation of both sides of Eq.(2.3), while term $e_{k\ell}$ of Eq.(2.12) continues to stand as an aggregated notation for $\left(\delta u_{k,\ell} + \delta u_{\ell,k}\right)/2$ quantity:

$$\delta_{\boldsymbol{u}} L_{\text{elastic}}\left[\boldsymbol{u}: \delta \boldsymbol{u}\right] = \frac{1}{2} c_{ijk\ell}\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}} e_{k\ell}\left(\delta u_{i,j} + \delta u_{j,i}\right) dV\ dt =$$

$$= \frac{1}{2} c_{ijk\ell} \int_{t_0}^{t_1} \iiint_{\Omega^{(0)}} e_{k\ell}\, \delta u_{i,j}\ dV\ dt + \frac{1}{2} c_{ijk\ell} \int_{t_0}^{t_1} \iiint_{\Omega^{(0)}} e_{k\ell}\, \delta u_{j,i}\ dV\ dt$$

by interchanging $i-j$ indices in the second term and $i-j$ minor symmetry of elastic stiffness tensor $\boldsymbol{c}$ we obtain

$$\delta_{\boldsymbol{u}} L_{\text{elastic}}[\boldsymbol{u}\colon \delta\boldsymbol{u}] = c_{jik\ell} \int_{t_0}^{t_1} \iiint_{\Omega^{(0)}} e_{k\ell}\, \delta u_{i,j}\ dV\ dt = [\text{spatial integration by parts}]$$

$$= c_{jik\ell} \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0)}} e_{k\ell}\, \delta u_i\, n_j^{(0)}\ dS\ dt - c_{jik\ell} \int_{t_0}^{t_1} \iiint_{\Omega^{(0)}} e_{k\ell,j}\, \delta u_i\ dV\ dt \tag{2.13}$$

<u>External force term</u>

$$\delta_{\boldsymbol{u}} W_{\Omega^{(0)}}^{\substack{\text{external}\\ \text{force}}} [\boldsymbol{u}\colon \delta\boldsymbol{u}] = \int_{t_0}^{t_1} \iiint_{\Omega^{(0)}} \hat{f}_i\, \delta u_i\ dV\,dt \tag{2.14}$$

<u>Boundary condition on $\partial\Omega_u^{(0)}$ where displacement is prescribed term</u>

$$I_{\partial\Omega_u^{(0)}}^{\text{given } u_i} [\boldsymbol{u};\boldsymbol{\lambda}] = \int_{t_0}^{t_1} \iint_{\partial\Omega_u^{(0)}} \left(u_i - \hat{u}_i\right)\lambda_i\ dS\,dt$$

$$\delta_{\boldsymbol{u}} I_{\partial\Omega_u^{(0)}}^{\text{given } u_i} [\boldsymbol{u};\boldsymbol{\lambda}\colon \delta\boldsymbol{u}] = \int_{t_0}^{t_1} \iint_{\partial\Omega_u^{(0)}} \lambda_i\, \delta u_i\ dS\,dt \tag{2.15}$$

$$\delta_{\lambda} I_{\partial\Omega_u^{(0)}}^{\text{given } u_i} [\boldsymbol{u};\boldsymbol{\lambda}\colon \delta\boldsymbol{\lambda}] = \int_{t_0}^{t_1} \iint_{\partial\Omega_u^{(0)}} \left(u_i - \hat{u}_i\right)\delta\lambda_i\ dS\,dt \tag{2.16}$$

<u>Boundary condition on the boundary $\partial\Omega_T^{(0)}$ where stress is prescribed term</u>

$$\delta_{\boldsymbol{u}} I_{\partial\Omega_T^{(0)}}^{\text{given } T_i} [\boldsymbol{u}\colon \delta\boldsymbol{u}] = \int_{t_0}^{t_1} \iint_{\partial\Omega_T^{(0)}} \hat{T}_i\, \delta u_i\ dS\,dt \tag{2.17}$$

As it was stated in the beginning of the present paragraph, as well as it has been shown in the calculation of the Gâteaux derivatives above, the independent fields with regard to which Gâteaux derivatives are calculated are the displacement field $\boldsymbol{u}$ and field $\boldsymbol{\lambda}$, whose physical meaning remains to be determined. Thus, variational equation (2.9) is written as

$$\delta\mathscr{H}[\boldsymbol{u};\boldsymbol{\lambda}:\delta\boldsymbol{u},\delta\boldsymbol{\lambda}] = 0 \Rightarrow \delta_{\boldsymbol{u}}\mathscr{H}[\boldsymbol{u};\boldsymbol{\lambda}:\delta\boldsymbol{u}] + \delta_{\boldsymbol{\lambda}}\mathscr{H}[\boldsymbol{u};\boldsymbol{\lambda}:\delta\boldsymbol{\lambda}] = 0 \tag{2.18}$$

Since variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{\lambda}$ are considered independent from one another, Eq.(2.18) is equivalent to

$$\delta_{\boldsymbol{u}}\mathscr{H}[\boldsymbol{u};\boldsymbol{\lambda}:\delta\boldsymbol{u}] = 0\,, \quad \delta_{\boldsymbol{\lambda}}\mathscr{H}[\boldsymbol{u};\boldsymbol{\lambda}:\delta\boldsymbol{\lambda}] = 0 \tag{2.19a,b}$$

From Eq.(2.19a), the following Euler-Lagrange equations are obtained

- $\rho_b\, \ddot{u}_i = c_{jik\ell}\, e_{k\ell,j} + \hat{f}_i$ over volume $\Omega^{(0)}$ (2.20a)

  which, **by substitution of Hooke's Law** (2.6), is Newton's Second Law for continuous media (2.3).

- $\hat{T}_i = c_{jik\ell}\, e_{k\ell}\, n_j^{(0)}$ on boundary $\partial\Omega_T^{(0)}$ (2.20b)

  which, **by substitution of Hooke's Law** (2.6), is Cauchy relation for stress (2.2) that gives the right boundary condition on boundary $\partial\Omega_T^{(0)}$ where stress vector $\boldsymbol{T}$ is prescribed.

- $\lambda_i = c_{jik\ell}\, e_{k\ell}\, n_j^{(0)}$ on boundary $\partial\Omega_u^{(0)}$ (2.20c)

  Eq.(2.20c) is Cauchy relation for stress (2.2) on boundary $\partial\Omega_u^{(0)}$, **by substitution of Hooke's Law** (2.6). This is not a needed boundary condition, but defines the auxiliary field $\boldsymbol{\lambda}$ as the stress over the boundary $\partial\Omega_u^{(0)}$ where the displacement $\boldsymbol{u}$ is prescribed.

From Eq.(2.19b), the following Euler-Lagrange equation is obtained

- $\hat{u}_i = u_i$ on boundary $\partial\Omega_u^{(0)}$ (2.20d)

  which is the right boundary condition on boundary $\partial\Omega_u^{(0)}$ where displacement is prescribed.

Thus, the proof of the variational principle (2.9) has been concluded and the meaning and the role of the various fields have been clarified.

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# 3. Linear Electrodynamics

## 3.1 Fields, Media and Constitutive Relations

*General References: Athanassoulis (2007), Balanis (1989), Griffiths (1999), Jackson (1998), Karlsson & Kristensson (1999), Landau & Lifshitz (1984), Solymar (1984).*

The way of introducing and interpreting the electromagnetic (E/M) field in the present work will be performed from a material perspective, since piezoelectric bodies are a special case of material media. That means that the Maxwell's Equations will be written with only the free charges and currents appearing explicitly ("macroscopic" Maxwell's Equations) using four E/M quantities $\boldsymbol{E}(\boldsymbol{x};t)$, $\boldsymbol{D}(\boldsymbol{x};t)$, $\boldsymbol{B}(\boldsymbol{x};t)$, $\boldsymbol{H}(\boldsymbol{x};t)$ (see below for definitions of these quantities).

By the term material media we call the compound physical entities of finite or infinite extent, characterized by their mass, electric charge and electric current, which are distributed and bound throughout their extent. For each material medium there is a natural state in vacuo, and a specific internal structure providing appropriate degrees of freedom to the charge and current distributed within the medium, so than the latter to be able to redistribute themselves under the action of an external E/M field or external electric charges or currents.

This definition of material media leads to a simple scheme for understanding E/M field inside material media using excitation-response terms:

1. An external E/M field, described by the *electric intensity vector field* $\boldsymbol{E}(\boldsymbol{x};t)$ and the *magnetic induction vector field* $\boldsymbol{B}(\boldsymbol{x};t)$, has been created somewhere outside the volume of interest and then entered the volume by propagation.

2. Under the action of the external E/M fields $\boldsymbol{E}(\boldsymbol{x};t)$ and $\boldsymbol{B}(\boldsymbol{x};t)$, the bound charges and currents within each of the different media that compose the total volume of interest are redistributed, giving rise to an additional electric vector field $\boldsymbol{P}(\boldsymbol{x};t)$ and an additional magnetic vector field $\boldsymbol{M}(\boldsymbol{x};t)$, inside each medium. Field $\boldsymbol{P}(\boldsymbol{x};t)$ is called *electric polarization* and field $\boldsymbol{M}(\boldsymbol{x};t)$ is called *magnetization*.

3. Thus the total electric and magnetic fields inside each medium are superpositions of the respective external ($\boldsymbol{E}(\boldsymbol{x};t)$ and $\boldsymbol{B}(\boldsymbol{x};t)$) and internal ($\boldsymbol{P}(\boldsymbol{x};t)$ and $\boldsymbol{M}(\boldsymbol{x};t)$) field. The total electric field is called *electric displacement vector field* $\boldsymbol{D}(\boldsymbol{x};t)$ and the total magnetic field is called *magnetic intensity vector field* $\boldsymbol{H}(\boldsymbol{x};t)$.

4. Since E/M fields $\boldsymbol{D}(\boldsymbol{x};t)$ and $\boldsymbol{H}(\boldsymbol{x};t)$ depend on material properties, matching conditions on the interfaces between the media have to be considered.

In the case of **linear dielectric and diamagnetic media** (see e.g. Griffiths 1999, Chs. 4 & 6 and Newnham 2005 Ch. 9 & 14), electric polarization $\boldsymbol{P}(\boldsymbol{x};t)$ and magnetization $\boldsymbol{M}(\boldsymbol{x};t)$ fields are linearly related to the external electric $\boldsymbol{E}(\boldsymbol{x};t)$ and magnetic field $\boldsymbol{H}(\boldsymbol{x};t)$ respectively, as shown in the following relations

$$P_i = \varepsilon_0 \chi_{ij}^{(e)} E_j, \tag{3.1}$$

$$M_i = \chi_{ij}^{(m)} H_j, \tag{3.2}$$

where $\varepsilon_0$ is the constant dielectric permittivity of vacuum and $\boldsymbol{\chi}^{(e)}$, $\boldsymbol{\chi}^{(m)}$ are the dielectric and magnetic susceptibilities respectively, which are material property tensors (2nd rank).

In compliance with step 3 of the above scheme, fields $\boldsymbol{D}(\boldsymbol{x};t)$ and $\boldsymbol{H}(\boldsymbol{x};t)$ are written as superpositions of the respective fields as

$$D_i = \varepsilon_0 E_i + P_i \tag{3.3}$$

$$H_i = \frac{1}{\mu_0} B_i - M_i, \tag{3.4}$$

where $\mu_0$ is the constant magnetic permeability of vacuum.

Substituting Eqs.(3.1) and (3.2) into Eqs(3.3) and (3.4) we obtain

$$D_i = \varepsilon_0 \left( \delta_{ij} + \chi_{ij}^{(e)} \right) E_j \equiv \varepsilon_{ij} E_j \tag{3.5}$$

$$B_i = \mu_0 \left( \delta_{ij} + \chi_{ij}^{(m)} \right) H_j \equiv \mu_{ij} H_j \tag{3.6}$$

where $\delta_{ij}$ is Kronecker's delta, $\varepsilon_{ij}$ is called dielectric permittivity and $\mu_{ij}$ is called magnetic permeability. Both $\varepsilon_{ij}$ and $\mu_{ij}$ are symmetric material properties tensors (2nd rank).

Eqs.(3.5) and (3.6) are called the **constitutive relations of electromagnetism for linear dielectric and diamagnetic materials**. They characterize the E/M response of this particular class of materials to external E/M excitation, modeling the scheme for understanding E/M field inside material media presented in the beginning of the present paragraph.
In the present work, it is the second time we encounter a constitutive relation; **generalized Hooke's Law** in chapter 2 was the elastic constitutive relation defining the strain response of linear elastic solids to stress excitation. It is important to underline the fact that *all constitutive relations are material properties, characterizing a certain class of materials*. If the class of materials changes (and this is going to happen when we consider piezoelectric materials) constitutive relations will be reconsidered.

### 3.2 Maxwell's "macroscopic" equations in integral form

Having defined E/M fields $(\boldsymbol{E},\boldsymbol{D};\,\boldsymbol{B},\boldsymbol{H})$ as above, we state that E/M phenomena taking place in a volume of interest $\Omega$ are fully described by *Maxwell's equations in integral form.*

Since the E/M fields defined are vector fields, a brief discussion of vector field integrals for a trial vector field $\boldsymbol{F}(\boldsymbol{x};\,t)$ precedes Maxwell's equations in integral form

- **Line integral** $\int_C F_i\, d\ell_i = \int_C F_i\, t_i\, d\ell$ expresses the generalized concept of "work" of vector $\boldsymbol{F}(\boldsymbol{x};\,t)$ along the path $C$. $F_i\, d\ell_i$ expresses the notion of work as a dot product of $\boldsymbol{F}(\boldsymbol{x};\,t)$ along the infinitesimal displacement $\boldsymbol{dl}$. For a clearer notation, infinitesimal displacement $\boldsymbol{dl}$ along curve $C$ can be written as $t_i\, d\ell$ where $\boldsymbol{t}$ is the tangent unit vector of $C$ and $d\ell$ a scalar curve element. This way of expressing $\boldsymbol{dl}$ gives rise to the second form of line integral, which can be easily interpreted as **the integration of the component of vector $\boldsymbol{F}(\boldsymbol{x};\,t)$ which is tangent to curve $C$ along curve $C$.**

- **Surface integral** $\iint_A F_i\, dS_i = \iint_A F_i\, n_i\, dS$ expresses the flux of vector $\boldsymbol{F}(\boldsymbol{x};\,t)$ through the surface $A$. Analogously to the modeling of work along curve $C$ by an integrand of dot product between field $\boldsymbol{F}(\boldsymbol{x};\,t)$ and tangent unit vector $\boldsymbol{t}$ of $C$ in line integral, the flux through surface $A$ is expressed by an integrand of dot product between field $\boldsymbol{F}(\boldsymbol{x};\,t)$ and outward normal unit vector $\boldsymbol{n}$ of surface $A$. Surface integration is then performed with regard to differential $dS$ which is a scalar surface element. Thus, the surface integral can be easily interpreted as **the integration of the component of vector $\boldsymbol{F}(\boldsymbol{x};\,t)$ which is vertical to surface $A$ on surface $A$.**

**Note 1:** When line and surface integrals are defined over closed lines and surfaces, symbols of integration $\int$ and $\iint$ are substituted by $\oint$ and $\oiint$ respectively. In this case, line integral models the "circulation" along the closed curve $C$ and surface integral models the flux through the boundary surface $\partial\Omega$ of a volume $\Omega$.

**Note 2:** As we have already mentioned, there are two unit vectors involved in line and surface integral expressions, tangent unit vector $\boldsymbol{t}$ of curve $C$ and outward normal unit vector $\boldsymbol{n}$ of surface $A$ respectively. In the special case where curve $C$ is the boundary curve of surface $A$ ($C \equiv \partial A$), the two vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ must comply with the right-hand rule. This is an essential condition in order to be able to pass from line to surface integrations and vice versa, using Kelvin – Stokes theorem (*see* below Eq.3.12).

After the previous revision of the mathematical entities to-be-used, we move on expressing *Maxwell's equations in integral form*:

*Gauss's Law for electrostatics:*

$$\oiint_{\partial\Omega} D_i\, n_i \; dS \;=\; \hat{Q}_f^{\text{enclosed}}\left(\Omega\right) \tag{3.7}$$

where $\oiint_{\partial\Omega} D_i\, n_i \; dS$ is the flux (surface integral) of electric displacement field through the closed surface $\partial\Omega$ (the boundary of volume $\Omega$) and $\hat{Q}_f^{\text{enclosed}}\left(\Omega\right)$ is the net free electric charge (scalar quantity) inside volume $\Omega$ (not including bound charge).

*Gauss's Law for magnetostatics:*

$$\oiint_{\partial\Omega} B_i\, n_i \; dS \;=\; 0 \tag{3.8}$$

where $\oiint_{\partial\Omega} B_i\, n_i \; dS$ is the magnetic flux (surface integral) through the closed surface $\partial\Omega$ (the boundary of volume $\Omega$).

*Faraday's Law of induction:*

$$\oint_{\partial\Gamma} E_i\, t_i \; d\ell \;=\; -\iint_{\Gamma} \dot{B}_i\, n_i \; dS \tag{3.9}$$

where $\oint_{\partial\Gamma} E_i\, t_i \; d\ell$ is the line integral of electric intensity field along the boundary $\partial\Gamma$ of a surface $\Gamma$ ($\partial\Gamma$ is always a closed curve) and $\iint_{\Gamma} B_i\, n_i \; dS$ the magnetic flux passing through surface $\Gamma$.

*Ampère's Law with Maxwell's extension:*

$$\oint_{\partial\Gamma} H_i\, t_i \; d\ell \;=\; \iint_{\Gamma} \dot{D}_i\, n_i \; dS + \hat{I}_f\left(\Gamma\right) \tag{3.10}$$

where $\oint_{\partial\Gamma} H_i\, t_i \; d\ell$ is the line integral of magnetic intensity field along the boundary $\partial\Gamma$ of a surface $\Gamma$ ($\partial\Gamma$ is always a closed curve), $\iint_{\Gamma} \dot{D}_i\, n_i \; dS$ is the flux of the first time derivative of electric displacement passing through surface $\Gamma$ and $\hat{I}_f(\Gamma)$ is the net free electrical current (scalar quantity) passing through surface $\Gamma$ (not including bound current).

### 3.3 Reformulating Maxwell's equations from integral to differential form

Since a complete discussion on how Maxwell's equation are derived and verified is clearly beyond the scope of the present work, we will consider Maxwell's equations in integral form as the axioms of electrodynamics that need no proof. What we are going to show in detail for the better understanding of Maxwell's system of equations is how we pass from the integral equations that capture the whole E/M phenomenon, to a system of PDEs, completed with the appropriate boundary conditions.
The reason to perform this is to show that *both PDEs and their boundary conditions*[1] *derive from Maxwell's equations in integral form.*

For the derivation of Maxwell's equation in differential form, two theorems of vector calculus will be used:

***Gauss's divergence theorem:*** Suppose $\Omega$ is a subset of $\mathbb{R}^n$ (in case of $n = 3$, $\Omega$ represents a volume in 3D space) which is compact and has a piecewise smooth boundary $\partial\Omega$. If $\boldsymbol{F}$ is a continuously differentiable vector field defined on a neighborhood of $\Omega$, then we have

$$\iiint\limits_{\Omega} F_{i,i}\, dV = \oiint\limits_{\partial\Omega} F_i n_i\, dS \tag{3.11}$$

where the left side is a volume integral over volume $\Omega$, the right side is the surface integral over the closed boundary surface $\partial\Omega$ of volume $\Omega$ and $\boldsymbol{n}$ is the outward normal vector of $\partial\Omega$.

***Kelvin – Stokes curl theorem:*** Let $\gamma : [a,b] \to \mathbb{R}^2$ be a piecewise smooth Jordan plane curve that bounds the domain $D \subset \mathbb{R}^2$. Suppose $\psi : D \to \mathbb{R}^3$ is smooth with $\Gamma := \psi[D]$ and $\partial\Gamma$ is the space curve defined by $\partial\Gamma(t) = \psi(\gamma(t))$. If $\boldsymbol{F}$ is a smooth vector field on $\mathbb{R}^3$, then we have

$$\iint\limits_{\Gamma} \epsilon_{ijk} F_{k,j} n_i\, dS = \oint\limits_{\partial\Gamma} F_i t_i\, d\ell \tag{3.12}$$

where the left side is a surface integral over surface $\Gamma$ and the right side is the surface integral over the closed boundary line $\partial\Gamma$ of surface $\Gamma$. $\epsilon_{ijk}$ is the Levi-Civita symbol.

[1] The term "matching conditions", instead of the term "boundary conditions", is more precise in the E/M context since E/M fields cannot be restricted to a certain volume and, also, in the present work where the volume of interest consists of different material media, the conditions on the interface between the media cannot be described as "boundary". Thus, from now on, we will use the term "matching conditions".

Observing Eqs.(3.11) and (3.12) we comment that these theorems, when applied, can increase (or reduce) the integrations performed by one, since both theorems relate integrals over one domain (volume or surface) with integrals on domain's boundary (surface or curve respectively). Thus, Eqs.(3.11) and (3.12) can be applied to the integral Maxwell's equations (3.7) – (3.10) so that the integrations appearing in each of the equations to be performed over the same geometrical domain.

*Gauss's Law for electrostatics*

Applying Gauss's divergence theorem (3.11) to the left side of Eq.(3.7) we obtain

$$\iiint_{\Omega} D_{i,i} \, dV = \hat{Q}_f^{\text{enclosed}}(\Omega). \tag{3.13}$$

Then we express the net free electric charge $\hat{Q}_f^{\text{enclosed}}(\Omega)$ inside volume $\Omega$ using the free charge volume density $\hat{\rho}_e = \hat{\rho}_e(\boldsymbol{x};t)$, $\boldsymbol{x} \in \Omega$ (scalar quantity)

$$\hat{Q}_f^{\text{enclosed}}(\Omega) = \iiint_{\Omega} \hat{\rho}_e \, dV \tag{3.14}$$

and Eq.(3.13) is written

$$\iiint_{\Omega} \left( D_{i,i} - \hat{\rho}_e \right) dV = 0 \tag{3.15}$$

Since Eq.(3.15) has to be simultaneously true for every volume $\Omega$ (enclosing free charge $\hat{Q}_f^{\text{enclosed}}$), it is necessary and sufficient for the integrand to be null everywhere

$$D_{i,i} = \hat{\rho}_e \tag{3.16}$$

<u>Eq. (3.16) is Gauss's Law for electrostatics in differential form.</u>

The respective matching condition is obtained through the following way: Let us consider a cylinder volume $\Omega_{cyl}$ whose center line coincides with the boundary between two dielectric media. Eq.(3.7) can be written for $\Omega_{cyl}$ as

$$\oiint_{\partial \Omega_{cyl}} D_i n_i \, dS = \iint_{\text{upper base}} D_i n_i \, dS + \iint_{\text{lower base}} D_i n_i \, dS + \iint_{\text{side}} D_i n_i \, dS = \hat{Q}_f^{\text{enclosed}}\left(\Omega_{cyl}\right) \tag{3.17}$$

Examining the limiting case when the height of cylinder $\Omega_{cyl}$ tends to be zero:

- Volume $\Omega_{cyl}$ degenerates to a surface $S$, which is the boundary between the two dielectric media, thus we obtain the desired matching condition.
- The side of $\Omega_{cyl}$ has zero area, and thus the respective electric displacement flux integral $\iint_{\text{side}} D_i\, n_i \; dS$ is null.
- Upper base coincides with lower base and since $\boldsymbol{n}$ is the outward unit vector:
$$n_i^{\text{lower}} = -\, n_i^{\text{upper}} \equiv -\, n_i$$
- Net free charge $\hat{Q}_f^{\text{enclosed}}\left(\Omega_{cyl}\right)$ is reduced to the net free charge on the boundary surface $\hat{Q}_f\left(S\right)$

Thus, Eq.(3.17) is written as

$$\iint_S \left(D_i^{(1)} - D_i^{(2)}\right) n_i \; dS = \hat{Q}_f\left(S\right), \tag{3.18}$$

with the superscripts (1) and (2) in components of $\boldsymbol{D}$ denoting the two different dielectric media (corresponding to the media of upper and lower base respectively).

By expressing net free surface charge $\hat{Q}_f\left(S\right)$ using the free charge surface density $\hat{\sigma}_e = \hat{\sigma}_e\left(\boldsymbol{x};\, t\right)$, $\boldsymbol{x} \in S$ (scalar quantity), Eq.(3.18) is reformulated as

$$\iint_S \left(D_i^{(1)} - D_i^{(2)}\right) n_i \; dS = \iint_S \hat{\sigma}_e \; dS \tag{3.19}$$

Since Eq.(3.19) has to be simultaneously true for every surface $S$, it is necessary and sufficient for the integrands to be equal everywhere

$$\left(D_i^{(1)} - D_i^{(2)}\right) n_i = \hat{\sigma}_e \tag{3.20}$$

<u>Eq.(3.20) is the matching condition with regard to electric displacement field $\boldsymbol{D}$ between two dielectric media.</u> It states that, on the boundary surface, the component of field $\boldsymbol{D}$ vertical to the boundary surface shows a jump equal to the net free charge surface density on the boundary. In absence of net free charge surface density, Eq.(3.20) is a continuity condition of the aforementioned component of field $\boldsymbol{D}$.

The procedure for obtaining *Gauss's Law for magnetostatics* in differential form and the matching condition for field $\boldsymbol{B}$ is identical to the previous procedure for Gauss's Law for electrostatics except for the absence of net magnetic charges in magnetostatics

$B_{i,i} = 0$ over volume $\Omega$ (3.21)

and $\left(B_i^{(1)} - B_i^{(2)}\right) n_i = 0$ on boundary $\partial\Omega$ (3.22)

The aforementioned absence of net magnetic charges results in the fact that Eqs.(3.21) and (3.22) are homogeneous. More specifically, Eq.(3.21) means that field $\boldsymbol{B}$ is solenoidal (divergence of vector equals to zero) and matching condition (3.22) means that, on the boundary surface, the component of field $\boldsymbol{B}$ vertical to the boundary surface $\partial\Omega$ is continuous.

*Ampère's Law with Maxwell's extension*
Applying Kelvin - Stokes curl theorem (3.12) to the left side of Eq.(3.10) we obtain

$$\iint_{\Gamma} \epsilon_{ijk} H_{k,j} n_i \, dS = \iint_{\Gamma} \dot{D}_i n_i \, dS + \hat{I}_f(\Gamma). \tag{3.23}$$

Then we express the net free electric current $\hat{I}_f(\Gamma)$ passing through surface $\Gamma$ using the free current volume density $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}(\boldsymbol{x};t)$, $\boldsymbol{x} \in \Omega$ (vector quantity)

$$\hat{I}_f(\Gamma) = \iint_{\Gamma} \hat{J}_i n_i \, dS \tag{3.24}$$

and Eq.(3.23) is written

$$\iint_{\Gamma} \left(\epsilon_{ijk} H_{k,j} - \dot{D}_i - J_i\right) n_i \, dS = 0 \tag{3.25}$$

Since Eq.(3.25) has to be simultaneously true for every surface $\Gamma$ (through which free current $\hat{I}_f$ passes), it is necessary and sufficient for the integrand to be null everywhere and so

$$\epsilon_{ijk} H_{k,j} = \dot{D}_i + J_i \tag{3.26}$$

<u>Eq. (3.26) is Ampère's Law with Maxwell's extension in differential form.</u>

The respective matching condition is obtained through the following way: Let us consider a parallelogram surface $\Gamma_{par}$ whose center line coincides with the boundary $S$ between two dielectric media. Eq.(3.10) can be written for $\Gamma_{par}$ as

$$\oint_{\partial\Gamma_{par}} H_i t_i \, d\ell =$$

$$= \int_{\text{upper side}} H_i t_i \, d\ell + \int_{\text{lower side}} H_i t_i \, d\ell + \int_{\text{left side}} H_i t_i \, d\ell + \int_{\text{right side}} H_i t_i \, d\ell =$$

$$= \iint_{\Gamma_{par}} \dot{D}_i n_i \, dS + \hat{I}_f(\Gamma) \tag{3.27}$$

Examining the limiting case when the height of parallelogram $\Gamma_{par}$ tends to be zero (while it continues to have non-zero width):

- Surface $\Gamma_{par}$ degenerates to a line $C$, that belongs to the boundary surface $S$ between the two dielectric media. Thus the electric displacement flux integral $\iint_{\Gamma_{par}} D_i n_i \, dS$ is null.
- The right and left side of $\Gamma_{par}$ have zero length, and thus the respective magnetic induction line integrals $\int_{\text{left side}} H_i t_i \, d\ell$, $\int_{\text{right side}} H_i t_i \, d\ell$ are null.
- Upper side coincides with lower side and since tangent unit vector $\boldsymbol{t}$ has a uniform direction for the whole closed curve $\partial\Gamma_{par}$:
$$t_i^{\text{lower}} = -\, t_i^{\text{upper}} \equiv -\, t_i \, .$$
It is also obvious that tangent unit vector $\boldsymbol{t}$ of the line $C$ is also tangent to boundary surface $S$.
- Net free current $\hat{I}_f\left(\Gamma_{par}\right)$ is reduced to the net free current on the boundary surface, which is the free current passing through line $C$ $\hat{I}_f(C)$.

Thus, Eq.(3.27) is written as

$$\int_C \left(H_i^{(1)} - H_i^{(2)}\right) t_i \, d\ell = \hat{I}_f(C), \tag{3.28}$$

with the superscripts (1) and (2) in components of $\boldsymbol{H}$ denoting the two different dielectric media (corresponding to upper and lower side respectively).

Expressing net free current $\hat{I}_f(C)$ using the free current surface density $\hat{\boldsymbol{K}} = \hat{\boldsymbol{K}}(\boldsymbol{x};t)$ (vector quantity) *of the boundary surface*, Eq.(3.28) is reformulated as

$$\int_C \left(H_i^{(1)} - H_i^{(2)}\right) t_i \, d\ell = \int_C \hat{K}_i b_i \, d\ell \tag{3.29}$$

where $\boldsymbol{b}$ is the binormal vector to outward normal unit vector $\boldsymbol{n}$ of the boundary surface $S$ and tangent unit vector $\boldsymbol{t}$, defined as $b_i = \epsilon_{ijk} n_j t_k$. The dot product of current surface density $\hat{\boldsymbol{K}}$ with binormal vector $\boldsymbol{b}$ is used since we have to calculate the current that flows in a direction through the loop $\partial\Gamma_{par}$ and thus vertical to both vector $\boldsymbol{n}$ and $\boldsymbol{t}$.

Since Eq.(3.29) has to be simultaneously true for every line $C \in S$, it is necessary and sufficient for the integrands to be equal everywhere in boundary surface $S$

$$\left(H_i^{(1)} - H_i^{(2)}\right) t_i = \hat{K}_i b_i \tag{3.30}$$

Eq.(3.30) is the matching condition with regard to magnetic induction field $\boldsymbol{H}$ between two dielectric media. It states that, on the boundary surface, the component of field $\boldsymbol{H}$ tangent to the boundary surface shows a jump analogous to the current surface density $\hat{\boldsymbol{K}}$ on the boundary. In absence of current surface density, Eq.(3.40) is a continuity condition of the aforementioned component of field $\boldsymbol{H}$.
For reasons of uniformity in expressions, Eq.(3.40) can be easily expressed using vector $\boldsymbol{n}$ instead of $\boldsymbol{t}$ and $\boldsymbol{b}$, as matching conditions (3.20) and (3.22). Using the equality $t_i = \epsilon_{ijk} b_j n_k$ that derives from the fact that $(\boldsymbol{n},\boldsymbol{t},\boldsymbol{b})$ consist a Cartesian base, Eq.(3.30) is written

$$\left(H_i^{(1)} - H_i^{(2)}\right) \epsilon_{ijk} b_j n_k = \hat{K}_i b_i \tag{3.31}$$

Recognizing that the left side of Eq.(3.31) is a scalar triple product, we perform a circular shift that leaves scalar triple product unaffected

$$b_i \epsilon_{ijk} n_j \left(H_k^{(1)} - H_k^{(2)}\right) = \hat{K}_i b_i \tag{3.32}$$

In order Eq.(3.32) to hold true for every boundary surface $S$, it is necessary and sufficient that

$$\epsilon_{ijk} n_j \left(H_k^{(1)} - H_k^{(2)}\right) = \hat{K}_i \tag{3.33}$$

Eq.(3.33) is then an alternative expression for the matching condition with regard to magnetic induction field $\boldsymbol{H}$ between two dielectric media. It is the expression of this matching condition which we will later use.

The procedure for obtaining *Faraday's Law of induction* in differential form and the matching condition for field $\boldsymbol{E}$ is identical to the previous procedure for Ampère's Law with Maxwell's extension except for the absence of net magnetic currents

$$\epsilon_{ijk} E_{k,j} = -\dot{B}_i \qquad \text{over volume } \Omega \tag{3.34}$$

and $\epsilon_{ijk}\, n_j \left( E_k^{(1)} - E_k^{(2)} \right) = 0$ on boundary $\partial\Omega$ (3.35)

The aforementioned absence of net magnetic currents results in the fact that Eqs.(3.34) and (3.35) are homogeneous. Matching condition (3.35) means that, on the boundary surface, the component of field $\boldsymbol{E}$ tangent to the boundary surface $\partial\Omega$ is continuous.

At this point, the work of deriving Maxwell's equations in differential form and the respective matching conditions from the integral form of equations seems complete. What we will also show before proceeding with the discussion over the Maxwell's PDEs, is that *the Principle of Conservation of electric charge is also derived from Maxwell's equations in integral form.* That underlines the argument that Maxwell's equations in integral form offer a complete modeling of electromagnetism.

More specifically, the principle of conservation of electric charge is embedded in the combination of Gauss's and Ampère – Maxwell's laws (as expected, since these are the laws which involve electric charges and currents) when applied to a certain geometry.
Let us consider Ampère's – Maxwell's law (3.10) applied to the boundary surface $\Gamma \in \partial\Omega$ of a balloon – like volume $\Omega$, with the balloon opening being the closed boundary curve $\partial\Gamma$ of surface $\Gamma$:

$$\oint_{\partial\Gamma} H_i\, t_i \; d\ell = \iint_{\Gamma} \dot{D}_i\, n_i \; dS + \iint_{\Gamma} \hat{J}_i\, n_i \; dS \tag{3.36}$$

Let us now think of the limiting case when "inflating" the balloon with air:

- The balloon opening curve $\partial\Gamma$ is degenerated to zero, thus line integral at the left side of Eq.(3.36) is null.
- The surface integrals at the right side of Eq.(3.36) are defined over a closed surface.

Thus, Eq.(3.36) becomes

$$\oint\!\!\oint_{\partial\Omega} \dot{D}_i\, n_i \; dS + \oint\!\!\oint_{\partial\Omega} \hat{J}_i\, n_i \; dS = 0 \tag{3.37}$$

Applying Gauss's law for electrostatics (3.7) at the first integral of Eq.(3.37), we obtain

$$\iiint_{\Omega} \dot{\rho}_e \; dV + \oint\!\!\oint_{\partial\Omega} \hat{J}_i\, n_i \; dS = 0 \tag{3.38}$$

Eq.(3.38) is the *Conservation of electric charge in integral form.* To obtain the respective differential form, we apply Gauss's divergence theorem at the second integral of Eq.(3.38)

$$\iiint\limits_{\Omega} \left( \dot{\hat{\rho}}_e + \hat{J}_{i,i} \right) dV = 0 \tag{3.39}$$

Since Eq.(3.39) has to be simultaneously true for every volume $\Omega$, it is necessary and sufficient for the integrand to be null everywhere

$$\dot{\hat{\rho}}_e + \hat{J}_{i,i} = 0 \tag{3.40}$$

Eq.(3.40) is the *Conservation of electric charge in differential form*.

With the argument of completeness of Maxwell's equations in integral form proven, we move on to the discussion of the system of Maxwell's PDEs in order to construct a variational formulation for them later on.

**3.4 Maxwell's "macroscopic" equations in differential form**

Let us summarize the system of *Maxwell's equations in differential form* derived before:

Gauss's Law for electrostatics: $$D_{i,i} = \hat{\rho}_e \tag{3.16}$$

Gauss's Law for magnetostatics: $$B_{i,i} = 0 \tag{3.21}$$

Faraday's Law of induction: $$\epsilon_{ijk} E_{k,j} = -\dot{B}_i \tag{3.34}$$

Ampère's Law with Maxwell's extension: $$\epsilon_{ijk} H_{k,j} = \dot{D}_i + \hat{J}_i \tag{3.26}$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol, $\hat{\rho}_e = \hat{\rho}_e(\boldsymbol{x};t)$ is the free charge volume density (scalar quantity) and $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}(\boldsymbol{x};t)$ is the free current volume density vector.

Maxwell's equations are supplemented by the constitutive relations between $\boldsymbol{D}$ - $\boldsymbol{E}$ and $\boldsymbol{H}$ - $\boldsymbol{B}$ fields, which, in the case of linear dielectric and diamagnetic media, are

$$D_i = \varepsilon_{ij} E_j \tag{3.5}$$

$$B_i = \mu_{ij} H_j \;\Rightarrow\; H_i = \mu_{ji}^{-1} B_j \tag{3.6}$$

where $\varepsilon_{ij}$ is dielectric permittivity, $\mu_{ij}$ is magnetic permeability $\mu_{ji}^{-1}$ and is the inverse magnetic permeability. Both $\varepsilon_{ij}$ and $\mu_{ij}$ are material properties tensors (2nd rank).

At the present work the system of charges $\hat{\rho}_e(\boldsymbol{x};t)$ and currents $\hat{\boldsymbol{J}}(\boldsymbol{x};t)$ is considered as external and given (that's why they appear with a hat above them). In spite of being given,

$\hat{\rho}_e(\boldsymbol{x};t)$ and $\hat{\boldsymbol{J}}(\boldsymbol{x};t)$ cannot be arbitrary, since they are interrelated by the following compatibility relation

$$\dot{\rho}_e + J_{i,i} = 0. \tag{3.40}$$

which is the *Conservation of Electric Charge*. Note, however, that there are problems in which the distribution of charges and currents are not known and their calculation is an essential part of the problem. Problems of this type will not be considered in the present work.

The (3.16) – (3.21) – (3.34) – (3.26) system of PDEs, denoted hereof as $\mathcal{S}$, is completed with the following matching conditions $\mathcal{M.C.}$ applied at the interfaces between the different material media:

Jump of $\boldsymbol{D}$-component vertical to interface: $\left(D_i^{(1)} - D_i^{(2)}\right)n_i = \hat{\sigma}_e$ (3.20)

Continuity of $\boldsymbol{B}$-component vertical to interface: $\left(B_i^{(1)} - B_i^{(2)}\right)n_i = 0$ (3.22)

Continuity of $\boldsymbol{E}$-component tangent to interface: $\epsilon_{ijk}\, n_j\left(E_k^{(1)} - E_k^{(2)}\right) = 0$ (3.35)

Jump of $\boldsymbol{H}$-component tangent to interface: $\epsilon_{ijk}\, n_j\left(H_k^{(1)} - H_k^{(2)}\right) = \hat{K}_i$ (3.33)

where $\boldsymbol{n}$ is the normal unit vector of interface, $\hat{\sigma}_e(\boldsymbol{x};t)$ is the surface charge density (scalar quantity) and $\hat{\boldsymbol{K}}(\boldsymbol{x};t)$ is the surface current density (vector quantity).

**Balance between number of equations and unknowns in $\mathcal{S}$**

| |
|---|
| Number of scalar equations: 8 |
| 1 scalar equation from Eq.(3.16) |
| 1 scalar equation from Eq.(3.21) |
| 3 scalar equations from Eq.(3.34) |
| 3 scalar equations from Eq.(3.26) |
| |
| Number of unknown scalar fields: 6 |
| 12 components of vectors $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{H}$ |
| unknowns reduced to 6 by constitutive relations (3.5), (3.6) |

Thus we have **6 unknown scalar fields to be determined by 8 scalar equations.**

Apparently, the system of Maxwell's equations seems to be over-determined since the number of equations exceeds the number of unknown fields. But it can be easily shown that *Maxwell's equations are not independent from one another* since the two scalar, time-independent equations (3.16) and (3.21) can almost be deduced from the two vector, time-dependent equations (3.34) and (3.26) in conjunction with the conservation law (3.40). For, if we apply the div operator on both sides of (3.34) and (3.26), we obtain

$$\overbrace{\cancel{\epsilon_{ijk} E_{k,ji}}}^{\equiv 0} = -\dot{B}_{i,i} \;\Rightarrow\; \dot{B}_{i,i} = 0 \tag{3.40a}$$

$$\overbrace{\cancel{\epsilon_{ijk} H_{k,ji}}}^{\equiv 0} = \dot{D}_{i,i} + \hat{J}_{i,i} \overset{(3.11)}{\Rightarrow} \dot{D}_{i,i} - \dot{\rho}_e = 0 \tag{3.40b}$$

Integrating Eqs.(3.40a,b) with regard to time variable we deduce that

$$B_{i,i} = \text{constant}\,, \tag{3.41a}$$

$$D_{i,i} = \rho_e + \text{constant}\,, \tag{3.41b}$$

that us the two scalar equations (3.16) and (3.21) up to an additive constant. The correct null value of this constant can be established, e.g., through the initial conditions.

The reduction (3.40) – (3.41) leads to a **well-balanced system of equations**. We shall now discuss a more systematic way out of the problem of redundancy of the set of Maxwell's equations $\mathcal{S}$, using *electromagnetic potentials.*

In $\mathcal{S}$, two out of four equations, namely (3.21) and (3.34) are homogeneous, due to the fact that net magnetic charges and currents do not exist (or haven been detected yet). We can use these two equations (which are, in fact, four scalar equations, as Eq.(3.21) is a scalar and Eq.(3.34) is a vector equation) in order to express the six fields $E_1, E_2, E_3, B_1, B_2, B_3$ in terms of four fields, the component of two EM potentials $\Phi^{e\ell}(\boldsymbol{x};t)$ and $\boldsymbol{A}(\boldsymbol{x};t)$.

Since $\boldsymbol{B}$ is a solenoidal field as $B_{i,i} = 0$, we can define $\boldsymbol{B}$ in terms of the rotation of another vector field $\boldsymbol{A}$:

$$B_i = \epsilon_{ijk} A_{k,j}\,. \tag{3.42a}$$

$\boldsymbol{A}$ is called the vector E/M potential.
Then, the other homogeneous equation (3.34), takes the form

$$\epsilon_{ijk} E_{k,j} + \epsilon_{ijk} \dot{A}_{k,j} = 0$$

or

$$\epsilon_{ijk}\left(E_k + \dot{A}_k\right)_{,j} = 0\,.$$

Thus, field $\boldsymbol{E} + \dot{\boldsymbol{A}}$ is irrotational (its div equals to zero) and can be represented by means of a scalar field, called the scalar E/M potential $\Phi^{e\ell}$, as

$$E_i + \dot{A}_i = -\Phi^{e\ell}_{,i}$$

or

$$E_i = -\Phi^{e\ell}_{,i} - \dot{A}_i . \tag{3.42b}$$

Thus, by means of the representation (3.42a,b), the six scalar fields $E_1$, $E_2$, $E_3$, $B_1$, $B_2$, $B_3$ have been represented by the four scalar fields $\Phi^{e\ell}$, $A_1$, $A_2$, $A_3$. At the same time, the two homogeneous equations (3.21) and (3.34) in $\mathcal{S}$ are identically (automatically) satisfied. This means that we have obtained a balanced system of 4 scalar equations (3.16) and (3.26) in terms of the 4 unknown fields $\Phi^{e\ell}$, $A_1$, $A_2$, $A_3$.

Thus, **by construction**, fields $\Phi^{e\ell}$ and $\boldsymbol{A}$ satisfy Eqs.(3.21) and (3.34) which can be written in terms of fields $\Phi^{e\ell}$ and $\boldsymbol{A}$ as follows:

Substituting Eq.(3.42a) into Eq.(3.21):

$$\epsilon_{ijk} A_{k,ji} = 0 \tag{3.43a}$$

Substituting Eqs.(3.42a,b) into (3.34):

$$\epsilon_{ijk}\left(-\Phi^{e\ell}_{,k} - \cancel{\dot{A}_k}\right)_{,j} = \cancel{-\epsilon_{ijk}\dot{A}_{k,j}} \Rightarrow$$

$$\epsilon_{ijk}\Phi^{e\ell}_{,kj} = 0 \tag{3.43b}$$

Eqs.(3.43a,b) are the requirements for fields $\Phi^{e\ell}$ and $\boldsymbol{A}$ **inside the volume of interest**. Eqs.(3.43a,b) have to be supplemented by continuity requirements for fields $\Phi^{e\ell}$ and $\boldsymbol{A}$ **on the interface surfaces,** which are obtained by substitution of Eqs.(3.42a,b) into homogeneous matching conditions (3.22), (3.35)

$$\epsilon_{ijk}\left(A^{(1)}_{k,j} - A^{(2)}_{k,j}\right)n_i = 0 \tag{3.44a}$$

$$\epsilon_{ijk} n_j\left(-\Phi^{e\ell(1)}_{,k} - \dot{A}^{(1)}_k + \Phi^{e\ell(2)}_{,k} + \dot{A}^{(2)}_k\right) = 0 \Rightarrow$$

$$-\epsilon_{ijk} n_j\left(\Phi^{e\ell(1)}_{,k} - \Phi^{e\ell(2)}_{,k}\right) - \epsilon_{ijk} n_j\left(\dot{A}^{(1)}_k - \dot{A}^{(2)}_k\right) = 0 \tag{3.44b}$$

Obviously, Eqs.(3.44a,b) can be obtained from

$$\Phi^{e\ell(1)} - \Phi^{e\ell(2)} = 0 \tag{3.45a}$$

and

$$\epsilon_{ijk} n_j\left(A^{(1)}_k - A^{(2)}_k\right) = 0 . \tag{3.45b}$$

Eqs. (3,45a,b) require that scalar E/M potential $\Phi^{e\ell}$ and the tangent component of vector E/M potential $\boldsymbol{A}$ are continuous on the interface surface between two media (1) and (2).
Thus, we state that field equations (3.43a,b) plus continuity requirements on the boundary (3.35a,b) are assumed to be satisfied by fields $\Phi^{e\ell}$ and $\boldsymbol{A}$ by construction.

Since we have defined E/M potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$ by the homogeneous Maxwell's equations and matching conditions, we move on expressing the two non-homogeneous equations Eqs.(3.16), (3.26) in terms of $\Phi^{e\ell}$ and $\boldsymbol{A}$, also implementing constitutive relations (3.5), (3.6):

*Gauss's law of electrostatics*

$$D_{i,i} = \hat{\rho}_e \overset{(3.5)}{\Rightarrow} \varepsilon_{ij} E_{j,i} = \hat{\rho}_e \overset{(3.42b)}{\Rightarrow}$$

$$\varepsilon_{ij}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right) = -\hat{\rho}_e \tag{3.46}$$

*Ampère's – Maxwell's law*

$$\epsilon_{ijk} H_{k,j} = \dot{D}_i + \hat{J}_i \underset{(3.6)}{\overset{(3.5)}{\Rightarrow}} \epsilon_{ijk}\mu^{-1}_{\ell k} B_{\ell,j} = \varepsilon_{ij}\dot{E}_j + \hat{J}_i \underset{(3.42b)}{\overset{(3.42a)}{\Rightarrow}}$$

$$\epsilon_{ijk}\epsilon_{\ell mn}\mu^{-1}_{\ell k} A_{n,mj} = -\varepsilon_{ij}\left(\dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j\right) + \hat{J}_i \tag{3.47}$$

Since the term $\epsilon_{ijk}\epsilon_{\ell mn}\mu^{-1}_{\ell k} A_{n,mj}$ is hard to be interpreted mathematically, Eq.(3.47) can be re-written using the relation for the product of Levi-Civita Symbols in terms of Kronecker's deltas:

$$\epsilon_{ijk}\epsilon_{\ell mn} = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & \delta_{in} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jn} \\ \delta_{k\ell} & \delta_{km} & \delta_{kn} \end{vmatrix} =$$

$$= \delta_{i\ell}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}\right) + \delta_{in}\left(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}\right) \tag{3.48}$$

Thus, we calculate the terms:

$$\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)\mu^{-1}_{\ell k} A_{n,mj} = \mu^{-1}_{\ell k} A_{k,jj} - \mu^{-1}_{\ell k} A_{j,kj} \Rightarrow$$

$$\delta_{i\ell}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)\mu^{-1}_{\ell k} A_{n,mj} = \delta_{i\ell}\left(\mu^{-1}_{\ell k} A_{k,jj} - \mu^{-1}_{\ell k} A_{j,kj}\right) = \mu^{-1}_{ik} A_{k,jj} - \mu^{-1}_{ik} A_{j,kj}$$

$$\left(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}\right)\mu^{-1}_{\ell k} A_{n,mj} = \mu^{-1}_{jk} A_{k,mj} - \mu^{-1}_{kk} A_{j,mj} \Rightarrow$$

$$\delta_{im}\left(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}\right)\mu^{-1}_{\ell k} A_{n,mj} = \delta_{im}\left(\mu^{-1}_{jk} A_{k,mj} - \mu^{-1}_{kk} A_{j,mj}\right) = \mu^{-1}_{jk} A_{k,ij} - \mu^{-1}_{kk} A_{j,ij}$$

$$\left(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}\right)\mu^{-1}_{\ell k} A_{n,mj} = \mu^{-1}_{jk} A_{n,kj} - \mu^{-1}_{kk} A_{n,jj} \Rightarrow$$

$$\delta_{in}\left(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}\right)\mu^{-1}_{\ell k} A_{n,mj} = \delta_{in}\left(\mu^{-1}_{jk} A_{n,kj} - \mu^{-1}_{kk} A_{n,jj}\right) = \mu^{-1}_{jk} A_{i,kj} - \mu^{-1}_{kk} A_{i,jj}$$

So, Eq.(3.47) can be written:

$$\mu_{ik}^{-1}\left(A_{k,jj} - A_{j,kj}\right) + \mu_{jk}^{-1}\left(A_{i,kj} - A_{k,ij}\right) + \mu_{kk}^{-1}\left(A_{j,ij} - A_{i,jj}\right) = -\varepsilon_{ij}\left(\dot{\Phi}_{,j}^{e\ell} + \ddot{A}_j\right) + \hat{J}_i \tag{3.49a}$$

In the special case of homogeneous diamagnetic media where magnetic permeability is a scalar quantity $\mu$ and not a tensor one $\mu_{ij}$ the complicated expression at the left side of Eq.(3.49) is simplified to

$$\mu^{-1}\left(A_{j,ij} - A_{i,jj}\right) = -\varepsilon_{ij}\left(\dot{\Phi}_{,j}^{e\ell} + \ddot{A}_j\right) + \hat{J}_i \tag{3.49b}$$

since Eq.(3.48) is simplified to $\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ when indices $k$ and $\ell$ coincide. Eq.(3.49b) is the form of equation discussed in most of the introductory lectures of electrodynamics, but in the present work we will proceed with the more general Eq.(3.49a).

We can also express the non-homogeneous matching conditions (3.20), (3.33) in terms of $\Phi^{e\ell}$ and $\boldsymbol{A}$, also implementing constitutive relations (3.5), (3.6):

*Jump of $\boldsymbol{D}$-component vertical to interface*

$$\begin{gathered}\left(D_i^{(1)} - D_i^{(2)}\right)n_i = \hat{\sigma}_e \overset{(3.5)}{\Rightarrow} \left(\varepsilon_{ij}^{(1)}E_j - \varepsilon_{ij}^{(2)}E_j\right)n_i = \hat{\sigma}_e \overset{(3.42b)}{\Rightarrow} \\ \left[\varepsilon_{ij}^{(1)}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right) - \varepsilon_{ij}^{(2)}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right)\right]n_i = -\hat{\sigma}_e\end{gathered} \tag{3.50}$$

*Jump of $\boldsymbol{H}$-component tangent to interface*

$$\begin{gathered}\epsilon_{ijk}n_j\left(H_k^{(1)} - H_k^{(2)}\right) = \hat{K}_i \overset{(3.6)}{\Rightarrow} \epsilon_{ijk}n_j\left[\left(\mu_{\ell k}^{-1}\right)^{(1)}B_\ell - \left(\mu_{\ell k}^{-1}\right)^{(2)}B_\ell\right] = \hat{K}_i \overset{(3.42a)}{\Rightarrow} \\ \epsilon_{ijk}\epsilon_{\ell mn}n_j\left(\mu_{\ell k}^{-1}\right)^{(1)}A_{n,m} - \epsilon_{ijk}\epsilon_{\ell mn}n_j\left(\mu_{\ell k}^{-1}\right)^{(2)}A_{n,m} = \hat{K}_i\end{gathered} \tag{3.51}$$

As Eq.(3.47), Eq.(3.51) can be re-written using the relation for the product of Levi-Civita Symbols in terms of Kronecker's deltas (3.48):

$$n_j\left\{\begin{bmatrix}\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,j} - A_{j,k}\right) - \\ -\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,j} - A_{j,k}\right)\end{bmatrix} + \begin{bmatrix}\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,k} - A_{k,i}\right) - \\ -\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,k} - A_{k,i}\right)\end{bmatrix} + \begin{bmatrix}\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,i} - A_{i,j}\right) - \\ -\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,i} - A_{i,j}\right)\end{bmatrix}\right\} = \hat{K}_i \tag{3.52a}$$

In the special case of homogeneous diamagnetic media characterized by a scalar magnetic permeability $\mu$, matching condition (3.52a) is simplified to

$$n_j \left[ \left(\mu^{-1}\right)^{(1)} \left(A_{j,i} - A_{i,j}\right) - \left(\mu^{-1}\right)^{(2)} \left(A_{j,i} - A_{i,j}\right) \right] = \hat{K}_i \, , \tag{3.52b}$$

following the same procedure as when deducting Eq.(3.49b) from Eq.(3.49a).

### 3.5 Variational Formulation of Electrodynamics: Hamilton's Principle

*Related References: Athanassoulis (2010), Panofsky & Phillips (1962) Ch. 24, Yourgrau & Mandelstam (1960).*

In the previous paragraph, we have finally reduced the system of Maxwell's equations to two, the scalar Eq.(3.46) and the vector Eq.(3.49a), along with two matching conditions, the scalar Eq.(3.50) and the vector Eq.(3.52a), will all equations been expressed in terms of the two E/M potentials, the scalar $\Phi^{e\ell}$ and the vector $\boldsymbol{A}$. In the present paragraph, we will formulate a Variational Principle for the final system of equations, *using as independent variables the two E/M potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$.*

Similarly to the way of work in the respective Sec. 2.2 for Elastodynamics, we will express the forms of energy appearing in the E/M problem throughout a domain $\Omega$

Electric Energy: $$U_{\text{electric}}\left(\Phi^{e\ell}, \boldsymbol{A}\right) = \frac{1}{2} \iiint\limits_{\Omega} D_i \, E_i \; dV \tag{3.53a}$$

Substituting electric displacement $\boldsymbol{D}$ from Eq.(3.5) (case of linear dielectric medium) into Eq.(3.53a) we obtain:

$$U_{\text{electric}}\left(\Phi^{e\ell}, \boldsymbol{A}\right) = \frac{1}{2} \iiint\limits_{\Omega} \varepsilon_{ij} \, E_j \, E_i \; dV \, . \tag{3.53b}$$

The components of electric intensity $\boldsymbol{E}$ appearing in Eq.(3.53b) can be expressed in terms of the E/M potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$ using Eq.(3.42b):

$$U_{\text{electric}}\left(\Phi^{e\ell}, \boldsymbol{A}\right) = \frac{1}{2} \iiint\limits_{\Omega} \varepsilon_{ij} \left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\left(\Phi^{e\ell}_{,i} + \dot{A}_i\right) dV \, . \tag{3.53c}$$

Magnetic Energy: $$U_{\text{magnetic}}\left(\boldsymbol{A}\right) = \frac{1}{2} \iiint\limits_{\Omega} H_i \, B_i \; dV \tag{3.54a}$$

Substituting magnetic intensity $\boldsymbol{H}$ from Eq.(3.6) (case of linear diamagnetic medium) into Eq.(3.54a) we obtain:

$$U_{\text{magnetic}}\left(\boldsymbol{A}\right) = \frac{1}{2}\iiint\limits_{\Omega} \mu_{ji}^{-1}\, B_j\, B_i\; dV\,. \tag{3.54b}$$

The components of magnetic induction $\boldsymbol{B}$ appearing in Eq.(3.54b) can be expressed in terms of the E/M vector potential $\boldsymbol{A}$ using Eq.(3.42b):

$$U_{\text{magnetic}}\left(\boldsymbol{A}\right) = \frac{1}{2}\iiint\limits_{\Omega} \mu_{ji}^{-1}\, \epsilon_{jmn}\, A_{n,m}\, \epsilon_{ik\ell}\, A_{\ell,k}\; dV\,. \tag{3.54c}$$

If we apply Eq.(3.48) for the product of Levi-Civita Symbols, we obtain, after some algebraic calculations:

$$U_{\text{magnetic}}\left(\boldsymbol{A}\right) = \frac{1}{2}\iiint\limits_{\Omega} \left(A_{\ell,k} - A_{k,\ell}\right)\left[\mu_{ii}^{-1}\, A_{\ell,k} + \mu_{ki}^{-1}\left(A_{i,\ell} - A_{\ell,i}\right)\right] dV \tag{3.54d}$$

<u>**Claim:**</u> In a time-varying E/M field, where both $\boldsymbol{E}$ and $\boldsymbol{B}$ are different from zero and vary in space and time in accordance to Maxwell's equations, the total E/M energy stored in the E/M field extended throughout a domain $\Omega$ is given as the sum of electric and magnetic energies defined above: $U_{\text{E/M}} = U_{\text{electric}} + U_{\text{magnetic}}$. This claim is neither obvious nor trivial.

Let us try to think of a variational formulation to electrodynamics by analogy to elastodynamics performed in Sec. 2.2:

| **Classical dynamics** | **Electrodynamics** |
|---|---|
| Kinetic Energy $U_{\text{kinetic}}$ | Electric Energy $U_{\text{electric}}$ |
| Elastic Energy $U_{\text{elastic}}$ | Magnetic Energy $U_{\text{magnetic}}$ |
| The sum is conserved | The sum is conserved |
| Lagrangian<br>$L_{CD} = U_{\text{kinetic}} - U_{\text{elastic}}$ | Lagrangian<br>$L_{ED} = U_{\text{electric}} - U_{\text{magnetic}}$ |
| Action functional<br>$\mathscr{L}_{CD} = \int_{t_0}^{t_1} L_{CD}\, dt$ | Action functional<br>$\mathscr{L}_{ED} = \int_{t_0}^{t_1} L_{ED}\, dt$ |

The claim that a Lagrangian function for electrodynamics might be taken as the difference of the electric minus the magnetic energy cannot be validated *a priori* by physical arguments. It is just a suggestive (heuristic) remark that can motivate a try to find an action functional appropriate to electrodynamics (which should be validated a posteriori, using the standard variational arguments).

Apart from heuristic way of defining a priori the action functional, another crucial point is the choice of the appropriate independent fields (the "degrees of freedom") that should be used as independent arguments in the electrodynamic action functional. In the beginning of the present paragraph we declared that we shall use E/M potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$ as independent variables of the variational principle and we have already calculated the energy quantities appearing in the problem in terms of these potentials. At this point, parenthetically, we will answer the obvious question why we will use E/M potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$ which have no physical meaning, since they are the result of a mathematical manipulation of fields $\boldsymbol{E}$ and $\boldsymbol{B}$, and not the $\boldsymbol{E}$ and $\boldsymbol{B}$ themselves.

Let us consider the action functional

$$\mathscr{L}[\boldsymbol{E},\boldsymbol{B}] = \frac{1}{2}\int_{t_0}^{t_1}\iiint\limits_{\Omega} \varepsilon_{ij}\, E_j\, E_i\; dV\,dt - \frac{1}{2}\int_{t_0}^{t_1}\iiint\limits_{\Omega} \mu_{ji}^{-1}\, B_j\, B_i\; dV\,dt$$

and let us calculate the Gâteaux derivate of $\mathscr{L}[\boldsymbol{E},\boldsymbol{B}]$:

$$\delta\mathscr{L}[\boldsymbol{E},\boldsymbol{B}\colon \delta\boldsymbol{E},\delta\boldsymbol{B}] = \delta_{\boldsymbol{E}}\mathscr{L}[\boldsymbol{E},\boldsymbol{B};\delta\boldsymbol{E}] + \delta_{\boldsymbol{B}}\mathscr{L}[\boldsymbol{E},\boldsymbol{B};\delta\boldsymbol{B}],$$

$$\delta_{\boldsymbol{E}}\mathscr{L}[\boldsymbol{E},\boldsymbol{B}\colon \delta\boldsymbol{E}] = \frac{1}{2}\frac{d}{d\xi}\int_{t_0}^{t_1}\iiint\limits_{\Omega} \varepsilon_{ij}\left(E_j+\xi\,\delta E_j\right)\left(E_i+\xi\,\delta E_i\right) dV\,dt\Bigg|_{\xi=0} =$$

$$\overset{\varepsilon_{ij}\ \text{symmetry}}{=} \int_{t_0}^{t_1}\iiint\limits_{\Omega} \varepsilon_{ij}\, E_j\,\delta E_i\; dV\,dt$$

$$\delta_{\boldsymbol{B}}\mathscr{L}[\boldsymbol{E},\boldsymbol{B}\colon \delta\boldsymbol{E}] = \frac{1}{2}\frac{d}{d\xi}\int_{t_0}^{t_1}\iiint\limits_{\Omega} \mu_{ji}^{-1}\left(B_j+\xi\,\delta B_j\right)\left(B_i+\xi\,\delta B_i\right) dV\,dt\Bigg|_{\xi=0} =$$

$$\overset{\mu_{ji}^{-1}\ \text{symmetry}}{=} \int_{t_0}^{t_1}\iiint\limits_{\Omega} \mu_{ji}^{-1}\, B_j\,\delta B_i\; dV\,dt$$

Thus, a variational equation of the form

$$\delta\mathscr{L}[\boldsymbol{E},\boldsymbol{B}\colon \delta\boldsymbol{E},\delta\boldsymbol{B}] = 0, \qquad \forall(\text{admissible})\,\delta\boldsymbol{E}, \qquad \forall(\text{admissible})\,\delta\boldsymbol{B},$$

will produce the uninteresting (and incorrect) results $\boldsymbol{E} = \boldsymbol{B} = 0$, a phenomenon called *variational crisis*. This could be expected since the six scalar fields $E_1$, $E_2$, $E_3$, $B_1$, $B_2$, $B_3$ are not independent from each other. Thus, we return to the process of constructing a variational principle using the four fields $\Phi^{e\ell}$, $A_1$, $A_2$, $A_3$ as independent variables.

At this point, just before formulating the variational principle, we have to define the "inner structure" of volume of reference $\Omega$, as we have already mentioned that it is composed by not only one dielectric and diamagnetic medium and we have already stated that E/M fields cannot be restricted to a finite volume.

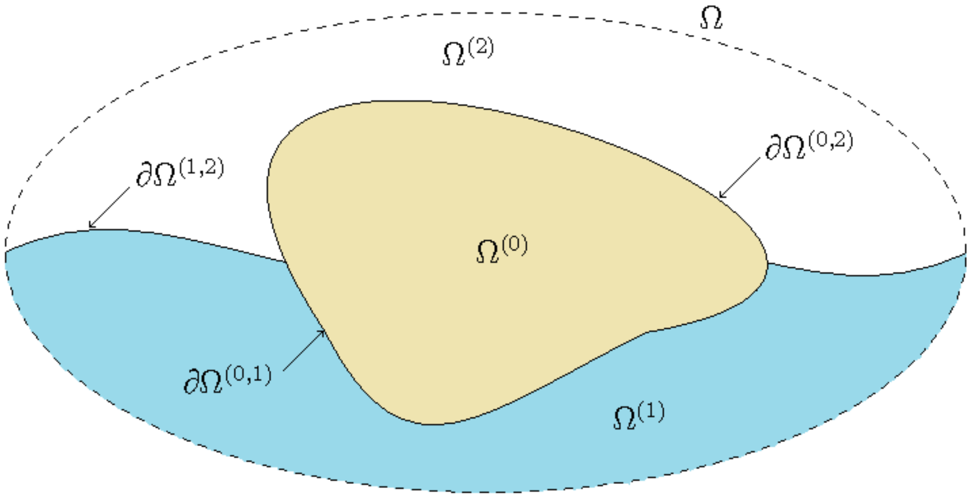


Fig. 3.1: Configuration of the three volumes considered

So, let us consider the volume $\Omega$ depicted in Fig. 3.1. It consists of three dielectric media, occupying volumes $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively. As we can see in Fig. 3.1, volume $\Omega^{(0)}$ is finite, and surrounded by ambient volumes $\Omega^{(1)}$ and $\Omega^{(2)}$ which are semi-infinitive. Thus, the interfaces between different media (denoted collectively as $\partial\Omega$), where matching conditions for E/M fields have to be expressed are $\partial\Omega^{(0,1)}$, $\partial\Omega^{(0,2)}$ and $\partial\Omega^{(1,2)}$ since there is no need for conditions on the remaining boundaries at infinity of $\Omega^{(1)}$ and $\Omega^{(2)}$ (denoted with dashed lines in Fig. 3.1).

We shall note that the action functional for the variational principle will be expressed with regard to the whole volume $\Omega$ and interfaces $\partial\Omega$. The partition of volumes and interfaces will be performed later, when calculating the partial Gâteaux derivatives

Hamilton's Principle for Linear Elastodynamics is expressed as:

$$\delta\mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A};\, \lambda,\, \boldsymbol{v} : \delta\Phi^{e\ell},\, \delta\boldsymbol{A},\, \delta\lambda,\, \delta v\right] = 0 \tag{3.55}$$

where $\mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A};\, \lambda,\, \boldsymbol{v}\right]$ is the functional of Eq.(3.56) and $\delta$ denotes the total Gâteaux functional derivative.

$$\begin{aligned}
\mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A};\, \lambda,\, \boldsymbol{v}\right] &= L\left[\Phi^{e\ell}, \boldsymbol{A}\right] - S_{\Omega}^{\substack{\text{free}\\\text{charge}}}\left[\Phi^{e\ell}\right] + S_{\Omega}^{\substack{\text{free}\\\text{current}}}\left[\boldsymbol{A}\right] + \\
&\quad + I_{\partial\Omega_\Phi}^{\text{given }\Phi^{e\ell}}\left[\Phi^{e\ell};\, \lambda\right] - I_{\partial\Omega_\sigma}^{\text{given }\sigma_e}\left[\Phi^{e\ell}\right] + I_{\partial\Omega_A}^{\text{given }A_t}\left[\boldsymbol{A};\, \boldsymbol{v}\right] + I_{\partial\Omega_K}^{\text{given }K}\left[\boldsymbol{A}\right] = \\
&= \int_{t_0}^{t_1}\left[U_{\text{electric}}\left(\Phi^{e\ell},\, \boldsymbol{A}\right) - U_{\text{magnetic}}\left(\boldsymbol{A}\right)\right] dt - \\
&\quad - \int_{t_0}^{t_1}\iiint_{\Omega}\hat{\rho}_e\,\Phi^{e\ell}\, dV\, dt + \int_{t_0}^{t_1}\iiint_{\Omega}\hat{J}_i\, A_i\, dV\, dt + \\
&\quad + \int_{t_0}^{t_1}\iint_{\partial\Omega_\Phi}\left(\Phi^{e\ell} - \hat{\Phi}^{e\ell}\right)\lambda\, dS\, dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_\sigma}\hat{\sigma}_e\,\Phi^{e\ell}\, dS\, dt - \\
&\quad - \int_{t_0}^{t_1}\iint_{\partial\Omega_A}\epsilon_{ijk}\, n_j\left(A_k - \hat{A}_k\right)\nu_i\, dS\, dt - \int_{t_0}^{t_1}\iint_{\partial\Omega_K}\hat{K}_i\, A_i\, dS\, dt
\end{aligned} \tag{3.56}$$

where $L\left[\Phi^{e\ell}, \boldsymbol{A}\right]$ is the action functional of the Lagrangian for electrodynamics previously defined as electric energy minus magnetic energy, $S_{\Omega}^{\substack{\text{free}\\\text{charge}}}\left[\Phi^{e\ell}\right]$ and $S_{\Omega}^{\substack{\text{free}\\\text{current}}}\left[\boldsymbol{A}\right]$ correspond to the sources of the given free charges $\hat{\rho}_e$ and currents $\hat{J}_i$ respectively which exist inside volume $\Omega$, $I_{\partial\Omega_\Phi}^{\text{given }\Phi^{e\ell}}\left[\Phi^{e\ell};\, \lambda\right]$ and $I_{\partial\Omega_A}^{\text{given }A_t}\left[\boldsymbol{A};\, \boldsymbol{v}\right]$ correspond to the boundary conditions on boundaries $\partial\Omega_\Phi$ and $\partial\Omega_A$ where scalar potential $\Phi^{e\ell}$ and tangent component of vector potential $\boldsymbol{A}$ are prescribed respectively, and $I_{\partial\Omega_\sigma}^{\text{given }\sigma_e}\left[\Phi^{e\ell}\right]$ and $I_{\partial\Omega_K}^{\text{given }K}\left[\boldsymbol{A}\right]$ correspond to the boundary conditions on boundaries $\partial\Omega_\sigma$ and $\partial\Omega_K$ where free surface charges $\hat{\sigma}_e$ and free surface currents $\hat{K}_i$ are prescribed respectively.

Auxiliary independent fields $\lambda$ and $\boldsymbol{v}$ appearing in terms $I_{\partial\Omega_\Phi}^{\text{given }\Phi^{e\ell}}\left[\Phi^{e\ell};\, \lambda\right]$ and $I_{\partial\Omega_A}^{\text{given }A_t}\left[\boldsymbol{A};\, \boldsymbol{v}\right]$ are Lagrange multipliers similar to auxiliary field $\lambda$ in action functional (2.10) for elastodynamics and their physical meaning will be identified after the variation (3.55).

To validate the above Hamilton's Principle, we perform the variation of Eq.(3.55) and re-obtain the governing equation of linear elastodynamics. For this, the partial Gâteaux derivatives for each part of the functional (3.56) follow.

**Discussion concerning the boundaries**

Since we will perform spatial integrations by parts for the calculation of the aforementined partial Gâteaux derivatives, we shall discuss the partition of the whole boundary $\partial\Omega$ in order to be able to identify the terms defined over the same boundary surface. Let us start with the boundaries of each constituent volume, namely $\partial\Omega^{(0)}$, $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ which will appear first is a spatial integration by parts. These boundaries can be written with regard of the interfaces $\partial\Omega^{(0,1)}$, $\partial\Omega^{(0,2)}$, $\partial\Omega^{(1,2)}$ in the following obvious way

$$\begin{aligned} \partial\Omega^{(0)} &= \partial\Omega^{(0,1)} \cup \partial\Omega^{(0,2)} \\ \partial\Omega^{(1)} &= \partial\Omega^{(0,1)} \cup \partial\Omega^{(1,2)} \\ \partial\Omega^{(2)} &= \partial\Omega^{(0,2)} \cup \partial\Omega^{(1,2)} \end{aligned} \tag{3.57a}$$

Let us now consider another three obvious partitions for interface $\partial\Omega$

$$\partial\Omega = \underbrace{\partial\Omega^{(0,1)} \cup \partial\Omega^{(0,2)} \cup \partial\Omega^{(1,2)}}_{\text{partition with regard to the neighbouring volumes}} = \underbrace{\partial\Omega_{\sigma} \cup \partial\Omega_{\Phi}}_{\substack{\text{partion for matching} \\ \text{conditions of } \Phi^{e\ell}}} = \underbrace{\partial\Omega_{K} \cup \partial\Omega_{\boldsymbol{A}}}_{\substack{\text{partion for matching} \\ \text{conditions of } \boldsymbol{A}}} \tag{3.57b}$$

From Eq.(3.57), we realize that we can part each of interfaces $\partial\Omega^{(0,1)}$, $\partial\Omega^{(0,2)}$, $\partial\Omega^{(1,2)}$ regarding the matching conditions for $\Phi^{e\ell}$ and $\boldsymbol{A}$, e.g.

$$\partial\Omega^{(0,1)} = \partial\Omega^{(0,1)}_{\sigma} \cup \partial\Omega^{(0,1)}_{\Phi} = \partial\Omega^{(0,1)}_{K} \cup \partial\Omega^{(0,1)}_{\boldsymbol{A}}. \tag{3.57c}$$

The last comment on interfaces, is that, apart from the notation of the surface of interface, we shall also use the outward normal unit vectors of the constituent volumes on the surfaces with a superscript denoting the respective volume e.g. $n_j^{(0)}$ for constituent volume $\Omega^{(0)}$. Since the outward unit vectors of two volumes are opposite on the interface between the two volumes, we shall use the following convention:

*We change the appearing outward unit vector so that its superscript to coincide with the first number of the superscript pair of the respective interface, e.g.*

$$n_j^{(1)} \text{ on } \partial\Omega^{(0,1)} \xrightarrow{\text{rewritten as}} -n_j^{(0)}. \tag{3.57d}$$

Eqs.(3.57a-d) offer a way of dealing with the boundaries to-be-appeared; For the first Gâteaux derivative calculation, all the steps will be presented in detail, while for the rest calculations, where will be only a reference to the (3.67a-c) set of relations.

Electric energy part of the Lagrangian

$$L_{\text{electric}}\left[\Phi^{e\ell},\boldsymbol{A}\right] = \int_{t_0}^{t_1} U_{\text{electric}}\left(\Phi^{e\ell},\,\boldsymbol{A}\right) dt \;=$$

$$= \frac{1}{2}\int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\left(\Phi^{e\ell}_{,i} + \dot{A}_i\right) dV\; dt$$

$$\delta_{\Phi^{e\ell}}\, L_{\text{electric}}\left[\Phi^{e\ell},\boldsymbol{A}\!:\delta\,\Phi^{e\ell}\right] =$$

$$= \frac{1}{2}\frac{d}{d\xi}\int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij}\left(\Phi^{e\ell}_{,j} + \xi\,\delta\,\Phi^{e\ell}_{,j} + \dot{A}_j\right)\left(\Phi^{e\ell}_{,i} + \xi\,\delta\,\Phi^{e\ell}_{,i} + \dot{A}_i\right) dV\; dt\Bigg|_{\xi=0} =$$

$$= \frac{1}{2}\int_{t_0}^{t_1} \iiint\limits_{\Omega} \left[\varepsilon_{ij}\left(\Phi^{e\ell}_{,i} + \dot{A}_i\right)\delta\,\Phi^{e\ell}_{,j} + \varepsilon_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\delta\,\Phi^{e\ell}_{,i}\right] dV\; dt \tag{3.58}$$

interchanging indices $i-j$ in the second term of the integrand of Eq.(3.58) and using the symmetry $\varepsilon_{ij} = \varepsilon_{ji}$ of dielectric permittivity property tensor we obtain

$$\delta_{\Phi^{e\ell}}\, L_{\text{electric}}\left[\Phi^{e\ell},\boldsymbol{A}\!:\delta\,\Phi^{e\ell}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\delta\,\Phi^{e\ell}_{,i}\; dV\; dt \;=$$

$$= \left[\text{partition of volume } \Omega\right] =$$

$$= \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \varepsilon^{(0)}_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\delta\,\Phi^{e\ell}_{,i}\; dV\; dt \;+$$

$$+ \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(1)}} \varepsilon^{(1)}_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\delta\,\Phi^{e\ell}_{,i}\; dV\; dt \;+$$

$$+ \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(2)}} \varepsilon^{(2)}_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\delta\,\Phi^{e\ell}_{,i}\; dV\; dt$$

$$= \left[\text{spatial integrations by parts}\right] =$$

$$= \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}} \varepsilon^{(0)}_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right) n^{(0)}_i\, \delta\,\Phi^{e\ell}\; dS\; dt - \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \varepsilon^{(0)}_{ij}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right)\delta\,\Phi^{e\ell}\; dV\; dt +$$

$$+ \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(1)}} \varepsilon^{(1)}_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right) n^{(1)}_i\, \delta\,\Phi^{e\ell}\; dS\; dt - \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(1)}} \varepsilon^{(1)}_{ij}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right)\delta\,\Phi^{e\ell}\; dV\; dt +$$

$$+\int_{t_0}^{t_1}\iint_{\partial\Omega^{(2)}} \varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(2)}\,\delta\Phi^{e\ell}\,dS\,dt-\int_{t_0}^{t_1}\iiint_{\Omega^{(2)}} \varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,ji}+\dot{A}_{j,i}\right)\delta\Phi^{e\ell}\,dV\,dt \tag{3.59}$$

Rearranging surface integrals on the boundaries according to Eq.(3.57a) we obtain

$$\begin{aligned}
&=\int_{t_0}^{t_1}\iint_{\partial\Omega^{(0,1)}}\left[\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(0)}+\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(1)}\right]\delta\Phi^{e\ell}\,dS\,dt+\\
&+\int_{t_0}^{t_1}\iint_{\partial\Omega^{(0,2)}}\left[\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(0)}+\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(2)}\right]\delta\Phi^{e\ell}\,dS\,dt+\\
&+\int_{t_0}^{t_1}\iint_{\partial\Omega^{(1,2)}}\left[\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(1)}+\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)n_i^{(2)}\right]\delta\Phi^{e\ell}\,dS\,dt-\\
&-\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}}\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,ji}+\dot{A}_{j,i}\right)\delta\Phi^{e\ell}\,dV\,dt-\\
&-\int_{t_0}^{t_1}\iiint_{\Omega^{(1)}}\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,ji}+\dot{A}_{j,i}\right)\delta\Phi^{e\ell}\,dV\,dt-\\
&-\int_{t_0}^{t_1}\iiint_{\Omega^{(2)}}\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,ji}+\dot{A}_{j,i}\right)\delta\Phi^{e\ell}\,dV\,dt
\end{aligned} \tag{3.60}$$

and using Eq.(3.57d) for outward normal unit vectors we finally obtain

$$\begin{aligned}
&\delta_{\Phi^{e\ell}}L_{\text{electric}}\left[\Phi^{e\ell},\boldsymbol{A}:\delta\Phi^{e\ell}\right]=\\
&=\int_{t_0}^{t_1}\iint_{\partial\Omega^{(0,1)}}n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)-\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)\right]\delta\Phi^{e\ell}\,dS\,dt+\\
&+\int_{t_0}^{t_1}\iint_{\partial\Omega^{(0,2)}}n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)\right]\delta\Phi^{e\ell}\,dS\,dt+\\
&+\int_{t_0}^{t_1}\iint_{\partial\Omega^{(1,2)}}n_i^{(1)}\left[\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,j}+\dot{A}_j\right)\right]\delta\Phi^{e\ell}\,dS\,dt-\\
&-\int_{t_0}^{t_1}\iiint_{\Omega^{(0)}}\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,ji}+\dot{A}_{j,i}\right)\delta\Phi^{e\ell}\,dV\,dt-
\end{aligned}$$

$$- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(1)}} \varepsilon_{ij}^{(1)} \left( \Phi^{e\ell}_{,ji} + \dot{A}_{j,i} \right) \delta \Phi^{e\ell} \; dV \; dt -$$

$$- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(2)}} \varepsilon_{ij}^{(2)} \left( \Phi^{e\ell}_{,ji} + \dot{A}_{j,i} \right) \delta \Phi^{e\ell} \; dV \; dt \tag{3.60}$$

$$\delta_A L_{\text{electric}} \left[ \Phi^{e\ell}, \boldsymbol{A} \colon \delta \boldsymbol{A} \right] =$$

$$= \frac{1}{2} \frac{d}{d\xi} \int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij} \left( \Phi^{e\ell}_{,j} + \dot{A}_j + \xi \delta \dot{A}_j \right) \left( \Phi^{e\ell}_{,i} + \dot{A}_i + \xi \delta \dot{A}_i \right) dV \; dt \Bigg|_{\xi = 0} =$$

$$= \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega} \left[ \varepsilon_{ij} \left( \Phi^{e\ell}_{,i} + \dot{A}_i \right) \delta \dot{A}_j + \varepsilon_{ij} \left( \Phi^{e\ell}_{,j} + \dot{A}_j \right) \delta \dot{A}_i \right] dV \; dt \tag{3.61}$$

interchanging indices $i - j$ in the second term of the integrand of Eq.(3.61) and using the symmetry $\varepsilon_{ij} = \varepsilon_{ji}$ of dielectric permittivity property tensor we obtain

$$\delta_A L_{\text{electric}} \left[ \Phi^{e\ell}, \boldsymbol{A} \colon \delta \boldsymbol{A} \right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij} \left( \Phi^{e\ell}_{,j} + \dot{A}_j \right) \delta \dot{A}_i \, dV \; dt =$$

$$= \left[ \text{temporal integration by parts} \right] =$$

$$= \underbrace{\cancel{\iiint\limits_{\Omega} \varepsilon_{ij} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) \delta A_i \, dV \Bigg|_{t_0}^{t_1}}}_{=0} - \int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) \delta A_i \, dV \; dt =$$

$$= \left[ \text{partition of volume } \Omega \right] =$$

$$= - \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \varepsilon_{ij}^{(0)} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) \delta A_i \, dV \; dt -$$

$$- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(1)}} \varepsilon_{ij}^{(1)} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) \delta A_i \, dV \; dt -$$

$$- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(2)}} \varepsilon_{ij}^{(2)} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) \delta A_i \, dV \; dt \tag{3.62}$$

Magnetic energy part of the Lagrangian

$$L_{\text{magnetic}}[\boldsymbol{A}] = \int_{t_0}^{t_1} U_{\text{magnetic}}(\boldsymbol{A})\, dt =$$

$$= \frac{1}{2}\int_{t_0}^{t_1} \iiint_{\Omega} \left(A_{\ell,k} - A_{k,\ell}\right)\left[\mu_{ii}^{-1} A_{\ell,k} + \mu_{ki}^{-1}\left(A_{i,\ell} - A_{\ell,i}\right)\right] dV\, dt$$

Calculating the Gâteaux derivative and after some algebraic manipulations and the use of symmetry $\mu_{ij}^{-1} = \mu_{ji}^{-1}$ of the magnetic permeability property tensor we obtain

$$\delta_{\boldsymbol{A}} L_{\text{magnetic}}[\boldsymbol{A}:\delta\boldsymbol{A}] =$$

$$= \int_{t_0}^{t_1} \iiint_{\Omega} \left[\mu_{kk}^{-1}\left(A_{i,j} - A_{j,i}\right) + \mu_{jk}^{-1}\left(A_{k,i} - A_{i,k}\right) + \mu_{ik}^{-1}\left(A_{j,k} - A_{k,j}\right)\right]\delta A_{i,j}\, dV\, dt$$

and by performing a spatial integration by parts and rearranging surface integrals according to Eqs.(3.57a-d) we obtain:

$$\delta_{\boldsymbol{A}} L_{\text{magnetic}}[\boldsymbol{A}:\delta\boldsymbol{A}] =$$

$$= \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,1)}} n_j^{(0)} \left\{\begin{aligned} &\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{i,j} - A_{j,i}\right) - \left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{i,j} - A_{j,i}\right)\right] + \\ &+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{k,i} - A_{i,k}\right) - \left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{k,i} - A_{i,k}\right)\right] + \\ &+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{j,k} - A_{k,j}\right) - \left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{j,k} - A_{k,j}\right)\right] \end{aligned}\right\} \delta A_i\, dS\, dt\ +$$

$$+ \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,2)}} n_j^{(0)} \left\{\begin{aligned} &\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{i,j} - A_{j,i}\right) - \left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{i,j} - A_{j,i}\right)\right] + \\ &+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{k,i} - A_{i,k}\right) - \left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{k,i} - A_{i,k}\right)\right] + \\ &+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{j,k} - A_{k,j}\right) - \left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{j,k} - A_{k,j}\right)\right] \end{aligned}\right\} \delta A_i\, dS\, dt\ +$$

$$+ \int_{t_0}^{t_1} \iint_{\partial\Omega^{(1,2)}} n_j^{(1)} \left\{\begin{aligned} &\left[\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{i,j} - A_{j,i}\right) - \left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{i,j} - A_{j,i}\right)\right] + \\ &+\left[\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{k,i} - A_{i,k}\right) - \left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{k,i} - A_{i,k}\right)\right] + \\ &+\left[\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{j,k} - A_{k,j}\right) - \left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{j,k} - A_{k,j}\right)\right] \end{aligned}\right\} \delta A_i\, dS\, dt\ -$$

$$
-\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(0)}}\left[\begin{array}{r}\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{i,jj}-A_{j,ij}\right)+\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{k,ij}-A_{i,kj}\right)+\\+\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{j,kj}-A_{k,jj}\right)\end{array}\right]\delta A_i\ dV\ dt-
$$

$$
-\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(1)}}\left[\begin{array}{r}\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{i,jj}-A_{j,ij}\right)+\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{k,ij}-A_{i,kj}\right)+\\+\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{j,kj}-A_{k,jj}\right)\end{array}\right]\delta A_i\ dV\ dt-
$$

$$
-\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(2)}}\left[\begin{array}{r}\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{i,jj}-A_{j,ij}\right)+\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{k,ij}-A_{i,kj}\right)+\\+\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{j,kj}-A_{k,jj}\right)\end{array}\right]\delta A_i\ dV\ dt \tag{3.63}
$$

<u>Source of free charges</u>

$$
\delta_{\Phi^{e\ell}}\overset{\text{free}}{S}{}_{\Omega}^{\text{charge}}\left[\Phi^{e\ell}:\delta\Phi^{e\ell}\right]=\int_{t_0}^{t_1}\iiint\limits_{\Omega}\hat{\rho}_e\,\delta\Phi^{e\ell}\ dV\,dt=\left[\text{partition of volumes}\right]=
$$

$$
=\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(0)}}\hat{\rho}_e^{(0)}\,\delta\Phi^{e\ell}\ dV\,dt+
$$

$$
+\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(1)}}\hat{\rho}_e^{(1)}\,\delta\Phi^{e\ell}\ dV\,dt+\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(2)}}\hat{\rho}_e^{(2)}\,\delta\Phi^{e\ell}\ dV\,dt \tag{3.64}
$$

<u>Source of free currents</u>

$$
\delta_{A}\overset{\text{free}}{S}{}_{\Omega}^{\text{current}}\left[\boldsymbol{A}:\delta\boldsymbol{A}\right]=\int_{t_0}^{t_1}\iiint\limits_{\Omega}\hat{J}_i\,\delta A_i\ dV\,dt=\left[\text{partition of volumes}\right]=
$$

$$
=\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(0)}}\hat{J}_i^{(0)}\,\delta A_i\ dV\,dt+
$$

$$
+\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(1)}}\hat{J}_i^{(1)}\,\delta A_i\ dV\,dt+\int_{t_0}^{t_1}\iiint\limits_{\Omega^{(2)}}\hat{J}_i^{(2)}\,\delta A_i\ dV\,dt \tag{3.65}
$$

<u>Term on $\partial\Omega_\Phi$ where scalar potential $\Phi^{e\ell}$ is prescribed</u>

$$
\delta_\lambda I_{\partial\Omega_\Phi}^{\text{given}\,\Phi^{e\ell}}\left[\Phi^{e\ell};\lambda\ \delta\lambda\right]=\int_{t_0}^{t_1}\iint\limits_{\partial\Omega_\Phi}\left(\Phi^{e\ell}-\hat{\Phi}^{e\ell}\right)\delta\lambda\ dS\,dt=\left[\text{partition of interfaces}\right]=
$$

$$
=\int_{t_0}^{t_1}\iint\limits_{\partial\Omega_\Phi^{(0,1)}}\left(\Phi^{e\ell}-\hat{\Phi}^{e\ell}\right)\delta\lambda\ dS\,dt+
$$

$$
+\int_{t_0}^{t_1}\iint\limits_{\partial\Omega_\Phi^{(0,1)}}\left(\Phi^{e\ell}-\hat{\Phi}^{e\ell}\right)\delta\lambda\ dS\,dt+\int_{t_0}^{t_1}\iint\limits_{\partial\Omega_\Phi^{(1,2)}}\left(\Phi^{e\ell}-\hat{\Phi}^{e\ell}\right)\delta\lambda\ dS\,dt \tag{3.66}
$$

$$\delta_{\Phi^{e\ell}} I_{\partial\Omega_\Phi}^{\text{given }\Phi^{e\ell}}\left[\Phi^{e\ell};\lambda\ \delta\Phi^{e\ell}\right] = \int_{t_0}^{t_1}\iint_{\partial\Omega_\Phi}\lambda\,\delta\Phi^{e\ell}\ dS\,dt = [\text{partition of interfaces}] =$$

$$= \int_{t_0}^{t_1}\iint_{\partial\Omega_\Phi^{(0,1)}}\lambda\,\delta\Phi^{e\ell}\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_\Phi^{(0,2)}}\lambda\,\delta\Phi^{e\ell}\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_\Phi^{(1,2)}}\lambda\,\delta\Phi^{e\ell}\ dS\,dt \qquad (3.67)$$

<u>Term on $\partial\Omega_\sigma$ where surface charge $\hat{\sigma}_e$ is prescribed</u>

$$\delta_{\Phi^{e\ell}} I_{\partial\Omega_\sigma}^{\text{given }\sigma_e}\left[\Phi^{e\ell}:\delta\Phi^{e\ell}\right] = \int_{t_0}^{t_1}\iint_{\partial\Omega_\sigma}\hat{\sigma}_e\,\delta\Phi^{e\ell}\ dS\,dt = [\text{partition of interfaces}] =$$

$$= \int_{t_0}^{t_1}\iint_{\partial\Omega_\sigma^{(0,1)}}\hat{\sigma}_e^{(0,1)}\,\delta\Phi^{e\ell}\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_\sigma^{(0,2)}}\hat{\sigma}_e^{(0,2)}\,\delta\Phi^{e\ell}\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_\sigma^{(1,2)}}\hat{\sigma}_e^{(1,2)}\,\delta\Phi^{e\ell}\ dS\,dt$$

$$(3.68)$$

<u>Term on $\partial\Omega_A$ where tangent component of vector potential $\boldsymbol{A}$ is prescribed</u>

$$\delta_\nu I_{\partial\Omega_A}^{\text{given }A_t}[\boldsymbol{A};\boldsymbol{\nu}:\delta\boldsymbol{\nu}] = \int_{t_0}^{t_1}\iint_{\partial\Omega_A}\epsilon_{ijk}\,n_j\left(A_k-\hat{A}_k\right)\delta\nu_i\ dS\,dt = [\text{partition of interfaces}] =$$

$$= \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(0,1)}}\epsilon_{ijk}\,n_j^{(0)}\left(A_k-\hat{A}_k\right)\delta\nu_i\ dS\,dt +$$

$$+ \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(0,2)}}\epsilon_{ijk}\,n_j^{(0)}\left(A_k-\hat{A}_k\right)\delta\nu_i\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(1,2)}}\epsilon_{ijk}\,n_j^{(1)}\left(A_k-\hat{A}_k\right)\delta\nu_i\ dS\,dt \qquad (3.69)$$

$$\delta_{\boldsymbol{A}} I_{\partial\Omega_A}^{\text{given }A_t}[\boldsymbol{A};\boldsymbol{\nu}:\delta\boldsymbol{A}] = \int_{t_0}^{t_1}\iint_{\partial\Omega_A}\epsilon_{kji}\,n_j\,\nu_k\,\delta A_i\ dS\,dt = [\text{partition of interfaces}] =$$

$$= \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(0,1)}}\epsilon_{kji}\,n_j^{(0)}\,\nu_k\,\delta A_i\ dS\,dt +$$

$$+ \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(0,1)}}\epsilon_{kji}\,n_j^{(0)}\,\nu_k\,\delta A_i\ dS\,dt + \int_{t_0}^{t_1}\iint_{\partial\Omega_A^{(1,2)}}\epsilon_{kji}\,n_j^{(1)}\,\nu_k\,\delta A_i\ dS\,dt \qquad (3.70)$$

<u>Term on $\partial \Omega_K$ where surface charge $\hat{\boldsymbol{K}}$ is prescribed</u>

$$\delta_A I^{\text{given}\, K}_{\partial\Omega_K}[\boldsymbol{A}: \delta \boldsymbol{A}] = \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_K} \hat{K}_i\, \delta A_i \; dS\, dt = [\text{partition of interfaces}] =$$

$$= \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_K^{(0,1)}} \hat{K}_i^{(0,1)}\, \delta A_i \; dS\, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_K^{(0,2)}} \hat{K}_i^{(0,2)}\, \delta A_i \; dS\, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_K^{(1,2)}} \hat{K}_i^{(1,2)}\, \delta A_i \; dS\, dt \tag{3.71}$$

As it was stated in the beginning of the present paragraph, as well as it has been shown in the calculation of the Gâteaux derivatives above, the independent fields with regard to which Gâteaux derivatives are calculated, are the E/M potential fields $\Phi^{e\ell}$, $\boldsymbol{A}$ and auxiliary independent fields $\lambda$ and $\nu$, whose physical meaning remains to be determined. Thus, variational equation (2.9) is written as

$$\begin{aligned} &\delta \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\Phi^{e\ell}, \delta\boldsymbol{A}, \delta\lambda, \delta\nu\right] = 0 \;\Rightarrow \\ &\delta_{\Phi^{e\ell}} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\Phi^{e\ell}\right] + \delta_{A} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\boldsymbol{A}\right] + \\ &\quad + \delta_{\lambda} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\lambda\right] + \delta_{\nu} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\nu\right] = 0 \end{aligned} \tag{3.72}$$

Since variations $\delta\Phi^{e\ell}$, $\delta\boldsymbol{A}$, $\delta\lambda$ and $\delta\nu$ are considered independent from one another, Eq.(2.18) is equivalent to

$$\delta_{\Phi^{e\ell}} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\Phi^{e\ell}\right] = 0, \quad \delta_{A} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\boldsymbol{A}\right] = 0 \tag{3.73a,b}$$

and

$$\delta_{\lambda} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\lambda\right] = 0, \quad \delta_{\nu} \mathscr{H}\left[\Phi^{e\ell}, \boldsymbol{A}; \lambda, \nu : \delta\nu\right] = 0 \tag{3.73c,d}$$

From Eq.(3.73a), the following Euler-Lagrange equations are obtained

- $$\varepsilon_{ij}^{(0)}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right) = -\hat{\rho}_e^{(0)} \quad \text{over } \Omega^{(0)} \tag{3.74a}$$

  $$\varepsilon_{ij}^{(1)}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right) = -\hat{\rho}_e^{(1)} \quad \text{over } \Omega^{(1)} \tag{3.74b}$$

  $$\varepsilon_{ij}^{(2)}\left(\Phi^{e\ell}_{,ji} + \dot{A}_{j,i}\right) = -\hat{\rho}_e^{(2)} \quad \text{over } \Omega^{(2)} \tag{3.74c}$$

  which are Eq.(3.46) (Gauss's Law for electrostatics in case of linear dielectric media) for each of the constituent volumes.

- $$n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(1)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\hat{\sigma}_e^{(0,1)} \text{ on } \partial\Omega_\sigma^{(0,1)} \tag{3.75a}$$
  $$n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\hat{\sigma}_e^{(0,2)} \text{ on } \partial\Omega_\sigma^{(0,2)} \tag{3.75b}$$
  $$n_i^{(1)}\left[\varepsilon_{ij}^{(1)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\hat{\sigma}_e^{(1,2)} \text{ on } \partial\Omega_\sigma^{(1,2)} \tag{3.75c}$$

  which are the matching condition (3.50) (Jump of $\boldsymbol{D}$-component vertical to interface) on each of the interfaces where surface charge density $\hat{\sigma}_e$ is prescribed.

- $$n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(1)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\lambda \quad \text{on } \partial\Omega_\Phi^{(0,1)} \tag{3.76a}$$
  $$n_i^{(0)}\left[\varepsilon_{ij}^{(0)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\lambda \quad \text{on } \partial\Omega_\Phi^{(0,2)} \tag{3.76b}$$
  $$n_i^{(1)}\left[\varepsilon_{ij}^{(1)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)-\varepsilon_{ij}^{(2)}\left(\Phi_{,j}^{e\ell}+\dot{A}_j\right)\right] = -\lambda \quad \text{on } \partial\Omega_\Phi^{(1,2)} \tag{3.76c}$$

  Eqs.(3.76) are the matching condition (3.50) on the interfaces where scalar potential $\Phi^{e\ell}$ is prescribed. This is not a needed matching condition, but defines the auxiliary field $\lambda$ as the free charge over the boundary $\partial\Omega_\Phi$ where the scalar potential $\Phi^{e\ell}$ is prescribed.

From Eq.(3.73b), the following Euler-Lagrange equations are obtained

- $$\begin{aligned}&\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{j,ij}-A_{i,jj}\right)+\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{i,kj}-A_{k,ij}\right)+\\&+\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{k,jj}-A_{j,kj}\right) = \hat{J}_i^{(0)}-\varepsilon_{ij}^{(0)}\left(\dot{\Phi}_{,j}^{e\ell}+\ddot{A}_j\right) \qquad \text{over } \Omega^{(0)}\end{aligned} \tag{3.77a}$$
  $$\begin{aligned}&\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,ij}-A_{i,jj}\right)+\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,kj}-A_{k,ij}\right)+\\&+\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,jj}-A_{j,kj}\right) = \hat{J}_i^{(1)}-\varepsilon_{ij}^{(1)}\left(\dot{\Phi}_{,j}^{e\ell}+\ddot{A}_j\right) \qquad \text{over } \Omega^{(1)}\end{aligned} \tag{3.77b}$$
  $$\begin{aligned}&\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,ij}-A_{i,jj}\right)+\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,kj}-A_{k,ij}\right)+\\&+\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,jj}-A_{j,kj}\right) = \hat{J}_i^{(2)}-\varepsilon_{ij}^{(2)}\left(\dot{\Phi}_{,j}^{e\ell}+\ddot{A}_j\right) \qquad \text{over } \Omega^{(2)}\end{aligned} \tag{3.77c}$$

  which are Eq.(3.49a) (Ampère's – Maxwell's Law in case of linear dielectric – diamagnetic media) for each of the constituent volumes.

- $$n_j^{(0)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix} = \hat{K}_i^{(0,1)} \qquad \text{on } \partial\Omega_K^{(0,1)}$$

(3.78a)

$$n_j^{(0)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix}=\hat{K}_i^{(0,2)}\qquad\text{on }\partial\Omega_K^{(0,2)}$$

(3.78b)

$$n_j^{(1)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix}=\hat{K}_i^{(1,2)}\qquad\text{on }\partial\Omega_K^{(1,2)}$$

(3.78c)

which are the matching condition (3.51) (Jump of $\boldsymbol{H}$ -component vertical to interface) on each of the interfaces where surface current density $\hat{\boldsymbol{K}}$ is prescribed.

- $$n_j^{(0)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix}=\epsilon_{kji}\,n_j^{(0)}\,\nu_k\quad\text{on }\partial\Omega_A^{(0,1)}$$

(3.79a)

$$n_j^{(0)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(0)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(0)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(0)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix}=\epsilon_{kji}\,n_j^{(0)}\,\nu_k\quad\text{on }\partial\Omega_A^{(0,2)}$$

(3.79b)

$$n_j^{(1)}\begin{Bmatrix}\left[\left(\mu_{kk}^{-1}\right)^{(1)}\left(A_{j,i}-A_{i,j}\right)-\left(\mu_{kk}^{-1}\right)^{(2)}\left(A_{j,i}-A_{i,j}\right)\right]+\\+\left[\left(\mu_{jk}^{-1}\right)^{(1)}\left(A_{i,k}-A_{k,i}\right)-\left(\mu_{jk}^{-1}\right)^{(2)}\left(A_{i,k}-A_{k,i}\right)\right]+\\+\left[\left(\mu_{ik}^{-1}\right)^{(1)}\left(A_{k,j}-A_{j,k}\right)-\left(\mu_{ik}^{-1}\right)^{(2)}\left(A_{k,j}-A_{j,k}\right)\right]\end{Bmatrix}=\epsilon_{kji}\,n_j^{(1)}\,\nu_k\quad\text{on }\partial\Omega_A^{(1,2)}$$

(3.79c)

Eqs.(3.79) are the matching condition (3.51) on the interfaces where tangent component of vector potential $\boldsymbol{A}$ is prescribed. This is not a needed matching condition, but defines the auxiliary field $\boldsymbol{v}$ since quantity $\epsilon_{kji} n_j v_k$ is identified as the free current over the boundary $\partial\Omega_A$ where tangent component of vector potential $\boldsymbol{A}$ is prescribed.

From Eq.(3.73c), the following Euler-Lagrange equation is obtained

- $\Phi^{e\ell} = \hat{\Phi}^{e\ell}$ on boundaries $\partial\Omega_\Phi^{(0,1)}$, $\partial\Omega_\Phi^{(0,2)}$ and $\partial\Omega_\Phi^{(1,2)}$ (3.80)

  which is the right matching condition on boundary $\partial\Omega_\Phi$ where scalar potential $\Phi^{e\ell}$ is prescribed.

From Eq.(3.73d), the following Euler-Lagrange equation is obtained

- $\epsilon_{ijk} n_j^{(0)} A_k = \epsilon_{ijk} n_j^{(0)} \hat{A}_k$ on boundaries $\partial\Omega_A^{(0,1)}$, $\partial\Omega_A^{(0,2)}$ and $\partial\Omega_A^{(1,2)}$ (3.81)

  which is the right matching condition on boundary $\partial\Omega_A$ where tangent component of vector potential $\boldsymbol{A}$ is prescribed.

Thus, the proof of the variational principle (3.56) has been concluded and the meaning and the role of the various fields have been clarified.

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# 4. Hydrodynamics

## 4.1 Discussion concerning the representation of the fluid domain – The material Derivative

*Related References: Athanassoulis & Belibassakis (2003) Ch. 4, Tsangaris (2005) Ch. 3*

Fluids are described as *continuous material media that occupy a time-dependent volume.* This initial statement gives the first hint for the dual representation of the fluid domain, as shows a one to one correspondence between a geometrical entity (volume) and a material one (continuous medium).
More specifically, we can think of the fluid domain in two different ways:

- Firstly, we consider the domain $\{\boldsymbol{x}\} = \Omega(t)$ which is the set of all geometrical points $\boldsymbol{x} = (x, y, z)$ of the volume $\Omega \in \mathbb{R}^3$ $\left(\text{or } \Omega \in \mathbb{R}^2\right)$ occupied by the fluid for every time moment $t$. Since in this description only geometrical terms are employed, domain $\{\boldsymbol{x}\} = \Omega(t)$ is call the *geometrical fluid domain.*

- We can also think of the fluid domain as the set of all material points $\{\delta m\}$ that make up the continuous fluid media. This set $\{\delta m\}$ is called the *material fluid domain*.

Under the obvious observation that each material point $\delta m$ is at one position $\boldsymbol{x}$ in every time moment $t$, we can see that the two domains, the material $\{\delta m\}$ and the geometrical one $\{\boldsymbol{x}\} = \Omega(t)$ are in one to one correspondence with each other and so each element of the material domain $\{\delta m\}$ can be expressed in terms of the geometrical domain $\{\boldsymbol{x}\} = \Omega(t)$ as

$$\delta m = \delta m(\boldsymbol{x}; t) \tag{4.1}$$

Eq. (4.1) seems to be obvious, but shows the fact that, when using the "geometrical" description, the principle of the conservation of the mass of each point inside the fluid is not satisfied a priori, despite being a simple assumption in a respective mechanical system with discrete masses.
The previous comment on Eq. (4.1) clarifies the way of working with regard to the fluid domain; The analytical description of the domain will be performed using the geometrical points $\boldsymbol{x} = (x, y, z)$ , while the physical quantities of the fluid will be calculated with regard to the material points $\delta m$.

The basic tool for calculating physical quantities with regard to material points while remaining in a geometrical description of the fluid domain is the **material (or total) derivative.** Material derivative, denoted as $D\bullet / Dt$, is the time derivate of a scalar physical field, e.g. $a = a(\boldsymbol{x}; t)$ on a moving material point $\delta m(\boldsymbol{x}; t)$ and it is calculated as follows:
Since the is a one to one correspondence between $\{\boldsymbol{x}\}$ and $\{\delta m\}$, the physical quantity can be rewritten as

$$a = a\big(\delta m(\boldsymbol{x};t)\big) \;\Leftrightarrow\; a = a(\boldsymbol{x};t) \tag{4.2}$$

We also note that after time $\delta t$, material point $\delta m$ will be in the position $\boldsymbol{x} + \delta\boldsymbol{x}$:

$$\delta m(\boldsymbol{x};t) \xrightarrow{\delta t} \delta m(\boldsymbol{x}+\delta\boldsymbol{x};t+\delta t) \tag{4.3}$$

Having Eqs. (4.2), (4.3) in mind we will express the material derivate of field $a = a(\boldsymbol{x};t)$ using the limit definition of derivatives:

$$\begin{aligned}\frac{Da(\boldsymbol{x};t)}{Dt} = \frac{Da\big(\delta m(\boldsymbol{x};t)\big)}{Dt} &= \lim_{\delta t\to 0}\frac{a\big(\delta m(\boldsymbol{x}+\delta\boldsymbol{x};t+\delta t)\big) - a\big(\delta m(\boldsymbol{x};t)\big)}{\delta t} = \\ &= \lim_{\delta t\to 0}\frac{a(\boldsymbol{x}+\delta\boldsymbol{x};t+\delta t) - a(\boldsymbol{x};t)}{\delta t}\end{aligned} \tag{4.4}$$

Expanding term $a(\boldsymbol{x}+\delta\boldsymbol{x};t+\delta t)$ in Taylor series regarding $\delta\boldsymbol{x}$ and $\delta t$, Eq. (4.4) is written as

$$\frac{Da(\boldsymbol{x};t)}{Dt} = \lim_{\delta t\to 0}\frac{1}{\delta t}\left(a(\boldsymbol{x};t) + \frac{\partial a(\boldsymbol{x};t)}{\partial\boldsymbol{x}}\delta\boldsymbol{x} + \frac{\partial a(\boldsymbol{x};t)}{\partial t}\delta t + O\big(\delta\boldsymbol{x}^2\big) + O\big(\delta t^2\big) - a(\boldsymbol{x};t)\right) \tag{4.5}$$

and calculating afterwards the limit in the above expression, we obtain

$$\frac{Da(\boldsymbol{x};t)}{Dt} = \frac{\partial a(\boldsymbol{x};t)}{\partial t} + \frac{\delta\boldsymbol{x}}{\delta t}\frac{\partial a(\boldsymbol{x};t)}{\partial\boldsymbol{x}} \tag{4.6}$$

Identifying in Eq. (4.6) the term $\delta\boldsymbol{x}/\delta t$ as velocity vector field $\boldsymbol{\upsilon}(\boldsymbol{x};t)$ and the term $\partial a(\boldsymbol{x};t)/\partial\boldsymbol{x}$ as the gradient $\nabla a(\boldsymbol{x};t)$, the final form for the material derivative definition relation is expressed as

$$\frac{Da(\boldsymbol{x};t)}{Dt} = \frac{\partial a(\boldsymbol{x};t)}{\partial t} + \boldsymbol{\upsilon}(\boldsymbol{x};t)\,\nabla a(\boldsymbol{x};t) \tag{4.7a}$$

or using index notation

$$a_{,t} = \dot{a} + \upsilon_i\, a_{,i} \tag{4.7b}$$

where $a_{,t}$ denotes the material derivative of scalar field $a$ in index notation.

### 4.2 Fields, Assumptions and Equations

*Related References: Athanassoulis & Belibassakis (2003) Ch. 4, Stoker (1957) Part 1 Ch. 1, Tsangaris (2005) Ch. 4*

The fields that are involved in the governing equations and boundary conditions of hydrodynamics are

- *Velocity vector field* $\boldsymbol{v}(\boldsymbol{x}\,;t)$ which will be reduced to *velocity potential scalar field* $\Phi^{f}(\boldsymbol{x}\,;t)$, under the assumption of irrotational flow (see Eqs. 4.9 and 4.10 below).

- *Stress tensor field* (2nd rank) $\boldsymbol{\sigma}(\boldsymbol{x};t)$ that model the internal forces developed between the material points of the fluid. As it will be shown below (see Eq. 4.8), under the assumption of inviscid fluid, stress tensor $\boldsymbol{\sigma}(\boldsymbol{x};t)$ is *reduced to pressure scalar field* $p(\boldsymbol{x};t)$.

The equations governing the hydrodynamic phenomena are the general physical laws of

1. the balance of mass
2. the balance of linear momentum
3. the balance of angular momentum
4. the balance of energy
5. the second law of thermodynamics (balance or increment of entropy)

applied at the fluid volume $\Omega(t)$.
Lastly, in order the above system of laws to be balanced, constitutive relations are also needed. These constitutive relations are of the same purpose with the respective relations in the elastodynamics chapter, since they relate strains to stresses in the fluid volume and they are a material characteristic.

The set of the five physical laws mentioned above can be drastically reduced by some **assumptions** that are valid approximations for the sea water case. Thus, the fluid is considered as

- *Incompressible* and so it has constant density $\rho_f = ct$.

- *Inviscid* which means that the internal forces of the fluid are caused only by normal stresses as

$$\sigma_{ij} = -p\delta_{ij} \tag{4.8}$$

holds true. Via Eq. (4.8), a reduction from stress tensor field $\boldsymbol{\sigma}(\boldsymbol{x};t)$ to pressure scalar field $p(\boldsymbol{x};t)$ is performed, as mentioned above.

- *Irrotational* which means that

$$\epsilon_{ijk}\,\upsilon_{k,j} = 0 \tag{4.9}$$

Eq. (4.9) leads to the same analysis made for the respective irrotational field $E_i + \dot{A}_i$ in electrodynamics, velocity vector field $\boldsymbol{\upsilon}(\boldsymbol{x}\,;t)$ can be expressed in terms of a *scalar velocity potential* $\Phi^f(\boldsymbol{x}\,;t)$ as

$$\upsilon_i = \Phi^{e\ell}_{,i} \tag{4.10}$$

Using Eq. (4.10), scalar velocity potential field $\Phi^f(\boldsymbol{x}\,;t)$ substitutes velocity vector field $\boldsymbol{\upsilon}(\boldsymbol{x}\,;t)$ in the set of variables of the problem, leading to a reduction of number of unknowns.

Furthermore, the first two assumptions of incompressibility and non-viscosity lead to the following simplifications for the set of laws + the constitutive relations that govern the hydrodynamic problem:

- The thermodynamic properties of the fluid are decoupled from the mechanical ones and thus, the second law of thermodynamics is no longer needed to be part of the system of equations, and the balance of energy involves only mechanical energies and thus it is automatically satisfied by the balance of linear momentum

- The balance of angular momentum is automatically satisfied.

- The two assumptions also substitute the constitutive relations needed.

Thus, the governing equations are reduced to the two following physical laws

1. the balance of mass
2. the balance of linear momentum

We shall move on expressing the above laws in integral form, using the concept of a fluid volume of interest $\Omega$ (with a closed boundary surface $\partial\Omega$) over which the balances of mass and linear momentum are performed.
Then, using the Gauss's divergence theorem of vector calculus (presented in the previous chapter, Eq. 3.11), the differential forms of the laws are obtained.

**The balance of mass**

The balance of mass is expressed as:

*The time derivative of fluid mass $m(t)$ included in volume $\Omega$ is equal to the amount of mass $m_{\text{in}}(\partial\Omega)$ entering the volume $\Omega$ through its boundary $\partial\Omega$ per unit time.*

Fluid mass $m(t)$ included in the volume $\Omega$ can be easily expressed using density of fluid $\rho_f(t)$

$$m(t) = \iiint_{\Omega} \rho_f(t)\,dV \tag{4.11}$$

and incoming mass per unit time $m_{\text{in}}(\partial\Omega)$ can be expressed as a spatial integral of mass flux vector $\left(j_m\right)_i = m\upsilon_i$ over the volume boundary $\partial\Omega$

$$m_{\text{in}}(\partial\Omega) = -\oiint_{\partial\Omega} \rho_f\,\upsilon_i\,n_i\,dS \tag{4.12}$$

where $\boldsymbol{n}$ is the *outward* normal unit vector of boundary surface $\partial\Omega$ and thus the minus sign is essential in order to calculate the *incoming* mass.

Using Eqs. (4.11) and (4.12), balance of mass for the fluid volume of reference $\Omega$ can be written in integral form as

$$\iiint_{\Omega} \dot{\rho}_f\,dV = -\oiint_{\partial\Omega} \rho_f\,\upsilon_i\,n_i\,dS \tag{4.13}$$

Using Gauss's divergence theorem (see Eq. 3.11) on the right side term of Eq. (4.13), we obtain

$$\iiint_{\Omega} \dot{\rho}_f\,dV = -\iiint_{\Omega} \left(\rho_f\,\upsilon_i\right)_{,i}\,dV \;\Rightarrow$$

$$\iiint_{\Omega} \left(\dot{\rho}_f + \left(\rho_f\,\upsilon_i\right)_{,i}\right) dV = 0 \tag{4.14}$$

Since Eq. (4.14) has to be simultaneously true for every volume $\Omega$, it is necessary and sufficient for the integrand to be null everywhere

$$\dot{\rho}_f + \left(\rho_f\,\upsilon_i\right)_{,i} = 0 \tag{4.15}$$

Eq. (4.15) is the balance of mass law in differential form for the general case of flow. Setting the density of fluid $\rho_f$ to be constant (and thus having zero time and spatial derivatives) under the assumption of incompressible fluid, Eq. (4.15) is simplified into

$$\upsilon_{i,i} = 0 \tag{4.16}$$

Velocity field $\boldsymbol{\upsilon}(\boldsymbol{x}\,;t)$ in Eq. (4.16) can be substituted by Eq.(4.10), leading an equation with regard to scalar velocity potential $\Phi^f(\boldsymbol{x}\,;t)$

$$\Phi^f_{,ii} = 0 \tag{4.17}$$

Eq. (4.17) is first equation derived for the description of the hydrodynamic problem and models the conservation of mass principle in the case of an incompressible, inviscid and irrotational fluid.

**The balance of linear momentum**

Working analogously to the balance of mass section above, balance of linear momentum is expressed as:

*The time derivative of linear momentum $\boldsymbol{J}(t)$ of the mass included in volume $\Omega$ is equal to the amount of linear momentum $\boldsymbol{J}_{\text{in}}(\partial\Omega)$ entering the volume $\Omega$ through its boundary $\partial\Omega$ per unit time plus the total force exercised over fluid volume $\Omega$.*

The previous expression for the balance of linear momentum is second Newton's law applied to the case of a fluid volume.

Working as before, we express via integrals:

$$J_i = \iiint\limits_{\Omega} \rho_f \upsilon_i \, dV \tag{4.18}$$

and

$$\left(J_{\text{in}}\right)_i = -\oiint\limits_{\partial\Omega} \rho_f \upsilon_i \upsilon_j n_j \, dS \tag{4.19}$$

As it is mentioned in the expression above, total force is also present in the balance of linear momentum. Total force is expressed as the total external forces minus the total internal forces in volume $\Omega$.

In the present work, the volume force $\boldsymbol{G}(\boldsymbol{x}\,;t)$ due to the homogeneous gravitational field with constant intensity $\boldsymbol{g}$ will be considered as the only external force, while the surface force $\boldsymbol{P}(\boldsymbol{x}\,;t)$ due to pressure $p(\boldsymbol{x};t)$ will be considered as the only internal force.

Thus, the volume force $\boldsymbol{G}(\boldsymbol{x}\,;t)$ can be expressed as an integral over volume $\Omega$

$$G_i = \iiint\limits_{\Omega} \rho_f \, g_i \, dV \tag{4.20a}$$

while the surface force $\boldsymbol{P}(\boldsymbol{x}\,;t)$ can be expressed as an integral over surface $\partial\Omega$

$$P_i = \oint\!\!\oint\limits_{\partial\Omega} p \, n_i \, dS \tag{4.20b}$$

Using Eqs. (4.18), (4.19) and (4.20a,b), as well as the assumption of incompressibility ($\rho_f = ct$) the balance of linear momentum is written as

$$\begin{gathered}
\dot{J}_i = \left(J_{\text{in}}\right)_i + G_i - P_i \;\Rightarrow \\
\iiint\limits_{\Omega} \rho_f \, \dot{\upsilon}_i \, dV = -\oint\!\!\oint\limits_{\partial\Omega} \rho_f \, \upsilon_i \, \upsilon_j \, n_j \, dS + \iiint\limits_{\Omega} \rho_f \, g_i \, dV - \oint\!\!\oint\limits_{\partial\Omega} p \, n_i \, dS
\end{gathered} \tag{4.21}$$

Eq. (4.21) is the balance of linear momentum in integral form for the case of incompressible inviscid fluid.
Applying Gauss's divergence theorem (see Eq. 3.11) to the first and last terms at the right side of equation (4.21), we obtain

$$\begin{gathered}
\iiint\limits_{\Omega} \rho_f \, \dot{\upsilon}_i \, dV = -\iiint\limits_{\Omega} \rho_f \left(\upsilon_i \, \upsilon_j\right)_{,j} dV + \iiint\limits_{\Omega} \rho_f \, g_i \, dV - \iiint\limits_{\Omega} p_{,i} \, dV \;\Rightarrow \\
\iiint\limits_{\Omega} \left[\rho_f \, \dot{\upsilon}_i + \rho_f \left(\upsilon_i \, \upsilon_j\right)_{,j} - \rho_f \, g_i + p_{,i}\right] dV = 0
\end{gathered} \tag{4.22}$$

Since Eq. (4.22) has to be simultaneously true for every volume $\Omega$, it is necessary and sufficient for the integrand to be null everywhere

$$\rho_f \, \dot{\upsilon}_i + \rho_f \left(\upsilon_i \, \upsilon_j\right)_{,j} - \rho_f \, g_i + p_{,i} = 0 \tag{4.23}$$

Expanding the second term in Eq. (4.23) we obtain

$$\rho_f \, \dot{\upsilon}_i + \rho_f \, \upsilon_{i,j} \, \upsilon_j + \rho_f \, \upsilon_i \, \upsilon_{j,j} - \rho_f \, g_i + p_{,i} = 0 \tag{4.24}$$

By implementing the balance of mass using Eq. (4.16), the third term in Eq. (4.24) is null, and thus we finally obtain

$$\rho_f \dot{\upsilon}_i + \rho_f \upsilon_{i,j} \upsilon_j - \rho_f g_i + p_{,i} = 0 \tag{4.25}$$

The second term in Eq.(4.25) can be written in a way that will be more convenient for the following analysis, using the following relation

$$\begin{aligned} \epsilon_{ijk} \upsilon_j \epsilon_{k\ell m} \upsilon_{m,\ell} &= \epsilon_{ijk} \epsilon_{k\ell m} \upsilon_j \upsilon_{m,\ell} = \\ &= \left(\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}\right) \upsilon_j \upsilon_{m,\ell} = \upsilon_j \upsilon_{j,i} - \upsilon_j \upsilon_{i,j} \end{aligned} \tag{4.26a}$$

Thus, the following sum can be calculated

$$\begin{aligned} 2\epsilon_{ijk} \upsilon_j \epsilon_{k\ell m} \upsilon_{m,\ell} + 2\upsilon_{i,j} \upsilon_j &= 2\left(\upsilon_j \upsilon_{j,i} - \upsilon_j \upsilon_{i,j}\right) + 2\upsilon_{i,j} \upsilon_j = \\ &= 2\upsilon_j \left(\upsilon_{j,i} - \cancel{\upsilon_{i,j}} + \cancel{\upsilon_{i,j}}\right) = 2\upsilon_j \upsilon_{j,i} = \left(\upsilon_j \upsilon_j\right)_{,i} \end{aligned} \tag{4.26b}$$

since the fluid is irrotational (Eq. 4.9), term $\epsilon_{ijk} \upsilon_j \epsilon_{k\ell m} \upsilon_{m,\ell}$ equals to null and thus (4.26b) is written as

$$\upsilon_{i,j} \upsilon_j = \frac{\left(\upsilon_j \upsilon_j\right)_{,i}}{2} \tag{4.26c}$$

Thus, Eq.(4.25) is rewritten as

$$\rho_f \dot{\upsilon}_i + \frac{\rho_f}{2}\left(\upsilon_j \upsilon_j\right)_{,i} - \rho_f g_i + p_{,i} = 0 \tag{4.27}$$

which is the balance of linear momentum in differential form for the case of an incompressible, inviscid and irrotational fluid.

Using once more the assumption that the fluid is irrotational, Eq.(4.27) can be written in terms of the scalar velocity potential $\Phi^f\left(\boldsymbol{x}\,;t\right)$ as

$$\dot{\Phi}^f_{,i} + \frac{1}{2}\left(\Phi^f_{,j} \Phi^f_{,j}\right)_{,i} - g_i + \frac{p_{,i}}{\rho_f} = 0 \tag{4.28}$$

Since the gravitational field is conservative (irrotational), if we choose a Cartesian set of axes in which the $z$ axis has the opposite direction for vector $\boldsymbol{g}$, the following relation holds true

$$g_i = -(g\,z)_{,i} \tag{4.29}$$

where $g$ is the norm of vector $g_i$.

Substituting Eq.(4.29) into Eq.(4.28) we obtain

$$\dot{\Phi}^f_{,i} + \frac{1}{2}\left(\Phi^f_{,j}\,\Phi^f_{,j}\right)_{,i} + (g\,z)_{,i} + \frac{p_{,i}}{\rho_f} = 0 \tag{4.30}$$

Since a spatial derivation of $i$ applies to all terms of Eq.(4.30), Eq.(4.30) can be rewritten as

$$\left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,j}\,\Phi^f_{,j} + g\,z + \frac{p}{\rho_f}\right)_{,i} = 0 \tag{4.31}$$

Spatially integrating Eq.(4.31) along a curve inside the fluid domain, we obtain

$$\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,j}\,\Phi^f_{,j} + g\,z + \frac{p}{\rho_f} = C(t) \tag{4.32}$$

with $C(t)$ being a time function that appears after performing the spatial integration.
Since scalar velocity potential $\Phi^f(\boldsymbol{x}\,;t)$ can absorb such a function as

$$\Phi^f_{\text{new}}(\boldsymbol{x}\,;t) = \Phi^f(\boldsymbol{x}\,;t) + \int_0^t C(\tau)\,d\tau \tag{4.33}$$

Eq. (4.32) can be written with function $C(t)$ set null, and solved with regard to pressure $p(\boldsymbol{x};t)$:

$$p = -\rho_f\,\dot{\Phi}^f - \rho_f\,\frac{1}{2}\Phi^f_{,j}\,\Phi^f_{,j} - \rho_f\,g\,z \tag{4.34}$$

Eq. (4.34), known as the **Bernoulli's equation**, is the second equation derived for the description of the hydrodynamic problem that relates the pressure field $p(\boldsymbol{x};t)$ with the velocity potential field $\Phi^f(\boldsymbol{x}\,;t)$ in the case of an incompressible, inviscid and irrotational fluid.
Observing Eq.(4.34), the pressure field $p(\boldsymbol{x};t)$ developed in the fluid has two components

- Hydrodynamic pressure $p_{\text{dynamic}} = -\rho_f\,\dot{\Phi}^f - \rho_f\,\frac{1}{2}\Phi^f_{,j}\,\Phi^f_{,j}$ that depends on velocity potential $\Phi^f(\boldsymbol{x}\,;t)$ and so it is a result of the movement of the fluid.

- Hydrostatic pressure $p_{\text{static}} = -\rho_f g z$ that depends on the depth $z$ where the pressure is measured and so it is the weight of the fluid mass above the measurement point.

### 4.3 Water wave problem

*Related References: Athanassoulis (2008) Ch. 2, Athanassoulis & Belibassakis (2003) Ch. 4 & 5, Stoker (1957) Part 1 Ch. 1, Tsangaris (2005) Ch. 4, Wehausen & Laitone (1960) Ch. B.*

After deriving the governing equations of hydrodynamics for the case of an incompressible, inviscid and irrotational fluid, we will move on to the definition of the sea wave problem (equations + boundary conditions).
For the better understanding of the configuration of the fluid domain, a vertical section of the domain is shown in Fig. 4.1. As it can be seen in Fig. 4.1, the boundary $\partial\Omega$ of liquid domain $\Omega$ consists of four subsets

- Seabed $\partial\Omega_{\Pi}$ $(z = -h_D)$ which is assumed as rigid and without vegetation in order to have negligible dissipation.
- The free surface $\partial\Omega_F$ $(z = \eta(x, y; t))$ which is the sea – air interface.
- A moving cliff $\partial\Omega_{c\ell}$ with displacement $u_i^{f,c\ell}$, on which the fluid pressure $\hat{p}$ and velocity $\hat{\upsilon}_i$ are prescribed.
- A boundary at infinity $\partial\Omega_{\infty}$, where we assume that all hydrodynamic fields remain bounded.

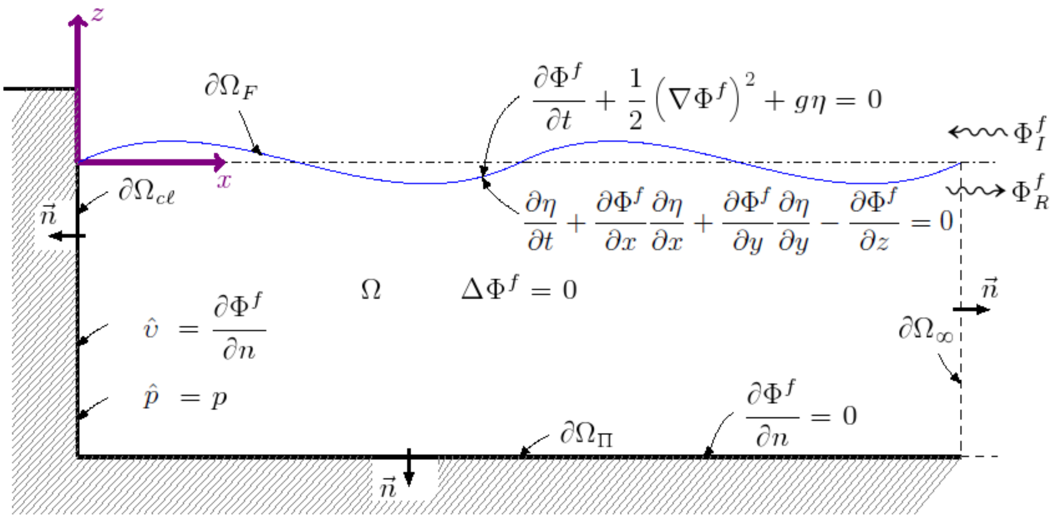


Fig. 4.1 Domain $\Omega$ and the hydrodynamic problem for $\Phi^f(\boldsymbol{x}; t)$

The orthogonal Cartesian coordinate system used, has its origin at a point on the intersection of the mean water level and the vertical cliff, with the $x$ axis lying on the mean water level and pointing towards the sea volume, the $y$ axis (perpendicular to the section shown in the figure) extending along the horizontal dimension of the vertical cliff, and the $z$ axis pointing vertically upwards.

As it has been shown in the above discussion and in Fig. 4.1, in the present problem one of the boundaries is the free surface $\partial\Omega_F$ of the fluid which is also moving. More specifically, free surface is a surface described by the implicit equation

$$F\left(x,y,z\,;\,t\right) \;=\; 0\,. \tag{4.35a}$$

Solving Eq.(4.35a) with regard to variable $z$, the expression

$$z \;=\; \eta\left(x,y\,;\,t\right) \tag{4.35b}$$

is obtained, with $\eta(x,y;t)$ being the elevation of free surface scalar field which measures the height difference between the free surface and the mean free surface ($z = 0$) of the fluid.
Thus, variable $\eta(x,y;t)$ will be involved in the expression of boundary conditions on free surface boundary $\partial\Omega_F$, while we shall keep in mind the two equivalent descriptions for the free surface

$$F\left(x,y,z,;\,t\right) \;\leftrightarrow\; z \;-\; \eta\left(x,y\,;\,t\right). \tag{4.35c}$$

Thus the sea wave problem is expressed as follows:

The *velocity potential* $\Phi^f\left(\boldsymbol{x};\,t\right)$ satisfies

- the **Laplace equation** inside fluid volume $\Omega$

$$\Phi^f_{,ii} \;=\; 0 \tag{4.36}$$

- the **kinematic free-surface condition** on free surface $\partial\Omega_F$

$$F_{,t} \;=\; 0 \tag{4.37a}$$

which models the fact that free surface is a material surface that consists of the same material points at every time $t$. Thus, calculating the material derivative $F_{,t}$ we obtain

$$\dot{F} + \upsilon_i\,F_{,i} \;=\; 0 \tag{4.37b}$$

in which velocity $\boldsymbol{v}(\boldsymbol{x};t)$ can be substituted by velocity potential $\Phi^f(\boldsymbol{x};t)$ using Eq.(4.10) since the fluid is irrotational

$$\dot{F} + \Phi^f_{,i} F_{,i} = 0 \tag{4.37c}$$

and, lastly, using equivalence of representations (4.35c), Eq.(4.37c) can be expressed using free surface elevation $\eta(x,y;t)$

$$\dot{\eta} + \Phi^f_{,x} \eta_{,x} + \Phi^f_{,y} \eta_{,y} - \Phi^f_{,z} = 0 \tag{4.37d}$$

where $\Phi^f_{,x} = \partial \Phi^f / \partial x$, $\Phi^f_{,y} = \partial \Phi^f / \partial y$ and $\Phi^f_{,z} = \partial \Phi^f / \partial z$.

- the **dynamic free-surface condition** on free surface $\partial \Omega_F$

$$\dot{\Phi}^f + \frac{1}{2} \Phi^f_{,i} \Phi^f_{,i} + g\eta = 0, \tag{4.38}$$

that expresses the continuity of pressure through the air-sea interface. It is the application of Bernoulli's equation (4.34) for $z = \eta(x,y;t)$ where fluid pressure is at its zero level since it is equal to atmospheric pressure.

- the **bottom boundary condition** on seabed $\partial \Omega_\Pi$

$$\Phi^f_{,i} n_i = 0 \tag{4.39}$$

that denotes non-penetration of sea water into the rigid, unmoving seabed

- the **kinematic boundary condition** on moving boundary $\partial \Omega_{c\ell}$

$$\left(\Phi^f_{,i} - \hat{\upsilon}_i\right) n_i = 0, \tag{4.40}$$

that denotes that normal fluid velocity $\boldsymbol{v}(\boldsymbol{x};t)\cdot\boldsymbol{n}$ over the moving boundary $\partial \Omega_{c\ell}$ is prescribed

- the **dynamic boundary condition** on moving boundary $\partial \Omega_{c\ell}$

$$p = \hat{p} \tag{4.41}$$

that denotes that fluid presure $p(\boldsymbol{x};t)$ over the moving boundary $\partial \Omega_{c\ell}$ is prescribed

- the **field behavior at infinity** on the boundary $\partial \Omega_\infty$

On the boundary at infinity $\partial\Omega_{\infty}$, we assume that all hydrodynamic fields remain bounded, and tend to those ones corresponding to a system of an incident and a reflected wave.

The above system (4.36) – (4.41) (plus field's behavior at infinity) of equation and boundary conditions is completed with **Bernoulli's law**, which defines water pressure $p(\boldsymbol{x};t)$ through velocity potential $\Phi^f(\boldsymbol{x};t)$. Bernoulli's law has been derived in the previous paragraph and it is also presented here for completeness's sake:

$$p = -\rho_f\dot{\Phi}^f - \rho_f\frac{1}{2}\Phi^f_{,j}\Phi^f_{,j} - \rho_f gz. \tag{4.34}$$

**Linearized water wave problem**

In accordance with the linear water-wave theory, the free-surface conditions (4.37) and (4.38) are applied on the mean water level $z = 0$, which will be also denoted by $\partial\Omega_{F_0}$, and they are simplified to

$$\dot{\eta} - \Phi^f_{,z} = 0, \tag{4.42}$$

and

$$\dot{\Phi}^f + g\eta = 0. \tag{4.43}$$

By solving Eq. (4.43) with regard to the free surface elevation $\eta$ and then substituting it in Eq.(4.42), **one closed free-surface condition for the velocity potential alone**

$$\ddot{\Phi}^f + g\Phi^f_{,z} = 0. \tag{4.44}$$

is obtained.

Linearized sea wave description can be solved more easily and thus it is used for the modeling of the application presented in the second part of the present work. Nevertheless, the variational description of water waves presented in the following paragraph will result in the equations of the non-linearized problem.

### 4.4 Variational Formulation of the water wave problem: Luke's Principle

*Related References: Athanassoulis (1982) Part 1, Ch. 2, Par. 6, Luke (1967).*

The goal of the present paragraph is to obtain a variational formulation for the non-linearized water wave problem modeled by Eq. (4.36) – (4.41).

The two fields considered as independent are velocity potential $\Phi^f(\boldsymbol{x};t)$ and fluid volume $\Omega(t)$.

One way of working is to follow the **Hamiltonian formulation,** already used in Sec. 3 and 4 for elastodynamics and electrodynamics respectively. In this formulation, kinetic energy minus potential energy is chosen as the Lagrangian functional.

Another way of working, which is the one to be followed here, is **Luke's Principle** (proposed in Luke 1967) in which the Lagrangian functional proposed is the volume integral of pressure $p$, expressed using Bernoulli's equation (4.34), over the fluid domain $\Omega(t)$.

It should also be noted that in Luke's variational principle, the admissible fields are free of any essential conditions, but velocity potential shall comply with its definition relation

$$\upsilon_i = \Phi^f_{,i} \qquad \text{(Eq. 4.10)}$$

Thus, Luke's functional that models the homogeneous non-linear water wave problem, assuming water as incompressible, inviscid and irrotational, reads as

$$\mathscr{L}\left[\Phi^f, \Omega(t)\right] = -\rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega(t)} \left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\,z\right) dV\,dt\,. \tag{4.45a}$$

Taking into consideration that in the configuration of the examined system, a moving boundary $\partial\Omega_{c\ell}$, with displacement $u_i^{f,c\ell}$,prescribed velocity $\hat{\upsilon}_i \equiv \partial u_i^{f,c\ell}/\partial t$ and prescribed pressure $\hat{p}$, exists, the previous functional (4.45a) is augmented to

$$\begin{aligned}\mathscr{L}_{\text{aug}}\left[\Phi^f, \Omega(t)\right] &= \mathscr{L}\left[\Phi^f, \Omega(t)\right] + \mathscr{L}_{\text{force}}\left[\Omega(t)\right] = \\ &= -\rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega(t)} \left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\,z\right) dV\,dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}} \hat{p}\,u_i^{f,c\ell}\,n_i\;dS\,dt\end{aligned} \tag{4.45b}$$

What remains to be proven is that the total Gâteaux derivative of functional (4.45b) produces the system of equations (4.36) – (4.41).

Firstly, the partial Gâteaux derivatives of functional (4.45b) regarding fields $\Phi^f(\boldsymbol{x};t)$ and $\eta(x,y;t)$ will be calculated as:

$$\delta_{\Phi^f}\mathscr{L}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = -\rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega(t)} \left(\delta\dot{\Phi}^f + \Phi^f_{,i}\,\delta\Phi^f_{,i}\right) dV\,dt \tag{4.46}$$

For better handling of the calculations, the partial Gâteaux derivative $\delta_{\Phi^f}\mathscr{L}\left[\Phi^f, \Omega(t); \delta\Phi^f\right]$ can be split up into two compontents

$$\delta_{\Phi^f}\mathscr{L}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = $$
$$= \delta_{\Phi^f}\mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] + \delta_{\Phi^f}\mathscr{L}_{\delta\Phi^f_{,i}}\left[\Phi^f, \Omega(t); \delta\Phi^f\right]$$

with regard to the type of differentiation (temporal or spatial) being applied to variation $\delta\Phi^f$. Thus the component with spatial derivative $\delta\Phi^f_{,i}$ can be expanded further using spatial integration by parts (Green's theorem):

$$\delta_{\Phi^f}\mathscr{L}_{\delta\Phi^f_{,i}}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = -\rho_f\int_{t_0}^{t_1}\iiint\limits_{\Omega(t)}\Phi^f_{,i}\,\delta\Phi^f_{,i}\;dV\,dt =$$
$$= -\rho_f\int_{t_0}^{t_1}\iint\limits_{\partial\Omega(t)}\Phi^f_{,i}\,n_i\,\delta\Phi^f\;dV\,dt + \rho_f\int_{t_0}^{t_1}\iiint\limits_{\Omega(t)}\Phi^f_{,ii}\,\delta\Phi^f\;dV\,dt \tag{4.47}$$

For the further calculation of the component with temporal derivative $\delta\dot{\Phi}^f$, the Reynolds transport theorem shall be used, since the volume $\Omega(t)$ is time-dependent. This imposes some complicacy to the calculations that has not been encountered at the previous chapters of elastodynamics and electrodynamics, since there the solid volumes were considered as time-independent.

***Reynolds transport theorem:*** For a (scalar or vector) function $\mathrm{A}(\boldsymbol{x};t)$, the time differentiation under the integral sign in the case of volume integration is performed as

$$\frac{d}{dt}\iiint\limits_{\Omega(t)}\mathrm{A}(\boldsymbol{x};t)dV = \iiint\limits_{\Omega(t)}\dot{\mathrm{A}}(\boldsymbol{x};t)dV + \iint\limits_{\partial\Omega(t)}\mathrm{A}(\boldsymbol{x};t)\upsilon_i^b\,n_i\,dS \tag{4.48}$$

where $\boldsymbol{\upsilon}^b$ is the *velocity of the surface element* and $\boldsymbol{n}$ is the outward normal unit vector of surface $\partial\Omega(t)$.

Thus, component $\delta_{\Phi^f}\mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f, \Omega(t); \delta\Phi^f\right]$ can be written with the aid of Reynolds transport theorem (4.48) as

$$\delta_{\Phi^f}\mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = -\rho_f\int_{t_0}^{t_1}\iiint\limits_{\Omega(t)}\delta\dot{\Phi}^f\;dV\,dt =$$

$$= -\rho_f \int_{t_0}^{t_1} \frac{d}{dt} \iiint\limits_{\Omega(t)} \delta\Phi^f \; dV\,dt + \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega(t)} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt =$$

$$= \cancel{-\rho_f \iiint\limits_{\Omega(t)} \delta\Phi^f \; dV \Bigg|_{t_0}^{t_1}}^{\,=0} + \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega(t)} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt$$

and so

$$\delta_{\Phi^f}\, \mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega(t)} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt \tag{4.49a}$$

Splitting up the surface integral of Eq.(4.49) into three integrals over the boundaries $\partial\Omega_F$, $\partial\Omega_\Pi$ and $\partial\Omega_{c\ell}$ we obtain

$$\begin{aligned}\delta_{\Phi^f}\, \mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] &= \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F(t)} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt + \\ &+ \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_\Pi} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt + \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}(t)} \upsilon_i^b\, n_i\, \delta\Phi^f \; dS\,dt\end{aligned} \tag{4.49b}$$

At this point, the normal velocity of surface elements $\boldsymbol{\upsilon}^b \cdot \boldsymbol{n}$ has to be defined for each of the boundaries $\partial\Omega_F$, $\partial\Omega_\Pi$ and $\partial\Omega_{c\ell}$:

- Since free surface boundary $\partial\Omega_F$ is a material surface of the fluid, normal surface velocity $\boldsymbol{\upsilon}^b \cdot \boldsymbol{n}$ can be expressed in terms of free surface function $F(x, y, z; t)$.
  We Commence from Eq.(4.37b) regarding the material derivative of free surface

  $$\dot{F} + \upsilon_i^b F_{,i} = 0$$

  which can be solved with regard to velocity $\upsilon_i^b$ and vector $\boldsymbol{F}_{,i}$ could appear as normalized

  $$\upsilon_i^b \frac{F_{,i}}{\sqrt{F_{,j}F_{,j}}} = -\frac{\dot{F}}{\sqrt{F_{,j}F_{,j}}}.$$

Since quantity $F_{,i} \big/ \sqrt{F_{,j}\, F_{,j}}$ is the normal unit vector $\boldsymbol{n}$ of the free surface, normal surface velocity $\boldsymbol{\upsilon}^{b} \cdot \boldsymbol{n}$ of the free surface can be expressed as

$$\upsilon_i^b\, n_i \;=\; -\,\frac{\dot{F}}{\sqrt{F_{,i}\, F_{,i}}}$$

- Since seabed $\partial\Omega_{\Pi}$ is a rigid surface, its normal surface velocity is prescribed and equals to null

$$\upsilon_i^b\, n_i \;=\; 0$$

- Since moving boundary $\partial\Omega_{c\ell}$ has its normal velocity prescribed, it holds true that

$$\upsilon_i^b\, n_i \;=\; \hat{\upsilon}_i\, n_i$$

Thus, Eq.(4.49b) is written as

$$\delta_{\Phi^f}\, \mathscr{L}_{\delta\dot{\Phi}^f}\left[\Phi^f,\, \Omega(t)\,;\, \delta\Phi^f\right] =$$
$$= \;-\,\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F(t)} \frac{\dot{F}}{\sqrt{F_{,i}\, F_{,i}}}\,\delta\Phi^f\; dS\, dt \,+\, \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}(t)} \hat{\upsilon}_i\, n_i\, \delta\Phi^f\; dS\, dt \tag{4.50}$$

Thus, combining Eq.(4.47) and (4.50), the partial derivative of Luke's functional regarding variable $\Phi^f$ reads

$$\delta_{\Phi^f}\, \mathscr{L}\left[\Phi^f,\, \Omega(t)\,;\, \delta\Phi^f\right] \;=\; -\,\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega(t)} \Phi^f_{,i}\, n_i\, \delta\Phi^f\; dV\, dt \,+$$
$$+\,\rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega(t)} \Phi^f_{,ii}\, \delta\Phi^f\; dV\, dt \,-\, \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F(t)} \frac{\dot{F}}{\sqrt{F_{,i}\, F_{,i}}}\,\delta\Phi^f\; dS\, dt \,+$$
$$+\,\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}(t)} \hat{\upsilon}_i\, n_i\, \delta\Phi^f\; dS\, dt \tag{4.51a}$$

and by splitting up the first surface integral to the three distinct boundaries $\partial\Omega_F$, $\partial\Omega_{\Pi}$ and $\partial\Omega_{c\ell}$ we obtain

$$\delta_{\Phi^f} \mathscr{L}\left[\Phi^f, \Omega(t); \delta\Phi^f\right] = +\rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega(t)} \Phi^f_{,ii}\, \delta\Phi^f \; dV\, dt \; -$$

$$-\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F(t)} \left( \Phi^f_{,i} n_i + \frac{\dot{F}}{\sqrt{F_{,j} F_{,j}}} \right) \delta\Phi^f \; dS\, dt \; +$$

$$+\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}(t)} \left( \hat{\upsilon}_i n_i - \Phi^f_{,i} n_i \right) \delta\Phi^f \; dS\, dt - \rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_\Pi} \Phi^f_{,i} n_i\, \delta\Phi^f \; dV\, dt \tag{4.51b}$$

Moving on to the calculation of partial Gâteaux derivative of Luke's function we have

$$\delta_{\Omega(t)} \mathscr{L}\left[\Phi^f, \Omega(t); \delta\Omega(t)\right] = -\rho_f \int_{t_0}^{t_1} \iiint\limits_{\delta\Omega(t)} \left( \dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\,z \right) dV\, dt \tag{4.52}$$

Let us now comment on the need for a variation of Luke's principle regarding the fluid volume $\Omega(t)$ over which the water wave problem is defined. Variation with regard to volume $\Omega(t)$ is performed since it is time-depended. The reason of volume's time dependence con be reduced to the presence of two non-rigid boundaries; free surface $\partial\Omega_F$, whose normal displacement is described as the free surface elevation $\eta$, and moving boundary $\partial\Omega_{c\ell}$, whose normal displacement is expressed as $u_i^{f,c\ell} n_i$. Since the displacements $\eta$ and $u_i^{f,c\ell} n_i$ of the two moving boundaries are considered independent from one another, $\delta_{\Omega(t)}$ variation of functional $\mathscr{L}$ can be split up into two variations $\delta_\eta$ and $\delta_{u_i^{f,c\ell}}$ over the two boundaries respectively, so

$$\begin{aligned}
\delta_{\Omega(t)} \mathscr{L}\left[\Phi^f, \Omega(t); \delta\,\Omega(t)\right] &= \delta_\eta \mathscr{L}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\eta\right] + \delta_{\boldsymbol{u}^{f,c\ell}} \mathscr{L}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\boldsymbol{u}^{f,c\ell}\right] = \\
&= -\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F} \left( \dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\eta \right) \delta\eta \; dS\, dt \; - \\
&\quad -\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}} \left( \dot{\Phi}^f + \frac{1}{2}\Phi^f_{,j}\Phi^f_{,j} + g\,z \right) n_i\, \delta\hat{u}_i \; dS\, dt
\end{aligned} \tag{4.53}$$

Having split up $\delta_{\Omega(t)}$ to variations $\delta_\eta$ and $\delta_{\boldsymbol{u}^{f,c\ell}}$, variation $\delta_{\Omega(t)} \mathscr{L}_{\text{force}}\left[\Omega(t); \delta\Omega(t)\right]$ can be also calculated easily as

$$\delta_{\Omega(t)} \mathscr{L}_{\text{force}}\left[\Omega(t); \delta\Omega(t)\right] = \delta_{\boldsymbol{u}^{f,c\ell}} \mathscr{L}_{\text{force}}\left[\boldsymbol{u}^{f,c\ell}; \delta\boldsymbol{u}^{f,c\ell}\right] = \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}} \hat{p}\, n_i\, \delta u_i^{f,c\ell} \; dS\, dt \tag{4.54}$$

Thus, by merging Eqs.(4.53) and (4.54), partial Gâteaux derivative regarding $\Omega(t)$ of Luke's functional is expressed as

$$\delta_{\Omega(t)} \mathscr{L}_{\text{aug}}\left[\Phi^f, \Omega(t); \delta\, \Omega(t)\right] = \delta_\eta \mathscr{L}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\eta\right] + \delta_{\boldsymbol{u}^{f,c\ell}} \mathscr{L}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\boldsymbol{u}^{f,c\ell}\right] =$$

$$= -\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_F} \left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\eta\right)\delta\eta\; dS\, dt -$$

$$- \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}} \left(\rho_f\dot{\Phi}^f + \rho_f\frac{1}{2}\Phi^f_{,j}\Phi^f_{,j} + \rho_f\, g\, z - \hat{p}\right) n_i\, \delta u_i^{f,c\ell}\; dS\, dt \tag{4.55}$$

After the calculation of the partial derivatives of Luke's functional regarding variations $\delta\Phi^f$ (Eq.4.51b), $\delta\eta$ and $\delta u_i^{f,c\ell}$ (Eq.4.55), we can apply Luke's variational principle that sets

$$\delta\mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\Phi^f, \delta\eta, \delta\boldsymbol{u}^{f,c\ell}\right] = \delta_{\Phi^f} \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\Phi^f\right] +$$

$$+ \delta_\eta \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\eta\right] + \delta_{\boldsymbol{u}^{f,c\ell}} \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\boldsymbol{u}^{f,c\ell}\right] = 0 \tag{4.56}$$

Since variations $\delta\Phi^f$, $\delta\eta$ and $\delta u_i^{f,c\ell}$ are considered independent from one another, Eq.(4.56) is equivalent to the following system of equations

$$\delta_{\Phi^f} \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\Phi^f\right] = 0 \tag{4.57a}$$

$$\delta_\eta \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\eta\right] = 0 \tag{4.57b}$$

$$\delta_{\boldsymbol{u}^{f,c\ell}} \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\boldsymbol{u}^{f,c\ell}\right] = 0 \tag{4.57c}$$

From Eq.(4.57a), the following Euler-Lagrange equations are obtained:

- $\Phi^f_{,ii} = 0$ over volume $\Omega(t)$ (4.58)

which is Laplace equation (4.36).

- $\Phi^f_{,i}\, n_i + \dfrac{\dot{F}}{\sqrt{F_{,j}\, F_{,j}}} = 0$ on free surface $\partial\Omega_F$ (4.59a)

Recognizing that normal unit vector $\boldsymbol{n}$ of free surface $\partial\Omega_F$ can be expressed in terms of free surface function $F$ as

$$n_i = \frac{F_{,i}}{\sqrt{F_{,j} F_{,j}}} \tag{4.59b}$$

and, by substituting Eq.(4.59b) into Eq.(4.59a), we obtain

$$\Phi^f_{,i} F_{,i} + \dot{F} = 0 \quad \text{on free surface } \partial\Omega_F \tag{4.59c}$$

which is the kinematic free surface condition (Eq.4.37c).

- $$\Phi^f_{,i} n_i = 0 \quad \text{on seabed } \partial\Omega_\Pi \tag{4.60}$$

which is the non-penetration of sea water into the rigid seabed (4.39)

- $$\left(\hat{\upsilon}_i - \Phi^f_{,i}\right) n_i = 0 \quad \text{on moving boundary } \partial\Omega_{c\ell} \tag{4.61}$$

which is denotes that normal velocity is prescribed on $\partial\Omega_{c\ell}$ (Eq.4.40)

So we observe that all equations for water-wave kinematics have been derived from setting $\delta_{\Phi^f} \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\Phi^f\right]$ equal to zero.

From Eq.(4.57b), the following Euler-Lagrange equation is obtained:

- $$\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g\eta = 0 \quad \text{on free surface } \partial\Omega_F \tag{4.62}$$

which is the dynamic free surface condition (Eq.4.38)

And lastly, from Eq.(4.57c), we obtain the following Euler-Lagrange equation:

- $$\rho_f\left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,i}\Phi^f_{,i} + g z\right) = \hat{p} \quad \text{on moving boundary } \partial\Omega_{c\ell} \tag{4.63}$$

which denotes that pressure is prescribed on $\partial\Omega_{c\ell}$ (Eq.4.41)

And so it is observed that all (non-linear) equations for water-wave dynamics have been derived from setting

$$\delta \mathscr{L}_{\text{aug}}\left[\Phi^f, \eta, \boldsymbol{u}^{f,c\ell}; \delta\Phi^f, \delta\eta, \delta\boldsymbol{u}^{f,c\ell}\right] = 0$$

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# 5. Coupled problem of Hydro/Piezo/Electricity

## 5.1 Introduction

In the present chapter, the way of work will be somehow the opposite of the way of work in the previous chapter 2 to 4: In these previous chapters, we have constructed the variational formulation of each un-coupled problem knowing a priori the governing equations. In this chapter, the governing equations of the whole hydro/piezo/electric problem will be generated by the variational principle whose functional is claimed be the sum of the Hamiltonian functional for linear elastodynamics, the Hamiltonian functional for linear electrodynamics and Luke's functional for hydrodynamics,

$$\mathscr{F} = \iiint\limits_{\substack{\text{piezoelectric}\\\text{volume}}} \left( U_{\text{kinetic}} - U_{\substack{\text{mechanical}\\\text{potential}}} \right) dV + \iiint\limits_{\substack{\text{piezoelectric}\\\text{and}\\\text{ambient volume}}} \left( U_{\substack{\text{energy of}\\\text{electric}\\\text{field}}} - U_{\substack{\text{energy of}\\\text{magnetic}\\\text{field}}} \right) dV + \iiint\limits_{\substack{\text{water}\\\text{volume}}} p \, dV +$$

$$+ \text{ source terms } + \text{ terms for boundary and matching conditions} \qquad (5.1)$$

with the coupling between the three constituent, initially un-coupled phenomena properly introduced. Functional $\mathscr{F}$ retains as independent variables elastic displacement $\boldsymbol{u}$ in piezoelectric volume, fluid volume and velocity potential $\Phi^f$ in water volume and electromagnetic potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$ throughout the whole volume of interest.

Thus, the way of work in the present chapter will be the specification of the couplings and their impact on the terms that appear in the functional proposed for the whole problem above. After all the terms of the functional are clarified, we will move on to the variational principle in order to obtain the governing equations of hydro/piezo/electricity.

## 5.2 Coupling Outline

In the previous chapters 2 to 4, we have stated the governing equations and formulated the respective variational principle for each of the constituent (and thus un-coupled) problems of hydro/piezo/electric phenomenon, which are: linear elastodynamics, electrodynamics and hydrodynamics. In order to come up with a description for the coupled problem, we have to introduce an interaction between the three constituent phenomena. The interactions introduced in the following paragraphs can be of two kinds:

1. *Different constitutive relations.* In each of the chapters 2 to 4, a discussion of the respective constitutive relations has been conducted. As we have mentioned in the previous chapters, a constitutive relation models the response of a certain class of materials to an external excitation. Thus, a constitutive relation is a relation between two physical quantities that does not profess a general physical law, but the behavior (defined by experiments) of a certain class of materials.

In the present work, a change of the elastodynamic and electrodynamics constitutive relations regarding the solid volume $\Omega^{(0)}$ will introduce the piezo-electric interaction. This change is valid since the solid material considered in the coupled problem is of the piezoelectric class which is different from the class of linear elastic and linear dielectric solid materials considered in the respective chapters of the un-coupled problems.

2. *Matching of physical fields on the boundaries.* In the expressions of boundary conditions of the constituent un-coupled problems, the boundary values of some physical quantities were considered as prescribed (externally defined). In the analysis of the coupled problem, these boundary conditions will be replaced by matching conditions that express the continuity of the respective physical quantity on the interface of the two different materials.

In the present work, such a replacement of boundary conditions with matching conditions will be performed to match the kinematic (velocity) and dynamical quantities (pressure) on the solid-liquid interface $\partial\Omega^{(0,1)}$.

Clearly, the above replacement of boundary conditions with the respective matching ones is obvious if we consider e.g. that elastic stress and hydrodynamic pressure refer to the same physical quantity and thus a continuity expression for this quantity has to be formulated on the interface of the two materials. The respective matching conditions on the interfaces of materials for the electromagnetic quantities are derived directly from the variational principle without any further analysis, since the electromagnetic part of the variational principle is defined by construction over the total volume $\Omega$ as a whole, not over each of the material volumes separately.

### 5.3 Domain Configuration for the coupled problem

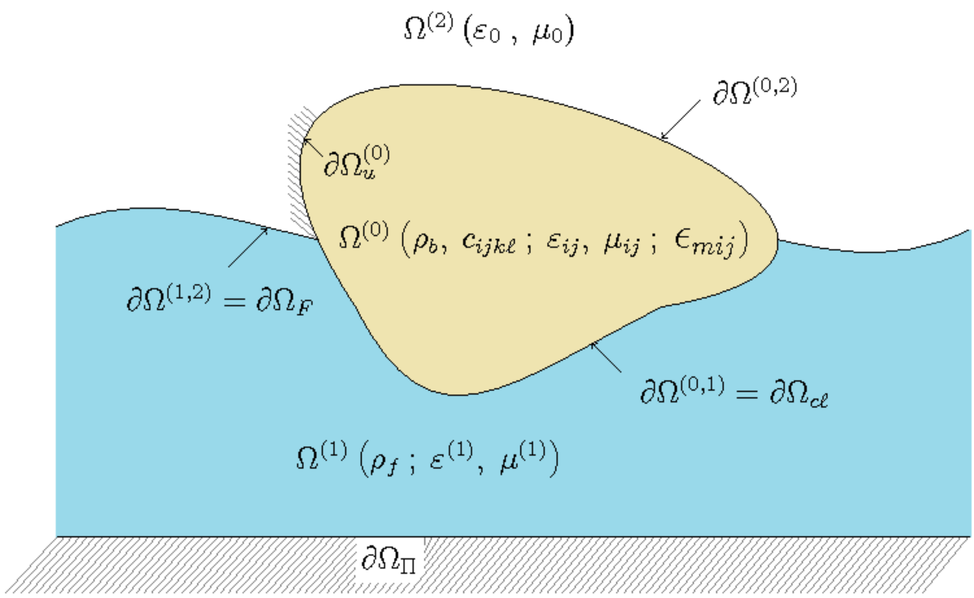


Fig. 5.1: Domain Configuration

In the previous chapters 2 to 4, a different domain configuration for each of the constituent problems has been set:

- Linear elastodynamics was examined over an elastic volume $\Omega^{(0)}$, whose boundary is split up in two parts, $\partial\Omega_T^{(0)}$ where the stress tensor is prescribed, and $\partial\Omega_u^{(0)}$ where displacement is prescribed.

- Linear electrodynamics was examined over a total volume $\Omega$ which consisted of three different materials $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ whose their material property tensors $\varepsilon_{ij}$ and $\mu_{ij}$ differed from one another and their interfaces where noted as $\partial\Omega^{(0,1)}$, $\partial\Omega^{(0,2)}$ and $\partial\Omega^{(1,2)}$.

- Hydrodynamics was examined over a liquid volume $\Omega^{(1)}$ whose boundaries was its free surface $\partial\Omega_F^{(1)}$, seabed $\partial\Omega_{\Pi}^{(1)}$, moving boundary $\partial\Omega_{c\ell}^{(1)}$ and boundary at infinity $\partial\Omega_{\infty}^{(1)}$.

At the present chapter, the volume $\Omega$ over which the whole hydro/piezo/electric phenomenon will be examined will consist of the volumes $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ previously mentioned which are specified as:

- Piezoelectric solid volume $\Omega^{(0)}$, with mass density $\rho_b$, elastic stiffness property tensor (4th rank) $\boldsymbol{c}$, dielectric permittivity tensor (2nd rank) $\boldsymbol{\varepsilon}$, magnetic permeability tensor (2nd rank) $\boldsymbol{\mu}$ and piezoelectric stress tensor (3rd rank) $\boldsymbol{\epsilon}$. The piezoelectric property tensor will be defined and discussed in the next paragraph.

- Liquid volume $\Omega^{(1)}$, with mass density $\rho_f$, dielectric permittivity constant (scalar) $\varepsilon^{(1)}$ and magnetic permeability constant (scalar) $\mu^{(1)}$.

- Air volume $\Omega^{(2)}$, whose electromagnetic properties are approximated by vacuum permittivity (scalar) $\varepsilon_0$ and vacuum permeability (scalar) $\mu_0$.

As it is also depicted in Fig. 5.1 the interfaces appearing in the coupled problem are the following

- Interfaces that compose the boundary $\partial\Omega^{(0)}$ of the piezoelectric volume: i) $\partial\Omega_u^{(0)}$ where the displacement is prescribed (in most cases the clamped boundary) ii) interface between piezoelectric material and air $\partial\Omega^{(0,2)}$ which is a boundary considered as free of stresses iii) interface between piezoelectric material and water $\partial\Omega^{(0,1)}$ where the two media (solid and liquid) exhibit the same velocity and pressure (matching condition).

- Boundaries $\partial\Omega^{(1)}$ of the liquid volume: i) liquid free surface $\partial\Omega_F$ which is by definition the liquid-air interface that is denoted by $\partial\Omega^{(1,2)}$ ii) moving cliff $\partial\Omega_{c\ell}$ which is identified as the piezoelectric material – liquid interface $\partial\Omega^{(0,1)}$ iii) rigid sea bottom $\partial\Omega_{\Pi}$ iv) boundary at infinity $\partial\Omega_{\infty}$.

In the following analysis, the more systematic notation of $\partial\Omega^{(*,*)}$ will be used for the boundaries, except for $\partial\Omega_{\Pi}$ and $\partial\Omega_{\infty}$.

## 5.4 Elastic-Electric Coupling

### *5.4.1 Constitutive Equations of Linear Piezoelectricity*

*General References: Jaffe et al. (1971), Bardzokas & Filshtinsky (2006) Parton & Kudryavtsev (1988)*

As we have stated in paragraph 5.2, the interaction between the elastic and electric quantities will be performed with the substitution of

Generalized Hooke's law: $\sigma_{ij} = c_{ijk\ell}\, e_{k\ell}$
and
Electric displacement definition relation for linear dielectrics: $D_i = \varepsilon_{ij} E_j$

with the two constitutive relations of linear piezoelectricity (see e.g. Meitzler et al. 1987):

$$\sigma_{ij} = c^{E}_{ijk\ell}\, e_{k\ell} - \epsilon_{mij} E_m \tag{5.2a}$$
and
$$D_i = \varepsilon^{S}_{ij} E_j + \epsilon_{ik\ell} e_{k\ell} \tag{5.2b}$$

As we have already mentioned, this change in constitutive relations reflects the change of class of the solid material considered to occupy volume $\Omega^{(0)}$: from the class of linear elastic and linear dielectric materials to the class of linear piezoelectric materials.

Observing the constitutive relations (5.2a), (5.2b) we can see that they are Hooke's law and electric displacement definition relation with an additional term of electric and elastic nature respectively. The material property tensor that relates the electric field to its stress response in Eq.(5.2a) is the same material property tensor (3rd rank) $\boldsymbol{\epsilon}$ that relates the elastic strain to its electric displacement response in Eq.(5.2b), a fact that categorizes piezoelectricity as a reversible phenomenon.
Moreover, by observation of indices in Eqs. (5.2a), (5.2b), we can state that property tensor $\boldsymbol{\epsilon}$, named piezoelectric stress tensor, exhibits a symmetry between the second and the third index

$$\epsilon_{kij} = \epsilon_{kji} \tag{5.3}$$

since these indices correspond to the indices of elastic quantity $\boldsymbol{e}$ (or $\boldsymbol{\sigma}$) which are symmetric. The first index of $\boldsymbol{\epsilon}$ corresponds to the index of electric quantity $\boldsymbol{E}$ (or $\boldsymbol{D}$).
In addition to the above observations, property tensors $\boldsymbol{c}$ (elastic stiffness) and $\boldsymbol{\varepsilon}$ dielectric permittivity appear in Eqs.(5.2a) and (5.2b) with superscripts $E$ and $S$ respectively, which imply that elastic stiffness components were measured under constant electric field (which is the second independent field in Eq.5.2a) and dielectric permittivity components were measured under constant elastic strain (which is the second independent field in Eq.5.2b). This need of clarification gives rise to the question:

*Do the values of property tensors* $\mathbf{c}$ *and* $\boldsymbol{\varepsilon}$ *change when measured under other constant fields (electric and elastic respectively)?*

This question can be easily answered though simple algebraic manipulations of Eqs.(5.2a), (5.2b):

Solving Eq.(5.2b) with regard to $\boldsymbol{E}$

$$E_m = \left(\varepsilon^{-1}\right)^S_{mi'} \left(D_{i'} - \epsilon_{i'k\ell}\, e_{k\ell}\right) \tag{5.4a}$$

(with $\left(\varepsilon^{-1}\right)^S$ being the inverse dielectric permittivity property tensor under constant strain)
and substituting into Eq.(5.2a) we obtain

$$\sigma_{ij} = \left(c^E_{ijk\ell} + \epsilon_{mij}\left(\varepsilon^{-1}\right)^S_{mi'} \epsilon_{i'k\ell}\right) e_{k\ell} - \epsilon_{mij}\left(\varepsilon^{-1}\right)^S_{mi'} D_{i'}. \tag{5.4b}$$

Eq.(5.4) is a new form of the constitutive equation that expresses $\boldsymbol{\sigma}$ in terms of $\boldsymbol{e}$ and $\boldsymbol{D}$. According to the previous explanation of the superscript notation and by observing Eq.(5.4b) we can introduce the material property tensor

$$c^D_{ijk\ell} = c^E_{ijk\ell} + \epsilon_{mij}\left(\varepsilon^{-1}\right)^S_{mi'} \epsilon_{i'k\ell} \tag{5.4c}$$

which is the elastic stiffness tensor under constant electric displacement. Thus, Eq.(5.4c) clearly shows that $c^D_{ijk\ell} \neq c^E_{ijk\ell}$ and so the superscript that shows the quantity held constant is essential.
Generalizing the concept of Eq.(5.4a) – (5.4c) we can state that since we have two elastic ($\boldsymbol{\sigma}$ and $\boldsymbol{e}$) and two electric ($\boldsymbol{D}$ and $\boldsymbol{E}$) quantities, we can generate 4 pairs of constitutive relations by choosing which elastic and which electric quantity are to be considered as independent. The presentation of these alternative constitutive relations is out of the scope of this paragraph, since the suitable constitutive relations for the following analysis are Eqs.(5.2a), (5.2b) which retain the fields $\boldsymbol{e}$ and $\boldsymbol{E}$ as independent, just as Hooke's law and electric displacement definition relation.

For completeness, the formulation of the alternative forms of constitutive relations and the nomenclature of the appearing property tensors will be presented in Appendix A.

### *5.4.2 Expression of the energy forms*

In the present paragraph we will consider the energy forms appearing in each of the volumes $\Omega^{(0)}$, $\Omega^{(1)}$ and $\Omega^{(2)}$ that compose $\Omega$ for the elastic-electric subproblem. Note that in the present paragraph, the hydrodynamic subproblem is not taken into account (and thus the water volume $\Omega^{(1)}$ is considered merely as a dielectric) since in the present its description does not follow the Hamiltonian formulation.

- Piezoelectric volume $\Omega^{(0)}$

*Kinetic Energy:* $$U_{\text{kinetic}} = \frac{1}{2}\rho_b \iiint_{\Omega^{(0)}} \dot{u}_i \dot{u}_i \, dV \tag{5.5}$$

same with Eq.(2.7b) of linear elastodynamics.

*Mechanical potential energy:* $$U_{\substack{\text{mechanical}\\ \text{potential}}} = \frac{1}{2}\iiint_{\Omega^{(0)}} \sigma_{ij}\, e_{ij}\, dV\,. \tag{5.6a}$$

By substituting piezoelectric constitutive relation (5.2a) into Eq.(5.6a) we obtain

$$U_{\substack{\text{mechanical}\\ \text{potential}}} = \frac{1}{2}\iiint_{\Omega^{(0)}} c^E_{ijk\ell}\, e_{k\ell}\, e_{ij}\, dV - \frac{1}{2}\iiint_{\Omega^{(0)}} \epsilon_{mij}\, E_m\, e_{ij}\, dV\,. \tag{5.6b}$$

We can see that mechanical potential energy in the case of piezoelectric media has two terms. The first term is purely elastic and is the same with mechanical potential energy in the case of linear elastodynamics (called elastic energy in chapter 2) and a second term of piezoelectric (coupling) nature, since it contains both electric and elastic quantities.

*Energy stored in the electric field:* $$U_{\substack{\text{energy of}\\ \text{electric}\\ \text{field}}} = \frac{1}{2}\iiint_{\Omega^{(0)}} D_i\, E_i\, dV\,. \tag{5.7a}$$

By substituting piezoelectric constitutive relation (5.2b) into Eq.(5.7a) we obtain

$$U_{\substack{\text{energy of}\\ \text{electric}\\ \text{field}}} = \frac{1}{2}\iiint_{\Omega^{(0)}} \varepsilon^S_{ij}\, E_j\, E_i\, dV + \frac{1}{2}\iiint_{\Omega^{(0)}} \epsilon_{ik\ell}\, e_{k\ell}\, E_i\, dV \tag{5.7b}$$

Similarly to Eq.(5.6b) we observe that the energy stored into the electric field has two terms. The first term is purely electric and is the same with energy stored into electric field in the case of linear dielectric media (called electric energy in chapter 3) and a second term of piezoelectric (coupling) nature, since it contains both electric and elastic quantities.

*Magnetic energy:* $U_{\text{magnetic}} = \dfrac{1}{2} \iiint\limits_{\Omega^{(0)}} \mu_{ji}^{-1} B_j B_i \, dV$ (5.8a)

Since piezoelectric volume is considered as a linear diamagnetic medium, the form of magnetic energy is the same with the one derived in Eq.(3.54b) in Chapter 3.

Having expressed all the energy quantities appearing inside piezoelectric volume $\Omega^{(0)}$, we can formulate the part of the Lagrangian functional that refers to volume $\Omega^{(0)}$

$$L_{\Omega^{(0)}} = U_{\text{kinetic}} - U_{\substack{\text{mechanical}\\ \text{potential}}} + U_{\substack{\text{energy of}\\ \text{electric}\\ \text{field}}} - U_{\text{magnetic}} =$$

$$= \frac{1}{2} \rho_b \iiint\limits_{\Omega^{(0)}} \dot{u}_i \dot{u}_i \, dV - \frac{1}{2} \iiint\limits_{\Omega^{(0)}} c_{ijk\ell}^{E} e_{k\ell} e_{ij} \, dV + \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \epsilon_{mij} E_m e_{ij} \, dV +$$

$$+ \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \varepsilon_{ij}^{S} E_j E_i \, dV + \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \epsilon_{ik\ell} e_{k\ell} E_i \, dV - \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \mu_{ji}^{-1} B_j B_i \, dV \tag{5.9a}$$

Summing the third and fifth term of Eq.(5.9a), functional $L_{\Omega^{(0)}}$ is written as

$$L_{\Omega^{(0)}} = U_{\text{kinetic}} - U_{\substack{\text{mechanical}\\ \text{potential}}} + U_{\substack{\text{energy of}\\ \text{electric}\\ \text{field}}} - U_{\text{magnetic}} =$$

$$= \frac{1}{2} \rho_b \iiint\limits_{\Omega^{(0)}} \dot{u}_i \dot{u}_i \, dV - \frac{1}{2} \iiint\limits_{\Omega^{(0)}} c_{ijk\ell}^{E} e_{k\ell} e_{ij} \, dV +$$

$$+ \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \varepsilon_{ij}^{S} E_j E_i \, dV - \frac{1}{2} \iiint\limits_{\Omega^{(0)}} \mu_{ji}^{-1} B_j B_i \, dV + \iiint\limits_{\Omega^{(0)}} \epsilon_{mij} E_m e_{ij} \, dV =$$

$$= U_{\text{kinetic}}[\dot{\boldsymbol{u}}] - U_{\text{elastic}}[\boldsymbol{e}] + U_{\text{electric}}[\boldsymbol{E}] - U_{\text{magnetic}}[\boldsymbol{B}] + U_{\text{piezoelectric}}[\boldsymbol{e}, \boldsymbol{E}] \tag{5.9b}$$

The rearranged and renamed energy terms in Eq.(5.9b) have two advantages: i) each one of the first four is solely of one nature (i.e. only electric or elastic terms involved) as it is suggested by their arguments ii) the first four terms are the same with the respective energy quantities formulated in Chapters 2 and 3 where linear elastic, dielectric and diamagnetic media were considered, while the last fifth term of Eq.(5.9b) captures the whole piezoelectric coupling in energy terms.

The Lagrangian functional of Eq.(5.9b) is verified by Lee (1991), although in that paper the starting point for the construction of the functional was a thermodynamic one, since the enthalpy functional was defined and used.

- Water volume $\Omega^{(1)}$

As we have already mentioned, water volume will be considered as a homogeneous linear dielectric and diamagnetic medium, and so the energy forms appearing in $\Omega^{(1)}$ are

*Electric Energy:* $U_{\text{electric}} = \dfrac{1}{2}\,\varepsilon^{(1)} \iiint\limits_{\Omega^{(0)}} E_i\; E_i\; dV\,.$ (5.10)

*Magnetic energy:* $U_{\text{magnetic}} = \dfrac{1}{2\,\mu^{(1)}} \iiint\limits_{\Omega^{(0)}} B_i\, B_i\; dV$ (5.11)

and so the part of the Lagrangian functional that refers to volume $\Omega^{(1)}$ is expressed as

$$L_{\Omega^{(1)}} = U_{\text{electric}} - U_{\text{magnetic}} = \frac{1}{2}\,\varepsilon^{(1)} \iiint\limits_{\Omega^{(0)}} E_i\; E_i\; dV - \frac{1}{2\,\mu^{(1)}} \iiint\limits_{\Omega^{(0)}} B_i\, B_i\; dV \tag{5.12}$$

- Air volume $\Omega^{(2)}$

As we have already mentioned, air volume will be considered as vacuum from the electromagnetic perspective, and the energy forms appearing in $\Omega^{(2)}$ are

*Electric Energy:* $U_{\text{electric}} = \dfrac{1}{2}\,\varepsilon_0 \iiint\limits_{\Omega^{(0)}} E_i\; E_i\; dV\,.$ (5.13)

*Magnetic energy:* $U_{\text{magnetic}} = \dfrac{1}{2\,\mu_0} \iiint\limits_{\Omega^{(0)}} B_i\, B_i\; dV$ (5.14)

and so the part of the Lagrangian functional that refers to volume $\Omega^{(1)}$ is expressed as

$$L_{\Omega^{(2)}} = U_{\text{electric}} - U_{\text{magnetic}} = \frac{1}{2}\,\varepsilon_0 \iiint\limits_{\Omega^{(0)}} E_i\; E_i\; dV - \frac{1}{2\,\mu_0} \iiint\limits_{\Omega^{(0)}} B_i\, B_i\; dV \tag{5.15}$$

Now, we claim that the total piezoelectric part of the Lagrangian functional is the sum of the functionals (5.9b), (5.12) and (5.15), and by writing aggregately the electric and magnetic energy terms over the whole volume $\Omega$ we obtain:

$$\begin{aligned} L_{\text{piezo}}\left[\boldsymbol{u},\, \boldsymbol{E},\, \boldsymbol{B}\right] &= L_{\Omega^{(1)}} + L_{\Omega^{(2)}} + L_{\Omega^{(3)}} = \\ &= \frac{1}{2}\,\rho_b \iiint\limits_{\Omega^{(0)}} \dot{u}_i\,\dot{u}_i\; dV - \frac{1}{2} \iiint\limits_{\Omega^{(0)}} c^{E}_{ijk\ell}\; e_{k\ell}\; e_{ij}\; dV \; + \\ &+ \frac{1}{2} \iiint\limits_{\Omega} \varepsilon_{ij}\, E_j\, E_i\; dV - \frac{1}{2} \iiint\limits_{\Omega} \mu^{-1}_{ji}\, B_j\, B_i\; dV \; + \\ &+ \iiint\limits_{\Omega^{(0)}} \epsilon_{mij}\, E_m\; e_{ij}\; dV = \\ &= L_{\substack{\text{linear elastodynamics}\\ \text{of Chapter 2}}}\left[\boldsymbol{u}\right] + L_{\substack{\text{linear electrodynamics}\\ \text{of Chapter 3}}}\left[\boldsymbol{E},\, \boldsymbol{B}\right] + U_{\text{piezoelectric}}\left[\boldsymbol{u},\, \boldsymbol{E}\right] \end{aligned} \tag{5.16}$$

Thus, from Eq.(5.16) we can see that the piezoelectric part of the functional referring to the hydro/piezo/electric problem for the whole volume $\Omega$ (piezoelectric body + ambient volume) is just the sum of the functionals obtained for linear elastodynamics in Chapter 2, linear electrodynamics for three dielectric and diamagnetic media in Chapter 3 and a new term of piezoelectric (coupling) energy on the volume $\Omega^{(0)}$.

**Remark 1:** Since the piezoelectric coupling was performed by the change in constitutive, we expect that the boundary terms appearing in the functionals of linear elastodynamics and electrodynamics to remain unaffected.
**Remark 2:** As in Chapter 3, piezoelectric action functional can be expressed in terms of the electromagnetic potentials $\Phi^{e\ell}$ and $\boldsymbol{A}$.

Applying the above remarks, the piezoelectric action functional can be expressed as:

$$
\begin{aligned}
&\mathscr{H}_{\text{piezo}}\left[\boldsymbol{u},\, \Phi^{e\ell},\, \boldsymbol{A}\,;\, \boldsymbol{\lambda},\, \lambda^{e\ell},\, \boldsymbol{v}\right] = \\
&= \mathscr{H}_{\substack{\text{linear}\\ \text{elastodynamics}}}\left[\boldsymbol{u};\, \boldsymbol{\lambda}\right] + \mathscr{H}_{\substack{\text{linear}\\ \text{electrodynamics}}}\left[\Phi^{e\ell}, \boldsymbol{A};\, \lambda^{e\ell},\, \boldsymbol{v}\right] - \mathscr{L}_{\substack{\text{piezoelectric}\\ \text{energy}}}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A}\right] = \\
&= \frac{1}{2}\,\rho_b \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \dot{u}_i\, \dot{u}_i \; dV\, dt - \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} c^{E}_{ijk\ell}\; e_{k\ell}\; e_{ij} \; dV\, dt + \\
&+ \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij} \left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\left(\Phi^{e\ell}_{,i} + \dot{A}_i\right) dV\, dt - \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega} \mu^{-1}_{ji}\, \epsilon_{jmn}\, A_{n,m}\, \epsilon_{ik\ell}\, A_{\ell,k} \; dV\, dt - \\
&- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{mij} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) e_{ij} \; dV\, dt + \\
&+ \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \hat{f}_i\, u_i \; dV\, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}_u} \left(u_i - \hat{u}_i\right)\lambda_i \; dS\, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}_T} \hat{T}_i\, u_i \; dS\, dt - \\
&- \int_{t_0}^{t_1} \iiint\limits_{\Omega} \hat{\rho}_e\, \Phi^{e\ell} \; dV\, dt + \int_{t_0}^{t_1} \iiint\limits_{\Omega} \hat{J}_i\, A_i \; dV\, dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{\Phi}} \left(\Phi^{e\ell} - \hat{\Phi}^{e\ell}\right)\lambda^{e\ell} \; dS\, dt + \\
&+ \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{\sigma}} \hat{\sigma}_e\, \Phi^{e\ell} \; dS\, dt - \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{A}} \epsilon_{ijk}\, n_j \left(A_k - \hat{A}_k\right) v_i \; dS\, dt - \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{K}} \hat{K}_i\, A_i \; dS\, dt
\end{aligned}
\tag{5.17}
$$

### 5.5 Hydro-Elastic Coupling

*Related Reference: Athanassoulis (1982) Part 1, Ch. 2, Par. 6.*

As we have already mentioned in paragraph 5.2, the hydro-elastic part of the coupling will be performed by matching the expressions for normal velocity and pressure of the liquid with the expressions for normal velocity and stress vector of the solid on the solid-liquid interface $\partial\Omega^{(0,1)}$.

More specifically, in the functional for linear elastodynamics (chapter 2), the term is included

$$I_{\partial\Omega_T^{(0)}}^{\text{given } T_i}[\boldsymbol{u}] = \int_{t_0}^{t_1} \iint_{\partial\Omega_T^{(0)}} \hat{T}_i u_i \; dS\, dt \tag{5.18a}$$

which, is the term that introduces the prescribed stress vector $\hat{\boldsymbol{T}}$ on the boundary $\partial\Omega_T^{(0)}$. Identifying this part of the boundary as the solid – liquid interface $\partial\Omega^{(0,1)}$, Eq.(5.18a) is written as

$$I_{\partial\Omega^{(0,1)}}^{\text{given } T_i}[\boldsymbol{u}] = \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,1)}} \hat{T}_i u_i \; dS\, dt \tag{5.18b}$$

The respective term of the functional for hydrodynamics (chapter 4) is

$$\mathscr{L}_{\text{force}}[\Omega(t)] = \int_{t_0}^{t_1} \iint_{\partial\Omega_{c\ell}} \hat{p}\, u_i^{f,c\ell}\, n_i \; dS\, dt \tag{5.19a}$$

which is the term that introduces the prescribed pressure $\hat{p}$ on the moving cliff boundary $\partial\Omega_{c\ell}$. Identifying this part of the boundary as the solid – liquid interface $\partial\Omega^{(0,1)}$, Eq.(5.19a) is written as

$$\mathscr{L}_{\text{force}}[\Omega(t)] = \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,1)}} \hat{p}\, u_i^{f,c\ell}\, n_i^{(1)} \; dS\, dt\,. \tag{5.19b}$$

Realizing that on the interface $\partial\Omega^{(0,1)}$ the displacement of the solid is equal to the displacement of the liquid ($u_i = u_i^{f,c\ell}$) and expressing outward normal unit vector $\boldsymbol{n}$ with respect to the solid volume $\partial\Omega^{(0)}$, Eq.(5.19b) is written as

$$\mathscr{L}_{\text{force}}\left[\Omega(t)\right] = \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0,1)}} \left(-\hat{p}\, n_i^{(0)}\right) u_i \; dS\, dt \,. \tag{5.19c}$$

Comparing Eq.(5.18b) with Eq.(5.19c) we can easily see that they both introduce the prescribed stress on the interface $\partial\Omega^{(0,1)}$, which in Eq.(5.18b) is expressed as $\hat{T}_i$ while in Eq.(5.19c) is expressed as $-\hat{p}\, n_i^{(0)}$. *Thus, in the functional used in the variational formulation for the total hydro/piezo/electricity, only the term of Eq.(5.18b) will appear.*

Apart from the above preparation of the functional terms in order to obtain the matching condition for pressure on $\partial\Omega^{(0,1)}$, we have to consider also the matching of velocity on $\partial\Omega^{(0,1)}$. This is done in an "implicit" way, since velocity is not considered as independent variable in neither the elastodynamic functional (in which the independent variable is elastic displacement $\boldsymbol{u}$) nor the hydrodynamic functional (in which the related independent variable is velocity potential $\Phi^f$).
Velocity on moving boundary $\partial\Omega_{c\ell}$ appears only in a term of Eq.(4.50) after we applied Reynolds transport theorem on partial Gâteaux derivative regarding $\delta\Phi^f$ of Luke's functional:

$$\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_{c\ell}(t)} \hat{\upsilon}_i\, n_i\, \delta\Phi^f \; dS\, dt \tag{5.20}$$

Thus we state that in the use of Reynolds transport theorem in the variational principle for the total hydro/piezo/electric phenomenon, prescribed velocity $\hat{\upsilon}_i$ of Eq.(5.20) will be substituted by the velocity of elastic solid $\dot{u}_i$.

*Summing up*, the hydro-elastic coupling is performed with the following actions:

- Considering the fluid displacement $\boldsymbol{u}^{f,c\ell}$ on $\partial\Omega^{(0,1)}$ (which is an independent variable in Luke's functional for hydrodynamics) as equal to the elastic displacement $\boldsymbol{u}$ on $\partial\Omega^{(0,1)}$ (which is an independent variable in Hamilton's functional for elastodynamics).

- Considering the fluid surface velocity $\boldsymbol{\upsilon}$ on $\partial\Omega^{(0,1)}$ as equal to the elastic velocity $\dot{\boldsymbol{u}}$ on $\partial\Omega^{(0,1)}$.

- Eliminating the prescribed pressure part of Luke's functional, since is a duplicate of term $\int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0,1)}} \hat{T}_i\, u_i \; dS\, dt$, which is the term that will introduce any externally prescribed stresses on the $\partial\Omega^{(0,1)}$, in the variational principle for the whole phenomenon.

**5.6 Variational Principle for the whole Hydro/Piezo/Electric problem**
Taking into consideration the remarks made for the hydrodynamic part of the functional in the previous paragraph, we claim that the functional for the variational formulation of the whole hydro/piezo/electric problem is the sum of functional (5.17) with Luke's functional for hydrodynamics proposed in Chapter 4:

$$
\begin{aligned}
&\mathscr{F}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A}, \Phi^{f}, \Omega^{(2)}(t)\,;\, \boldsymbol{\lambda}, \lambda^{e\ell}, \boldsymbol{v}\right] = \\
&= \mathscr{H}_{\text{piezo}}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A}\,;\, \boldsymbol{\lambda}, \lambda^{e\ell}, \boldsymbol{v}\right] + \mathscr{L}_{\text{hydrodynamic}}\left[\Phi^{f}, \Omega^{(2)}(t)\right] = \\
&= \frac{1}{2}\rho_b \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \dot{u}_i \dot{u}_i \, dV\,dt - \frac{1}{2}\int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} c^{E}_{ijk\ell}\, e_{k\ell}\, e_{ij}\, dV\,dt + \\
&+ \frac{1}{2}\int_{t_0}^{t_1} \iiint\limits_{\Omega} \varepsilon_{ij}\left(\Phi^{e\ell}_{,j} + \dot{A}_j\right)\left(\Phi^{e\ell}_{,i} + \dot{A}_i\right) dV\,dt - \frac{1}{2}\int_{t_0}^{t_1} \iiint\limits_{\Omega} \mu^{-1}_{ji}\,\epsilon_{jmn} A_{n,m}\,\epsilon_{ik\ell} A_{\ell,k}\, dV\,dt - \\
&- \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{mij}\left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) e_{ij}\, dV\,dt - \rho_f \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(2)}(t)} \left(\dot{\Phi}^{f} + \frac{1}{2}\Phi^{f}_{,i}\Phi^{f}_{,i} + g z\right) dV\,dt + \\
&+ \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \hat{f}_i u_i \, dV\,dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}_u} \left(u_i - \hat{u}_i\right)\lambda_i \, dS\,dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0,1)}} \hat{T}_i u_i \, dS\,dt - \\
&- \int_{t_0}^{t_1} \iiint\limits_{\Omega} \hat{\rho}_e \Phi^{e\ell}\, dV\,dt + \int_{t_0}^{t_1} \iiint\limits_{\Omega} \hat{J}_i A_i \, dV\,dt + \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_\Phi} \left(\Phi^{e\ell} - \hat{\Phi}^{e\ell}\right)\lambda^{e\ell}\, dS\,dt + \\
&+ \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_\sigma} \hat{\sigma}_e \Phi^{e\ell}\, dS\,dt - \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_A} \epsilon_{ijk}\, n_j\left(A_k - \hat{A}_k\right) v_i \, dS\,dt - \int_{t_0}^{t_1} \iint\limits_{\partial\Omega_K} \hat{K}_i A_i \, dS\,dt
\end{aligned}
\tag{5.21}
$$

where strain $e_{ij}$ is not be considered as an independent variable, barely an aggregated notation for the quantity $\left(u_{i,j} + u_{j,i}\right)/2$, as in Chapter 2. Also, tensors $\varepsilon_{ij}$ and $\mu^{-1}_{ji}$ have to be seen as $\varepsilon\delta_{ij}$ and $\delta_{ji}/\mu$ respectively in the case of the homogeneous dielectric and diamagnetic media $\Omega^{(1)}$ and $\Omega^{(2)}$.

And thus the variational principle for hydro/piezo/electricity is expressed as

$$
\delta\mathscr{F}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A}, \Phi^{f}, \Omega^{(2)}(t)\,;\, \boldsymbol{\lambda}, \lambda^{e\ell}, \boldsymbol{v} : \delta\boldsymbol{u}, \delta\Phi^{e\ell}, \delta\boldsymbol{A}, \delta\Phi^{f}, \delta\Omega^{(2)}(t), \delta\boldsymbol{\lambda}, \delta\lambda^{e\ell}, \delta\boldsymbol{v}\right] = 0 \tag{5.22}
$$

Since the Gâteaux derivatives for most of the terms of functional (5.21) have been calculated in detail in the previous chapters, we will move on calculating the derivatives of the new coupling term as well as re-calculating $\delta_{\boldsymbol{u}} \mathscr{L}_{\text{hydrodynamic}}\left[\Phi^{f}, \eta, \boldsymbol{u} ; \delta \boldsymbol{u}\right]$, taking into consideration the solid-fluid coupling.

$$\delta_{\boldsymbol{u}} \mathscr{L}_{\substack{\text{piezoelectric}\\ \text{energy}}}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A} : \delta \boldsymbol{u}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) \delta e_{i j} \, dV \, dt =$$

$$= \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) \frac{\delta u_{i,j} + \delta u_{j,i}}{2} \, dV \, dt =$$

$$= \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) \delta u_{i,j} \, dV \, dt + \frac{1}{2} \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m j i} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) \delta u_{i,j} \, dV \, dt =$$

$$= \left[i - j \text{ symmetry of property tensor } \boldsymbol{\epsilon}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) \delta u_{i,j} \, dV \, dt =$$

$$= \left[\text{spatial integration by parts}\right] =$$

$$= \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) n^{(0)}_j \, \delta u_i \, dS \, dt - \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \left(\Phi^{e\ell}_{,m,j} + \dot{A}_{m,j}\right) \delta u_i \, dV \, dt \tag{5.23}$$

$$\delta_{\Phi^{e\ell}} \mathscr{L}_{\substack{\text{piezoelectric}\\ \text{energy}}}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A} : \delta \Phi^{e\ell}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \, e_{i j} \, \delta \Phi^{e\ell}_{,m} \, dV \, dt =$$

$$= \left[\text{spatial integration by parts}\right] =$$

$$= \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0)}} \epsilon_{m i j} \, e_{i j} \, n^{(0)}_m \, \delta \Phi^{e\ell} \, dS \, dt - \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \, e_{i j,m} \, \delta \Phi^{e\ell} \, dV \, dt \tag{5.24}$$

$$\delta_{\boldsymbol{A}} \mathscr{L}_{\substack{\text{piezoelectric}\\ \text{energy}}}\left[\boldsymbol{u}, \Phi^{e\ell}, \boldsymbol{A} : \delta \boldsymbol{A}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{m i j} \, e_{i j} \, \delta \dot{A}_m \, dV \, dt =$$

$$\left[\text{temporal integration by parts}\right] = \int_{t_0}^{t_1} \iiint\limits_{\Omega^{(0)}} \epsilon_{i m j} \, \dot{e}_{m j} \, \delta A_i \, dV \, dt \tag{5.25}$$

$$\delta_{\boldsymbol{u}} \mathscr{L}_{\text{hydrodynamic}}\left[\Phi^{f}, \eta, \boldsymbol{u} : \delta \boldsymbol{u}\right] = -\rho_f \int_{t_0}^{t_1} \iint\limits_{\partial\Omega^{(0,1)}} \left(\dot{\Phi}^{f} + \frac{1}{2} \Phi^{f}_{,j} \Phi^{f}_{,j} + g z\right) n^{(1)}_i \, \delta u_i \, dS \, dt \tag{5.26a}$$

Eq. (5.26) is just Gâteaux derivative calculated in Eq.(4.55), written under the understanding that on the solid-fluid interface, the surface displacement with respect to the fluid $\boldsymbol{u}^{f,c\ell}$ coincides with the surface displacement with respect to the solid $\boldsymbol{u}$.
Eq.(5.26) can be re-written using the outward normal unit vector of piezoelectric volume $\Omega^{(0)}$

$$\delta_{\boldsymbol{u}} \mathscr{L}_{\text{hydrodynamic}}\left[\Phi^f, \eta, \boldsymbol{u} : \delta \boldsymbol{u}\right] = \rho_f \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,1)}} \left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,j}\Phi^f_{,j} + g\,z\right) n_i^{(0)}\,\delta u_i \; dS\,dt \tag{5.26b}$$

and in order the hydrodynamic pressure (normal stress) to have conformity with the general notion for stress used in elasticity, Kronecker's delta is used

$$\delta_{\boldsymbol{u}} \mathscr{L}_{\text{hydrodynamic}}\left[\Phi^f, \eta, \boldsymbol{u} : \delta \boldsymbol{u}\right] = \rho_f \int_{t_0}^{t_1} \iint_{\partial\Omega^{(0,1)}} \left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,m}\Phi^f_{,m} + g\,z\right) \delta_{ij}\, n_j^{(0)}\,\delta u_i \; dS\,dt \tag{5.26c}$$

Thus, the Euler-Lagrange equations obtained from $\delta_{\boldsymbol{u}} \mathscr{F} = 0$ are:

- $\rho_b\, \ddot{u}_i = c^E_{jik\ell}\, e_{k\ell,j} + \epsilon_{mij}\left(\Phi^{e\ell}_{,m,j} + \dot{A}_{m,j}\right) + \hat{f}_i$ over volume $\Omega^{(0)}$ (5.27a)

  which, **by substitution of piezoelectric constitutive relation** (5.2a), is Newton's Second Law for continuous media, as expressed in Eq.(2.3).

- $\left[c^E_{jik\ell}\, e_{k\ell} + \epsilon_{mij}\left(\Phi^{e\ell}_{,m} + \dot{A}_m\right)\right] n_j^{(0)} = 0$ on boundary $\partial\Omega^{(0,2)}$ (5.27b)

  which, **by substitution of piezoelectric constitutive relation** (5.2a), is Cauchy relation for stress (2.2) modelling that piezoelectric boundary is free of stresses on its solid-air interface part.

- $\left[c^E_{jik\ell}\, e_{k\ell} + \epsilon_{mij}\left(\Phi^{e\ell}_{,m} + \dot{A}_m\right) - \rho_f\left(\dot{\Phi}^f + \frac{1}{2}\Phi^f_{,m}\Phi^f_{,m} + g\,z\right)\delta_{ij}\right] n_j^{(0)} = \hat{T}_i$

  on boundary $\partial\Omega^{(0,1)}$ (5.27c)

  which, **by substitution of piezoelectric constitutive relation** (5.2a) and **Bernoulli's equation for hydrodynamic pressure** (4.34), is the matching condition for pressure on the solid- fluid interface $\partial\Omega^{(0,1)}$.

- $\lambda_i = \left[c^E_{jik\ell}\, e_{k\ell} + \epsilon_{mij}\left(\Phi^{e\ell}_{,m} + \dot{A}_m\right)\right] n_j^{(0)}$ on boundary $\partial\Omega^{(0)}_u$ (5.27d)

Eq.(5.27d) is Cauchy relation for stress (2.2) on boundary $\partial\Omega_u^{(0)}$, **by substitution of piezoelectric constitutive relation** (5.2a). This is not a needed boundary condition, but defines the auxiliary field $\boldsymbol{\lambda}$ as the stress over the boundary $\partial\Omega_u^{(0)}$ where the displacement $\boldsymbol{u}$ is prescribed.

The Euler-Lagrange equation obtained from $\delta_{\lambda}\mathscr{F} = 0$ is

- $\hat{u}_i = u_i$ on boundary $\partial\Omega_u^{(0)}$ (5.28)

  which is the right boundary condition on boundary $\partial\Omega_u^{(0)}$ where displacement is prescribed.

The Euler-Lagrange equations obtained from $\delta_{\Phi^{e\ell}}\mathscr{F} = 0$ are

- $$\varepsilon_{ij}^{S}\left(\Phi_{,ji}^{e\ell} + \dot{A}_{j,i}\right) - \epsilon_{mij}\, e_{ij,m} = -\hat{\rho}_e^{(0)} \quad \text{over } \Omega^{(0)} \tag{5.29a}$$
  $$\varepsilon^{(1)}\left(\Phi_{,ii}^{e\ell} + \dot{A}_{i,i}\right) = -\hat{\rho}_e^{(1)} \quad \text{over } \Omega^{(1)} \tag{5.29b}$$
  $$\varepsilon_0\left(\Phi_{,ii}^{e\ell} + \dot{A}_{i,i}\right) = -\hat{\rho}_e^{(2)} \quad \text{over } \Omega^{(2)} \tag{5.29c}$$

  which are Gauss's Law for electrostatics for each of the constituent volumes.

- $$n_i^{(0)}\left[\varepsilon_{ij}^{S}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right) - \epsilon_{ik\ell}\, e_{k\ell} - \varepsilon^{(1)}\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\hat{\sigma}_e^{(0,1)} \text{ on } \partial\Omega_{\sigma}^{(0,1)} \tag{5.30a}$$
  $$n_i^{(0)}\left[\varepsilon_{ij}^{S}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right) - \epsilon_{ik\ell}\, e_{k\ell} - \varepsilon_0\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\hat{\sigma}_e^{(0,2)} \text{ on } \partial\Omega_{\sigma}^{(0,2)} \tag{5.30b}$$
  $$n_i^{(1)}\left[\varepsilon^{(1)}\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right) - \varepsilon_0\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\hat{\sigma}_e^{(1,2)} \text{ on } \partial\Omega_{\sigma}^{(1,2)} \tag{5.30c}$$

  which are the matching condition (3.50) (Jump of $\boldsymbol{D}$-component vertical to interface) on each of the interfaces where surface charge density $\hat{\sigma}_e$ is prescribed.

- $$n_i^{(0)}\left[\varepsilon_{ij}^{S}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right) - \epsilon_{ik\ell}\, e_{k\ell} - \varepsilon^{(1)}\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\lambda^{e\ell} \text{ on } \partial\Omega_{\Phi}^{(0,1)} \tag{5.31a}$$
  $$n_i^{(0)}\left[\varepsilon_{ij}^{S}\left(\Phi_{,j}^{e\ell} + \dot{A}_j\right) - \epsilon_{ik\ell}\, e_{k\ell} - \varepsilon_0\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\lambda^{e\ell} \text{ on } \partial\Omega_{\Phi}^{(0,2)} \tag{5.31b}$$
  $$n_i^{(1)}\left[\varepsilon^{(1)}\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right) - \varepsilon_0\left(\Phi_{,i}^{e\ell} + \dot{A}_i\right)\right] = -\lambda^{e\ell} \text{ on } \partial\Omega_{\Phi}^{(1,2)} \tag{5.31c}$$

  which are the matching condition (3.50) on the interfaces where scalar potential $\Phi^{e\ell}$ is prescribed. This is not a needed matching condition, but defines the auxiliary field $\lambda^{e\ell}$ as the free charge over the boundary $\partial\Omega_{\Phi}$ where the scalar potential $\Phi^{e\ell}$ is prescribed.

The Euler-Lagrange equations obtained from $\delta_A \mathscr{F} = 0$ are

- $$\mu_{kk}^{-1}\left(A_{j,ij} - A_{i,jj}\right) + \mu_{jk}^{-1}\left(A_{i,kj} - A_{k,ij}\right) + \\ + \mu_{ik}^{-1}\left(A_{k,jj} - A_{j,kj}\right) = \hat{J}_i^{(0)} - \varepsilon_{ij}^{S}\left(\dot{\Phi}_{,j}^{e\ell} + \ddot{A}_j\right) + \epsilon_{im\ell}\,\dot{e}_{m\ell} \quad \text{over } \Omega^{(0)} \tag{5.32a}$$

$$\frac{1}{\mu^{(1)}}\left(A_{j,ij} - A_{i,jj}\right) = -\varepsilon^{(1)}\left(\dot{\Phi}_{,i}^{e\ell} + \ddot{A}_i\right) + \hat{J}_i^{(1)} \quad \text{over } \Omega^{(1)} \tag{5.32b}$$

$$\frac{1}{\mu_0}\left(A_{j,ij} - A_{i,jj}\right) = -\varepsilon_0\left(\dot{\Phi}_{,i}^{e\ell} + \ddot{A}_i\right) + \hat{J}_i^{(2)} \quad \text{over } \Omega^{(2)} \tag{5.32c}$$

which are Ampère's – Maxwell's Law for each of the constituent volumes.

- $$n_j^{(0)}\left[\begin{matrix}\mu_{kk}^{-1}\left(A_{j,i} - A_{i,j}\right) + \mu_{jk}^{-1}\left(A_{i,k} - A_{k,i}\right) + \\ + \mu_{ik}^{-1}\left(A_{k,j} - A_{j,k}\right) - \dfrac{1}{\mu^{(1)}}\left(A_{j,i} - A_{i,j}\right)\end{matrix}\right] = \hat{K}_i^{(0,1)} \quad \text{on } \partial\Omega_K^{(0,1)} \tag{5.33a}$$

$$n_j^{(0)}\left[\begin{matrix}\mu_{kk}^{-1}\left(A_{j,i} - A_{i,j}\right) + \mu_{jk}^{-1}\left(A_{i,k} - A_{k,i}\right) + \\ + \mu_{ik}^{-1}\left(A_{k,j} - A_{j,k}\right) - \dfrac{1}{\mu_0}\left(A_{j,i} - A_{i,j}\right)\end{matrix}\right] = \hat{K}_i^{(0,1)} \quad \text{on } \partial\Omega_K^{(0,2)} \tag{5.33a}$$

$$n_j^{(1)}\left[\frac{1}{\mu^{(1)}}\left(A_{j,i} - A_{i,j}\right) - \frac{1}{\mu_0}\left(A_{j,i} - A_{i,j}\right)\right] = \hat{K}_i^{(1,2)} \quad \text{on } \partial\Omega_K^{(1,2)} \tag{5.33c}$$

which are the matching condition (3.51) (Jump of $\boldsymbol{H}$-component vertical to interface) on each of the interfaces where surface current density $\hat{\boldsymbol{K}}$ is prescribed.

- $$n_j^{(0)}\left[\begin{matrix}\mu_{kk}^{-1}\left(A_{j,i} - A_{i,j}\right) + \mu_{jk}^{-1}\left(A_{i,k} - A_{k,i}\right) + \\ + \mu_{ik}^{-1}\left(A_{k,j} - A_{j,k}\right) - \dfrac{1}{\mu^{(1)}}\left(A_{j,i} - A_{i,j}\right)\end{matrix}\right] = \epsilon_{kji}\, n_j^{(0)}\, \nu_k \quad \text{on } \partial\Omega_K^{(0,1)} \tag{5.34a}$$

$$n_j^{(0)}\left[\begin{matrix}\mu_{kk}^{-1}\left(A_{j,i} - A_{i,j}\right) + \mu_{jk}^{-1}\left(A_{i,k} - A_{k,i}\right) + \\ + \mu_{ik}^{-1}\left(A_{k,j} - A_{j,k}\right) - \dfrac{1}{\mu_0}\left(A_{j,i} - A_{i,j}\right)\end{matrix}\right] = \epsilon_{kji}\, n_j^{(0)}\, \nu_k \quad \text{on } \partial\Omega_K^{(0,2)} \tag{5.34a}$$

$$n_j^{(1)}\left[\frac{1}{\mu^{(1)}}\left(A_{j,i} - A_{i,j}\right) - \frac{1}{\mu_0}\left(A_{j,i} - A_{i,j}\right)\right] = \epsilon_{kji} n_j^{(0)} \nu_k \qquad \text{on } \partial\Omega_K^{(1,2)} \qquad (5.34c)$$

which are the matching condition (3.51) on the interfaces where tangent component of vector potential $\boldsymbol{A}$ is prescribed. This is not a needed matching condition, but defines the auxiliary field $\boldsymbol{\nu}$ since quantity $\epsilon_{kji} n_j \nu_k$ is identified as the free current over the boundary $\partial\Omega_A$ where tangent component of vector potential $\boldsymbol{A}$ is prescribed.

The Euler-Lagrange equation obtained from $\delta_{\lambda^{e\ell}} \mathscr{F} = 0$ is

- $\Phi^{e\ell} = \hat{\Phi}^{e\ell}$ on boundaries $\partial\Omega_\Phi^{(0,1)}$, $\partial\Omega_\Phi^{(0,2)}$ and $\partial\Omega_\Phi^{(1,2)}$ (5.35)

  which is the right matching condition on boundary $\partial\Omega_\Phi$ where scalar potential $\Phi^{e\ell}$ is prescribed.

The Euler-Lagrange equation obtained from $\delta_{\nu} \mathscr{F} = 0$ is

- $\epsilon_{ijk} n_j^{(0)} A_k = \epsilon_{ijk} n_j^{(0)} \hat{A}_k$ on boundaries $\partial\Omega_A^{(0,1)}$, $\partial\Omega_A^{(0,2)}$ and $\partial\Omega_A^{(1,2)}$ (5.36)

  which is the right matching condition on boundary $\partial\Omega_A$ where tangent component of vector potential $\boldsymbol{A}$ is prescribed.

The Euler-Lagrange equations obtained from $\delta_{\Phi^f} \mathscr{F} = 0$ are

- $\Phi^f_{,ii} = 0$ over volume $\Omega^{(1)}(t)$ (5.37a)

which is Laplace equation (4.36).

- $\Phi^f_{,i} F_{,i} + \dot{F} = 0$ on free surface $\partial\Omega^{(1,2)}$ (5.37b)

which is the kinematic free surface condition (Eq.4.37c).

- $\Phi^f_{,i} n_i = 0$ on seabed $\partial\Omega_\Pi$ (5.37c)

which is the non-penetration of sea water into the rigid seabed (4.39)

- $\left(\dot{u}_i - \Phi^f_{,i}\right) n_i = 0$ on moving boundary $\partial\Omega^{(0,1)}$ (5.37d)

which is the velocity matching condition on the solid-fluid interface

The Euler-Lagrange equation obtained from $\delta_\eta \mathscr{F} = 0$ (part of the variation $\delta_{\Omega^{(1)}(t)} \mathscr{F}$ ) is

- $\dot{\Phi}^f + \frac{1}{2} \Phi^f_{,i} \Phi^f_{,i} + g\eta = 0$ on free surface $\partial\Omega^{(1,2)}$ (5.37e)

Thus, the above system of equations (5.27a) – (5.37e) is the complete system of equations that models the hydro/piezo/electric phenomenon. An interesting comment on this system of equations is to consider *a posteriori* how piezoelectric coupling appears in Newton's law (5.27a), Gauss's law for electrostatics (5.29a) and Ampère's – Maxwell's law (5.32a) for volume $\Omega^{(0)}$.

*Newton's Law:* $$\rho_b \ddot{u}_i = c^E_{jik\ell} e_{k\ell,j} + \underline{\underline{\epsilon_{mij} \left( \Phi^{e\ell}_{,m,j} + \dot{A}_{m,j} \right)}} + \hat{f}_i$$

The underlined term is the piezoelectric term that appears in addition to the terms of Newton's law in the case of linear elastodynamics and thus, this term can be seen as an **apparent piezoelectric force** should the media considered as linear elastic

$$f_i^{\text{piezo}} = \epsilon_{mij} \left( \Phi^{e\ell}_{,m,j} + \dot{A}_{m,j} \right) \tag{5.38}$$

*Gauss's law:* $$\varepsilon^S_{ij} \left( \Phi^{e\ell}_{,ji} + \dot{A}_{j,i} \right) = -\hat{\rho}_e^{(0)} + \underline{\underline{\epsilon_{mij} e_{ij,m}}}$$

In the same way of analysis as in Newton's law above, the underlined term is added to the Gauss's law for linear dielectric and thus this term can be seen as an **apparent piezoelectric electric charge** should the media considered as linear dielectric

$$\rho^{\text{piezo}} = -\epsilon_{mij} e_{ij,m} \tag{5.39a}$$

*Ampère's – Maxwell's law:* $$\begin{aligned} &\mu^{-1}_{kk} \left( A_{j,ij} - A_{i,jj} \right) + \mu^{-1}_{jk} \left( A_{i,kj} - A_{k,ij} \right) + \\ &+ \mu^{-1}_{ik} \left( A_{k,jj} - A_{j,kj} \right) = \hat{J}^{(0)}_i - \varepsilon^S_{ij} \left( \dot{\Phi}^{e\ell}_{,j} + \ddot{A}_j \right) + \underline{\underline{\epsilon_{im\ell} \dot{e}_{m\ell}}} \end{aligned}$$

In the case of Ampère's – Maxwell's law, the underlined term can be seen as an **apparent piezoelectric electric current** should the media considered as linear dielectric

$$J_i^{\text{piezo}} = \epsilon_{im\ell} \dot{e}_{m\ell} \tag{5.39b}$$

The validity of the above interpretation of the additional piezoelectric terms in the electromagnetic equations is valid, since the apparent electric charges $\rho^{\text{piezo}}$ and currents $J_i^{\text{piezo}}$ satisfy the conservation of electric charge principle

$$\dot{\rho}^{\text{piezo}} + J^{\text{piezo}}_{i,i} = -\epsilon_{im\ell} \dot{e}_{m\ell,i} + \epsilon_{im\ell} \dot{e}_{m\ell,i} = 0 \tag{5.40}$$

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## 6. Quasi – static approximation for piezoelectricity

The quasi – static case is a common approximation in electrodynamics that neglects the magnetic field, leading to equations similar to the electrostatic ones. In the present chapter, the quasi – static approximation shall be performed for the equations of linear piezoelectricity derived in the previous chapter. The present chapter follows the analysis of Bardzokas & Filshtinsky (2006) par. 1.2.8 and Parton & Kudryavtsev (1988) par. 1.3.

Let us consider Faraday's Law of induction (Eq.3.34) and consider the time variation of the magnetic field as sufficiently small

$$\epsilon_{ijk} E_{k,j} = -\dot{B}_i \cong 0 . \tag{6.1}$$

Eq.(6.1) leads to the electrostatic expression of field $\boldsymbol{E}$ using only one (scalar) E/M potential $\Phi^{e\ell}$

$$E_i = -\Phi^{e\ell}_{,i} . \tag{6.2}$$

Now, Eq.(6.2) can be used for the expression of field $\boldsymbol{E}$ in Newton's law and Gauss's law for the electric field as:

$$\rho_b \ddot{u}_i = c^E_{jik\ell} e_{k\ell,j} - \epsilon_{mij} E_m + \hat{f}_i \Rightarrow$$
$$\rho_b \ddot{u}_i = c^E_{jik\ell} e_{k\ell,j} + \epsilon_{mij} \Phi^{e\ell}_{,m,j} + \hat{f}_i \tag{6.3a}$$

and

$$\varepsilon^S_{ij} E_{j,i} = -\hat{\rho}^{(0)}_e + \epsilon_{mij} e_{ij,m} \Rightarrow \varepsilon^S_{ij} \Phi^{e\ell}_{,ji} = -\hat{\rho}^{(0)}_e + \epsilon_{mij} e_{ij,m} \tag{6.3b}$$

as well as the respective boundary and matching conditions

$$\left( c^E_{jik\ell} e_{k\ell} + \epsilon_{mij} \Phi^{e\ell}_{,m} \right) n^{(0)}_j = \hat{T}_i \qquad \text{or} \qquad \hat{u}_i = u_i \tag{6.4a}$$

and

$$n^{(0)}_i \left( \varepsilon^S_{ij} \Phi^{e\ell}_{,j} - \epsilon_{ik\ell} e_{k\ell} - \varepsilon^{(1)} \Phi^{e\ell}_{,i} \right) = -\hat{\sigma}_e \qquad \text{or} \qquad \Phi^{e\ell} = \hat{\Phi}^{e\ell} \tag{6.4b}$$

As it was expected, Eqs.(6.3) and (6.4) are electro-elastic, with no magnetic term, due to the decoupling between electric and magnetic terms performed in Eq.(6.1). Thus, Gauss's law for the magnetic field $B_{i,i} = 0$ does not contribute in the solution of the system of Eqs.(6.3) and (6.4) and thus can be omitted.

The question arising naturally at this point is the following:

*Since the quasi – static approximation performed using Faraday's law of induction (6.1) led to a system of two PDEs (6.3) and (6.4) with two unknown fields* $u_i$ *and* $\Phi^{e\ell}$ *that is uncoupled to the magnetostatic problem* $B_{i,i} = 0$ *and thus can be solved independently, how does the Ampère's – Maxwell's law appear in the quasi – static problem?*

In order to answer the question, we shall return to the physical phenomenon that is modelled by Ampère's – Maxwell's law $\epsilon_{ijk} H_{k,j} = \dot{D}_i + \hat{J}_i$. As it can be seen, it relates a magnetic field term $\epsilon_{ijk} H_{k,j}$ and the time variation of electric displacement $\dot{D}_i$ with the presence of external electric current $\hat{J}_i$. Thus the need of Ampère's – Maxwell's equation is based solely on the presence of electric currents on the examined problem.

- In the case of a conservative problem where no external electric currents $\hat{\boldsymbol{J}}$ appear, the problem is a purely electrostatic system modelled by a closed system of equations and thus there is no need for defining $\dot{\boldsymbol{D}}$.

- In the non-conservative case of a system where current flow $\hat{\boldsymbol{J}}$ is exhibited, a simplified version of Ampère's – Maxwell's relation, such as the definition relation of the displacement current is used in order to express electric current in terms of the unknown fields of Eq.(6.3) and (6.4) Such a $\boldsymbol{J} = \boldsymbol{J}\left(\boldsymbol{u}, \boldsymbol{\Phi}^{el}\right)$ expression can be then used in an (externally imposed) relation involving currents (e.g. Ohm's law) in order to provide the system of Eqs.(6.3), (6.4) with a closure condition.

A case of non-conservative piezoelectric system will be examined thoroughly in the next part of the present work.

**References**

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# 7. Voigt notation

### 7.1 Introducing Voigt notation in elastic fields $\boldsymbol{\sigma}$ and $\boldsymbol{e}$

*General References: Parton & Kudryavtsev (1988) par. 1.1, Wikipedia article on Voigt notation*

As it can be seen on both references of the present chapter, the Voigt notation is inherited by piezoelectricity from it elastic subproblem, and more specifically, from the fact that the two tensors (3rd rank) of elastic stress $\boldsymbol{\sigma}$ and elastic strain $\boldsymbol{e}$ are symmetric, and thus the can be represented by vectors as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \Rightarrow \tilde{\boldsymbol{\sigma}} = \left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\right) \equiv \left(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_6\right) \tag{i}$$

$$\boldsymbol{e} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \Rightarrow \tilde{\boldsymbol{e}} = \left(e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12}\right) \equiv \left(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6\right) \tag{ii}$$

Note that the newly-introduced stress vector $\tilde{\boldsymbol{\sigma}}$ has as components the six distinct element of stress tensor $\boldsymbol{\sigma}$. So vector $\tilde{\boldsymbol{\sigma}}$ is essentially equivalent to tensor $\boldsymbol{\sigma}$, and thus the tilde sign over stress vector components can be omitted, with the following relation holding true

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \equiv \begin{pmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{pmatrix} \tag{7.1}$$

On the other hand, strain vector $\tilde{\boldsymbol{e}}$ consists of the diagonal components of strain tensor $\boldsymbol{e}$ as well as of its off-diagonal components multiplied by two. Thus, a helpful relation between the components of strain tensor and the components of strain vector is

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij} &= e_{11} + e_{12} + e_{13} + e_{21} + e_{22} + e_{23} + e_{31} + e_{32} + e_{33} = \left[\text{symmetric tensor}\right] = \\ &= e_{11} + e_{22} + e_{33} + 2e_{23} + 2e_{13} + 2e_{12} = \left[\text{Voigt definition relation (ii)}\right] = \\ &= \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 + \tilde{e}_5 + \tilde{e}_6 = \sum_{b=1}^{6} \tilde{e}_b \end{aligned} \tag{7.2}$$

So, with the understanding of Eq.(7.2), the tilde sign over strain vector component can also be omitted.

Starting from the definition relations (i) and (ii) of stress and strain vectors in Voigt notation we have concluded that Eqs.(7.1) and (7.2) relate stress and strain vectors with the respective symmetric tensors of stress and strain. Regarding these relations, the following observations can be made

- Eq. (7.1) can only be applied after performing the summations over indices $i$ and $j$, for the individual components of stress tensor to appear and then to be replaced with their one-index equivalents.

- Eq. (7.2) can be applied directly in this contracted sigma form, without the calculation of the summations over indices $i$ and $j$.

### 7.2 Rewriting the quasi-static equations of piezoelectricity in Voigt notation

Considering the simple domain configuration of one piezoelectric volume $\Omega^{(0)}$ surrounded by a non-elastic dielectric volume $\Omega^{(1)}$, while no mass forces and electric charge sources are considered and no part of the boundary $\partial\Omega^{(0,1)}$ has its elastic displacement $\boldsymbol{u}$ or its electrostatic potential $\Phi^{e\ell}$ prescribed, the quasi-static equations of piezoelectricity can be expressed

Newton's Law: $$c^E_{ijk\ell}\, e_{k\ell,j} + \epsilon_{mij}\, \Phi^{e\ell}_{,mj} = \rho_b\, \ddot{u}_i \tag{7.3}$$

Gauss's Law: $$-\varepsilon^{(0)}_{ji}\, \Phi^{e\ell}_{,ij} + \epsilon_{jik}\, e_{ik,j} = 0 \tag{7.4}$$

as well as the respective condition over the boundary $\partial\Omega^{(0,1)}$

elastic boundary condition $$\left[c^E_{ijk\ell}\, e_{k\ell} + \epsilon_{mij}\, \Phi^{e\ell}_{,m}\right] n^{(0)}_j = \hat{\sigma}_{ji}\, n^{(0)}_j \tag{7.5}$$

electric matching condition $$\left[-\varepsilon^{(0)}_{ji}\, \Phi^{e\ell}_{,i} + \epsilon_{jik}\, e_{ik}\right] n^{(0)}_j = -\varepsilon^{(1)}_{ji}\, \Phi^{e\ell}_{,i}\, n^{(0)}_j . \tag{7.6}$$

For the derivation of Eqs. (7.3) – (7.6) see Eqs. (6.3) – (6.4) of the previous chapter.

As we can see in Newton's law (7.3) and the elastic boundary condition (7.5), index $i$ is free (not repeated). Thus, Eqs. (7.3) and (7.5) are vector equations regarding index $i$ and each one of them is equivalent to three scalar equations, one for every value of $i = 1,2,3$.
So the scalar elastic equations are

$$c^E_{1jk\ell}\, e_{k\ell,j} + \epsilon_{m1j}\, \Phi^{e\ell}_{,mj} = \rho_b\, \ddot{u}_1 \tag{7.7}$$
$$c^E_{2jk\ell}\, e_{k\ell,j} + \epsilon_{m2j}\, \Phi^{e\ell}_{,mj} = \rho_b\, \ddot{u}_2 \tag{7.8}$$
$$c^E_{3jk\ell}\, e_{k\ell,j} + \epsilon_{m3j}\, \Phi^{e\ell}_{,mj} = \rho_b\, \ddot{u}_3 \tag{7.9}$$

and the scalar elastic boundary conditions are

$$\left[c^{E}_{1jk\ell}\, e_{k\ell} + \epsilon_{m1j}\, \Phi^{e\ell}_{,m}\right] n^{(0)}_j \;=\; \hat{\sigma}_{j1}\, n^{(0)}_j \tag{7.10}$$

$$\left[c^{E}_{2jk\ell}\, e_{k\ell} + \epsilon_{m2j}\, \Phi^{e\ell}_{,m}\right] n^{(0)}_j \;=\; \hat{\sigma}_{j2}\, n^{(0)}_j \tag{7.11}$$

$$\left[c^{E}_{3jk\ell}\, e_{k\ell} + \epsilon_{m3j}\, \Phi^{e\ell}_{,m}\right] n^{(0)}_j \;=\; \hat{\sigma}_{j3}\, n^{(0)}_j \tag{7.12}$$

As we can see, the 4th rank elastic stiffness tensor $c^{E}_{ijk\ell}$ under constant electric field and the 3rd rank piezoelectric stress tensor $\epsilon_{mk\ell}$ appear in the equations of piezoelectricity. These material property tensors relate the fields of elastic stress $\sigma_{ij}$ elastic strain $e_{k\ell}$ and electrostatic field $E_m \equiv \Phi^{e\ell}_{,m}$ with one another. As it is explained in Chs. 2 and 5

- The first two indices of $c^{E}_{ijk\ell}$ are the same with the indices of elastic stress $\sigma_{ij}$ and its second two indices are the same with the indices o elastic strain $e_{k\ell}$.

- The first index of $\epsilon_{mk\ell}$ is the same with the index of electrostatic field $E_m \equiv \Phi^{e\ell}_{,m}$ with the indices of elastic strain $e_{k\ell}$.

The above remarks lead to the conclusion that, by applying the Voigt notation on elastic tensors $\sigma_{ij}$ and $e_{k\ell}$, the 4th rank elastic stiffness tensor $c^{E}_{ijk\ell}$ is transformed into the symmetric 2nd rank tensor $c^{E}_{ab}$ with $a = 1,2,\ldots,6$ , $b = 1,2,\ldots,6$ and 3rd rank piezoelectric stress tensor $\epsilon_{mk\ell}$ into the 2nd rank tensor $\epsilon_{mb}$ with $m = 1,2,3$ , $b = 1,2,\ldots,6$ . Thus, by the careful application of Voigt notation on $\sigma_{ij}$, $e_{k\ell}$ and the corresponding indices of $c^{E}_{ijk\ell}$ and $\epsilon_{mk\ell}$ that appear in equations (7.7), (7.8), (7.9) and (7.4) as well as the conditions (7.10), (7.11), (7.12) and (7.6) on the boundary, we shall not only reduce the number of indices appearing in these relations but also define the relations between the components of tensors $c^{E}_{ijk\ell}$ and $\epsilon_{mk\ell}$ and the components of he corresponding tensors $c^{E}_{ab}$ and $\epsilon_{mb}$.

All equations and conditions on the boundary are expressed using Einstein summation convention. That means that in these relations, a repeated index implies a summation over all its possible values, which for the indices used in these relations are the integers 1, 2, 3. Einstein summation convention, helpful t may be in order to write lengthy relations in contracted form, it is not convenient when performing the transition to Voigt notation, since the newly-introduced indices $a$ and $b$ , that get entangled with the indices $i, j, k, \ell, m$ have as possible values the integers 1 to 6. In order to avoid any confusion, we shall rewrite the equations and the conditions on the boundary using sigma notation for indicating summation before introducing Voigt notation.

Commencing with Gauss's law (7.4) we obtain

$$-\varepsilon_{ji}^{(0)}\Phi_{,ij}^{e\ell} + \epsilon_{jik}\,e_{ik,j} = 0 \;\Rightarrow\; -\sum_{j=1}^{3}\sum_{i=1}^{3}\varepsilon_{ji}^{(0)}\Phi_{,ij}^{e\ell} + \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\epsilon_{jik}\,e_{ik,j} = 0 \;\Rightarrow$$

$$[\text{using Eq.(7.2)}] \;\Rightarrow\; -\sum_{j=1}^{3}\sum_{i=1}^{3}\varepsilon_{ji}^{(0)}\Phi_{,ij}^{e\ell} + \sum_{j=1}^{3}\sum_{b=1}^{6}\epsilon_{jb}\,e_{b,j} = 0 \tag{7.13}$$

Eq.(7.2) is applied with the understanding that the components of the appearing tensor $\epsilon_{jb}$ are defined as follows

$$\epsilon_{jb} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} & \epsilon_{14} & \epsilon_{15} & \epsilon_{16} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} & \epsilon_{24} & \epsilon_{25} & \epsilon_{26} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} & \epsilon_{34} & \epsilon_{35} & \epsilon_{36} \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} \epsilon_{111} & \epsilon_{122} & \epsilon_{133} & \epsilon_{123} & \epsilon_{113} & \epsilon_{112} \\ \epsilon_{211} & \epsilon_{222} & \epsilon_{233} & \epsilon_{223} & \epsilon_{213} & \epsilon_{212} \\ \epsilon_{311} & \epsilon_{322} & \epsilon_{333} & \epsilon_{323} & \epsilon_{313} & \epsilon_{312} \end{pmatrix} \tag{7.14}$$

Eq.(7.13) can be simplified further, if we assume a Cartesian system of axes that results into a tensor of dielectric permittivities $\boldsymbol{\varepsilon}^{(0)}$ that is diagonal. Such a system can always be found, see Newnham (2005) par. 9.3. So

$$\varepsilon_{11}^{(0)}\Phi_{,11}^{e\ell} + \varepsilon_{22}^{(0)}\Phi_{,22}^{e\ell} + \varepsilon_{33}^{(0)}\Phi_{,33}^{e\ell} = \sum_{j=1}^{3}\sum_{b=1}^{6}\epsilon_{jb}\,e_{b,j} \tag{7.15}$$

Electric matching condition (7.6) can be expressed using Voigt notation if the same with Eq.(7.14) chain of actions is followed. The result is

$$\left[-\varepsilon_{ji}^{(0)}\Phi_{,i}^{e\ell} + \epsilon_{jik}\,e_{ik}\right] n_j^{(0)} = -\varepsilon_{ji}^{(1)}\Phi_{,i}^{e\ell}\,n_j^{(0)} \;\Rightarrow$$

$$\Rightarrow \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\epsilon_{jik}\,e_{ik}\,n_j^{(0)} = \sum_{j=1}^{3}\sum_{i=1}^{3}\left(\varepsilon_{ji}^{(0)}\Phi_{,i}^{e\ell} - \varepsilon_{ji}^{(1)}\Phi_{,i}^{e\ell}\right) n_j^{(0)} \;\Rightarrow$$

$$\Rightarrow \sum_{j=1}^{3}\sum_{b=1}^{6}\epsilon_{jb}\,e_{b}\,n_j^{(0)} = \sum_{j=1}^{3}\sum_{i=1}^{3}\left(\varepsilon_{ji}^{(0)}\Phi_{,i}^{e\ell} - \varepsilon_{ji}^{(1)}\Phi_{,i}^{e\ell}\right) n_j^{(0)} \tag{7.16}$$

In order to simplify Eq.(7.16) with regard to its electric components in a similar way with the simplification made in Eq.(7.13), we have to make assumptions not only for the form of dielectic tensor $\boldsymbol{\varepsilon}^{(0)}$ of the piezoelectric body, but also for the form of dielectric tensor $\boldsymbol{\varepsilon}^{(1)}$ that refers to the electric properties of the ambient volume. More specifically, since we use the same Cartesian coordinates for both the piezoelectric body and its ambient volume, we can always choose a system of axes that diagonalises tensor $\boldsymbol{\varepsilon}^{(0)}$ (as in Eq.7.15) but such a choice does not mean that also diagonalises tensor $\boldsymbol{\varepsilon}^{(1)}$. Thus, Eq.(7.16) can be simplified only if we neglect the electric field in the ambient volume, and *thus approximating the electric matching condition with one electric boundary condition*:

$$\varepsilon_{11}^{(0)}\Phi_{,1}^{e\ell}n_1^{(0)} + \varepsilon_{22}^{(0)}\Phi_{,2}^{e\ell}n_2^{(0)} + \varepsilon_{33}^{(0)}\Phi_{,3}^{e\ell}n_3^{(0)} = \sum_{j=1}^{3}\sum_{b=1}^{6}\epsilon_{jb}\,e_b\,n_j^{(0)} \tag{7.17}$$

Now we can move on performing the transition to Voigt notation with regard to the elastic equation (7.7) *(it is the equation involving acceleration component $\ddot{u}_1$):*

$$c_{1jk\ell}^{E}e_{k\ell,j} + \epsilon_{m1j}\Phi_{,mj}^{e\ell} = \rho_b\ddot{u}_1 \Rightarrow \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{\ell=1}^{3}c_{1jk\ell}^{E}e_{k\ell,j} + \sum_{k=1}^{3}\sum_{j=1}^{3}\epsilon_{k1j}\Phi_{,kj}^{e\ell} = \rho_b\ddot{u}_1 \Rightarrow$$

$$[\text{using Eq.(7.2)}] \Rightarrow \sum_{j=1}^{3}\sum_{b=1}^{6}c_{1jb}^{E}e_{b,j} + \sum_{k=1}^{3}\sum_{j=1}^{3}\epsilon_{k1j}\Phi_{,kj}^{e\ell} = \rho_b\ddot{u}_1 \tag{7.18}$$

To proceed further with Eq.(7.18), two lemmas will be stated and proven:

**Lemma 1.** $\displaystyle\sum_{j=1}^{3}\sum_{b=1}^{6}c_{1jb}^{E}e_{b,j} = \sum_{b=1}^{6}c_{1b}^{E}e_{b,1} + \sum_{b=1}^{6}c_{5b}^{E}e_{b,3} + \sum_{b=1}^{6}c_{6b}^{E}e_{b,2}$

*Proof.* Writing explicitly the summation over $j$ we obtain

$$\sum_{j=1}^{3}\sum_{b=1}^{6}c_{1jb}^{E}e_{b,j} = \sum_{b=1}^{6}c_{11b}^{E}e_{b,1} + \sum_{b=1}^{6}c_{12b}^{E}e_{b,2} + \sum_{b=1}^{6}c_{13b}^{E}e_{b,3}$$

Since the first two indices of tensor $c_{ijb}^{E}$ are the same with the indices of elastic stress tensor $\sigma_{ij}$, we can directly apply Voigt notation on these indices, on the right side of the above relation, on the basis of Eq.(7.1)

$$\sum_{j=1}^{3}\sum_{b=1}^{6}c_{1jb}^{E}e_{b,j} = \sum_{b=1}^{6}c_{1b}^{E}e_{b,1} + \sum_{b=1}^{6}c_{6b}^{E}e_{b,2} + \sum_{b=1}^{6}c_{5b}^{E}e_{b,3} \qquad \square$$

**Lemma 2.** $$\sum_{k=1}^{3}\sum_{j=1}^{3}\epsilon_{k1j}\,\Phi^{e\ell}_{,kj} = \sum_{k=1}^{3}\epsilon_{k1}\,\Phi^{e\ell}_{,k1} + \sum_{k=1}^{3}\epsilon_{k6}\,\Phi^{e\ell}_{,k2} + \sum_{k=1}^{3}\epsilon_{k5}\,\Phi^{e\ell}_{,k3}$$

*Proof.* Writing explicitly the summation over $j$ we obtain

$$\sum_{k=1}^{3}\sum_{j=1}^{3}\epsilon_{k1j}\,\Phi^{e\ell}_{,kj} = \sum_{k=1}^{3}\epsilon_{k11}\,\Phi^{e\ell}_{,k1} + \sum_{k=1}^{3}\epsilon_{k12}\,\Phi^{e\ell}_{,k2} + \sum_{k=1}^{3}\epsilon_{k13}\,\Phi^{e\ell}_{,k3} =$$
$$= \sum_{k=1}^{3}\epsilon_{k1}\,\Phi^{e\ell}_{,k1} + \sum_{k=1}^{3}\epsilon_{k6}\,\Phi^{e\ell}_{,k2} + \sum_{k=1}^{3}\epsilon_{k5}\,\Phi^{e\ell}_{,k3}$$

since tensor $\epsilon_{kb}$ is defined according to Eq.(7.14).

Using lemmas 1 and 2, Eq.(7.18) can be written as

$$\sum_{b=1}^{6}c^{E}_{1b}\,e_{b,1} + \sum_{b=1}^{6}c^{E}_{6b}\,e_{b,2} + \sum_{b=1}^{6}c^{E}_{5b}\,e_{b,3} + $$
$$+ \sum_{k=1}^{3}\epsilon_{k1}\,\Phi^{e\ell}_{,k1} + \sum_{k=1}^{3}\epsilon_{k6}\,\Phi^{e\ell}_{,k2} + \sum_{k=1}^{3}\epsilon_{k5}\,\Phi^{e\ell}_{,k3} = \rho_b\,\ddot{u}_1 \tag{7.19}$$

Following the same chain of action, elastic boundary condition (7.10) can be written as

$$\sum_{b=1}^{6}c^{E}_{1b}\,e_b\,n^{(0)}_1 + \sum_{b=1}^{6}c^{E}_{6b}\,e_b\,n^{(0)}_2 + \sum_{b=1}^{6}c^{E}_{5b}\,e_b\,n^{(0)}_3 + \sum_{k=1}^{3}\epsilon_{k1}\,\Phi^{e\ell}_{,k}\,n^{(0)}_1 + \sum_{k=1}^{3}\epsilon_{k6}\,\Phi^{e\ell}_{,k}\,n^{(0)}_2 + $$
$$+ \sum_{k=1}^{3}\epsilon_{k5}\,\Phi^{e\ell}_{,k}\,n^{(0)}_3 = \hat{\sigma}_{11}\,n^{(0)}_1 + \hat{\sigma}_{21}\,n^{(0)}_2 + \hat{\sigma}_{31}\,n^{(0)}_3 \tag{7.20}$$

Using Eq.(7.1) on the right side of Eq.(7.20) we obtain

$$\sum_{b=1}^{6}c^{E}_{1b}\,e_b\,n^{(0)}_1 + \sum_{b=1}^{6}c^{E}_{6b}\,e_b\,n^{(0)}_2 + \sum_{b=1}^{6}c^{E}_{5b}\,e_b\,n^{(0)}_3 + \sum_{k=1}^{3}\epsilon_{k1}\,\Phi^{e\ell}_{,k}\,n^{(0)}_1 + \sum_{k=1}^{3}\epsilon_{k6}\,\Phi^{e\ell}_{,k}\,n^{(0)}_2 + $$
$$+ \sum_{k=1}^{3}\epsilon_{k5}\,\Phi^{e\ell}_{,k}\,n^{(0)}_3 = \hat{\sigma}_1\,n^{(0)}_1 + \hat{\sigma}_6\,n^{(0)}_2 + \hat{\sigma}_5\,n^{(0)}_3 \tag{7.21}$$

Following the same procedure with the rest scalar elastic equations (7.8), (7.9) and the rest scalar elastic boundary conditions we obtain

- For equation (7.8) *(it is the equation involving acceleration component* $\ddot{u}_2$*):*

$$\sum_{b=1}^{6} c^{E}_{6b}\, e_{b,1} + \sum_{b=1}^{6} c^{E}_{2b}\, e_{b,2} + \sum_{b=1}^{6} c^{E}_{4b}\, e_{b,3} + \\ + \sum_{k=1}^{3} \epsilon_{k6}\, \Phi^{e\ell}_{,k1} + \sum_{k=1}^{3} \epsilon_{k2}\, \Phi^{e\ell}_{,k2} + \sum_{k=1}^{3} \epsilon_{k4}\, \Phi^{e\ell}_{,k3} = \rho_b\, \ddot{u}_2 \tag{7.22}$$

- For boundary condition (7.11):

$$\sum_{b=1}^{6} c^{E}_{6b}\, e_b\, n^{(0)}_1 + \sum_{b=1}^{6} c^{E}_{2b}\, e_b\, n^{(0)}_2 + \sum_{b=1}^{6} c^{E}_{4b}\, e_b\, n^{(0)}_3 + \sum_{k=1}^{3} \epsilon_{k6}\, \Phi^{e\ell}_{,k}\, n^{(0)}_1 + \sum_{k=1}^{3} \epsilon_{k2}\, \Phi^{e\ell}_{,k}\, n^{(0)}_2 + \\ + \sum_{k=1}^{3} \epsilon_{k4}\, \Phi^{e\ell}_{,k}\, n^{(0)}_3 = \hat{\sigma}_6\, n^{(0)}_1 + \hat{\sigma}_2\, n^{(0)}_2 + \hat{\sigma}_4\, n^{(0)}_3 \tag{7.23}$$

- For equation (7.9) *(it is the equation involving acceleration component* $\ddot{u}_3$*):*

$$\sum_{b=1}^{6} c^{E}_{5b}\, e_{b,1} + \sum_{b=1}^{6} c^{E}_{4b}\, e_{b,2} + \sum_{b=1}^{6} c^{E}_{3b}\, e_{b,3} + \\ + \sum_{k=1}^{3} \epsilon_{k5}\, \Phi^{e\ell}_{,k1} + \sum_{k=1}^{3} \epsilon_{k4}\, \Phi^{e\ell}_{,k2} + \sum_{k=1}^{3} \epsilon_{k3}\, \Phi^{e\ell}_{,k3} = \rho_b\, \ddot{u}_3 \tag{7.24}$$

- For boundary condition (7.11):

$$\sum_{b=1}^{6} c^{E}_{5b}\, e_b\, n^{(0)}_1 + \sum_{b=1}^{6} c^{E}_{4b}\, e_b\, n^{(0)}_2 + \sum_{b=1}^{6} c^{E}_{3b}\, e_b\, n^{(0)}_3 + \sum_{k=1}^{3} \epsilon_{k5}\, \Phi^{e\ell}_{,k}\, n^{(0)}_1 + \sum_{k=1}^{3} \epsilon_{k4}\, \Phi^{e\ell}_{,k}\, n^{(0)}_2 + \\ + \sum_{k=1}^{3} \epsilon_{k3}\, \Phi^{e\ell}_{,k}\, n^{(0)}_3 = \hat{\sigma}_5\, n^{(0)}_1 + \hat{\sigma}_4\, n^{(0)}_2 + \hat{\sigma}_3\, n^{(0)}_3 \tag{7.25}$$

For equations (7.19), (7.22), (7.24) and boundary conditions (7.21), (7.23), (7.25) to be well defined, we have to define explicitly the components of the newly-introduced tensor $c^{E}_{ab}$. Like the definition relation (7.14) for tensor $\epsilon_{jb}$, the components of tensor $c^{E}_{ab}$ must compatible with Eq.(7.2).

Thus:

$$c_{ab}^{E} = \begin{pmatrix} c_{11}^{E} & c_{12}^{E} & c_{13}^{E} & c_{14}^{E} & c_{15}^{E} & c_{16}^{E} \\ c_{21}^{E} & c_{22}^{E} & c_{23}^{E} & c_{24}^{E} & c_{25}^{E} & c_{26}^{E} \\ c_{31}^{E} & c_{32}^{E} & c_{33}^{E} & c_{34}^{E} & c_{35}^{E} & c_{36}^{E} \\ c_{41}^{E} & c_{42}^{E} & c_{43}^{E} & c_{44}^{E} & c_{45}^{E} & c_{46}^{E} \\ c_{51}^{E} & c_{52}^{E} & c_{53}^{E} & c_{54}^{E} & c_{55}^{E} & c_{56}^{E} \\ c_{61}^{E} & c_{62}^{E} & c_{63}^{E} & c_{64}^{E} & c_{65}^{E} & c_{66}^{E} \end{pmatrix} \equiv \begin{pmatrix} c_{1111}^{E} & c_{1122}^{E} & c_{1133}^{E} & c_{1123}^{E} & c_{1113}^{E} & c_{1112}^{E} \\ c_{2211}^{E} & c_{2222}^{E} & c_{2233}^{E} & c_{2223}^{E} & c_{2213}^{E} & c_{2212}^{E} \\ c_{3311}^{E} & c_{3322}^{E} & c_{3333}^{E} & c_{3323}^{E} & c_{3313}^{E} & c_{3312}^{E} \\ c_{2311}^{E} & c_{2322}^{E} & c_{2333}^{E} & c_{2323}^{E} & c_{2313}^{E} & c_{2312}^{E} \\ c_{1311}^{E} & c_{1322}^{E} & c_{1333}^{E} & c_{1323}^{E} & c_{1313}^{E} & c_{1312}^{E} \\ c_{1211}^{E} & c_{1222}^{E} & c_{1233}^{E} & c_{1223}^{E} & c_{1213}^{E} & c_{1212}^{E} \end{pmatrix} \tag{7.26}$$

**References**

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Wikipedia article on Voigt notation: http://en.wikipedia.org/wiki/Voigt_notation

# PART II.

## A NON-CONSERVATIVE HYDRO/PIEZO/ELECTRIC SYSTEM

# 1. Introduction

The second part of the present thesis is presented somewhat differently than the first one. One obvious reason for this is the wish to incorporate the published paper in the corpus of the present thesis as it is. The other, more fundamental reason is that, while the first part deals with the expression of the governing equations for a conservative hydro/piezo/electric system in general, the aim of part II is to obtain solutions for a non-conservative hydro/piezo/electric system.

This gives rise to two issues:

the issue of **specifying system's configuration** (rather than referring to a general configuration of volumes as in part I), since there is a number of configuration parameters to be determined:

- The internal structure of the piezoelectric body (one bulk body or many individual elements, electrical wiring between the elements)
- The position of the piezoelectric body inside the fluid domain.
- The external circuit choice (AC or DC)

and the issue of the need for **analytic solution,** in order to obtain easily an accurate solution. Thus, the following analysis is restricted to the simplified case of piezoelectric coupling in which:

*Of all components of the piezoelectric property tensor $\boldsymbol{\epsilon}$, only the component $\epsilon_{333} \equiv \epsilon_{33}$ has a non-zero value (practically its value is sufficiently higher that the values of the rest of the components). This simplification determines that the only piezoelectric effect exhibited is between the normal strain $e_{33}$ and the electric field component $E_3$. The reduction of piezoelectricity to the coupling between these directions of the elastic and the electric fields only is called **3-3 mode** or **thickness mode.**[1]*

As it can be seen in the next chapter, under the assumption of thickness mode and by defining the configuration parameters, we were able to obtain results for a simple case of a non-conservative hydro/piezo/electric system.

The last chapter of part II deals with bridging the distributed model of piezoelectricity used throughout the present work with the lumped model for piezoelements encountered in bibliography (e.g. Lefeuvre *et al.* 2010). Through a Lagrangian formulation considering both

[1] Thickness mode is certainly not the only choice; according to each material's tensor $\boldsymbol{\epsilon}$, that mode can or cannot be a valid simplification. In many piezoelectric materials, the component dominating tensor $\boldsymbol{\epsilon}$ is $\epsilon_{311} \equiv \epsilon_{31}$. In these materials, the analogous simplification of considering $\epsilon_{31}$ as the sole non-zero component of $\boldsymbol{\epsilon}$ is called 3-1 mode or thickness-shear mode.

elastic and electric subproblems, it is proven that the lumped model proposed in bibliography for small piezoelements operating in 3-3 mode is equivalent to the distributed one.

**Reference**

Lefeuvre, E., Lallart, M., Richard, C. and Guyomar, D. (2010), Chapter 9 in (Eds., Gómez, E.S. *et. al.*),*Piezoelectric Ceramics,* SCIYO. (Free online edition available at www.sciyo.com).

# Modeling and analysis of a cliff-mounted piezoelectric sea-wave energy absorption system

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**Abstract.** Sea waves induce significant pressures on coastal surfaces, especially on rocky vertical cliffs or breakwater structures (Peregrine 2003). In the present work, this hydrodynamic pressure is considered as the excitation acting on a piezoelectric material sheet, installed on a vertical cliff, and connected to an external electric circuit (on land). The whole hydro/piezo/electric system is modeled in the context of linear wave theory. The piezoelectric elements are assumed to be small plates, possibly of stack configuration, under a specific wiring. They are connected with an external circuit, modeled by a complex impedance, as usually happens in preliminary studies (Liang and Liao 2011). The piezoelectric elements are subjected to thickness-mode vibrations under the influence of incident harmonic water waves. Full, kinematic and dynamic, coupling is implemented along the water-solid interface, using propagation and evanescent modes (Athanassoulis and Belibassakis 1999). For most energetically interesting conditions the long-wave theory is valid, making the effect of evanescent modes negligible, and permitting us to calculate a closed-form solution for the efficiency of the energy harvesting system. It is found that the efficiency is dependent on two dimensionless hydro/piezo/electric parameters, and may become significant (as high as 30 – 50%) for appropriate combinations of parameter values, which, however, corresponds to exotically flexible piezoelectric materials. The existence or the possibility of constructing such kind of materials formulates a question to material scientists.

**Keywords:** renewable energy; piezoelectricity; sea wave energy

## 1. Introduction

Ocean waves carry huge amount of energy propagating in a thin layer near the surface of the sea and, eventually, impinging on the coastline. Being a surface phenomenon, sea waves consist one of the most intense natural energy resources. Nowadays this resource has been very well documented throughout the world ocean. See, e.g., Pontes, Athanassoulis *et al*. (1995, 1996), Cavaleri, Athanassoulis, Barstow (1999), Barstow and Mørk *et al*. (2003), Barstow *et al*. (2009), Mørk and Barstow *et al*. (2010), which describe the results of three European Commission–funded projects (WERATLAS, EUROWAVES and WORLDWAVES) studying offshore and nearshore wave conditions and wave energy resource.

In the open sea, especially in the northern oceans, the mean wave power may be more than per meter of the wave front (100 kW/m). Of course, as the waves approach the coast, shoaling

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causes breaking on the free surface and dissipation in the seabed boundary layer, resulting in lower figures for the available mean wave power per wave-front meter. Even thought, when the shoreline has the form of an (almost) vertical cliff, either rocky or manmade, with appreciable depth in front of it, waves impinge on it exerting large pressure loads. The wave climate in such sites has been extensively studied, mainly to provide information for the design of breakwaters or for the study of the erosive effects on natural coasts, as well as for assessing the available wave potential for nearshore and onshore wave energy devices. An extended list of many existing wave energy devices can be found in Wikipedia (http://en.wikipedia.org/wiki/Wave_power). In depth discussions of the physics principles and the technological aspects of the various devices are provided by the relevant papers in the technical literature; modern guides to this huge literature are the recently published books by Cruz (2008) and Khaligh and Onar (2010). Types of power take-off include: hydraulic ram, elastomeric hose pump, pump-to-shore, hydroelectric turbine, oscillating water columns in conjunction with air turbine, linear electrical generator, etc. In the present work an alternative point of view is adopted. We are going to investigate if it is possible to take off wave power directly through a piezoelectric material placed on the cliff.

Piezoelectricity, known since 1880 thanks to the experimental work by the brothers Pierre and Jacques Curie, has been intensively exploited in the recent years for designing energy harvesting devices, mainly in microscale. See, e.g., the recent review articles Sodano *et al.* (2004), Anton and Sodano (2007), Priya (2007), and the books by Priya and Inman (2009), Erturk and Inman (2011). Most of devices studied or reviewed in the literature are vibration–based energy harvesters, transducing the energy of mechanical vibrations to electric power supply of small electronic devices. Some concepts appropriate for converting energy from ambient fluid flow into useful electrical energy have appeared in the last decade or so. For example, Priya *et al.* (2005) and Myers *et al.* (2007) designed and tested a piezoelectric windmill, transducing wind energy into electricity; Taylor *et al.* (2001) and Pobering and Schwesinger (2004) studied piezoelectric flag generators, consisting of a flexible sheet placed downstream of a bluff body and excited by the von Kármán vortex sheet.

The subject of direct piezoelectric conversion of ocean wave energy is rather undeveloped. The main reason for this seems to be the very low frequency regime of sea waves (below to 0.5 Hz). Early concepts of piezoelectric wave harvesters, based on piezoelectric films or ropes made of Polyvinylidene fluoride (PVDF) (Taylor and Burns 1983, Haeusler and Stein 1985), have not been practically applied. The concept of a floating wave carpet, proposed by Koola and Ibragimov (2003) could be interesting when combined with an appropriate modeling and analysis of a flexible piezo-electric material. Murray and Rastegar (2009) proposed a two-stage piezoelectric wave energy harvester, consisting of a primary, low frequency, subsystem (e.g., a heaving buoy), which excites a secondary subsystem vibrating at its natural frequency, the latter being orders of magnitude higher than the frequency of the primary subsystem. The aforementioned piezo-electric wave energy harvesters, as well as other existing variants of them, all belong to the classes of point absorbers or attenuators.

The goal of the present paper is to investigate a terminator-type piezoelectric system that could extract electric energy from the direct impact of sea waves, impinging upon a vertical cliff. This seems to be the simplest possible configuration of a hydro/piezo/electric system, that could be deployed in large scale on the cliffs, especially those ones formed by breakwaters or floating breakwaters. Wave energy impinging upon such kind of structures induces large loads that can have only catastrophic effects. If a part of it could be transduced into electricity, two advantages would be realized: relaxing the exerted loads and gaining useful energy.

The structure of the paper is as follows: In Sec. 2 the whole system, consisting of three distinct subsystems (the hydrodynamic and the piezoelectric ones, and an external electrical circuit), is described in detail. In Sec. 3 the 3-3 mode of the piezoelectric vibration of a single piezoelement and the whole piezoelectric sheet covering the cliff, under a specific wiring, are studied. In Sec. 4 the hydrodynamic problem is formulated and a complete modal representation of the wave potential in the vicinity of the vertical cliff is given. Results from Sec. 3 and 4 are exploited in Sec. 5, where the coupling of the two subsystems is implemented through the interfacial, fluid–solid, matching conditions, taking the form of an infinite system of algebraic equations with respect to the modal coefficients. In the same section an approximate (yet accurate) closed form expression is obtained for the wave reflection coefficient, which controls the energetic coupling of the three subsystems. Finally, in Sec. 6, the ohmic resistance of the external circuit optimizing the efficiency of the hydro/piezo/electric harvester is found. The optimized efficiency is calculated analytically and investigated numerically. It is shown that efficiency may become significant (as high as 30 – 50%) for appropriate combinations of two dimensionless hydro/piezo/ electric parameters. To practically exploit this high efficiency new piezoelectric materials are needed, exhibiting much higher flexibility than the usual ones, and high values of the energy conversion factor. The possibility of manufacturing such kind of materials remains an open question.

## 2. System configuration

Before proceeding to the consideration of a specific, piezoelectric, wave-energy harvesting system, a description of the "virgin site" where this system could be installed in, seems to be appropriate. The virgin site would be any vertical cliff, either natural or manmade, as, e.g., a rocky cliff, a breakwater or a floating breakwater, with appreciable sea depth in front of it, so that the shoaling and dissipation effects to remain mild. Under these conditions, incoming waves induce large pressure loads on the vertical cliffs, which can be considered as rigid (non-deformable) bodies. The impinging wave energy partly dissipates (due to wave breaking and bottom friction), and partly is reflected back to the sea. The vertical cliff, being rigid and not moving, acts as a perfect barrier of the energy flow. The proposed concept of wave energy harvesting relies on the following observations: if a deformable body is interposed between the rigid vertical cliff and the incoming waves, the presence of both pressure on and deformation of the fluid-solid interface would result into an energy flow from sea waves to this body. If, in addition, the deformable body exhibits piezoelectric properties, part of the energy flowing through the fluid-solid interface would be transformed into electrical energy, which could be stored in (or consumed by) an external electric circuit, without the intervention of any other mechanical parts.

Since the present paper aims at a preliminary assessment of such an energy harvesting system, we focus on the basic physics facts, disregarding many technical details. Even though, we have to make a complete (yet simplified) modeling of three distinct subsystems: the hydrodynamic subsystem, i.e., the hydrodynamic wave field in the vicinity of the cliff, the piezoelectric subsystem, i.e., the material layer posed on the cliff and facing the action of sea waves, and an external electrical circuit, located on land.

### *2.1 The hydrodynamic subsystem: sea waves impinging into the cliff*

Waves impinging into the cliff produce a complicated, nonlinear, slightly dissipative, impact phenomenon, resulting in the development of a fluctuating hydrodynamic pressure pattern on the fluid-solid interface. Realistic, wind-generated, sea waves are usually modeled as random waves, characterized by means of their spectrum. The angular frequencies $\omega$ may range from $0.314\,\text{rad/sec}$ to $3.14\,\text{rad/sec}$ (corresponding to periods $2\,\text{sec} < T < 20\,\text{sec}$), the actual range being strongly case and site dependent. The complete modeling of this phenomenon is an extremely difficult problem, not fully understood yet, which is out of the scope of the present paper. A general description of the phenomenon along with a survey of earlier works has been presented by Peregrine (2003). Some aspects of the nonlinear water-wave impact problem on rigid vertical surfaces have been recently studied by Molin *et al.* (2005), Jamois *et al.* (2006), Molin *et al.* (2010). Advanced methods of numerical simulation of such problems, using moving particles techniques and taking into account both nonlinearities and dissipation effects, have also been developed recently; see, e.g., Khayyer and Gotoh (2009).

For reasons explained above, we shall restrict ourselves to a reasonably convenient mathematical formulation of the hydrodynamic problem, namely the linear water-wave theory; see, e.g., Wehausen and Laitone (1960), Sec. 11. We shall also make the assumptions that the vertical cliff has an appreciable horizontal extent and the front of the incident wave is almost aligned to it, which permit us to treat the hydrodynamic problem as two-dimensional (2D). In addition, to simplify the hydrodynamic analysis, we assume that the seabed is horizontal. A vertical section of the fluid domain $\Omega$ is shown in Fig. 1. In the same figure it is also shown the Cartesian coordinate system used in the hydrodynamic analysis. The not shown $y$ axis (perpendicular to the paper) extends along the horizontal dimension of the vertical cliff.

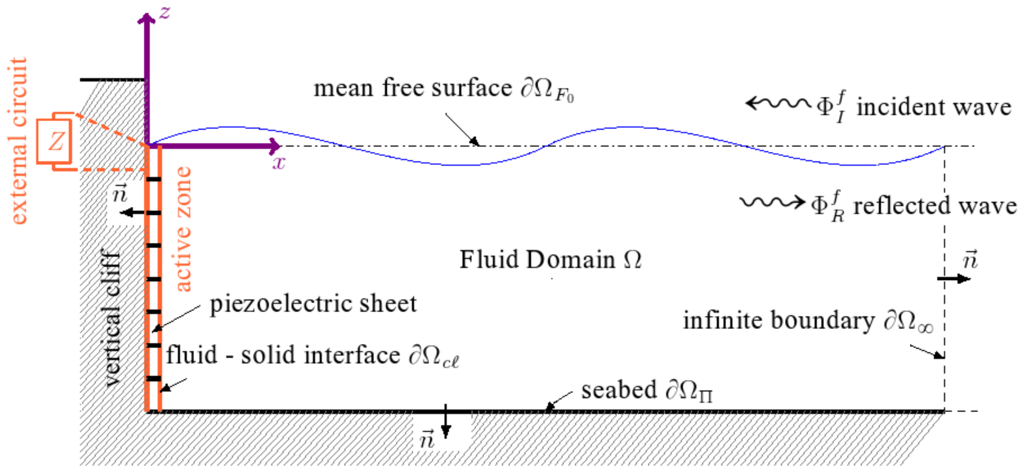


Fig. 1 Geometric configuration of the system

Under the assumption of linearity, the superposition principle is valid, which permits us to synthesize any (linear) wave pattern from the monochromatic (frequency domain) solution. Thus, focusing on the monochromatic case, we can assume that the velocity field is derived by a velocity potential $\Phi^f(x,z;t)$, which is expressed in terms of the complex phasor $\Phi^f(x,z;\omega)$ by means of the equation

$$\begin{aligned}\Phi^f(x,z;t) &= \mathrm{Re}_j\left\{\Phi^f(x,z;\omega)\exp(j\omega t)\right\} = \\ &= \mathrm{Re}_j\left\{\left(\Phi_I^f(x,z;\omega)+\Phi_R^f(x,z;\omega)+\Phi_{loc}^f(x,z;\omega)\right)\exp(j\omega t)\right\}\end{aligned} \tag{1a}$$

where

$$\Phi_I^f(x,z;\omega) = \frac{jg}{\omega}\frac{H}{2}\frac{\cosh\left[k_0\left(h_D+z\right)\right]}{\cosh(k_0 h_D)}\exp(jk_0 x) \tag{1b}$$

is the incident wave, having amplitude $H/2$

$$\Phi_R^f(x,z;\omega) = W\frac{jgH}{2\omega}\frac{\cosh\left[k_0\left(h_D+z\right)\right]}{\cosh(k_0 h_D)}\exp(-jk_0 x) \tag{1c}$$

is the reflected wave, and $\Phi_{\ell oc}^f(x,z;\omega)$ is a local wave field, vanishing exponentially far from the cliff. (The exact form of $\Phi_{\ell oc}^f(x,z;\omega)$ will be given in Sec.4). In Eqs. (1), $j=\sqrt{-1}$ is the imaginary unit, $g$ is the acceleration due to gravity, $\omega$ is the frequency of the monochromatic incident wave, $k_0$ is the corresponding wave number, $h_D$ is the sea depth in front of the vertical cliff and $W$ is the reflection coefficient. The latter is, in general, complex valued, $W = |W|\cdot e^{j\,\mathrm{Arg}(W)}$, $|W|$ being the amplitude attenuation factor and $\mathrm{Arg}(W)$ being the phase shift with respect to the incident wave.

The hydrodynamic pressure field in the fluid, $p(x,z;\omega)$, is given by the linearized Bernoulli's law:

$$p(x,z;\omega) = -j\rho_f\,\omega\,\Phi^f(x,z;\omega) \tag{2}$$

where $\rho_f$ is the mass density of sea water. Note that, when the nonlinear effects are taken into account, the total hydrodynamic pressure induced on the vertical cliff exhibits, in general, larger values than those obtained by means of the linear theory.

### *2.2 The piezoelectric subsystem: energy harvesting elements on the cliff*

Piezoelectricity, initially detected in some crystalline solid materials, is a phenomenon according to which an electric field is developed in the material in response to externally applied mechanical stresses. It is a reversible process; when an external electric field is applied to the piezoelectric material, the latter exhibits deformations. Linear piezoelectricity is quantified macroscopically by means of the piezoelectric constitutive equations, connecting mechanical stress $\{\sigma_i\}_{i=1}^{i=6} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\}$ and electric displacement $D_k$ to mechanical strain $\{e_i\}_{i=1}^{i=6} = \{e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{31}, 2e_{12}\}$ and electric intensity $E_k$

$$\sigma_i = c_{ik}^E e_k - \epsilon_{ki} E_k, \qquad D_i = \epsilon_{ik} e_k + \varepsilon_{ij}^S E_j \tag{3a,b}$$

where $c_{ik}^E$ is the elastic stiffness tensor under constant electric intensity, $\varepsilon_{ij}^S$ is the dielectric permittivity tensor under constant strain, and $\epsilon_{ik}$ is the piezoelectric stress tensor. The latter contains null elements since the piezoelectric effect disappears for certain crystallographic and limiting point symmetry groups. (Newnham 2005, Ch. 12.3).

The widely used piezoelectric materials largely consist of two classes; the piezoelectric ceramics (e.g., PZT family) and the electroactive polymers (EAP), as PVDF. Piezoelectric ceramics dominate the transducer applications, showing strong piezoelectric effect but are stiff and brittle and thus inappropriate, in the form of bulk materials, for energy harvesting applications where flexibility is needed (Brockmann 2009, Ch.4). On the other hand, traditional EAP show relatively improved flexibility but moderate piezoelectric coefficients (Bar-Cohen 2010). In between the two aforementioned classes of materials, lie the piezoelectric composites which combine high coupling factors with relatively high mechanical flexibility (Uchino 2010), but their properties cannot differ substantially from the properties of their constituent materials. Also interesting materials are those manufactured by the newer developments in EAP, such as relaxor ferroelectric copolymers and cellular polymers (Bauer and Bauer 2008). Probably the most promising materials are the dielectric electroactive polymers (DEAP), exhibiting the ability of large deformations along with high values of energy conversion ratio (Carpi *et. al.* 2008). Let it be noted, however, that the electromechanical properties of DEAP do not fit well in the classical piezoelectric modeling, followed in this work, since they show viscoelastic behavior and they are practically incompressible.

Piezoelectric materials are available either in small solid pieces or in the form of films or ropes. In this conjunction, and in order to exploit the thickness-mode oscillations, the piezo-elements considered in the present study are assumed to be small plates with transverse dimensions $\ell_1$, $\ell_2$, of order of magnitude of some centimeters, and thickness $h$, of order of magnitude of some millimeters. One of their surfaces $S = \ell_1 \times \ell_2$ is clamped on the vertical cliff and the other is free to oscillate under the influence of the wave impact. Piezoelements are installed contiguously from the sea bottom to the mean free surface and are electrically connected in series, forming a vertical array of $M_1$ piezoelements; see Fig. 1. The repetition of this array for an appreciable length $L_2 = M_2\, \ell_2$ in the direction of the $y-$axis (horizontally along the cliff), in conjunction with a parallel electric connection between the vertical arrays, results in a two dimensional active zone of piezoelements, which is also called the piezoelectric sheet; Fig. 2.

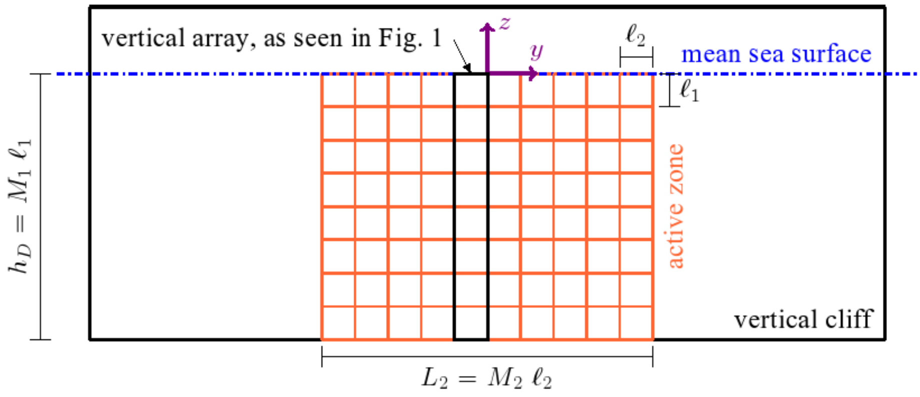


Fig. 2 The two-dimensional, cliff-mounded, piezoelectric active zone

Two basic technical issues, relevant to the formation and the installation of the piezoelectric sheet, are the insulation from the ambient sea water and the fixation on the vertical cliff. Both issues are strongly material dependent and they are out of the scope of the present work, which aims at a feasibility study of the basic concept.

### *2.3 The external electrical circuit*

In order to take off power from the impinging waves, the output terminals of the system of piezoelements should be plugged in an external electrical circuit. A typical choice for the latter is the so-called standard energy harvesting (SEH) circuit, including a diode rectifier and a smoothing capacitor; see, e.g., Gyomar *et al.* (2005) and Shu and Lien (2006). A simpler choice, which is the usual one in most of the literature emphasizing on the mechanical part of the system, is a standard AC circuit, characterized by its impedance

$$Z(\omega) = R + j\,X(\omega) \tag{4}$$

where $R$ models the total resistance and $X(\omega)$ models the total reactance. A thorough discussion concerning the effect of the external circuit on the energy flow in piezoelectric harvesters can be found in Liang and Liao (2011). In the context of linear theory, the angular frequency $\omega$ comes from the monochromatic wave excitation, having a very low value. By studying the considered hydro/piezo/electric system connected to the above described circuit, it is possible to find a closed form expression for the net (time average) power taken off from the waves, which reveals the main (dimensionless) parameters affecting the energy harvesting phenomenon.

## 3. The piezoelectric problem

### *3.1 The piezoelectric problem for a single piezoelement*

For each piezoelement, a local, $(x_1\, x_2\, x_3)$ – Cartesian coordinate system is introduced, with $x_i$-axis coinciding with the corresponding principal piezoelectric axis; see Fig. 3. Each piezo-element is considered geometrically symmetric with respect to the coordinate planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Face $\gamma\delta$ is clamped (on the vertical cliff), while face $\alpha\beta$ is free to oscillate under the influence of incoming sea waves. [Note that in the physical position, faces $\alpha\beta$ and $\gamma\delta$ of each piezoelement are vertical; cf. Fig. 1]. Both faces $\alpha\beta$ and $\gamma\delta$ are electroded.

In this paper the thickness-mode vibration is considered, in which the resulting electric polarization vector has the same direction as the applied stress (thus $i = j = 3$). Thus, the constitutive Eqs. (3(a) and (b)) take the form

$$\sigma_3(x_3;t) = c_{33}^E e_3(x_3;t) - \epsilon_{33} E_3(x_3;t) \tag{5a}$$

$$D_3(x_3;t) = \epsilon_{33} e_3(x_3;t) + \varepsilon_3^S E_3(x_3;t) \tag{5b}$$

where $\varepsilon_3^S \equiv \varepsilon_{33}^S$. The external excitation (tensile) stress $\hat{\sigma}_3$, applied to the electroded face $\alpha\beta$, equals to $-p$, where $p$ is the hydrodynamic pressure; the presence of minus sign is due to the fact that $p$ is always compressive. The applied excitation $\hat{\sigma}_3$ gives rise to mechanical displacements $u_3(x_3;t)$ and voltage difference

$$\Delta V(t) = V_1(t) - V_0(t) = \Phi^{e\ell}\left(h/2;\, t\right) - \Phi^{e\ell}\left(-h/2;\, t\right) \tag{6}$$

between the two faces $\alpha\beta$ and $\gamma\delta$, where $\Phi^{e\ell}(x_3;t)$ is the electric potential field developed inside the piezoelement.

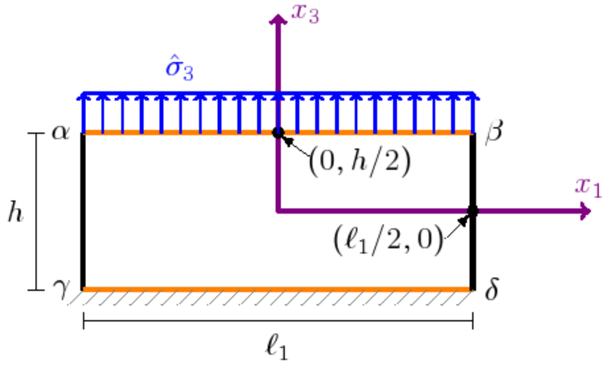


Fig. 3 Mode 3-3 vibration of a single piezoelement with $\alpha\beta$ and $\gamma\delta$ faces electroded

As the physical length (width) of each piezoelement is small in comparison with both the depth $h_D$ of the sea in front of the vertical cliff and the wavelength of sea waves, the applied stress $\hat{\sigma}_3$ (due to sea waves) can be considered almost constant on the face $\alpha\beta$ of each piezoelement. Thus, we can consider $\hat{\sigma}_3$ equal to the mean value of the hydrodynamic pressure $-p$ over the face $\alpha\beta$, and simplify the piezoelectric problem assuming that all quantities are dependent only on $x_3$ coordinate. In this way, the piezoelectric phenomenon to be studied becomes essentially one dimensional (1D).

The equations governing the piezoelectric phenomenon are Newton's second law for deformable bodies, Maxwell's equations, the constitutive equations of piezoelectricity, and appropriate boundary conditions. See, e.g., Parton and Kudryavtsev (1988), Ch 1, Bardzokas and Filshtinsky (2006), Ch. 2, Meitzler *et. al.* (1987). Note that mechanical and dielectric dissipative phenomena are ignored in this study.

Since the frequency range of sea waves (exciting the piezoelements) is very low in comparison with electromagnetic waves frequencies, the equations governing the piezoelectric phenomenon are the quasi-static ones. In addition, under the assumption of monochromatic excitation, with circular frequency $\omega$, all quantities can be represented by the corresponding phasors, i.e., $u_3\left(x_3;\, t\right) \;=\; \mathrm{Re}_j \left\{u_3\left(x_3;\, \omega\right) \exp\left(j\,\omega t\right)\right\}$. Then, for the present case of 1D linear problem in the frequency domain, the set of governing equations and boundary conditions takes the form

$$c^E_{33}\,\frac{\partial^2 u_3}{\partial x_3^2}\left(x_3;\omega\right) + \epsilon_{33}\,\frac{\partial^2 \Phi^{e\ell}}{\partial x_3^2}\left(x_3;\omega\right) \;=\; -\rho_b\,\omega^2\,u_3\left(x_3;\omega\right) \tag{7}$$

$$\varepsilon^S_3\,\frac{\partial^2 \Phi^{e\ell}}{\partial x_3^2}\left(x_3;\omega\right) \;=\; \epsilon_{33}\,\frac{\partial^2 u_3}{\partial x_3^2}\left(x_3;\omega\right) \tag{8}$$

$$u_3\left(-h/2;\omega\right) \;=\; 0 \tag{9a}$$

$$c^E_{33}\,\frac{\partial u_3}{\partial x_3}\left(h/2;\omega\right) + \epsilon_{33}\,\frac{\partial \Phi^{e\ell}}{\partial x_3}\left(h/2;\omega\right) \;=\; \hat{\sigma}_3\,(\omega) \tag{9b}$$

$$\Phi^{e\ell}\left(-h/2;\omega\right) \;=\; V_0\,(\omega) \tag{10a}$$

$$\Phi^{e\ell}\left(h/2;\omega\right) \;=\; V_1\left(\omega\right) \tag{10b}$$

where $\rho_b$ is the mass density of the piezoelement.

Eq. (7) is Newton's second law for deformable bodies containing also an electric term due to constitutive Eqs. (5(a) and (8)) is Gauss's law for the electric field containing also an elastic term due to constitutive Eq. (5(b)). Eqs. (9(a) and (b)) are mechanical boundary conditions. The problem is supplemented by the electrostatic boundary conditions Eqs. (10(a) and (b)). Eq. (10(a)) is a gauge condition which sets the level value of the potential. $V_0(\omega)$ is arbitrarily chosen, the quantity having physical meaning being the voltage difference $\Delta V(\omega)$. Eq. (10(b)) relates the unknown quantity $V_1(\omega) = \Delta V(\omega) + V_0(\omega)$ with the also unknown quantity $\Phi^{e\ell}\left(h/2\,;\,\omega\right)$. Accordingly, boundary conditions Eqs. (10(a) and (b)) do not specify boundary data; they just specify relations between unknown quantities. As a consequence, the solution of the boundary problem (7) - (10) is not unique. As we shall see in the sequel, this lack of boundary data will result in an undermined coefficient, the true value of which will be obtained later on by using information from the external electric circuit. This complicacy makes the present (direct) piezoelectric problem different from the (inverse) piezoelectric problem of free mechanical vibrations under the influence of known voltage difference (more extensively studied in the literature, see, e.g., Yang (2006b)), where the corresponding boundary conditions do not contain unknown quantities, rendering the problem uniquely solvable Ieşan (1990).

The boundary-value problem (7) - (10) is easily solved, as follows. Integrating twice Eq. (8) and using boundary conditions Eqs. (9(a)) and (10(a)), we express $\Phi^{e\ell}\left(x_3;\omega\right)$ in terms of $u_3\left(x_3\,;\,\omega\right)$ and an unknown coefficient $A(\omega)$, in the form

$$\Phi^{e\ell}\left(x_3\,;\,\omega\right) = \frac{\epsilon_{33}}{\varepsilon_3^S}\, u_3\left(x_3\,;\,\omega\right) + A(\omega)\left(x_3 - \frac{h}{2}\right) + V_0(\omega)\,.$$

Substituting this expression for $\Phi^{e\ell}\left(x_3\,;\,\omega\right)$ in boundary condition Eq. (9(b)) and in Eq. (7), we formulate a boundary value problem for $u_3\left(x_3\,;\,\omega\right)$, containing also $A(\omega)$. Solving the latter, $u_3\left(x_3;\,\omega\right)$ is calculated in terms of $A(\omega)$. Applying the resulting solution to $x_3 = h/2$ and invoking Eq. (6), we get

$$u_3\left(x_3 = h/2;\,\omega\right) = \left(\hat{\sigma}_3(\omega) - \epsilon_{33}\,A(\omega)\right)\frac{h}{c_{33}^D}\,\frac{\tan(\tilde{\omega})}{\tilde{\omega}} \tag{11}$$

$$\Delta V(\omega) = V_1(\omega) - V_0(\omega) = \frac{\epsilon_{33}\,\hat{\sigma}_3(\omega) - \epsilon_{33}^2\,A(\omega)}{\varepsilon_3^S}\,\frac{h}{c_{33}^D}\,\frac{\tan(\tilde{\omega})}{\tilde{\omega}} + A(\omega)\,h \tag{12}$$

where $$c_{33}^D = c_{33}^E + \epsilon_{33}^2/\varepsilon_3^S \qquad \text{and} \qquad \tilde{\omega} \equiv \omega h\sqrt{\rho_b/c_{33}^D} \tag{13a,b}$$

are the elastic stiffness coefficient under constant electric displacement, and the dimensionless frequency, respectively.

Eqs. (11) and (12) can be simplified further, by observing that $\tilde{\omega}$ is a very small quantity in the context of the considered application. In fact, taking into account that for common piezo-elements the orders of magnitude of the involved quantities are: $\rho_b \sim O(10^4 kg/m^3)$, $c_{33}^D \sim O(10^{10} Pa)$, $h \le h_{\max} \sim O(10^{-2} m)$ (see, e.g., Bauer and Bauer 2008, Bloomfield 1994, Smith and Auld 1991, APC 2002), and that for sea waves $\omega \sim O(1 rad/s)$, we find that $\tilde{\omega} \sim O(10^{-7})$.

Thus, $\tan(\tilde{\omega})/\tilde{\omega} \approx 1$, and Eqs. (11) and (12) can be safely simplified to

$$u_3\left(x_3 = h/2; \omega\right) = \left[\hat{\sigma}_3(\omega) - \epsilon_{33} A(\omega)\right] \frac{h}{c_{33}^D} \tag{14}$$

$$\Delta V(\omega) = V_1(\omega) - V_0(\omega) = \frac{\left[\epsilon_{33}\hat{\sigma}_3(\omega) - \epsilon_{33}^2 A(\omega)\right]}{\varepsilon_3^S} \frac{h}{c_{33}^D} + A(\omega) h \tag{15}$$

The undetermined coefficient $A(\omega)$ will be now expressed in terms of the current $I = I(\omega)$ flowing through the piezoelement (and through the external circuit).

To bring the electric current into play, we need some (simplified) electrodynamic equation, not included in the quasi-static problem (7) - (10). Since the magnetic field is negligible, the additional electrodynamic equation can be thought in various different ways, e.g., either as the time derivative of Gauss's law (Erturk and Inman 2011, Sec. 3.1.3, Parton and Kudryavtsev 1988, Sec. 1.3), or as the conservation of electric charge, or as a degenerate form of Ampère's law which provide the definition of displacement current. In any case, for the present 1D piezoelectric problem, the additional equation for the electric current has the form

$$I/S = \dot{D}_3 \tag{16}$$

where $S$ is the area of each of the electroded surfaces of the piezoelement. Using constitutive relation Eq.(5(b)) and Eq. (16) is written as

$$I = j\omega S\left(\epsilon_{33} e_3 + \varepsilon_3^S E_3\right) \tag{17}$$

Recalling that $e_3 = \partial u_3/\partial x_3$ and $E_3 = -\partial \Phi^{el}/\partial x_3$, we obtain, for the case of $\tilde{\omega} << 1$

$$e_3 = \frac{\hat{\sigma}_3(\omega) - \epsilon_{33} A(\omega)}{c_{33}^D}, \qquad E_3 = -\frac{\epsilon_{33}\hat{\sigma}_3(\omega) - \epsilon_{33}^2 A(\omega)}{\varepsilon_3^S c_{33}^D} - A(\omega) \tag{18a,b}$$

Substituting Eqs. (18(a) and (b)) into Eq. (17) we get

$$I(\omega) = -j\omega\varepsilon_3^S S A(\omega) \tag{19}$$

Now, since the voltage, Eq. (15), and the current, Eq. (19), are both expressed in terms of coefficient $A(\omega)$, it is clear that coupling of the piezoelement with an external electrical circuit would provide us with a specific value of $A(\omega)$ and, thus, a complete solution of the piezoelectric problem.

For comparison purposes we consider here the two limiting cases, namely, the open-electrode piezoelement ($I(\omega)=0$), and the short-circuit piezoelement ($\Delta V(\omega)=0$). In the first case, $A(\omega)=0$ and thus, from Eq. (15), we get $\Delta V(\omega) = \left(\epsilon_{33}/\varepsilon_3^S\right)\left(h/c_{33}^D\right)\hat{\sigma}_3(\omega)$, a result also given by APC (2002), Table 1.8. In the second case, combining $\Delta V(\omega)=0$ with Eq. (15), we obtain $A(\omega) = -k_t^2\,\hat{\sigma}_3(\omega)\big/\epsilon_{33}\left(1-k_t^2\right)$, where

$$k_t^2 = \frac{\epsilon_{33}^2}{\varepsilon_3^S\, c_{33}^D} \tag{20}$$

is the energy conversion (or coupling) factor. Introducing the above expression for $A(\omega)$ into Eq. (14) we obtain $u_3(\omega)=\hat{\sigma}_3(\omega)h/((1-k_t^2)c_{33}^D)=(h/c_{33}^E)\hat{\sigma}_3(\omega)$, in accordance with Preumont (2011), Sec. 3.6.2. For a physical interpretation of $k_t^2$ and the derivation of equation $c_{33}^E=\left(1-k_t^2\right)c_{33}^D$, see Jaffe *et. al.* (1971), Ch. 3, Sec. C.1. Furthermore, in Jaffe *et. al.* (1971), Ch. 2, Sec. 2, it is proven that $0<k_t^2<1$.

### *3.2 The system of piezoelements on the vertical cliff in series connection*

We shall now proceed to considering the whole active zone. Various connections are possible between the electrodes of adjacent piezoelements that form each vertical array. In the present work a series connection has been selected, as depicted in Fig. 4.

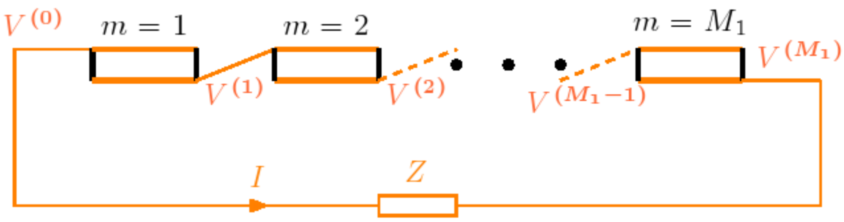


Fig. 4 Series connection of piezoelements forming one vertical array

The results obtained in previous subsection, for a single piezoelement, can be applied to each piezoelement of the group. All quantities associated with the $m-th$ piezoelement, e.g., $u_3\left(h/2,\,\omega\right)$, $\hat{\sigma}_3(\omega)$, etc., will be now distinguished by a superscript $m$ in parenthesis, e.g.,

$u_3^{(m)}(h/2,\,\omega)$, $\hat{\sigma}_3^{(m)}(\omega)$, etc.. Considering all piezoelements being of the same material and of the same dimensions, we do not use the $m$ superscript for material properties and element dimensions. On the basis of the series connection of adjacent piezoelements, $I^{(m)}(\omega) = I(\omega)$, $V_0^{(m)} = V^{(m-1)}$ and $V_1^{(m)} = V^{(m)}$, $m = 1, 2, \ldots, M_1$. Using Eq. (11), the voltages $V^{(m)}(\omega)$ at the output electrode of each piezoelement are given by (see also Fig. 4)

$$V^{(m)}(\omega) - V^{(m-1)}(\omega) = \frac{\left[\epsilon_{33}\,\hat{\sigma}_3^{(m)}(\omega) - \epsilon_{33}^2\,A^{(m)}(\omega)\,h\right]}{\varepsilon_3^S\,c_{33}^D} + A^{(m)}(\omega)\,h \tag{21}$$

where

$$\hat{\sigma}_3^{(m)}(\omega) = \frac{1}{\ell_1}\int_{\substack{m-th\\ piezoelement}} \hat{\sigma}_3(z;\omega)dz \tag{22}$$

and $\hat{\sigma}_3(z;\omega) = -\hat{p}(z;\omega), -h_D \le z \le 0$, $z$ being the global vertical coordinate; see Fig. 1.

Setting $V^{(0)} = 0$ on the first electrode of the first piezoelement, and summing up all Eqs. (21), we find the total voltage difference

$$\Delta V(\omega) = V^{(M_1)}(\omega) = \frac{\left(\epsilon_{33}\sum_{m=1}^{M_1}\hat{\sigma}_3^{(m)}(\omega) - \epsilon_{33}^2\sum_{m=1}^{M_1}A^{(m)}(\omega)\right)h}{\varepsilon_3^S\,c_{33}^D} + h\sum_{m=1}^{M_1}A^{(m)}(\omega) \tag{23}$$

Applying Ohm's law to the external circuit, represented here by an equivalent impedance $Z(\omega)$, we get the following equation for the current

$$I(\omega) = \frac{\Delta V(\omega)}{Z(\omega)} = \frac{\left(\epsilon_{33}\sum_{m=1}^{M_1}\hat{\sigma}_3^{(m)}(\omega) - \epsilon_{33}^2\sum_{m=1}^{M_1}A^{(m)}(\omega)\right)h}{\varepsilon_3^S\,c_{33}^D\,Z(\omega)} + \frac{h}{Z(\omega)}\sum_{m=1}^{M_1}A^{(m)}(\omega) \tag{24}$$

As the piezoelements are connected in series, the current $I(\omega)$ is common over the whole circuit. Thus, Eq. (19), applied to each piezoelement, takes the form

$$I(\omega) = -\,j\omega\varepsilon_3^S S A^{(m)}(\omega) \;\Rightarrow\; A^{(1)} = \ldots = A^{(m)} = \ldots = A^{(M_1)} = A \tag{25}$$

By introducing the piezoelectric constant

$$C_0 = \frac{\varepsilon_3^S\,S}{h} \tag{26}$$

called the clamped capacitance (of each piezoelement), [for a physical interpretation see, e.g., Lefeuvre *et. al.* (2010), Guyomar *et. al.* (2005)], and setting

$$\mathscr{E}_t(\omega) = \frac{k_t^2}{(1-k_t^2) + j\omega\,(C_0 / M_1)\,Z(\omega)} \tag{27}$$

the system of Eqs. (24) and (25) provides the following solution for the common value of $A$'s, $A(\omega)$

$$A(\omega) = -\frac{\mathscr{E}_t(\omega)}{\epsilon_{33}} \frac{1}{M_1} \sum_{m=1}^{M_1} \hat{\sigma}_3^{(m)}(\omega) \tag{28}$$

Substituting Eq. (28) into Eq. (24), we obtain the total voltage output

$$\Delta V(\omega) = I(\omega)\,Z(\omega) = j\omega\,C_0\,Z(\omega)\frac{\mathscr{E}_t(\omega)}{\epsilon_{33}}\,h\,\frac{1}{M_1}\sum_{m=1}^{M_1}\hat{\sigma}_3^{(m)}(\omega) \tag{29}$$

Eq. (29) provides us with the solution of the piezoelectric system connected with an external circuit of equivalent impedance $Z(\omega)$, in terms of the applied stress $\hat{\sigma}_3^{(m)}(\omega)$. However, in order to implement the coupling of this (sub)system with the water wave impinging on the cliff (given by Eqs. (42) and (43)), we need to find a relation between the piezoelements face velocities $j\omega\,u_3^{(m)}\left(x_3 = h/2;\omega\right)$ and the excitation stresses $\hat{\sigma}_3^{(m)}(\omega)$. To this aim, we come back to the Eq. (14), written for each piezoelement, and substitute $A(\omega)$ from Eq. (28)

$$u_3^{(m)}(x_3 = h/2;\omega) = \frac{h}{c_{33}^D}\hat{\sigma}_3^{(m)}(\omega) + \frac{h\mathscr{E}_t(\omega)}{c_{33}^D}\frac{1}{M_1}\sum_{m=1}^{M_1}\hat{\sigma}_3^{(m)}(\omega), \quad m = 1,\dots,M \tag{30}$$

Since the vertical width $\ell_1$ of each piezoelement is only a small fraction of the water-wave length, the stress variation over the face $\alpha\beta$ of each piezoelement is negligible, which implies that the mean excitation stress $\hat{\sigma}_3^{(m)}$ is approximately equal to $\hat{\sigma}_3(z;\omega) = -\hat{p}(z;\omega)$, with $z$ restricted to vary over the face $\alpha\beta$ of the $m-th$ piezoelement. Moreover, the sum of the mean stresses applied over the totality of piezoelements, can be written, using Eq. (22), as

$$\frac{1}{M}\sum_{m=1}^{M_1}\hat{\sigma}_3^{(m)}(\omega) = \frac{1}{h_D}\int_{-h_D}^{1}\hat{\sigma}_3(z;\omega)dz = \overline{\hat{\sigma}_3(\omega)} \tag{31}$$

where $\overline{\hat{\sigma}_3(\omega)}$ is the mean excitation stress over the whole vertical cliff. Thus, Eq. (30) can be reformulated in a continuous fashion in the form

$$\hat{u}_3(z;\omega) \;=\; \frac{h}{c_{33}^{D}}\,\hat{\sigma}_3(z;\omega) \;+\; \frac{h}{c_{33}^{D}}\,\mathscr{E}_t(\omega)\,\overline{\hat{\sigma}_3(\omega)}, \qquad -h_D \le z \le 0 \tag{32}$$

where $\hat{u}_3(z;\omega) \approx u_3^{(m)}(x_3 = h/2;\omega)$, for $z$ varying over the face $\alpha\beta$ of the $m-th$ piezo-element. Eq. (32) shows that the mechanical displacement of the interface $\partial\Omega_{c\ell}$ (the outer face of the piezoelectric sheet covering the cliff) comes from two terms, a local one and a global one. The first is of elastic nature, having a local dependence on the applied pressure and stiffness coefficient $c_{33}^{D} = c_{33}^{E} + \epsilon_{33}^{2} / \varepsilon_3^{S}$. The presence of $c_{33}^{D}$, which is greater than the standard $c_{33}^{E}$ coefficient appearing in the constitutive Eq. (5(a)), models the piezoelectric stiffening phenomenon. (For a general discussion and mathematical formulation of piezoelectric stiffening see Auld (1969) and Yang (2006a), Sec. 2.2.1.) The second term is of purely piezoelectric nature, it has a global dependence on the applied pressure, and is also dependent on the external electric circuit characteristics through the factor $\mathscr{E}_t(\omega)$.

Let it be noted that, by substituting Eq. (30) into Eq. (29) for the case of the single piezoelement ($M_1 = 1$), the following relation between $\Delta V(\omega)$ and $\hat{u}_3(\omega)$ is obtained

$$\Delta V(\omega) \;=\; j\,\omega\, a\, Z(\omega)\;\hat{u}_3(\omega) / \big(1 + j\,\omega\, C_0\, Z(\omega)\big), \qquad \text{with} \;\; a = \epsilon_{33}\, S / h \tag{33}$$

a result also found by Guyomar *et. al.* (2005).

### *3.3 Power flow relations*

Our main goal in this paper is to investigate the possibility of extracting power from the incoming sea waves and deliver it to an external circuit, through the intervention of a piezoelectric sheet covering the cliff. We are now focusing on the calculation of this power flow. The net (time average) power flowing through the piezoelectric sheet covering an area $h_D \times L_2$ of the cliff (see Fig. 2) is given by the equation

$$\mathrm{P}_{c\ell}^{\text{piezo}}(\omega) \;=\; L_2\,\frac{1}{T}\int_{t=0}^{t=T}\int_{z=-h_D}^{z=0} \frac{\partial \hat{u}_3(z;t)}{\partial t}\;\hat{\sigma}_3(z;t)\; dz dt \tag{34}$$

where $T = 2\pi/\omega$ is the period of the oscillating system (the same as the period of the incoming wave), and the factor $L_2$ accounts for the horizontal extent of the piezoelectric sheet. Using phasors, the net power flow is written in the form

$$\mathrm{P}_{c\ell}^{\text{piezo}} \;=\; \mathrm{P}_{c\ell}^{\text{piezo}}(\omega) \;=\; L_2\,\frac{1}{2}\mathrm{Re}_j\left\{\int_{z=h_D}^{z=0} j\omega\,\hat{u}_3(z;\omega)\;\hat{\sigma}_3^{*}(z;\omega)\; dz\right\} \tag{35}$$

where the asterisk denotes the complex conjugate. Using Eq. (32), we easily see that the elastic part of the velocity ( $j\omega\left(h/c_{33}^{D}\right)\hat{\sigma}_3(z;\,\omega)$ ) does not contribute to the power flow (as expected), which takes finally the form

$$\mathrm{P}_{c\ell}^{\text{piezo}}(\omega) \;=\; -\frac{1}{2}\,\omega\,\frac{h}{c_{33}^{D}}\;\mathrm{Im}\left\{\mathscr{E}_t(\omega)\right\}\,(h_D L_2)\,\left|\overline{\hat{\sigma}_3(\omega)}\right|^2 \tag{36}$$

Using Eq. (27) and decomposing the complex impedance $Z(\omega) = R + jX(\omega)$, it can be checked that $\mathrm{P}_{c\ell}^{\text{piezo}}(\omega) > 0 \;\Leftrightarrow\; R > 0$, the latter being always valid.

Besides, the net electric power $\mathrm{P}_Z(\omega)$ consumed by the external circuit is calculated in terms of the electric quantities $\Delta V_{\text{tot}}(\omega)$ and $I_{\text{tot}}(\omega)$. Due to the parallel electrical connection between the vertical arrays and the identical electrical quantities of each array, it holds true that $\Delta V_{\text{tot}}(\omega) = \Delta V(\omega)$ and $I_{\text{tot}} = M_2 I = L_2 I/\ell_2$, where $M_2$ is the number of vertical arrays that form the active zone. Thus, $\mathrm{P}_Z(\omega)$ is calculated as

$$\begin{aligned}\mathrm{P}_Z(\omega) &= \frac{1}{2}\mathrm{Re}_j\left\{\Delta V_{tot}(\omega) I_{tot}^{*}(\omega)\right\} = M_2\frac{1}{2}\mathrm{Re}_j\left\{\Delta V(\omega) I^{*}(\omega)\right\} = \\ &= M_2\frac{1}{2}\mathrm{Re}_j\left\{\Delta V(\omega)\frac{\Delta V^{*}(\omega)}{Z^{*}(\omega)}\right\} = M_2\frac{1}{2}\left|\frac{\Delta V^{*}(\omega)}{Z^{*}(\omega)}\right|^2 \mathrm{Re}_j\left\{Z(\omega)\right\}\end{aligned} \tag{37}$$

Using Eqs. (27) and (29), we get

$$\mathrm{P}_Z(\omega) \;=\; M_2\,\frac{1}{2}\,\omega^2\,C_0^2\,\frac{h^2}{\epsilon_{33}^2}\left|\mathscr{E}_t(\omega)\right|^2\,\left|\overline{\hat{\sigma}_3(\omega)}\right|^2 R \tag{38}$$

Calculating $\mathrm{Im}\left\{\mathscr{E}_t(\omega)\right\}$ and $\left|\mathscr{E}_t(\omega)\right|^2$, and recalling the definitions of the quantities $C_0$ and $k_t^2$ (see Eqs. (26) and (20)), we can show that $\mathrm{P}_{c\ell}^{\text{piezo}}(\omega) = \mathrm{P}_Z(\omega)$. Thus, the whole net power flowing through the piezoelectric sheet is delivered at the external circuit. This is a statement of the conservation of energy, since we have neglected the dissipation within the piezoelements.

## 4. The hydrodynamic problem

### *4.1 Mathematical formulation of the hydrodynamic boundary-value problem*

The 2D liquid domain $\Omega$ extends from the seabed $\partial\Omega_{\Pi}(z=-h_D)$ up to the free surface $\partial\Omega_F\left(z=\eta(x;t)\right)$, and from the vertical cliff $\partial\Omega_{c\ell}$ up to infinity $\partial\Omega_{\infty}$; see Fig.1. Two hydrodynamic fields are involved in the problem: the velocity potential field $\Phi^f\left(x,z;t\right)$, and the pressure field $p(x, z; t)$. Both fields are independent from the $y$ coordinate, in accordance with the 2D character of the problem, as discussed in subsection 2.2. In the context of linear water-wave theory, the domain of definition of velocity potential $\Phi^f(x, z; t)$ is restricted to the fixed domain $\Omega_0 = \{-h_D < z < 0,\ 0 < x < +\infty\}$, i.e., to a half strip with plane boundaries.

The complete, linearized, boundary-value problem for the total wave potential $\Phi^f(x, z; \omega)$, in the frequency domain, is formulated as follows (see, e.g., Wehausen and Laitone (1960), Sec. 11, Stoker (1957), Sec. 3.1, or Mei *et al*. (2005), Sec. 1.4)

$$\Delta \Phi^f(x, z; \omega) = 0, \quad \text{in } \Omega_0 \tag{39}$$

$$\frac{\partial \Phi^f}{\partial z}(x, z=0; \omega) - \mu_0 \Phi^f(x, z=0; \omega) = 0, \qquad \mu_0 = \omega^2 / g, \qquad (\text{on } \partial\Omega_{F_0}) \tag{40}$$

$$\frac{\partial \Phi^f}{\partial z}(x, z=-h_D; \omega) = 0, \qquad (\text{on } \partial\Omega_{\Pi}) \tag{41}$$

$$\frac{\partial \Phi^f}{\partial x}(x=0, z; \omega) = j\omega \hat{u}_3(z; \omega), \qquad (\text{on } \partial\Omega_{c\ell}) \tag{42}$$

$$p(x=0, z; \omega) = -\hat{\sigma}_3(z; \omega), \qquad (\text{on } \partial\Omega_{c\ell}) \tag{43}$$

$$\Phi^f \to \Phi_I^f + \Phi_R^f, \qquad \text{when} \qquad x \to +\infty,\ (i.e., \text{ at } \partial\Omega_\infty) \tag{44}$$

where $\Phi_I^f = \Phi_I^f(x, z; \omega)$ is the incident wave, and $\Phi_R^f = \Phi_R^f(x, z; \omega)$ is the reflected wave, already prescribed by Eqs. (1(b) and (c)). The pressure $p(x, z; \omega)$ is given by the linearized Bernoulli's law, Eq. (2).

Conditions (42) and (43) are matching conditions, ensuring the continuity of normal velocity and normal pressure through the fluid-solid interface $\partial\Omega_{c\ell}$, respectively. These conditions match the hydrodynamic quantities $\Phi^f$ and $p$ with the elastodynamic quantities $\hat{u}_3$ and $\hat{\sigma}_3$ and, through them, with the piezo-electric problem.

### *4.2 Modal representation of the wave potential*

The solution of the coupled problem is greatly facilitated by means of the following modal representation of the wave potential:

**Modal Representation Theorem of the Wave Potential:** Every function $\Phi^f$ defined in the half strip $\Omega_0$, satisfying the Laplace Eq. (39) therein, the free-surface boundary condition (40) on $\partial\Omega_{F_0}(z=0)$ , the seabed boundary condition (41) on $\partial\Omega_{\Pi}(z=-h_D)$ , and the condition $\left|\Phi^f(x,z;\omega)\right| \le M = const.,$ , in $\Omega_0$, admits of the following representation

$$\Phi^f(x,z;\omega) = \frac{j\,g\,H}{2\omega} Z_0(z)\exp\left(j\,k_0\,x\right) + W\,\frac{j\,g\,H}{2\omega} Z_0(z)\exp\left(-j\,k_0\,x\right) + \Phi^f_{\ell oc}(x,z;\omega) \qquad (45)$$

where the first term in the right-hand side of the above equation represents the incident wave, the second term represents the reflected wave, and the third term, $\Phi^f_{\ell oc}(x,z;\omega)$, represents a local wave field, vanishing exponentially far from the cliff, which can be expanded in the form of an infinite series of evanescent modes, as follows

$$\Phi^f_{\ell oc}(x,z;\omega) = \sum_{n=1}^{\infty} \Phi^f_n(x,z;\omega) = \sum_{n=1}^{\infty} C_n Z_n(z)\exp(-k_n x) \qquad (46)$$

$Z_0(z)$, $Z_n(z)$, $n = 1, 2, 3, \ldots$, are the vertical eigenfunctions of the water-wave problem, given by the equations

$$Z_0(z) = \frac{\cosh\left[k_0\left(h_D+z\right)\right]}{\cosh\left(k_0\,h_D\right)}, \qquad Z_n(z) = \frac{\cos\left[k_n\left(h_D+z\right)\right]}{\cos\left(k_n\,h_D\right)}, \quad n = 1, 2, \ldots. \qquad (47a,b)$$

The constants $k_0$ and $k_n$, $n = 1, 2, 3, \ldots$, appearing in the above equations are the positive roots of the dispersion relation

$$\frac{\mu_0}{k_0} = \tanh\left(k_0\,h_D\right), \qquad \frac{\mu_0}{k_n} = -\tan\left(k_n\,h_D\right) \qquad (48a,b)$$

where $h_D$ is the (constant) sea depth.

The coefficients $W, C_n$, $n = 1, 2, 3, \ldots$, are free; they can be determined by means of the boundary (matching) conditions imposed on the vertical boundary surfaces $\partial\Omega_{c\ell}, \partial\Omega_{\infty}$. The above representation theorem traces back to Kreisel (1949). It is also discussed by Wehausen & Laitone (1960), Sec. 17 and Mei (2005), Sec. 8.4.1, and it has been extensively used in the study of various water wave problems over a locally varying bathymetry (see, e.g., Bai and Yeung (1974), Lenoir and Tounsi (1988), Athanassoulis and Belibassakis (1999), Belibassakis and Athanassoulis (2005)).

Using Eqs. (45) and (46) we easily obtain representations of the horizontal wave velocity $\hat{\Phi}^f_{,x}(z;\omega) = \partial\Phi^f(x=0,z;\omega)/\partial x$ and the pressure $\hat{p}(z;\omega) = p(x=0,z;\omega)$ on the fluid-solid interface $\partial\Omega_{c\ell}$ , in terms of the unknown coefficients $W, C_n$ , $n = 1, 2, 3, \ldots$ . These representations will be exploited in the next section in order to solve the coupled problem.

### *4.3 Power flow relations*

The net (time average) power flowing towards the cliff through a vertical section at any position $x = a$ within the liquid domain (having horizontal extent $L_2$, normally to the wave front) is given by the equation

$$\mathrm{P}_a^f \;=\; -L_2 \frac{1}{T} \int_{t=0}^{t=T} \int_{z=-h_D}^{z=0} p(x=a,z;t)\; \Phi_{,x}^f(x=a,z;t)\; dzdt \tag{49}$$

where $\Phi_{,x}^f(x=a,z;t) = \partial\Phi^f(x=a,z;t)/\partial x$ and $T = 2\pi/\omega$ is the period of the wave. Passing to phasors, Eq. (49) takes the form

$$\mathrm{P}_a^f(\omega) \;=\; L_2 \frac{1}{2} \operatorname{Re}_j \left\{ j\omega\,\rho_f \int_{-h_D}^{0} \Phi^f(x=a,z;\omega) \left(\Phi_{,x}^f(x=a,z;\omega)\right)^* dz \right\} \tag{50}$$

As expected from energy considerations (and it can be proved by using Green's Theorem) the above quantity is independent from the position $x = a$ of the considered section. Accordingly, the easiest way to calculate $\mathrm{P}_a^f(\omega)$ (in terms of hydrodynamic quantities) is by letting $a \to \infty$ and using Eq. (45) keeping only the first two (non-evanescent) modes. After straightforward calculations, we obtain

$$\mathrm{P}_a^f(\omega) \;=\; \frac{1}{8}\,\rho_f\, g\, H^2 L_2\, \omega \frac{k_0}{\mu_0} \left(1 - |W|^2\right) \| Z_0 \|^2 \tag{51}$$

On the other hand, if we apply Eq. (49) to $x = 0$, and take into account the matching conditions (42) and (43), we readily see that $\mathrm{P}_{a=0}^f(\omega)$ is exactly the power flowing through the fluid-solid interface $\partial\Omega_{c\ell}$ towards the piezoelements, which, finally, is consumed by the external circuit; see Eq. (38). The equations

$$\mathrm{P}_\infty^f(\omega) \;=\; \mathrm{P}_a^f(\omega) \;=\; \mathrm{P}_{a=0}^f(\omega) \;=\; \mathrm{P}_{c\ell}^{\text{piezo}}(\omega) \;=\; \mathrm{P}_Z(\omega) \tag{52}$$

express the conservation of energy under the idealized conditions that the dissipation during the propagation of the sea waves as well as the dissipation in the piezoelements are negligible.

## 5. Solution of the coupled problem

The dynamical coupling between the piezoelectric and hydrodynamic problem is realized by means of the matching conditions (42) and (43). Combining these two equations with Eq. (32), we obtain the following condition on the fluid-solid interface $\partial\Omega_{c\ell}$

$$\hat{\Phi}^{f}_{,x}(z;\omega) + j\omega \frac{h}{c^{D}_{33}} \hat{p}(z;\omega) + j\omega \frac{h}{c^{D}_{33}} \frac{\mathscr{E}_{t}(\omega)}{h_D} \int_{-h_D}^{0} \hat{p}(z;\omega) dz = 0 \tag{53}$$

This is a non-local (because of the last term) condition connecting the hydrodynamic fields $\Phi^{f}\left(x, z; \omega\right)$ and $p\left(x, z; \omega\right)$ at $x = 0$.

### *5.1 Formulation of infinite system of equations with respect to the modal coefficients*

The modal expansion, given by Eqs. (45) and (46), permits us to obtain modal expansions for the functions $\Phi^{f}_{,x}(x=0,z;t) = \partial\Phi^{f}(x=0,z;t)/\partial x$ and $\hat{p}\left(z; \omega\right) = p\left(x=0, z; \omega\right)$, in terms of the expansion coefficients $W$, $C_n$, $n = 1,2,3,\ldots$. Substituting these modal expansions into Eq. (53), and performing the appropriate algebraic manipulations, we finally obtain

$$\begin{aligned} &\left\{a_0 Z_0(z) + \beta_0 Z_0(z) + \gamma_0 \mathscr{T}_0\right\} W + \sum_{n=1}^{\infty} \left\{a_n Z_n(z) + \beta Z_n(z) + \gamma \mathscr{T}_n \right\} = \\ &= a_0 Z_0(z) + \beta_0 Z_0(z) + \gamma_0 \mathscr{T}_0, \quad -h_D \le z \le 0 \end{aligned} \tag{54}$$

where $\alpha_0 = \dfrac{g k_0}{\omega} \dfrac{H}{2}$, $\quad \alpha_n = -k_n$, $\quad \beta_0 = j\omega \rho_f g \dfrac{h}{c^{D}_{33}} \dfrac{H}{2}$, $\quad \beta = \rho_f \omega^2 \dfrac{h}{c^{D}_{33}}$ (55)

and $\gamma_0 = j\omega \rho_f g \dfrac{H}{2}, \quad \dfrac{h}{c^{D}_{33}} \dfrac{\mathscr{E}_t(\omega)}{h_D}, \quad \gamma = \rho_f \omega^2 \dfrac{h}{c^{D}_{33}} \dfrac{\mathscr{E}_t(\omega)}{h_D}, \quad \mathscr{T}_n = \int_{-h_D}^{0} Z_n(z) dz$ (56)

Recall now that the vertical eigenfunctions $Z_0\left(z\right)$, $Z_n\left(z\right)$, $n = 1, 2, \ldots$, as defined by Eq. (47 (a) and (b)), constitute an orthogonal system of functions, complete in the Hilbert space $L^2\left(-h_D, 0\right)$. Accordingly, by projecting both members of Eq. (54) on each one of the basis functions $Z_0\left(z\right)$, $Z_n\left(z\right)$, $n = 1, 2, \ldots$, we obtain the following infinite system of equations with respect to the unknown coefficients $W$, $C_n$

$$\left\{\mathrm{K}^{+}_{00} + \gamma_0 \Lambda_{00}\right\} W + \sum_{n=1}^{\infty} \gamma \Lambda_{n0} C_n = \mathrm{K}^{-}_{00} - \gamma_{00} \Lambda_{00} \tag{57a}$$

$$\gamma_0 \Lambda_{0m} W + \sum_{n=1}^{\infty} \left\{\mathrm{K}_{nn} \delta_{nm} + \gamma \Lambda_{nm}\right\} C_n = -\lambda_0 \Lambda_{0m}, \quad m = 1,2.3,\ldots, \tag{57b}$$

where

$$\mathrm{K}_{00}^{+} = \left(\alpha_0 + \beta_0\right) \left\| Z_0 \right\|^2, \quad \mathrm{K}_{00}^{-} = \left(\alpha_0 - \beta_0\right) \left\| Z_0 \right\|^2, \quad \Lambda_{n0} = \Lambda_{0n} = \mathscr{T}_n \mathscr{T}_0 \tag{58}$$

$$\mathrm{K}_{nn} = \left(\alpha_n + \beta\right) \left\| Z_n \right\|^2, \quad \Lambda_{nm} = \Lambda_{mn} = \mathscr{T}_n \mathscr{T}_m, \tag{59}$$

and $\left\|Z_n\right\|^2 = \int_{-h_D}^{0} Z_n^2(z)\, dz$ is the square of the norm of $Z_n(z)$ in the space $L^2\left(-h_D, 0\right)$.

Using Eq. (57(a)) we eliminate $W$ from Eq. (57(b)), which then take the form

$$\sum_{n=1}^{\infty} \left\{ \mathrm{K}_{nn} \delta_{nm} + \gamma \Lambda_{nm} - \frac{\gamma \gamma_0 \Lambda_{00} \Lambda_{nm}}{\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}} \right\} C_n = \gamma_0 \Lambda_{0m} \frac{\mathrm{K}_{00}^{+} + \mathrm{K}_{00}^{-}}{\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}}, \quad m = 1, 2, \ldots \tag{60}$$

For our purposes, the most important coefficient is the reflection coefficient $W$, which is expressed, in terms of $C_n$, by means of the equation (obtained from (57(a))

$$W = -\sum_{n=1}^{\infty} \frac{\gamma \Lambda_{n0}}{\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}} C_n + \frac{\mathrm{K}_{00}^{-} - \gamma_0 \Lambda_{00}}{\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}} \tag{61}$$

Since the coefficients $\gamma \Lambda_{n0} / \left(\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}\right)$, multiplying $C_n$ in Eq. (61), are about four orders of magnitude smaller than the $C_n$ – independent term $\left(\mathrm{K}_{00}^{-} - \gamma_0 \Lambda_{00}\right) / \left(\mathrm{K}_{00}^{+} + \gamma_0 \Lambda_{00}\right)$, it is expected that the effect of the $C_n$ – dependent terms on $W$ should be small. This has been definitely verified by means of detailed numerical calculations, shown that the effect of the $C_n$ – dependent series on the values of $W$ is less than 0.1%. Thus, it is safe to proceed with our analysis by keeping only the second ($C_n$ – independent) term in Eq. (61). This approximation is compatible with the long-wave theory for water waves.

### *5.2 Closed-form solution for the reflection coefficient*

Under the (numerically confirmed) simplification that the $C_n$ coefficients do not practically affect the reflection coefficient $W$, the second term of the right-hand side of Eq. (61) provides us with a closed-form solution for $W$. Recalling the definitions (58) and (59) of the quantities $\mathrm{K}_{00}^{+}$, $\Lambda_{n0}$, $\Lambda_{00}$, in conjunction with Eqs. (55) and (56) defining $\alpha_0$, $\beta_0$, $\gamma_0$, the closed-form expression for the reflection coefficient $W$ can be written in the form

$$W = \frac{1 - j\,\mathscr{H}\,\dfrac{h}{c_{33}^D}\,(1 + \mathscr{Y}\,\mathscr{E}_t(\omega))}{1 + j\,\mathscr{H}\,\dfrac{h}{c_{33}^D}\,(1 + \mathscr{Y}\,\mathscr{E}_t(\omega))} \tag{62}$$

where $\mathscr{H} = \rho_f \, g \tanh(k_0 \, h_D)\,, \qquad \mathscr{Y} = (\mathscr{T}_0\,\mathscr{T}_0)\,/\,(h_D \, \|Z_0\|^2)$ (63a,b)

are two purely hydrodynamic, real-valued (positive), quantities. $\mathscr{H}$ is analogous to the specific weight of sea water (also affected by the sea depth), while $\mathscr{Y}$ represents the effect of the vertical structure of the hydrodynamic pressure. Note that the numerical values of these two hydrodynamic quantities satisfy the following estimates

$$\mathscr{H} \sim O(\rho_f\, g) = O(10^4 \, Pa\,/\,m) \qquad \text{and} \quad 0 < \mathscr{Y} \le 1 \tag{64a,b}$$

Inequality $\mathscr{Y} \le 1$ is obtained by applying the Cauchy-Schwartz inequality to the functions $Z_0(z)$ and 1, $z \in [-h_D, 0]$.

As is seen from Eq. (62), the reflection coefficient $W$ is dependent only on the following (dimensionless) coefficients

$$\varpi = \mathscr{H}\frac{h}{c_{33}^D} > 0 \qquad \text{and} \qquad \lambda = \mathscr{Y}\,\mathscr{E}_t(\omega) \in \mathbb{C} \tag{65a,b}$$

which realize the energetic coupling between the three subsystems (hydrodynamic, piezoelectric and external circuit). We shall call these coefficients hydro/piezo/electric compliances. From the definition of quantity $\mathscr{E}_t(\omega)$, Eq. (27), we obtain

$$\mathscr{E}_t(\omega) = \frac{k_t^2\,\Pi}{\Pi^2 + \chi^2} - j\,\frac{k_t^2\,\chi}{\Pi^2 + \chi^2} \tag{66}$$

where $\Pi \sim 1 - k_t^2 - \omega(C_0\,/\,M_1)\; X(\omega)$ and $\chi = \omega(C_0\,/\,M_1)\; R > 0$

Furthermore, since the external inductance $X(\omega)$ is expected to take small values, we can assume that $\Pi \approx 1 - k_t^2 > 0$. Taking into account Eqs. (64(b)) and (65), and keeping track of the dependence on $\chi$, the real and the imaginary parts of $\lambda$ are expressed as follows

$$\lambda_R(\chi) = \mathscr{Y}\,\mathrm{Re}_j\{\mathscr{E}_t\} = \sigma\,\frac{\Pi}{\Pi^2 + \chi^2}\,, \quad \lambda_J(\chi) = \mathscr{Y}\,\mathrm{Im}_j\{\mathscr{E}_t\} = -\sigma\,\frac{\chi}{\Pi^2 + \chi^2} \tag{68a,b}$$

where $\sigma \equiv \mathscr{Y} k_t^2$, is a partial hydro/piezo/electric compliance. Using the notation introduced above, we can write $|W|^2$ in the form

$$|W|^2(\chi) = \frac{1 + 2\varpi\lambda_J(\chi) + \varpi^2[1+\lambda_R(\chi)]^2 + \varpi^2\lambda_J^2(\chi)}{1 - 2\varpi\lambda_J(\chi) + \varpi^2[1+\lambda_R(\chi)]^2 + \varpi^2\lambda_J^2(\chi)} \equiv \frac{F(\varpi,\lambda_R(\chi),\lambda_J(\chi))}{G(\varpi,\lambda_R(\chi),\lambda_J(\chi))}. \tag{69}$$

## 6. Optimization and efficiency of the hydro/piezo/electric harvester

Combining Eqs. (51), (52) with Eq. (69), we readily see that the ratio of the total power taken off the impinging waves over the incident wave power, that is, the efficiency of the hydro/piezo/electric harvester described in Sec. 2, can be expressed as

$$\mathrm{P}^f_{a=0}(\omega,\chi)/\mathrm{P}^f_I(\omega) = 1 - |W(\chi)|^2 \tag{70}$$

where $\mathrm{P}^f_I(\omega) = \frac{1}{2}\rho_f\, g \left(\frac{H}{2}\right)^2 L_2\, \omega \frac{k_0}{\mu_0} \|Z_0\|^2$ is the incident wave power. Thus, it is clear that the coupling phenomenon between the hydrodynamic wave field, the piezoelectrically vibrating elements and the external electric circuit is solely modeled by $1 - |W(\chi)|^2$. Since in the variable $\chi = \omega\left(C_0/M_1\right)R$, the easily adjustable ohmic resistance $R$ of the external circuit is involved, it is expedient to maximize $1 - |W(\chi)|^2$ (equivalently, the taken-off power) with regard to $\chi$, following the common practice in piezoelectric harvesters (Guyomar *et al.* 2005, Lefeuvre *et al.* 2010). Using the first derivative test, we have to solve the equation

$$d\left[1-|W|^2(\chi)\right]/d\chi = 0 \iff \left\{\frac{dF}{d\chi}G - F\frac{dG}{d\chi}\right\} = 0 \tag{71}$$

After some algebraic manipulations, we find that Eq. (71) reduces to

$$\left(\chi^2 + \Pi^2\right)^2\left(\chi^2 - \Pi^2\left(1+\mu^2\right)\right) = 0 \tag{72}$$

where

$$\mu^2 \equiv \mu^2\left(\frac{\sigma}{\Pi}, \varpi\right) = \frac{\sigma}{\Pi}\left(\frac{\sigma}{\Pi} + 2\right)\frac{\varpi^2}{1+\varpi^2} \tag{73}$$

That is, Eq. (71) has the double negative root $\chi^2_{1,2} = -\Pi^2$, which is of no importance for our purposes, and the positive root $\chi_3^2 = \Pi^2\left(1+\mu^2\right)$, where

$$\frac{\sigma}{\Pi} = \frac{\mathscr{Y}\, k_t^2}{1-k_t^2-\omega\left(C_0/M_1\right)X(\omega)} \approx \frac{\mathscr{Y}\, k_t^2}{1-k_t^2} \tag{74}$$

[The second (simplified) form of $\sigma/\Pi$ is valid since the reactance $X(\omega)$ is expected to be much smaller than $(1-k_t^2)/\omega(C_0/M_1) \sim O(10^{12}\Omega)$ .]

Thus, the value $\chi = \chi_{\mathrm{opt}} = \omega\left(C_0/M_1\right)R_{\mathrm{opt}} > 0$ which maximizes the taken-off power is given by the formula $\chi_{\mathrm{opt}} = \Pi\sqrt{1+\mu^2}$, which leads to the following optimal external ohmic resistance value $R_{\mathrm{opt}}$

$$R_{opt} = \left(\frac{1-k_t^2}{\omega(C_0/M_1)} - X(\omega)\right)\sqrt{1+\mu^2} \approx \frac{1-k_t^2}{\omega(C_0/M_1)}\sqrt{1+\mu^2} \tag{75}$$

Introducing $\chi_{\mathrm{opt}}$ in Eq. (69), the following form for the electrically optimized efficiency is obtained

$$1-\left|W\right|_{opt}^2 = \mathscr{W}\left(\frac{\sigma}{\Pi},\varpi\right) =$$

$$= \frac{4\omega\dfrac{\sigma}{\Pi}\sqrt{1+\mu^2\left(\sigma/\Pi,\varpi\right)}\Big/\left(2+\mu^2\left(\sigma/\Pi,\varpi\right)\right)}{1+2\varpi\dfrac{\sigma}{\Pi}\dfrac{\sqrt{1+\mu^2\left(\sigma/\Pi,\varpi\right)}}{2+\mu^2\left(\sigma/\Pi,\varpi\right)} + \dfrac{\varpi^2}{2+\mu^2\left(\sigma/\Pi,\varpi\right)}\left[2+\mu^2\left(\sigma/\Pi,\varpi\right)+2\dfrac{\sigma}{\Pi}+(\dfrac{\sigma}{\Pi})^2\right]} \tag{76}$$

It should be stressed that the optimum value $1-\left|W\right|_{\mathrm{opt}}^2$ is dependent only on the two dimensionless, positive-valued quantities $\varpi$ and $\sigma/\Pi$, which appropriately combine the hydrodynamic, the piezoelectric and the circuit characteristics affecting the energetic coupling of the system. Furthermore, taking into account the definitions of $\sigma$ and $\Pi$, and the facts that $\mathscr{Y} \in (0,1]$ and (for many interesting materials) $k_t^2 \in (0.01, 0.5)$, we easily find that $\sigma/\Pi$ ranges (for all realistic situations) from 0 to (approximately) 1.0.

The quantity $1-\left|W\right|_{\mathrm{opt}}^2$ as a function of the two arguments $\varpi$ and $\sigma/\Pi$ is shown in Fig. 5. By this figure, it is seen that, for every value of $\sigma/\Pi$, the efficiency of the system is maximized for values of $\varpi \sim O(10^0)$, and that the system absorbs appreciable energy in the range $O(10^{-1}) < \varpi < O(10^1)$.

The dependence of the efficiency $1-\left|W\right|_{\mathrm{opt}}^2$ on $\sigma/\Pi$ is monotonically increasing; the higher the value $\sigma/\Pi$ the better the efficiency is. Since $\varpi \equiv \mathscr{H}\, h/c_{33}^D$ and $\mathscr{H} \sim O(10^4\, Pa/m)$, it is concluded that the piezoelectric material needed for an efficient harvester would be characterized by $h / c_3^D \sim O(10^{-4} m / Pa)$, having also $k_t^2$ as higher as possible in order that the parameter $\sigma / \Pi$ has a relatively high value.

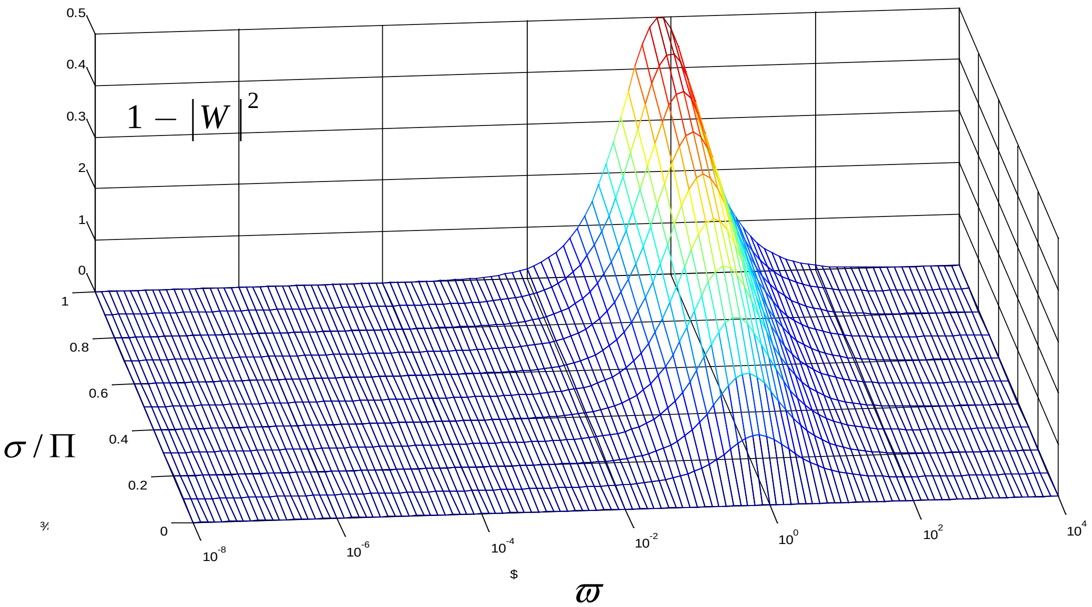


Fig. 5 The efficiency $1 - |W|^2_{\text{opt}}$ of the hydro/piezo/electric harvester, as a function of the two dimensionless quantities $\varpi$ and $\sigma / \Pi$

To get a first idea concerning the feasibility of the above requirements in relation with existing materials, we have compiled Table 1, showing the corresponding properties of some piezoelectric materials. From this Table we see that materials do not meet the flexibility requirement for an efficient harvester. An improvement of the flexibility coefficient $h / c_{33}^D$ by (approximately) three orders of magnitude is necessary in order that the piezoelectric sheet absorbs enough energy from the impinging waves.

Table 1 Piezoelectric properties of some common materials, assuming *h* = 0.1 m

| | PZT ceramics | PVDF polymers | 1-3 ceramic(PZT)-polymer composites | Cellular polypropylenes | Silicone dow corning HS3 (DEAP) |
|---|---|---|---|---|---|
| $h/c_{33}^D$ (m/Pa) | $10^{-13} - 10^{12}$ [i] | (1 – 5), $10^{-11}$ [ii), iii)] | $10^{-12} - 10^{-11}$ [vi)] | $O\,(5 \times 10^{-8})$ [ii)] | $O\,(8 \times 10^{-7})$ [vii)] |
| $k_t^2$ | 0.22 – 0.40 [i)] | 0.012 – 0.023 [iv), v)] | 0.25 – 0.42 [vi)] | $O\,(3.6 \times 10^{-3})$ [iv)] | 0.65 [vii)] |

[i)] Sherman and Butler (2007), Appendix A.5, [ii)] Bauer and Bauer (2008), Table 6.1, [iii)] Bloomfield (1994), Table 1, [iv)] Döring *et al.* (2008), Table 2,[v)] Splitt (1996), Table 1, [vi)] Smith & Auld (1991), Figs. 3 and 4., [vii)]Carpi *et. al.* (2008), Ch. 4, Table 4.1

## 7. Conclusions

In the present work, a sea-wave energy absorption system using, as energy harvester, an active zone of thickness-oscillating piezoelements installed on a vertical cliff is studied. The considered active zone is formed by parallel-connected vertical arrays, each one consisting of piezoelements connected in series. The active zone is then connected to an external AC electric circuit modeling the consumer load. The analysis of the system performed was restricted to the linear theory for both piezoelectric and hydrodynamic subproblems, and has led to a closed form efficiency coefficient, optimized with respect to the *external resistive load*. The main conclusions drawn from the obtained solution and its numerical study can be summarized as follows:

- There are two dimensionless parameters governing the efficiency of the system, namely $\sigma / \Pi$ and $\varpi$. Each of these dimensionless parameters is the product of two factors, one of piezoelectric and one of hydrodynamic nature.
- System's efficiency $\mathscr{W}$ is strongly affected by the value of parameter $\varpi$. In fact, $\mathscr{W}$ exhibits a resonance pattern around the value of $\varpi = 1$.
- System's efficiency $\mathscr{W}$ is mildly dependent on the parameter $\sigma / \Pi$, exhibiting a monotonically increasing behavior.
- The optimal resistive load takes a large value since $R_{opt} \propto 1/C \sim O(10^{12}\Omega)$. Similarly large values of optimal resistance have been obtained by Guyomar *et. al.* (2005).

Evaluating the feasibility of the studied system, we state the following:

- The elastic flexibility $h / c_{33}^{D}$ of the common piezoelectric materials (*see* Table 1) is not large enough for parameter $\varpi$ to reach the resonant value. Clearly, it is a question towards the material scientists if the advances in material manufacturing could lead to electroactive materials exhibiting large flexibility and appreciable coupling factor.
- Dielectric Electroactive Polymers (DEAP), could offer a solution to this problem. In cases of using such materials the modeling of the whole system should be adapted to the physics of DEAP.

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## Appendix. Nomenclature

*Latin Symbols*

| | |
|---|---|
| $A(\omega)$ | coeff. in the piezoelectric solution defined by Eq. (28) |
| $a$ | piezoelectric force factor of one piezoelement |
| $C_0$ | clamped capacitance of one piezoelement |
| $C_n$ | Coeffs. of evanescent sea waves |
| $c_{ij}^E$ | elastic stiffness coeffs under constant electric intensity |
| $c_{ij}^D$ | elastic stiffness coeffs under constant electric displacement |
| $D_i(x_3;\omega)$ | electric displacement components |
| $E_i(x_3;\omega)$ | electric intensity components |
| $e_i(x_3;\omega)$ | mechanical strain components |
| $\mathscr{E}_t(\omega)$ | generalized energy conversion factor; see Eq. (27) |
| $g$ | acceleration due to gravity |
| $H/2$ | incident wave amplitude |
| $h$ | thickness of one piezoelement |
| $h_D$ | sea depth in front of the vertical cliff |
| $\mathscr{H}$ | hydrodynamic coefficient; see Eq. (63a) |
| $I(\omega)$ | electric current |
| $\mathscr{T}_n$ | integrals of $Z_n(z)$'s over $h_D$ |
| $j$ | the imaginary unit |
| $k_n$ | eigenvalues of the water wave problem (wavenumbers) |
| $k_t^2$ | piezoelectric energy conversion (or coupling) factor |
| $L_2$ | length of the active zone |
| $\ell_1,\ \ell_2$ | transverse dimensions of one piezoelement |
| $M_1$ | number of piezoelements in the vertical direction |
| $M_2$ | number of piezoelements in the lateral direction |
| $\vec{n}$ | outward normal unit vector on the sea volume boundaries |
| $\mathrm{P}_\infty^f(\omega)$ | net sea wave power flow at a liquid section away from the vertical cliff |
| $\mathrm{P}_a^f(\omega)$ | net sea wave power flow at the liquid section $x=a$ |
| $\mathrm{P}_{c\ell}^{\text{piezo}}(\omega)$ | net power flowing through the piezoelectric sheet |
| $\mathrm{P}_Z(\omega)$ | net electric power consumed by the external circuit |
| $p(x,z;\omega)$ | hydrodynamic pressure field in the fluid |
| $R$ | total resistance of the external AC electric circuit |
| $S$ | surface of one piezoelement |
| $T$ | period of the oscillating system |
| $t$ | time variable |
| $u_3(x_3;\omega)$ | mechanical displacement |
| $(x\,y\,z)$ | Cartesian axes for the hydrodynamic problem (global Cartesian axes) |
| $(x_1\,x_2\,x_3)$ | Cartesian axes for each piezoelement (local Cartesian axes) |
| $X(\omega)$ | total reactance of the external AC electric circuit |
| $V_0(\omega),\ V_1(\omega)$ | voltages at the clamped and the free surface of a piezoelement |
| $W$ | reflection coefficient of sea waves |
| $\mathscr{W}^{*}$ | electrically optimized efficiency of the system; see Eq. (60) |
| $\mathscr{Y}$ | hydrodynamic coefficient; see Eq. (63b) |
| $Z(\omega)$ | total impedance of the external AC electric current |
| $Z_n(z)$ | eigenfunctions of the water wave problem |
| $\lVert Z_n(z)\rVert$ | norm of $Z_n(z)$ in the space $L^2\left(-h_D,0\right)$ |

*Greek Symbols*

| | |
|---|---|
| $\alpha\beta,\ \gamma\delta$ | surfaces of one piezoelement |
| $\Delta V(\omega)$ | voltage difference between the surfaces of a piezoelement |
| $\Delta\Phi^f(x,z;\omega)$ | Laplacian of the hydrodynamic potential |
| $\epsilon_{ij}$ | piezoelectric stress coefficients |
| $\varepsilon_{ij}^S$ | dielectric permittivity coefficients under constant strain |
| $\lambda,\lambda_R,\lambda_J$ | see Eqs. (65b) and (68a,b) |
| $\mu_0=\omega^2/g$ | sea wave frequency parameter |
| $\mu^2$ | see Eq. (73) |
| $\Pi$ | see Eq. (67a) |
| $\varpi$ | see Eq. (65a) |
| $\rho_b$ | mass density of piezoelectric material |
| $\rho_f$ | mass density of sea water |
| $\sigma_i(x_3;\omega)$ | mechanical stress components |

$\sigma = \mathscr{Y} k_t^2$ partial hydro/piezo/electric compliance,

$\Phi^{e\ell}(x_3;\omega)$ electric potential inside each piezoelement

$\Phi^f(x, z;\omega)$ hydrodynamic velocity potential field

$\Phi_I^f(x, z;\omega)$ velocity potential of the incident wave

$\Phi_R^f(x, z;\omega)$ velocity potential of the reflected wave

$\Phi_n^f(x, z;\omega)$ velocity potential of the evanescent waves

$\chi$ see Eq. (67a)

$\Omega, \Omega_0$ sea volume; see Fig. 1

$\partial\Omega_{c\ell}, \partial\Omega_{F_0}, \partial\Omega_{\Pi}, \partial\Omega_{\infty}$ boundaries of the sea volume; see Fig. 1

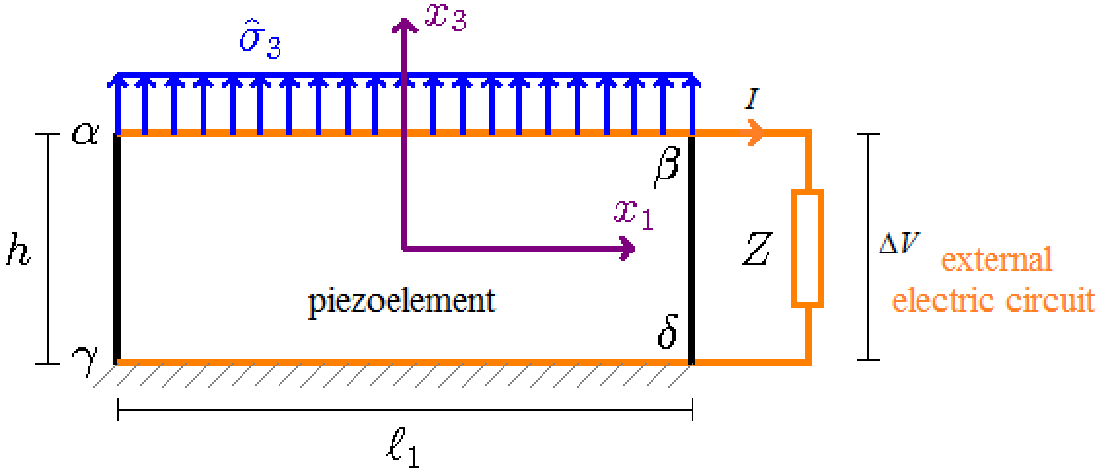
$\omega$ frequency of the oscillating system

$\tilde{\omega}$ nondimensionalized frequency; Eq. (13b)

# 3. Lagrangian formulation of the lumped piezoelement

## 3.1 Introduction

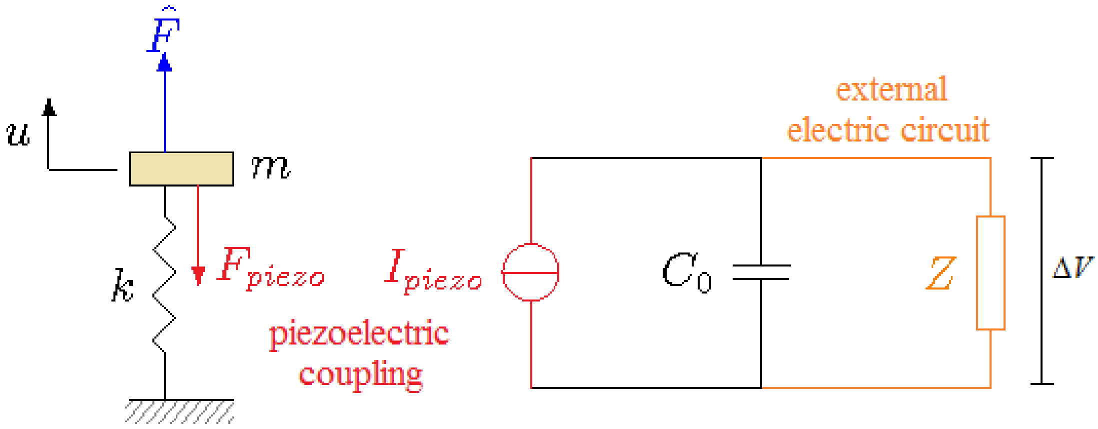
In the paper presented in the previous chapter, the expressions of mechanical displacement $u_3(x_3\,;\omega)$, voltage $\Delta V(\omega)$ difference between the electroded surfaces and electric current $I(\omega)$ where obtained for a piezoelement operating in thickness mode, having one of its electrode surfaces clamped on a non-moving, rigid base. These expressions were obtained by solving the quasi-static piezoelectric equations for thickness mode under the presence of an external electric circuit $Z$, which provided the equations with the closure condition regarding the displacement electric current flowing from piezoelement. Thus, the model describing the piezoelement is a distributed one and it is summarized in Fig. 1.




- $u_3\left(x_3 = -\,h/2;\,\omega\right) \;=\; 0$ (clamped side $\gamma\delta$)
- $u_3\left(x_3 = h/2\,;\,t\right) \;=\; \dfrac{h}{c_{33}^{D}}\,\hat{\sigma}_3(\omega) + \dfrac{h}{c_{33}^{D}}\,\mathcal{E}_t(\omega)\,\hat{\sigma}_3(\omega)$ (free side $\alpha\beta$)
- $\Delta V(\omega) \;=\; j\,\omega C_0\,Z(\omega)\,\dfrac{h}{\epsilon_{33}}\,\mathcal{E}_t\,\hat{\sigma}_3$
- $\mathcal{E}_t \;\equiv\; \dfrac{k_t^2}{\left(1-k_t^2\right)+j\,\omega\,C_0\,Z(\omega)}, \quad k_t^2 \;\equiv\; \dfrac{\epsilon_{33}^{2}}{\varepsilon_3^{S}\;c_{33}^{D}}$

Fig. 1: Distributed model of one piezoelement and the solutions obtained for the fields.

In bibliography though (e.g. Guyomar *et al.* 2005, Lefeuvre *et al.* 2010), the solutions for the fields appearing in the piezoelement are found using a lumped model consisting of a one DOF mechanical oscillator coupled with an electric circuit, as can be seen in Fig. 2.




$$m \equiv \rho_b S h, \quad k \equiv c_{33}^E S/h, \quad C_0 \equiv \varepsilon_3 S/h, \quad a \equiv \epsilon_{33} S/h$$

$$F_{piezo} = aV, \quad I_{piezo} = a\dot{u}$$

- $\hat{F} - ku - aV = m\ddot{u}$
- $a\dot{u} - C_0\dot{V} = \dfrac{V}{Z}$

Fig. 2: Lumped model of one piezoelement and the solutions obtained for the fields.

The scope thus of the present chapter is to show that the two models (distributed and lumped) are equivalent by constructing a Lagrangian description for the lumped model and manipulating the obtained solutions algebraically afterwards, in order to re-obtain the expressions of the fields as determined by the distributed model.

**3.2 Lagrangian description of the elastic subproblem**

Let us consider the classical case of a one-DOF mechanical oscillator under the excitation of an external force $\hat{F}$, as shown in Fig. 3.

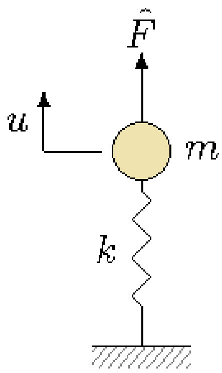



Fig. 3: Conservative one-DOF mechanical oscillator

The only independent variable considered in this problem is the mechanical displacement $u$ of mass $m$ and thus the energy forms appearing in this problem can be expressed as

$$U_{\text{kinetic}}[\dot{u}] = \frac{1}{2} m \dot{u}^2 \tag{3.1a}$$

$$U_{\text{elastic}}[u] = \frac{1}{2} k u^2 \tag{3.1b}$$

and thus the Lagrangian function of this system is

$$L_{\text{mechanical}}[t, u, \dot{u}] = U_{\text{kinetic}}[\dot{u}] - U_{\text{elastic}}[u] = \frac{1}{2} m \dot{u}^2 - \frac{1}{2} k u^2 . \tag{3.2}$$

Since on mass $m$ an external non-potential force $\hat{F}$ is exercised, the governing equation of the system can be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = \hat{F} \Rightarrow m\ddot{u} + ku = \hat{F} \Rightarrow$$
$$\Rightarrow \hat{F} - ku = m\ddot{u} \tag{3.4}$$

Eq.(3.4) is the expected **Newton's Second Law**, with the forces applied on the discrete mas $m$ being the external force $\hat{F}$ and the spring force $k u$.

**3.3 Lagrangian description of the electric subproblem**

Similarly to the classical Lagrangian description for the elastic subproblem shown in the previous paragraph, we can construct a Lagrangian formulation for the electric subproblem shown in Fig. 4.

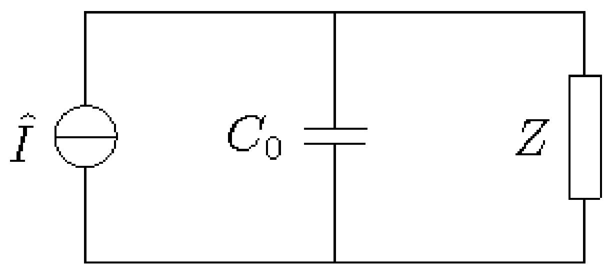



Fig. 4: The elastic subproblem

The elastic subproblem shown in Fig. 4 consists of a current source $\hat{I}$, a capacitor $C_0$ and an impedance $Z$ connected in parallel.

In accordance with Haas *et al.* (2000) and Preumont (2010), every electrical network admits a Lagrangian formulation with electric flux $\lambda$ as an independent variable. Flux linkage $\lambda$ is a physical field that has not being encountered in the previous chapters of the present work, and its first time derivative is defined using the known field of electric potential as

$$\dot{\lambda} = V \tag{3.5}$$

The choice of using flux linkage as independent variable is somewhat peculiar, but since the Lagrangian consists of the energy forms, and one element of the system is current source $\hat{I}$, it is easily proven by dimensional analysis that the product $\hat{I} \cdot \lambda$ is an energy expression, whereas the product $\hat{I} \cdot V$ is an expression of power and thus not suitable:

$$[\hat{I} \cdot \lambda] = [\hat{I}] \cdot [\lambda] = \mathrm{A} \times \mathrm{V} \times \mathrm{m} \equiv \mathrm{W} \times \mathrm{m} = \mathrm{J} \tag{3.6}$$

The energy forms appearing in electric subproblem is the energy stored in capacitor $C_0$:

$$U_{\text{capacitor}}\left[\dot{\lambda}\right] = \frac{1}{2} C_0 V^2 = \frac{1}{2} C_0 \dot{\lambda}^2 \tag{3.7}$$

and the energy of the current source $\hat{I}$:

$$U_{\text{source}}\left[\lambda\right] = \hat{I} \lambda \tag{3.8}$$

Thus the Lagrangian function of the problem reads

$$L_{\text{electrical}}\left[t, \lambda, \dot{\lambda}\right] = U_{\text{capacitor}}\left[\dot{\lambda}\right] + U_{\text{source}}\left[\lambda\right] = \frac{1}{2} C_0 \dot{\lambda}^2 + \hat{I} \lambda . \tag{3.9}$$

In addition to the above the energy dissipation element of impedance $Z$, can be modeled as an external non-potential generalized force using the following Rayleigh dissipation function (see Haas *et al.* 2000)

$$D = \frac{1}{2Z} V^2 = \frac{1}{2Z} \dot{\lambda}^2 \tag{3.10}$$

Eq.(3.8) is derived from the energy expression for the dissipative element $Z$. Under the quasi-static approach, which will be used uniformly for the coupled system, impedance $Z$ can be assumed as constant.

Thus the governing equation of the electrical subproblem is derived as follows

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}} - \frac{\partial L}{\partial \lambda} = -\frac{\partial D}{\partial \dot{\lambda}} \Rightarrow C_0 \ddot{\lambda} - \hat{I} = -\frac{\dot{\lambda}}{Z} \Rightarrow$$
$$\Rightarrow \hat{I} = C_0 \ddot{\lambda} + \frac{\dot{\lambda}}{Z} \tag{3.11}$$

Eq.(3.11) is the expected **Kirchhoff's circuit law** for the electrical network of Fig.4. Note that **Kirchhoff's voltage law** is considered to be satisfied automatically, since the admissible variations of $\lambda$ are defined in such a manner.

### 3.4 Lagrangian description of the lumped piezoelectric problem

Let us now consider a system that has as Lagrangian function the sum of the Lagrangians of the previous two subproblems (mechanical and electrical)

$$\begin{aligned} L\left[t, u, \dot{u}, \lambda, \dot{\lambda}\right] &= L_{\text{mechanical}}[t, u, \dot{u}] + L_{\text{electrical}}\left[t, \lambda, \dot{\lambda}\right] = \\ &= U_{\text{kinetic}}[\dot{u}] - U_{\text{elastic}}[u] + U_{\text{capacitor}}\left[\dot{\lambda}\right] + U_{\text{source}}[\lambda] = \\ &= \frac{1}{2} m\dot{u}^2 - \frac{1}{2} ku^2 + \frac{1}{2} C_0 \dot{\lambda}^2 + \hat{I}\lambda \end{aligned} \tag{3.12}$$

The Lagrangian of the whole piezoelectric problem as the sum of the respective constituent problems is in accordance with the familiar procedure of Part I, Ch.5.

The *introduction of the electric-elastic coupling* will be performed by the substitution of current source $\hat{I}$ with

$$\hat{I} = a\dot{u} \tag{3.13}$$

This substitution of current source $\hat{I}$ justifies why current source was treated as an element of the electrical network rather than an external generalized force I the previous paragraph. Thus Lagrangian (3.12) reads

$$\begin{aligned} L\left[t, u, \dot{u}, \lambda, \dot{\lambda}\right] &= L_{\text{mechanical}}[t, u, \dot{u}] + L_{\text{electrical}}\left[t, \lambda, \dot{\lambda}\right] = \\ &= U_{\text{kinetic}}[\dot{u}] - U_{\text{elastic}}[u] + U_{\text{capacitor}}\left[\dot{\lambda}\right] + U_{\text{source}}[u, \lambda] = \\ &= \frac{1}{2} m\dot{u}^2 - \frac{1}{2} ku^2 + \frac{1}{2} C_0 \dot{\lambda}^2 + a\dot{u}\lambda \end{aligned} \tag{3.12}$$

Thus, the Euler-Lagrange equations are the following two, since an external mechanical forcing $\hat{F}$ and an electrical dissipative element $Z$ with Rayleigh function $D$ are also present (see Fig. 2)

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} &= \hat{F} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}} - \frac{\partial L}{\partial \lambda} &= -\frac{\partial D}{\partial \dot{\lambda}} \end{aligned}\right\} \Rightarrow \qquad \left.\begin{aligned} m\ddot{u} + a\dot{\lambda} + ku &= \hat{F} \\ C_0\ddot{\lambda} - a\dot{u} &= -\frac{\dot{\lambda}}{Z} \end{aligned}\right\} \Rightarrow$$

$$[\text{using Eq.(3.5)}] \qquad \Rightarrow \qquad \left.\begin{aligned} \hat{F} - aV - ku &= m\ddot{u} \\ a\dot{u} - C_0\dot{V} &= \frac{V}{Z} \end{aligned}\right\} \tag{3.12}$$

### 3.5 Equivalence between distributed and lumped models

Eqs.(3.12) are the same equations stated by Guyomar *et al.* (2005) for governing the lumped piezoelement. Up to this point, we have not verified the equivalence between the distributed and the lumped models, we have only verified that Eqs.(3.12) govern the lumped system described in Fig.2. In this paragraph we will derive the equations of distributed model from Eqs.(3.12) under:

i) the quasi-static approximation, thus kinetic energy in the Lagrangian function is neglected and the equations are simplified to

$$\hat{F} - aV - ku = 0 \tag{3.13a}$$

and

$$a\dot{u} - C_0\dot{V} = \frac{V}{Z}, \tag{3.13b}$$

ii) the following claims for the definition relations of the elements of the lumped model with regard to material properties and geometry of the distributed model

| | |
|---|---|
| $k \equiv c_{33}^E S/h$ | (3.14a) |
| $a \equiv \epsilon_{33} S/h$ | (3.14b) |
| $C_0 \equiv \varepsilon_{33}^S S/h.$ | (3.14c) |
| where $S$ is the area of one electroded surface and $h$ is the thickness of the piezoelement | |

iii) the identification of variables $u$ and $V$ of the lumped model as the variables $u(t) = u_3(x_3 = h/2\,;\,t)$ and $\Delta V(t)$ of the distributed model respectively.

Assuming time harmonic state, Eqs.(3.13) read

$$\hat{F}(\omega) - ku(\omega) - aV(\omega) = 0 \tag{3.15a}$$

and

$$V(\omega) = \frac{j\omega a Z(\omega)}{1 + j\omega C_0 Z(\omega)}\, u(\omega) \tag{3.15b}$$

Substituting Eq.(3.15b) into Eq.(3.15a), Eq.(3.15a) can be solved regarding voltage $V(\omega)$:

$$\left(k+\frac{j\omega a^2 Z(\omega)}{1+j\omega C_0 Z(\omega)}\right)u(\omega) = \hat{F}(\omega) \Rightarrow \left[\frac{k+j\omega\left(kC_0+a^2\right)Z(\omega)}{1+j\omega C_0 Z(\omega)}\right]u(\omega) = \hat{F}(\omega) \quad (3.16)$$

Using the following lemma

**Lemma:** $k_t^2 = \dfrac{a^2}{kC_0+a^2}$

*Proof:*

$$k_t^2 = \frac{a^2}{kC_0+a^2} = \frac{\epsilon_{33}^2 S^2/h^2}{c_{33}^E \varepsilon_3^S S^2/h^2+\epsilon_{33}^2 S^2/h^2} = \frac{\epsilon_{33}^2}{\varepsilon_3^S\left(c_{33}^E+\epsilon_{33}^2/\varepsilon_3^S\right)} = \frac{\epsilon_{33}^2}{\varepsilon_3^S c_{33}^D}$$

Eq.(3.16) is written as

$$\left(k+j\omega\frac{a^2}{k_t^2}Z(\omega)\right)u = \hat{F}(\omega)\left(1+j\omega C_0 Z(\omega)\right) \Rightarrow$$

$$\left(k+j\omega\frac{a^2}{k_t^2}Z(\omega)\right)\frac{\cancel{1+j\omega C_0 Z(\omega)}}{j\omega a Z(\omega)}V(\omega) = \hat{F}(\omega)\cancel{\left(1+j\omega C_0 Z(\omega)\right)} \Rightarrow$$

$$\frac{k_t^2 k+j\omega a^2 Z(\omega)}{j\omega k_t^2 a Z(\omega)}V(\omega) = \hat{F}(\omega) \Rightarrow V = j\omega C_0 Z(\omega)\frac{k_t^2 a}{k_t^2 C_0 k+j\omega a^2 C_0 Z(\omega)}\hat{F}(\omega) \Rightarrow$$

$$V(\omega) = j\omega C_0 Z(\omega)\frac{h}{\epsilon_{33}}\frac{k_t^2}{\left(1-k_t^2\right)+j\omega C_0 Z(\omega)}\frac{\hat{F}(\omega)}{S} \quad (3.17)$$

Eq.(3.17) is the final expression for voltage difference, and coincides with the respective expression (29) of the previous chapter for the case of one piezoelement ($M_1=1$):

$$\Delta V(\omega) \equiv V(\omega) = j\omega C_0 Z(\omega)\frac{h}{\epsilon_{33}}\mathscr{E}_t\,\hat{\sigma}_3 \quad (3.18)$$

Substituting solution (3.18) into Eq.(3.15b) we obtain

$$u_3\left(x_3=h/2\,;\,\omega\right) \equiv u(\omega) = \frac{h}{c_{33}^D}\hat{\sigma}_3(\omega)+\frac{h}{c_{33}^D}\mathscr{E}_t(\omega)\,\hat{\sigma}_3(\omega) \quad (3.19)$$

which coincides with expression (30) of the previous chapter for the case of one piezoelement ($M_1=1$).

Thus the equivalence between the distributed and the lumped models under the quasi-static approximation and definition relations (3.14) for the lumped elements, has been proven.

**References**

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# Appendix A. Alternative forms of the piezoelectric constitutive relations

The purpose of the present Appendix is to derive the alternative forms of piezoelectric constitutive relations shown in par. 2.6 of Meitzler *et. al.* (1987) which is the IEEE Standard for Piezoelectricity.
Firstly all the fields and material property tensors involved are defined, and the symmetries of material property tensors are outlined. Then, starting from the standard pair of piezoelectric constitutive relations that has as independent variables the elastic strain tensor $e_{k\ell}$ and the electric intensity field $E_m$, the alternative forms of the pair of constitutive relations are derived through algebraic manipulations. With this process, the relations between material property tensors that refer to the same material property measured under different constant field (e.g. $c^E_{ijk\ell}$ and $c^D_{ijk\ell}$) are also clarified.

The fields appearing in the constitutive relations are:

$\sigma_{ij}$: stress tensor (2nd rank) field
$e_{k\ell}$: strain tensor (2nd rank) field
$D_n$: electric displacement vector field
$E_m$: electric intensity vector field

The elastic material properties included in the constitutive relations are:
$c^E_{ijk\ell}$: elastic stiffness coefficients (4th rank tensor) at constant electric field
$c^D_{ijk\ell}$: elastic stiffness coefficients (4th rank tensor) at constant electric displacement
$s^E_{ijk\ell}$: elastic compliance coefficients (4th rank tensor) at constant electric field
$s^D_{ijk\ell}$: elastic compliance coefficients (4th rank tensor) at constant electric displacement

The relation between the elastic stiffness and the elastic compliance, both measured under the same electric field kept constant is: $s_{ijk\ell} = c^{-1}_{ijk\ell}$

The symmetries of the above elastic property tensors are:
- the major symmetry: $i\,j$ can be interchanged with $k\,\ell$,
- the minor symmetries: $i$ can be interchanged with $j$ and $k$ with $\ell$.

The electric material properties in the constitutive relations are:
$\varepsilon^S_{nm}$: dielectric permittivity matrix at constant strain
$\varepsilon^T_{nm}$: dielectric permittivity matrix at constant stress
$\beta^S_{nm}$: dielectric impermittivity (inverse permittivity) matrix at constant strain
$\beta^T_{nm}$: dielectric impermittivity (inverse permittivity) matrix at constant stress

The relation between the dielectric permittivity and the dielectric impermittivity, both measured under the same elastic field kept constant is: $\beta_{nm} = \varepsilon_{nm}^{-1}$

The symmetry of the above electric properties is the interchangeability of $n$ and $m$.

The piezoelectric material properties (3rd rank) in the constitutive relations are:
$\epsilon_{mij}$: piezoelectric stress-electric displacement constants
$d_{mij}$: piezoelectric strain-electric displacement constants
$h_{mij}$: piezoelectric stress-electric intensity constants
$g_{mij}$: piezoelectric strain-electric intensity constant

Symmetries of piezoelectric constants: $i$ and $j$ interchangeability, coming from the minor symmetries of the elastic constants.

The pair of constitutive relations that have $\left(\sigma_{ij}, D_n\right)$ as dependent variables and $\left(e_{k\ell}, E_m\right)$ as independent variables is:

$$\sigma_{ij} = c^{E}_{ijk\ell}\, e_{k\ell} - \epsilon_{mij}\, E_m$$
$$D_n = \varepsilon^{S}_{nm}\, E_m + \epsilon_{nk\ell}\, e_{k\ell}$$

The pair of constitutive relations that have $\left(e_{k\ell}, D_n\right)$ as dependent variables and $\left(\sigma_{ij}, E_m\right)$ as independent variables can be derived by the previous pair as:

$$e_{k\ell} = s^{E}_{ijk\ell}\, \sigma_{ij} + s^{E}_{ijk\ell}\, \epsilon_{mij}\, E_m$$

$$D_n = \varepsilon^{S}_{nm}\, E_m + \epsilon_{nk\ell}\left(s^{E}_{ijk\ell}\, \sigma_{ij} + s^{E}_{ijk\ell}\, \epsilon_{mij}\, E_m\right) \Rightarrow$$
$$D_n = \left(\varepsilon^{S}_{nm} + \epsilon_{nk\ell}\, s^{E}_{ijk\ell}\, \epsilon_{mij}\right) E_m + s^{E}_{ijk\ell}\, \epsilon_{nk\ell}\, \sigma_{ij}$$

Defining:
$$d_{mk\ell} = s^{E}_{ijk\ell}\, \epsilon_{mij}$$
$$\varepsilon^{T}_{nm} = \varepsilon^{S}_{nm} + \epsilon_{nk\ell}\, s^{E}_{ijk\ell}\, \epsilon_{mij}$$

The constitutive relations can be written:

$$e_{k\ell} = s^{E}_{ijk\ell}\, \sigma_{ij} + d_{mk\ell}\, E_m$$
$$D_n = \varepsilon^{T}_{nm}\, E_m + d_{nij}\, \sigma_{ij}$$

The pair of constitutive relations that have $\left(\sigma_{ij}, E_m\right)$ as dependent variables and $\left(e_{k\ell}, D_n\right)$ as independent variables is:

$$E_m = \beta^S_{nm} D_n - \beta^S_{nm} \epsilon_{nk\ell} e_{k\ell}$$

$$\sigma_{ij} = c^E_{ijk\ell} e_{k\ell} - \epsilon_{mij}\left(\beta^S_{nm} D_n - \beta^S_{nm} \epsilon_{nk\ell} e_{k\ell}\right) \Rightarrow$$
$$\sigma_{ij} = \left(c^E_{ijk\ell} + \epsilon_{mij} \beta^S_{nm} \epsilon_{nk\ell}\right) e_{k\ell} - \epsilon_{mij} \beta^S_{nm} D_n$$

Defining:
$$h_{mk\ell} = \beta^S_{nm} \epsilon_{nk\ell}$$
$$c^D_{ijk\ell} = c^E_{ijk\ell} + \epsilon_{mij} \beta^S_{nm} \epsilon_{nk\ell}$$

The constitutive relations can be written

$$\sigma_{ij} = c^D_{ijk\ell} e_{k\ell} - h_{nij} D_n$$
$$E_m = \beta^S_{nm} D_n - h_{mk\ell} e_{k\ell}$$

The pair of constitutive relations that have $\left(e_{k\ell}, E_m\right)$ as dependent variables and $\left(\sigma_{ij}, D_n\right)$ as independent variables is:

$$e_{k\ell} = s^D_{ijk\ell} \sigma_{ij} + s^D_{ijk\ell} \epsilon_{mij} \beta^S_{nm} D_n$$

$$E_m = \beta^S_{nm} D_n - \beta^S_{nm} \epsilon_{nk\ell}\left(s^D_{ijk\ell} \sigma_{ij} + s^D_{ijk\ell} \epsilon_{mij} \beta^S_{nm} D_n\right) \Rightarrow$$
$$E_m = \left(\beta^S_{nm} - \beta^S_{n'm} \epsilon_{n'k\ell} s^D_{ijk\ell} \epsilon_{m'ij} \beta^S_{nm'}\right) D_n - \beta^S_{nm} \epsilon_{nk\ell} s^D_{ijk\ell} \sigma_{ij}$$

As we have denoted:
$$\varepsilon^T_{nm} = \varepsilon^S_{nm} + \epsilon_{nk\ell} s^E_{ijk\ell} \epsilon_{mij} \Rightarrow \beta^T_{nm} = \frac{1}{\varepsilon^S_{nm} + \epsilon_{nk\ell} s^E_{ijk\ell} \epsilon_{mij}} \Rightarrow$$
$$\beta^T_{nm} = \frac{1}{\dfrac{1}{\beta^S_{nm}} + \dfrac{\epsilon_{nk\ell} \epsilon_{mij}}{c^E_{ijk\ell}}} = \frac{\beta^S_{nm} c^E_{ijk\ell}}{c^E_{ijk\ell} + \beta^S_{nm} \epsilon_{nk\ell} \epsilon_{mij}}$$

Also:
$$\beta^S_{nm} - \beta^S_{n'm} \epsilon_{n'k\ell} s^D_{ijk\ell} \epsilon_{m'ij} \beta^S_{nm'} = \beta^S_{nm} - \frac{\beta^S_{n'm} \epsilon_{n'k\ell} \epsilon_{m'ij} \beta^S_{nm'}}{c^E_{ijk\ell} + \epsilon_{mij} \beta^S_{nm} \epsilon_{nk\ell}}$$
$$= \frac{\beta^S_{nm} c^E_{ijk\ell} \cancel{+ \beta^S_{n'm} \epsilon_{n'k\ell} \epsilon_{m'ij} \beta^S_{nm'}} \cancel{- \beta^S_{n'm} \epsilon_{n'k\ell} \epsilon_{m'ij} \beta^S_{nm'}}}{c^E_{ijk\ell} + \epsilon_{mij} \beta^S_{nm} \epsilon_{nk\ell}}$$

Thus: $\beta^T_{nm} = \beta^S_{nm} - \beta^S_{n'm} \epsilon_{n'k\ell} s^D_{ijk\ell} \epsilon_{m'ij} \beta^S_{nm'}$

And the constitutive relation can be written:

$$e_{k\ell} = s^{D}_{ijk\ell}\,\sigma_{ij} + s^{D}_{ijk\ell}\,\epsilon_{mij}\,\beta^{S}_{nm}\,D_n$$
$$E_m = \beta^{T}_{nm}\,D_n - \beta^{S}_{nm}\,\epsilon_{nk\ell}\,s^{D}_{ijk\ell}\,\sigma_{ij}$$

Defining:
$$g_{mij} = \beta^{S}_{nm}\,\epsilon_{nk\ell}\,s^{D}_{ijk\ell}$$

The constitutive relation can be written:

$$e_{k\ell} = s^{D}_{ijk\ell}\,\sigma_{ij} + g_{nk\ell}\,D_n$$
$$E_m = \beta^{T}_{nm}\,D_n - g_{mij}\,\sigma_{ij}$$

**Reference**

Meitzler, A.H. (chair) et. al. (1987), *IEEE Standard on Piezoelectricity,* IEEE Inc.